# On the Existence of Positive Periodic Solutions to Second Order Linear Functional Differential Equations

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For linear second order functional differential equations, the periodic boundary value problem is investigated (see, for example, [1–5]). We will find unimprovable conditions for the existence of a positive solution in two cases:

1. the Green function of the periodic problem can change its sign (Theorems 2, 3, 4, Corollary 1);
2. right-hand side functions $f$ of the equations are not necessary non-negative or non-positive (Theorems 2, 5, 6, Corollary 2).

Consider the periodic boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t) \ \text{ for almost all } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{1}$$

where $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $f \in \mathbf{L}[0,1]$, a solution $x : [0,1] \to \mathbb{R}$ has an absolutely continuous derivative, $\mathbf{C}[0,1]$ is the space of all continuous functions $x : [0,1] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{C}} = \max\limits_{t\in[0,1]} |x(t)|$, $\mathbf{L}[0,1]$ is the space of all integrable functions $z : [0,1] \to \mathbb{R}$ with the norm $\|x\|_{\mathbf{L}} = \int\limits_0^1 |z(t)|\,dt$.

**Assumption 1.** *Let non-negative functions $q$, $r \in \mathbf{L}[0,1]$ be given,*

$$p \equiv q - r,$$

$$\mathcal{P} \equiv \int\limits_0^1 p(t)\,dt \neq 0, \ \ \widetilde{p} \equiv p/\mathcal{P}.$$

*We suppose that the operator $T$ has a representation*

$$T = T^+ - T^-,$$

*where $T^+$, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ are linear bounded operators such that*

$$T^+\mathbf{1} = q, \quad T^-\mathbf{1} = r,$$

$\mathbf{1}$ *is the unit function, the operators $T^+$, $T^-$ are positive (that is, they map nonnegative functions from $\mathbf{C}[0,1]$ into almost everywhere non-negative functions from $\mathbf{L}[0,1]$).*

**Definition 1.** For every $t_1$, $t_2$ $(0 \le t_1 \le t_2 \le 1)$, define the piecewise linear function

$$g_{t_1,t_2}(s) \equiv G(t_2,s) - G(t_1,s), \ \ s \in [0,1],$$

where

$$G(t,s)=\begin{cases} t(s-1) & \text{if } 0\le t\le s\le 1,\\ s(t-1) & \text{if } 0\le s<t\le 1,\end{cases}$$

is the Green function of the Dirichlet problem $\ddot{x}(t)=z(t)$, $t\in[0,1]$, $x(0)=0$, $x(1)=0$.

For every function $z\in\mathbf{L}[0,1]$, we denote

$$g_{t_1,t_2,z}(s)\equiv g_{t_1,t_2}(s)-\int\limits_0^1 z(\tau)g_{t_1,t_2}(\tau)\,d\tau,\ \ s\in[0,1],$$

$$[z]^+(s)\equiv\frac{z(s)+|z(s)|}{2},\quad [z]^-(s)\equiv\frac{|z(s)|-z(s)}{2},\ \ s\in[0,1].$$

**Theorem 1.** *Let*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{p}}]^+(t)+r(t)[g_{t_1,t_2,\widetilde{p}}]^-(t)\Big)\,dt<1. \tag{2}$$

*Then periodic problem* (1) *has a unique solution.*

**Assumption 2.** *Suppose further that* $\int\limits_0^1 f(s)\,ds\neq 0$. *Define* $\mathcal{F}\equiv\int\limits_0^1 f(s)\,ds$, $\widetilde{f}\equiv f/\mathcal{F}$.

**Theorem 2.** *Let inequality* (2) *be fulfilled.*

*If*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{f}}]^+(t)+r(t)[g_{t_1,t_2,\widetilde{f}}]^-(t)\Big)\,dt<1 \tag{3}$$

*and*

$$\max_{0\le t_1\le t_2\le 1}\int\limits_0^1\Big(q(t)[g_{t_1,t_2,\widetilde{f}}]^-(t)+r(t)[g_{t_1,t_2,\widetilde{f}}]^+(t)\Big)\,dt<1, \tag{4}$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{F}\mathcal{P})\,x(t)>0\ \text{ for all }\ t\in[0,1]. \tag{5}$$

**Definition 2.** Let $\mu\ge 1$. Define the set

$$S_\mu\equiv\Big\{h\in\mathbf{L}[0,1]:\ \operatorname{vrai\,sup}_{t\in[0,1]}h(t)\le\mu\operatorname{vrai\,inf}_{t\in[0,1]}h(t)>0\Big\}.$$

