On the Existence of Positive Solutions of Nonlinear Elliptic Boundary Value Problems

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1. Introduction. This paper was stimulated by a number of recent studies [2–4, 6–9, 16–18] of mildly non-linear elliptic boundary and eigenvalue problems. Problems of this type arise for example in the theory of nonlinear diffusion processes and in nuclear or chemical reactor theory (*e.g.* [3, 4, 8, 9]). In this connection only positive solutions are of interest.

More specifically, we consider problems of the form

$$
\begin{aligned}
Lu &= f(x, u) \quad \text{in} \quad \Omega, \\
Bu &= \varphi(x, u) \quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where L is a uniformly elliptic second order differential operator with negative definite principal part and B is a linear first-order boundary operator. Moreover, f and φ may depend on an eigenvalue parameter λ. We are interested in the existence of non-negative classical solutions of (1.1).

This problem has been considered by several authors [2–4, 6–9, 16–18] where in all papers except [8] it was assumed that φ was independent of u. In addition to the (physically motivated) assumption

$$
f(x, 0) \geqq 0, \quad \varphi(x, 0) \geqq 0 \tag{1.2}
$$

several hypotheses upon f and φ, *e.g.* monotonicity and convexity, where shown to imply the existence of a non-negative solution of (1.1). These proofs were based on iteration procedures which require certain growth restrictions upon the functions f and φ.

A simple geometric interpretation of these theorems suggests that it should be possible to prove the existence of a non-negative solution of (1.1) imposing, in addition to (1.2), a very weak assumption which is immediately seen to be necessary too, namely that there exists a non-negative function v satisfying

$$
\begin{aligned}
Lv &\geqq f(x, v) \quad \text{in} \quad \Omega, \\
Bv &\geqq \varphi(x, v) \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{1.3}
$$

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The main purpose of this paper is to show that, under the hypothesis (1.2), the assumption (1.3) is in fact a necessary and sufficient condition for the existence of a non-negative solution of (1.1). Moreover, this fact is proved without any additional assumptions upon f and φ besides Hölder-continuity, which is well-known to be necessary for the existence of a classical solution even in the linear case where f and φ are independent of u.

This result has important consequences. Indeed, using this theorem it is not only possible to rederive most of the special cases cited above but it can also easily be used to prove existence and even non-existence theorems for non-linear elliptic boundary and eigenvalue problems under much more general and natural hypotheses. Several applications of this type are included here.

After having finished the first draft of this paper the author became aware of the paper [16] of Shampine. In this paper the sufficiency of condition (1.3) was claimed in the very special case where L is self-adjoint, $\varphi \equiv 0$, and f satisfies a certain growth condition which allows the use of an iteration argument. (For the validity of this claim stronger continuity hypotheses than those given in that paper have to be made).

In the following section we formulate our assumptions and state the main results. Section 3 contains the proofs of some *a priori*-inequalities which are needed for the proof of a convergence theorem which in turn will play an important role in our existence proof. This convergence theorem is proved in Section 3 too.

Section 4 contains the proof of our main theorem, namely the statement that (1.2) and (1.3) imply the existence of a non-negative solution. In fact, a somewhat more general theorem is proved which contains the above assertion as a special case. First, under an additional growth hypothesis upon f we prove a constructive version of our main theorem. This result is then used to prove the general result by means of a non-linear iteration argument.

Finally, the last section contains the proofs of the remaining statements of Section 2. These are the applications of our main existence theorem mentioned above.

2. Assumptions and Statement of Results. Let $\alpha \in (0, 1)$ be fixed. Denote by Ω a bounded domain of real N-space $\mathbf{R}^N$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. We assume that $\partial\Omega$ belongs to the class $C^{2+\alpha}$.

We consider the second order linear differential operator

$$Lu \equiv -\sum_{i,k=1}^{N} a_{ik} \frac{\partial^2 u}{\partial x^i \partial x^k} + \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x^i} + au$$

with real coefficients $a_{ik} \in C^{2+\alpha}(\bar{\Omega})$, $a_i \in C^{1+\alpha}(\bar{\Omega})$, $a \in C^{\alpha}(\bar{\Omega})$ where we assume that, for all $x = (x^1, \cdots, x^N) \in \Omega$,

$$a(x) \geqq 0.$$

In some cases under consideration it suffices to assume that a_{ik}, a_i, $a \in C^{\alpha}(\bar{\Omega})$;

see Remark 4.3. Moreover, L is supposed to be uniformly elliptic, *i.e.* there exists a constant $\gamma_0 > 0$ such that, for all $x \in \Omega$ and all $\xi \in \mathbf{R}^N$,

$$\sum_{i,k=1}^{N} a_{ik}\xi^i\xi^k \geqq \gamma_0 \sum_{i=1}^{N} (\xi^i)^2.$$

For every $x \in \partial\Omega$ we denote by $\nu(x) \equiv (\nu^1(x), \cdots, \nu^N(x))$ the outer normal to $\partial\Omega$ at x. Let $\beta_1, \cdots, \beta_N \in C^{1+\alpha}(\partial\Omega)$ be given in such a way that, for all $x \in \partial\Omega$,

$$\sum_{i=1}^{N} \beta_i(x)\nu^i(x) > 0.$$

Let ϵ denote a variable which assumes the values 0 and 1 only. Then we define *boundary operators* B_ϵ by

$$B_\epsilon u \equiv \beta u + \epsilon \sum_{i=1}^{N} \beta_i \frac{\partial u}{\partial x^i},$$

where $\beta \in C^{2-\epsilon+\alpha}(\partial\Omega)$ and satisfies, for all $x \in \partial\Omega$,

$$\beta(x) > 0 \quad \text{if} \quad \epsilon = 0,$$

and

$$\beta(x) \geqq 0 \quad \text{if} \quad \epsilon = 1.$$

Moreover, a and β are not both identically zero.

According to these assumptions B_0 is a Dirichlet boundary operator whereas B_1 corresponds either to a Neumann or to a regular oblique derivative boundary value problem.

Our hypotheses imply a *maximum-principle, i.e.* $u \in C^2(\Omega) \cap C^\epsilon(\bar{\Omega})$ and

$$(2.1) \qquad \begin{aligned} Lu &\geqq 0 \quad \text{in} \quad \Omega, \\ B_\epsilon u &\geqq 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

imply $u(x) \geqq 0$ for all $x \in \bar{\Omega}$. Moreover, $u(x) > 0$ for all $x \in \Omega$ provided the strict inequality sign in (2.1) holds at least at one point $x \in \bar{\Omega}$ (*e.g.* [11, 13]).

Finally, it is well-known that, for arbitrary $f \in C^\alpha(\bar{\Omega})$ and $\varphi \in C^{2-\epsilon+\alpha}(\partial\Omega)$, the boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in} \quad \Omega, \\ B_\epsilon u &= \varphi \quad \text{on} \quad \partial\Omega, \end{aligned}$$

has a unique solution $u \in C^{2+\alpha}(\bar{\Omega})$ (*e.g.* [10]).

Let Δ be a subset of $\mathbf{R}^N$ and consider an arbitrary function $f: \Delta \to \mathbf{R}$. Then we shall write $f \geqq 0$ if $f(x) \geqq 0$ for all $x \in \Delta$ and we shall say $f > 0$ provided $f \geqq 0$ but $f \neq 0$.

Let S be a subset of $\mathbf{R}$. For any real-valued function on $\Delta \times S$ we denote by the corresponding capital letter the *Nemytskii operator* defined by that function, *i.e.* if $f: \Delta \times S \to \mathbf{R}$ then, for every function $u: \Delta \to S$, $F(u)$ is defined by

$$F(u)(x) \equiv f(x, u(x)) \quad \text{for all} \quad x \in \Delta.$$

Finally, functions denoted by Latin letters are defined on all of $\bar{\Omega}$ whereas functions defined on $\partial\Omega$ only are denoted by Greek letters.

Set $\mathbf{R}_+ \equiv [0, \infty)$ and consider functions $\varphi_0 \in C^{2+\alpha}(\partial\Omega)$ and $\varphi_1 \in C^{1+\alpha}(\partial\Omega \times \mathbf{R}_+)$. In the following we shall always use the unified representation

$$\varphi_\epsilon = (1 - \epsilon)\varphi_0 + \epsilon\varphi_1 \in C^{2-\epsilon+\alpha}(\partial\Omega \times \mathbf{R}_+).$$

Finally, let $f: \bar{\Omega} \times \mathbf{R}_+ \to \mathbf{R}$ be given. Then, we consider *nonlinear boundary value problems* of the form

$$\begin{aligned} Lu &= F(u) \quad \text{in} \quad \Omega, \\ B_\epsilon u &= \Phi_\epsilon(u) \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

where by a solution of (2.2) we always mean a function $u \in C^{2+\alpha}(\bar{\Omega})$ which satisfies (2.2) identically.

We emphasize that in the case of the Dirichlet boundary value problem ($\epsilon = 0$) the boundary function φ_0 is independent of u.

Our main result is given by the following theorem (where we set $\varphi_0(x, 0) \equiv \varphi_0(x)$ for all $x \in \partial\Omega$).

Theorem A. *Let $f \in C^\alpha(\bar{\Omega} \times \mathbf{R}_+)$, $\varphi_0 \in C^{2+\alpha}(\partial\Omega)$ and $\varphi_1 \in C^{1+\alpha}(\partial\Omega \times \mathbf{R}_+)$ be given and assume*

$$f(x, 0) \geqq 0 \quad \textit{for all} \quad x \in \Omega,$$

and

$$\varphi_\epsilon(x, 0) \geqq 0 \quad \textit{for all} \quad x \in \partial\Omega.$$

Then, a necessary and sufficient condition for the existence of a non-negative solution of the boundary value problem (2.2) is the existence of a non-negative $v \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\begin{aligned} Lv &\geqq F(v) \quad \textit{in} \quad \Omega, \\ B_\epsilon v &\geqq \Phi_\epsilon(v) \quad \textit{on} \quad \partial\Omega. \end{aligned}$$

Moreover, if this condition is satisfied, there exist a maximal non-negative solution $\hat{u} \leqq v$ and a minimal non-negative solution $\bar{u} \leqq v$ in the sense that, for every non-negative solution $u \leqq v$ of (2.2), the inequality

$$\bar{u} \leqq u \leqq \hat{u}$$

holds.

If we assume, in addition to the hypotheses of Theorem A, that there exists a non-negative function $m \in C^\alpha(\bar{\Omega})$ such that, for all $x \in \Omega$ and $\xi \geqq \eta \geqq 0$,

$$f(x, \xi) - f(x, \eta) \geqq -m(x)(\xi - \eta),$$

then the maximal and the minimal solution can be computed iteratively. This statement is a special case of Theorem 1 which will be proved in Section 4.

In order to indicate the importance of Theorem A, in the rest of this paragraph we state some existence and non-existence theorems for non-linear elliptic boundary and eigenvalue problems. These theorems are applications of Theorem A and they will be proved in Section 5.

By basing the proofs of the following theorems in this section on the constructive version of Theorem A mentioned above one easily obtains constructive versions of these theorems too. For the sake of simplicity these applications are omitted.

We begin with the following simple existence theorem.

Theorem B. *Let the assumptions of Theorem A be satisfied. Moreover, assume that there exist a* $g \in C^{\alpha}(\bar{\Omega})$ *and a* $\psi \in C^{1+\alpha}(\partial\Omega)$ *such that*

$$f(x, \xi) \leqq g(x) \quad on \quad \Omega \times \mathbf{R}_+ ,$$

and

$$\varphi_1(x, \xi) \leqq \psi(x) \quad on \quad \partial\Omega \times \mathbf{R}_+ .$$

Then (2.2) *has at least one non-negative solution.*

This theorem generalizes Theorem 4.1 of [8]. Moreover, it is a simple consequence of Theorem B that the boundary value problem considered by Cohen and Laetsch in [4] has a positive solution under the sole assumptions $H - 1$ to $H - 3$ of [4]. Indeed, these assumptions imply that f is majorized by a constant.