**Theorem 3.** *Let inequality* (2) *be fulfilled,* $f\in S_\mu$.

*If*

$$\min\Big\{\operatorname{vrai\,sup}_{t\in[0,1]}q(t),\operatorname{vrai\,sup}_{t\in[0,1]}r(t)\Big\}+\\ +\mu\max\Big\{\operatorname{vrai\,sup}_{t\in[0,1]}q(t),\operatorname{vrai\,sup}_{t\in[0,1]}r(t)\Big\}\le 8(1+\sqrt{\mu})^2,$$

*and*

$$q+\mu r\not\equiv 8(1+\sqrt{\mu})^2,\quad r+\mu q\not\equiv 8(1+\sqrt{\mu})^2,$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t)>0\ \text{ for all }\ t\in[0,1].$$

**Theorem 4.** *Let inequality* (2) *be fulfilled,* $f \in S_\mu$.
*If*

$$\min\bigg\{\int\limits_0^1 q(t)\,dt, \int\limits_0^1 r(t)\,dt\bigg\} + \sqrt{\mu}\max\bigg\{\int\limits_0^1 q(t)\,dt, \int\limits_0^1 r(t)\,dt\bigg\} \le 4(1+\sqrt{\mu}),$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \ \textit{for all} \ \ t \in [0,1].$$

**Corollary 1.** *Let* $q \equiv 0$ *or* $r \equiv 0$.
*If*

$$\operatorname{vrai\,sup}_{t\in[0,1]} |p(t)| \le 32\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big)^2, \quad |p| \not\equiv 32\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big)^2,$$

*or*

$$\int\limits_0^1 |p(t)|\,dt \le 8\Big(1 - \frac{\sqrt{\mu}-1}{2\sqrt{\mu}}\Big),$$

*then for each* $f \in S_\mu$ *a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})x(t) > 0 \ \ \textit{for all} \ \ t \in [0,1].$$

**Definition 3.** Let $\rho > 1$. Define the set

$$\Lambda_\rho \equiv \bigg\{h \in \mathbf{L}[0,1]: \ h \not\equiv 0, \ \int\limits_0^1 [h]^+(t)\,dt \ge \rho \int\limits_0^1 [h]^-(t)\,dt\bigg\}.$$

**Theorem 5.** *Let inequality* (2) *be fulfilled,* $f \in \Lambda_\rho$.
*If*

$$\max\Big\{\operatorname{vrai\,sup}_{t\in[0,1]} q(t), \operatorname{vrai\,sup}_{t\in[0,1]} r(t)\Big\} \le 8\,\frac{\rho-1}{\rho+1}\,, \ \ r \not\equiv 8\,\frac{\rho-1}{\rho+1}, \ \ q \not\equiv 8\,\frac{\rho-1}{\rho+1}\,,$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \ \textit{for all} \ \ t \in [0,1].$$

**Theorem 6.** *Let inequality* (2) *be fulfilled,* $f \in \Lambda_\rho$.
*If*

$$\rho\max\bigg\{\int\limits_0^1 q(t)\,dt, \int\limits_0^1 r(t)\,dt\bigg\} - \min\bigg\{\int\limits_0^1 q(t)\,dt, \int\limits_0^1 r(t)\,dt\bigg\} \le 4\,(\rho-1),$$

*then a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})\,x(t) > 0 \ \ \textit{for all} \ \ t \in [0,1].$$

**Corollary 2.** *Let* $q \equiv 0$ *or* $r \equiv 0$.
*If*

$$\operatorname{vrai\,sup}_{t\in[0,1]} |p(t)| \le 8\,\frac{\rho-1}{\rho+1}\,, \quad |p| \not\equiv 8\,\frac{\rho-1}{\rho+1}\,,$$

*or*

$$\int\limits_0^1 |p(t)|\,dt \le 4\Big(1 - \frac{1}{\rho}\Big),$$

*then for each* $f \in \Lambda_\rho$ *a unique solution to problem* (1) *satisfies the inequality*

$$-\operatorname{sgn}(\mathcal{P})x(t) > 0 \ \ \textit{for all} \ \ t \in [0,1].$$

**Remark.** All inequalities in all these theorems and corollaries are sharp. In particular, if inequality (2) is not fulfilled, then there exists an operator $T$ such that Assumption 1 is satisfied and problem (1) does not have a unique solution. If inequality (3) or (4) is not fulfilled, then there exist an operator $T$ and a function $f$ such that Assumption 1 is satisfied and problem (1) has a solution which does not satisfy (5).

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