Next we consider non-linear eigenvalue problems. Following Keller and Cohen [9] we call the set of all real values λ for which the boundary value problem

$$\begin{aligned} Lu &= \lambda F(u) \quad \text{in} \quad \Omega, \\ B_\epsilon u &= \lambda \Phi_\epsilon(u) \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.3}$$

has non-negative solutions the *spectrum* of (2.3). Then the following general theorem holds.

Theorem C. *Let* $f \in C^{\alpha}(\bar{\Omega} \times \mathbf{R}_+)$, $\varphi_0 \in C^{2+\alpha}(\partial\Omega \times \mathbf{R}_+)$, *and* $\varphi_1 \in C^{1+\alpha}(\partial\Omega \times \mathbf{R}_+)$ *be non-negative. Then, if* $\lambda_1 > 0$ *belongs to the spectrum of* (2.3) *the whole interval* $0 \leqq \lambda \leqq \lambda_1$ *is in the spectrum too. Moreover, the spectrum is a non-degenerate interval with* 0 *as left end point.*

This theorem characterizes the spectrum of (2.3) completely. It generalizes results of Keller and Cohen [9] who proved Theorem C under the assumptions $\Phi_\epsilon = 0$, L and B_ϵ self-adjoint, $f(x, 0) > 0$ on Ω and $f(x, \cdot)$ strictly increasing [9, Theorem 3.1 and Corollary 3.3.1].

Throughout the rest of this paragraph we shall assume that L *is self-adjoint with boundary operator* B_ϵ. In particular, this implies that

$$\beta_i = \sum_{i=1}^{N} a_{ik}\nu^k, \qquad i = 1, 2, \cdots, N.$$

For every positive function $r \varepsilon C^\alpha(\bar{\Omega})$ we denote by $\lambda_0(r)$ the lowest eigenvalue of

$$Lu - \lambda ru = 0 \quad \text{in} \quad \Omega,$$

$$B_\epsilon u = 0 \quad \text{on} \quad \partial\Omega,$$

and state the following non-existence theorem.

Theorem D. *Let $f \varepsilon C^\alpha(\bar{\Omega} \times \mathbf{R}_+)$ be given and assume there exist positive functions f_0, $r \varepsilon C^\alpha(\bar{\Omega})$ such that*

(2.4) $$f(x, \xi) \geqq f_0(x) + r(x)\xi, \quad (x, \xi) \varepsilon \Omega \times \mathbf{R}_+ .$$

Then, the non-linear eigenvalue problem

(2.5) $$Lu = \lambda F(u) \quad \textit{in} \quad \Omega,$$

$$B_\epsilon u = \quad \textit{on} \quad \partial\Omega,$$

has no non-negative solution for any $\lambda \geqq \lambda_0(r)$.

Theorem D furnishes an upper bound for the spectrum of (2.3). It generalizes certain results of Keller and Cohen [9, p. 1370] where $f(x, \cdot)$ was supposed to be strictly convex.

Our last theorem indicates that Theorem A may be applied to even more general eigenvalue problems. Indeed, we shall prove that, under suitable assumptions, the non-linear eigenvalue problem

(2.6) $$Lu - \lambda ru = F(u, \lambda) \quad \text{in} \quad \Omega,$$

$$B_\epsilon u = 0 \quad \text{on} \quad \partial\Omega,$$

has non-negative solutions. For this purpose, we denote, for any $b \varepsilon C^\alpha(\bar{\Omega})$ such that $b \leqq a$ (a is the coefficient of u in L), by $\mu_0(b)$ the lowest eigenvalue of

$$(L - b)u - \lambda ru = 0 \quad \text{in} \quad \Omega,$$

$$B_\epsilon u = 0 \quad \text{on} \quad \partial\Omega.$$

Then we have the following existence theorem.

Theorem E. *For every $\lambda \varepsilon \mathbf{R}_+$ let $f(\cdot\,, \cdot\,, \lambda) \varepsilon C^\alpha(\bar{\Omega} \times \mathbf{R}_+)$ satisfy $f(\cdot\,, 0, \lambda) \geqq 0$. Moreover, assume there exist non-negative functions g, $h \varepsilon C^\alpha(\bar{\Omega})$ with $h \leqq a$ such that*

(2.7) $$f(x, \xi, \lambda) \leqq g(x) + h(x)\xi, \; (x, \xi) \varepsilon \Omega \times \mathbf{R}_+ , \; \lambda \varepsilon \mathbf{R}_+ .$$

Then, for every positive $r \varepsilon C^\alpha(\bar{\Omega})$, the non-linear eigenvalue problem (2.6) *has at least one non-negative solution provided $\lambda < \mu_0(h)$.*

This theorem is in some sense a generalization of a theorem of Keller [7, Theorem 3.2] where $f(x, \cdot, \lambda)$ was assumed to be strictly concave. For the sake of simplicity we restricted the hypotheses of Theorem E more than necessary. It is easily seen that our statement remains here under more general hypotheses of the type considered in [7].

3. Some Auxiliary Results. Since $\partial\Omega \in C^{2+\alpha}$ we can find a finite open covering $\{U_1, \cdots, U_m\}$ such that there exist diffeomorphisms τ_k, $k = 1, \cdots, m$, of class $C^{2+\alpha}$ mapping $\bar{U}_k$ onto the closed unit ball $\bar{K}$ of $\mathbf{R}^N$. Moreover, the image of $U_k \cap \Omega$ under τ_k coincides with $K_+ \equiv \{y \in \mathbf{R}^N \mid |y| < 1, y^N > 0\}$ and the image of $U_k \cap \partial\Omega$ is given by $\Sigma \equiv \{y \in \mathbf{R}^N \mid |y| < 1, y^N = 0\}$.

By means of this covering we define, for every $\varphi \in C(\partial\Omega)$ and every $p > 1$, the norm

$$||\varphi||_{L_p(\partial\Omega)} \equiv \left(\sum_{k=1}^m ||\varphi_k||^p_{L_p(\Sigma)}\right)^{1/p}, \qquad \varphi_k \equiv \varphi \circ \tau_k^{-1}.$$

Then, we begin with the following

Lemma 3.1. *Let $\varphi \in C(\partial\Omega)$ be given. Then there exists a constant γ independent of φ such that*

$$\inf ||u||_{W_p^1(\Omega)} \leqq \gamma\, ||\varphi||_{L_p(\partial\Omega)}$$

where the infimum is taken with respect to all $u \in C^1(\bar{\Omega})$ satisfying $u \mid \partial\Omega = 0$ and $\partial u/\partial\nu = \varphi$.

Proof. It is enough to show that there exists an admissible function u and a constant γ which is independent of φ such that

$$||u||_{W_p^1(\Omega)} \leqq \gamma\, ||\varphi||_{L_p(\partial\Omega)}\,.$$

Let $V_1, \cdots, V_m$ be open subsets of $U_1, \cdots, U_m$, respectively, such that $V_k \cap \partial\Omega = U_k \cap \partial\Omega$ and let e_k, $k = 1, \cdots, m$, be a partition of unity subordinate to $\{V_1, \cdots, V_m\}$. For each $k = 1, \cdots, m$, let $u_k \in C^1(V_k \cap \bar{\Omega})$ be given in such a way that $u_k \mid (V_k \cap \partial\Omega) = 0$ and $(\partial u_k/\partial\nu) \mid (V_k \cap \partial\Omega) = \varphi$. Then it is easily verified that

$$u \equiv \begin{cases} \sum_{k=1}^m u_k e_k & \text{in} \quad \bar{\Omega} \cap \bigcup_{k=1}^m V_k \\ 0 & \text{otherwise,} \end{cases}$$

has the desired properties.

It is well-known (*e.g.* [12, Theorem 3.6.4]) that $u \mid \partial\Omega = 0$ implies

$$\int_\Omega |u|^p\, dx \leqq \gamma \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x^i}\right|^p dx.$$

(Here and in the following we denote by γ any positive constant whose exact value is unimportant.) Using these facts and local coordinates it is easily seen that it suffices to prove the following assertion: There exists a constant γ such that, for every $\psi \in C(\bar{\Sigma})$ we can find a $v \in C^1(P)$ satisfying

$$v \mid \Sigma = 0, \qquad \frac{\partial v}{\partial y^N} \mid \Sigma = \psi, \tag{3.1}$$

and the inequalities

$$\int_P \left| \frac{\partial v}{\partial y^i} \right|^p dy \leqq \gamma \int_\Sigma |\psi|^p \, d\bar{y}, \qquad i = 1, \cdots, N, \tag{3.2}$$

where P denotes the "pyramid"

$$P \equiv \{ y \in K_+ \cup \Sigma \mid \bar{y} + y^N \equiv (y^1 + y^N, \cdots, y^{N-1} + y^N) \in \Sigma \}.$$

For every $y \equiv (\bar{y}, t) \in P$ define v by

$$v(y) \equiv t^{2-N} \int_{\bar{y}}^{\bar{y}+t} \psi(\bar{\eta}) \, d\bar{\eta}$$

with an obvious abbreviation for the $(N - 1)$-fold iterated integral. Evidently, $v \in C^1(P)$ and satisfies (3.1).

By means of Hölder's inequality it is easily seen that, for every $f \in C[a, b]$ and for every $t \in (0, b - a)$,

$$\int_a^{b-t} \left| \frac{1}{t} \int_x^{x+t} f(\xi) \, d\xi \right|^p dx \leqq \int_a^b |f(x)|^p \, dx.$$

Using this inequality and Hölder's inequality repeatedly, one proves finally that v satisfies an inequality of the form (3.2). Q.E.D.

We shall use Lemma 3.1 to prove an *a priori*-estimate for the boundary value problem

$$Lu = f \text{ in } \Omega,$$
$$B_1 u = \varphi \text{ on } \partial\Omega.$$

A priori inequalities of related type have been given by Schechter [15] for much more general problems but under very strong smoothness hypotheses. Since it is not easy to see how Schechter's inequalities apply to our problem we shall outline a proof of the needed inequality following some ideas found in [15].

Lemma 3.2. *There exists a constant γ such that, for every $u \in C^2(\bar{\Omega})$,*

$$\|u\|_{W_p^1(\Omega)} \leqq \gamma(\|Lu\|_{L_q(\Omega)} + \|B_1 u\|_{L_p(\partial\Omega)} + \|B_1 u\|_{L_q(\partial\Omega)})$$

where $q \equiv p/(p - 1)$.

Proof. Denote by L^* and B_1^* the adjoint differential and boundary operator, respectively. The differentiability assumptions upon the coefficients of L and B_1 assure that the *a priori* estimates of Agmon–Douglis–Nirenberg [1, Theorem 15.2] hold for both (L, B_1) and (L^*, B_1^*). Hence

$$||u||_{W_p^2(\Omega)} \leqq \gamma\, ||Lu||_{L_p(\Omega)} \quad \text{if} \quad B_1 u = 0,$$

and

$$(3.3) \qquad ||u||_{W_p^2(\Omega)} \leqq \gamma\, ||L^*u||_{L_p(\Omega)} \quad \text{if} \quad B_1^* u = 0,$$

since both problems admit at most one solution.

We define $V \equiv \{u \in C^2(\bar{\Omega}) \mid B_1 u = 0\}$ and $V' \equiv \{u \in C^2(\bar{\Omega}) \mid B_1^* u = 0\}$ and set, for $k = 1, 2$,

$$|u|_{-k,q} \equiv \sup_{v \in V'} \frac{|(u, v)|}{||v||_{W_p^k(\Omega)}}, \qquad u \in C^1(\bar{\Omega}),$$

where $(\cdot\, ,\, \cdot)$ denotes the inner product in $L_2(\Omega)$.

Now, (3.3) implies

$$|Lu|_{-2,q} = \sup_{v \in V'} \frac{|(Lu, v)|}{||v||_{W_p^2(\Omega)}} = \sup_{v \in V'} \frac{|(u, L^*v)|}{||v||_{W_p^2(\Omega)}}$$

$$\geqq \gamma^{-1} \sup_{v \in V'} \frac{|(u, L^*v)|}{||L^*v||_{L_p(\Omega)}} \geqq \gamma^{-1}\, ||u||_{L_p(\Omega)}\,.$$

Hence, for every $u \in C^2(\bar{\Omega})$,

$$||u||_{L_p(\Omega)} \leqq \gamma\, |Lu|_{-2,q}\,.$$

Following Schechter [14, p. 7], by means of the theory of interpolation spaces one deduces

$$(3.4) \qquad ||u||_{W_p^1(\Omega)} \leqq \gamma\, |Lu|_{-1,q}\,, \quad u \in V.$$

(It is easily checked that [14, Lemma 2.4] holds in case of our assumptions upon $\partial\Omega$).

Let $u \in C^2(\bar{\Omega})$ be arbitrary. Denote by $u_0 \in C^2(\bar{\Omega})$ any function satisfying

$$(3.5) \qquad \begin{aligned} B_1(u - u_0) &= 0, \\ u_0 \mid \partial\Omega &= 0. \end{aligned}$$

Hence $u - u_0 \in V$ and (3.4) implies

$$||u||_{W_p^1(\Omega)} \leqq \gamma\, |Lu|_{-1,q} + \gamma \inf\, (|Lu_0|_{-1,q} + ||u_0||_{W_p^1(\Omega)})$$

where the infimum is taken with respect to all functions u_0 satisfying (3.5).

By partial integration one finds, for every $v \in V'$,

$$(Lu_0\,, v) = \int_\Omega \left(\sum_{i,k=1}^N a_{ik} \frac{\partial u_0}{\partial x^i} \frac{\partial v}{\partial x^k} + \sum_{i=1}^N b_i \frac{\partial u_0}{\partial x^i} v + a u_0 v \right) dx + \int_{\partial\Omega} \delta v\, B_1 u_0\, d\sigma$$

with $b_i \in C(\bar{\Omega})$ and $\delta \in C(\partial\Omega)$. Hence

$$|(Lu_0\,, v)| \leqq \gamma\left(||u_0||_{W_q^1(\Omega)}\, ||v||_{W_p^1(\Omega)} + \int_{\partial\Omega} |vB_1 u_0|\, d\sigma \right).$$

But

$$\int_{\partial\Omega} |vB_1u_0|\, d\sigma \leqq \gamma\, ||B_1u_0||_{L_q(\partial\Omega)}\, ||v||_{L_p(\partial\Omega)}$$

and, according to a result of Gagliardo [5, Teorema 1.I]

$$||v||_{L_p(\partial\Omega)} \leqq \gamma\, ||v||_{W_p^1(\Omega)} .$$

This implies that

$$|Lu_0|_{-1,q} \leqq \gamma(||u_0||_{W_q^1(\Omega)} + ||B_1u_0||_{L_q(\partial\Omega)})$$

and, with (3.4), we obtain

$$||u||_{W_p^1(\Omega)} \leqq \gamma(|Lu|_{-1,q} + ||B_1u||_{L_q(\partial\Omega)}) + \gamma \inf ||u_0||_{W_p^1(\Omega)} .$$

Finally, by means of local coordinates, it is easily seen that there exists a $\delta \in C(\partial\Omega)$ such that the two sets

$$\mathfrak{B}_1 \equiv \{u \in C^2(\bar\Omega) \mid B_1u = \varphi, \quad u \mid \partial\Omega = 0\}$$

and

$$\mathfrak{B}_0 \equiv \left\{u \in C^2(\Omega) \,\middle|\, \frac{\partial u}{\partial \nu} = \delta\varphi,\, u|\, \partial\Omega = 0\right\}$$

are equal. Hence, by Lemma 3.1,

$$\inf_{\mathfrak{B}_1} ||u_0||_{W_p^1(\Omega)} = \inf_{\mathfrak{B}_0} ||u_0||_{W_p^1(\Omega)} \leqq \gamma\, ||B_1u||_{L_p(\partial\Omega)}$$

and the statement follows. Q.E.D.

It should be remarked that the inequalities of Lemma 3.1 and Lemma 3.2 are not sharp. In order to obtain sharp inequalities one has to introduce "intermediate" spaces of distributions.

By means of Lemma 3.2 we shall prove a convergence theorem which will be of utmost importance for the proof of our main theorem. But first we shall collect some known results.

Let k assume the values 1 and 2. Following Agmon–Douglis–Nirenberg [1] we define boundary norms

$$||\varphi||_{k-1/p} \equiv \inf ||u||_{W_p^k(\Omega)}$$

where the infimum is taken with respect to all functions $u \in C^k(\bar\Omega)$ which equal φ on $\partial\Omega$. Let $\mu \in (1/q, 1)$ be given. Then as a consequence of a theorem of Gagliardo [5, Teorema 1.I], there exists a constant γ such that, for all $\varphi \in C^\mu(\partial\Omega)$,

$$||\varphi||_{1-1/p} \leqq \gamma\, ||\varphi||_{C^\mu(\partial\Omega)} . \tag{3.6}$$

Finally, it is well-known that, for any $p > N$, the Sobolev space $W_p^k(\Omega)$ is continuously imbedded in $C^{k-1+\mu}(\bar\Omega)$ with $\mu \equiv 1 - N/p$, *i.e.* there exists a constant γ such that, for all $u \in W_p^k(\Omega)$, (3.7) $$||u||_{C^{k-1+\mu}(\bar\Omega)} \leqq \gamma\, ||u||_{W_p^k(\Omega)}\,.$$

(As an immediate consequence of [12, Theorem 3.6.6] we obtain $W_p^k(\Omega) \subset C^{k-1+\mu}(\bar\Omega)$. The continuity of this imbedding is now a simple consequence of the closed graph theorem).

Let $\bar v, \hat v \in C^{2+\alpha}(\bar\Omega)$ be given such that $\bar v \leqq \hat v$. Then we define

$$\bar\Omega \times [\bar v, \hat v] \equiv \{(x, \xi) \mid x \in \bar\Omega,\ \bar v(x) \leqq \xi \leqq \hat v(x)\}$$

and similar definitions apply to $\partial\Omega \times [\bar v, \hat v]$, $\bar\Omega \times [\bar v, \hat v]^2$ and $\partial\Omega \times [\bar v, \hat v]^2$.

Proposition 3.3. *Let $\{g_n\}$ be a bounded sequence in $C^\alpha(\bar\Omega \times [\bar v, \hat v]^2)$ converging pointwise to a function g and let $\{\psi_n\}$ be a bounded sequence in $C^{1+\alpha}(\partial\Omega \times [\bar v, \hat v]^2)$ converging pointwise to a function ψ. Moreover, let $\{u_n\}$ be a sequence in $C^{2+\alpha}(\bar\Omega)$ satisfying*

$$\bar v \leqq u_n \leqq \hat v$$

which converges pointwise to a function u. Assume that, for every $n \geqq 2$,

(3.8) $$Lu_n = G_n(u_n, u_{n-1}) \quad in \quad \Omega,$$
$$B_\epsilon u_n = \epsilon\Psi_n(u_n, u_{n-1}) + (1-\epsilon)\varphi \quad on \quad \partial\Omega,$$

where $\varphi \in C^{2+\alpha}(\partial\Omega)$.

Then, $u \in C^{2+\alpha}(\bar\Omega)$ and satisfies

(3.9) $$Lu = G(u, u) \quad in \quad \Omega,$$
$$B_\epsilon u = \epsilon\Psi(u, u) + (1-\epsilon)\varphi \quad on \quad \partial\Omega,$$

and, moreover, the sequence $\{u_n\}$ converges in $C^2(\bar\Omega)$ to u.

Proof. Evidently $C^{2+\alpha}(\bar\Omega) \subset W_p^2(\Omega)$ for arbitrary $p > 1$. Hence, for $p = N/(1-\alpha)$, according to (3.7) we have, for all n,

(3.10) $$||u_n||_{C^{1+\alpha}(\bar\Omega)} \leqq \gamma\, ||u_n||_{W_p^2(\Omega)}\,.$$

Now, we consider the two cases $\epsilon = 0$ and $\epsilon = 1$ separately.

i) $\epsilon = 0$. Since u_n satisfies (3.8), the L_p-estimate of Agmon–Douglis–Nirenberg [1, Theorem 15.2] applies, *i.e.*

$$||u_n||_{W_p^2(\Omega)} \leqq \gamma\, (||G_n(u_n, u_{n-1})||_{L_p(\Omega)} + ||\varphi||_{2-1/p}).$$

According to our assumptions the sequence $\{G_n(u_n, u_{n-1})\}$ is bounded in $L_p(\Omega)$, hence $\{u_n\}$ is bounded in $W_p^2(\Omega)$ and therefore, by (3.10), $\{u_n\}$ is bounded in $C^{1+\alpha}(\bar\Omega)$. This implies that $\{G_n(u_n, u_{n-1})\}$ is a bounded sequence in $C^\alpha(\bar\Omega)$ and, hence, the Schauder estimate (*e.g.* [1], [10], [11])

$$||u_n||_{C^{2+\alpha}(\bar\Omega)} \leqq \gamma\, (||G_n(u_n, u_{n-1})||_{C^\alpha(\bar\Omega)} + ||\varphi||_{C^{2+\alpha}(\partial\Omega)})$$

implies the boundedness of $\{u_n\}$ in $C^{2+\alpha}(\bar\Omega)$.

ii) $\epsilon = 1$. In this case the L_p-estimate of Agmon–Douglis–Nirenberg has the form

$$||u_n||_{W_p^2(\Omega)} \leqq \gamma\ (||G_n(u_n\ ,\ u_{n-1})||_{L_p(\Omega)} + ||\Psi_n(u_n\ ,\ u_{n-1})||_{1-1/p}).$$

Hence (3.6) implies

$$||u_n||_{W_p^2(\Omega)} \leqq \gamma\ (||G_n(u_n\ ,\ u_{n-1})||_{L_p(\Omega)} + ||\Psi_n(u_n\ ,\ u_{n-1})||_{C^\mu(\partial\Omega)})$$

with $1/q < \mu < 1$. Therefore, in addition to our considerations in i), we have to show that $\{\Psi_n(u_n\ ,\ u_{n-1})\}$ is bounded in $C^\mu(\partial\Omega)$. According to our assumptions this will be proved provided we show that $\{u_n\}$ is a bounded sequence in $C^\mu(\Omega)$. Hence, by (3.7), it suffices to show that $\{u_n\}$ is bounded in $W^1_{p_1}(\Omega)$ with $p_1 = N/(1 - \mu) > Np$. But this is now a simple consequence of Lemma 3.2 since $\{\Psi_n(u_n\ ,\ u_{n-1})\}$ is obviously bounded in $C(\partial\Omega)$. Hence again, $\{u_n\}$ is a bounded sequence in $W^2_p(\Omega)$ and therefore in $C^{1+\alpha}(\bar{\Omega})$. By our assumptions it follows that $\{\Psi_n(u_n\ ,\ u_{n-1})\}$ is bounded in $C^{1+\alpha}(\partial\Omega)$ and therefore the Schauder-type inequality (*e.g.* [1], [10], [11])

$$||u_n||_{C^{2+\alpha}(\bar{\Omega})} \leqq \gamma\ (||G_n(u_n\ ,\ u_{n-1})||_{C^\alpha(\bar{\Omega})} + ||\Psi_n(u_n\ ,\ u_{n-1})||_{C^{1+\alpha}(\partial\Omega)})$$

implies the boundedness of $\{u_n\}$ in $C^{2+\alpha}(\bar{\Omega})$.

Now again, we consider both cases $\epsilon = 0$ and $\epsilon = 1$ simultaneously. The above considerations show that $\{u_n\}$ is a bounded sequence in $C^{2+\alpha}(\bar{\Omega})$. Hence, by the Arzéla–Ascoli theorem, there exists a subsequence which converges in $C^2(\bar{\Omega})$ to an element $v \in C^{2+\alpha}(\bar{\Omega})$. But, since $\{u_n\}$ converges pointwise to u, we have $v = u$ and, moreover, the whole sequence converges in $C^2(\bar{\Omega})$ to u. Hence, $Lu_n \to Lu$ and $B_\epsilon u_n \to B_\epsilon u$ uniformly. Finally,

$$\begin{aligned}|G_n(u_n\ ,\ u_{n-1}) - G(u,\ u)| &\leqq |G_n(u_n\ ,\ u_{n-1}) - G_n(u,\ u)| + |G_n(u,\ u) - G(u,\ u)| \\ &\leqq \gamma(|u_n - u|^\alpha + |u_{n-1} - u|^\alpha) + |G_n(u,\ u) - G(u,\ u)|,\end{aligned}$$

which shows that $G_n(u_n\ ,\ u_{n-1}) \to G(u,\ u)$ pointwise. An analogue inequality implies $\Psi_n(u_n\ ,\ u_{n-1}) \to \Psi(u,\ u)$ pointwise and the remaining statement (3.9) follows. Q.E.D.

Remark 3.4. It will be important to notice that only in the case $\epsilon = 1$ Lemma 3.2 was used. Moreover, if the Ψ_n are independent of u_n and u_{n-1}, then even in this case Lemma 3.2 is not needed.

4. Proof of the Main Theorem. In the following we shall consider somewhat more general assumptions than in the statement of Theorem A, namely the

Hypothesis (H): *There exist functions $\bar{v}$, $\hat{v} \in C^{2+\alpha}(\bar{\Omega})$ satisfying $\bar{v} \leqq \hat{v}$ and functions $f \in C^\alpha(\bar{\Omega} \times [\bar{v},\ \hat{v}])$, $\varphi_0 \in C^{2+\alpha}(\partial\Omega)$, and $\varphi_1 \in C^{1+\alpha}(\partial\Omega \times [\bar{v},\ \hat{v}])$ such that*

$$L\bar{v} \leqq F(\bar{v}) \quad in \quad \Omega, \quad B_\epsilon\bar{v} \leqq \Phi_\epsilon(\bar{v}) \quad on \quad \partial\Omega,$$

and

$$L\hat{v} \geqq F(\hat{v}) \quad in \quad \Omega, \quad B_\epsilon\hat{v} \geqq \Phi_\epsilon(\hat{v}) \quad on \quad \partial\Omega.$$

We are interested in the existence of solutions of the non-linear elliptic boundary value problem

$$
\begin{aligned}
Lu &= F(u) \quad \text{in} \quad \Omega, \\
B_\epsilon u &= \Phi_\epsilon(u) \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{4.1}
$$

We shall begin with the following *constructive existence theorem* which is a generalization of [8, Theorem 4.1] and [16, Theorem 1]. The proof of this theorem is based upon arguments developed by Cohen and Keller [2, 3, 8, 9].

Theorem 1. *Let Hypothesis (H) be satisfied. Moreover, assume there exists a non-negative $m \in C^\alpha(\bar\Omega)$ such that, for all $x \in \bar\Omega$ and all ξ, η satisfying*

$$
\begin{gathered}
\check{v}(x) \leqq \eta \leqq \xi \leqq \hat{v}(x), \\
f(x, \xi) - f(x, \eta) \geqq -m(x)(\xi - \eta).
\end{gathered}
\tag{4.2}
$$

Let $\mu \in \mathbf{R}$ satisfy

$$
\mu \geqq \max_{\partial\Omega \times [\check{v}, \hat{v}]} \left| \frac{\partial \varphi_1}{\partial \xi} \right|
\tag{4.3}
$$

and define sequences $\{u_n\} \subset C^{2+\alpha}(\bar\Omega)$ by

$$
\begin{aligned}
Lu_n + mu_n &= F(u_{n-1}) + mu_{n-1} \quad \textit{in} \quad \Omega, \\
B_\epsilon u_n + \epsilon\mu u_n &= \Phi_\epsilon(u_{n-1}) + \epsilon\mu u_{n-1} \quad \textit{on} \quad \partial\Omega.
\end{aligned}
$$

Then, if $u_0 = \hat{v}$, the sequence $\{u_n\}$ converges monotonically from above to a solution $\hat{u}$ of (4.1) and, if $u_0 = \check{v}$, $\{u_n\}$ converges monotonically from below to a solution $\check{u}$ of (4.1). In any case the sequence $\{u_n\}$ converges in $C^2(\bar\Omega)$. Moreover, any solution u of (4.1) with $\check{v} \leqq u \leqq \hat{v}$ satisfies

$$
\check{v} \leqq \check{u} \leqq u \leqq \hat{u} \leqq \hat{v}.
$$

Proof. Set $u_0 = \hat{v}$. Then u_1 is well-defined and (H) implies

$$
\begin{aligned}
L(u_1 - \hat{v}) + m(u_1 - \hat{v}) &= F(\hat{v}) - L\hat{v} \leqq 0 \text{ in } \Omega, \\
B_\epsilon(u_1 - \hat{v}) + \epsilon\mu(u_1 - \hat{v}) &= \Phi_\epsilon(\hat{v}) - B_\epsilon\hat{v} \leqq 0 \text{ on } \partial\Omega.
\end{aligned}
$$

Hence, by the maximum-principle, $u_1 \leqq \hat{v}$. This inequality and (4.2, 3) imply

$$
\begin{aligned}
L(u_1 - \check{v}) &\geqq F(\hat{v}) - F(\check{v}) + m(\hat{v} - u_1) \geqq m(\check{v} - u_1), \\
B_\epsilon(u_1 - \check{v}) &\geqq \Phi_\epsilon(\hat{v}) - \Phi_\epsilon(\check{v}) + \epsilon\mu(\hat{v} - u_1) \geqq \epsilon\mu(\check{v} - u_1),
\end{aligned}
$$

or

$$
\begin{aligned}
L(u_1 - \check{v}) + m(u_1 - \check{v}) &\geqq 0 \quad \text{in} \quad \Omega, \\
B_\epsilon(u_1 - \check{v}) + \epsilon\mu(u_1 - \check{v}) &\geqq 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
$$

Hence, again by the maximum-principle, $u_1 \geqq \check{v}$. Now, by means of an induction argument, it is easily seen that the sequence $\{u_n\}$ is well-defined, belongs to $C^{2+\alpha}(\bar{\Omega})$ and satisfies

$$\bar{v} \leqq \cdots \leqq u_n \leqq u_{n-1} \leqq \cdots \leqq u_1 \leqq \hat{v}.$$

Therefore, $\{u_n\}$ converges pointwise to some function $\hat{u}$ on $\bar{\Omega}$. Finally, for all n, set

$$g_n(x, \xi, \eta) \equiv f(x, \eta) + m(x)(\eta - \xi), \; (x, \xi, \eta) \in \bar{\Omega} \times [\bar{v}, \hat{v}]^2$$

$$\psi_n(x, \xi, \eta) \equiv \varphi_1(x, \eta) + \mu(\eta - \xi), \; (x, \xi, \eta) \in \partial\Omega \times [\bar{v}, \hat{v}]^2$$

and apply Proposition 3.3 to prove the statement of the theorem concerning the convergence of the sequence $\{u_n\}$ with $u_0 = \hat{v}$.

In an analogous way it is shown that, in case $u_0 = \bar{v}$, we have

$$\bar{v} \leqq u_1 \leqq \cdots \leqq u_{n-1} \leqq u_n \leqq \cdots \leqq \hat{v}$$

and that this sequence converges in $C^2(\bar{\Omega})$ to a solution $\bar{u}$ of (4.1) with $\bar{v} \leqq \bar{u} \leqq \hat{v}$.

Now, let u with $\bar{v} \leqq u \leqq \hat{v}$ be any solution of (4.1). Then, in the considerations above, we can take u instead of $\bar{v}$ and we find $u \leqq \hat{u} \leqq \hat{v}$ and, similarly, by taking u instead of $\hat{v}$ we find $\bar{v} \leqq \bar{u} \leqq u$. This proves the theorem.

Next we need the following uniqueness result which generalizes [8, Theorem 3.2].

Theroem 2. *Let $\bar{v}$, $\hat{v}$: $\bar{\Omega} \to \mathrm{R}$ satisfy $\bar{v} \leqq \hat{v}$. Moreover, assume that, for every $x \in \bar{\Omega}$ and ξ, η with $\bar{v}(x) \leqq \xi \leqq \eta \leqq \hat{v}(x)$,*

$$f(x, \xi) \geqq f(x, \eta), \tag{4.4}$$
$$\varphi_1(x, \xi) \geqq \varphi_1(x, \eta).$$

Then (4.1) *has at most one solution u satisfying $\bar{v} \leqq u \leqq \hat{v}$.*

Proof. Let u_1, u_2 be two solutions with $\bar{v} \leqq u_1, u_2 \leqq \hat{v}$. Set $\Omega_1 \equiv \{x \in \Omega \mid u_1(x) > u_2(x)\}$ and observe that the monotonicity assumption (4.4) implies

$$L(u_1 - u_2) \leqq 0 \quad \text{in} \quad \Omega_1,$$

$$B_\epsilon(u_1 - u_2) \leqq 0 \quad \text{on} \quad \partial\Omega_1 \cap \partial\Omega,$$

$$u_1 - u_2 = 0 \quad \text{on} \quad \partial\Omega_1 \cap \Omega.$$

Therefore, by the maximum principle (applied to every component of Ω_1), we have $u_1 \leqq u_2$ in $\bar{\Omega}_1$, hence in $\bar{\Omega}$. Similarly, it is shown that $u_1 \geqq u_2$, hence $u_1 = u_2$. Q.E.D.

Let $\gamma > 0$ denote any bound for the Hölder-constant of f, *i.e.* for all (x, ξ), $(y, \eta) \in \bar{\Omega} \times [\bar{v}, \hat{v}]$,

$$|f(x, \xi) - f(y, \eta)| \leqq \gamma(|x - y|^\alpha + |\xi - \eta|^\alpha),$$

and define functions $\bar{f}$, $\hat{f} \varepsilon C^{\alpha}(\bar{\Omega} \times [\bar{v}, \hat{v}])$ by

$$\bar{f}(x, \xi) \equiv f(x, \bar{v}(x)) - \gamma(\xi - \bar{v}(x))^{\alpha}$$

and

$$\hat{f}(x, \xi) \equiv f(x, \hat{v}(x)) + \gamma(\hat{v}(x) - \xi)^{\alpha}.$$

Let μ satisfy (4.3) and define $\bar{\varphi}_1$, $\hat{\varphi}_1 \varepsilon C^{1+\alpha}(\partial\Omega \times [\bar{v}, \hat{v}])$ by

$$\bar{\varphi}_1(x, \xi) \equiv \varphi_1(x, \bar{v}(x)) - \mu(\xi - \bar{v}(x))$$

and

$$\hat{\varphi}_1(x, \xi) \equiv \varphi_1(x, \hat{v}(x)) + \mu(\hat{v}(x) - \xi).$$

Finally, set $\bar{\varphi}_0 = \hat{\varphi}_0 = \varphi_0$ and observe that

$$\bar{f} \leqq f \leqq \hat{f} \quad \text{and} \quad \bar{\varphi}_\epsilon \leqq \varphi_\epsilon \leqq \hat{\varphi}_\epsilon . \tag{4.5}$$

Lemma 4.1. Let Hypothesis (H) be satisfied. Moreover, assume there exists a constant $\delta > 0$ *such that*

$$L\bar{v} - \bar{F}(\bar{v}) \leqq -\delta \text{ in } \Omega, \quad B_\epsilon\bar{v} - \bar{\Phi}_\epsilon(\bar{v}) \leqq -(\beta + \epsilon)\delta \text{ on } \partial\Omega,$$

and

$$L\hat{v} - \hat{F}(\hat{v}) \geqq \delta \text{ in } \Omega, \quad B_\epsilon\hat{v} - \hat{\Phi}_\epsilon(\hat{v}) \geqq (\beta + \epsilon)\delta \text{ on } \partial\Omega.$$

Then, the boundary value problems

$$Lu = \bar{F}(u) \text{ in } \Omega, \quad B_\epsilon u = \bar{\Phi}_\epsilon(u) \text{ on } \partial\Omega, \tag{4.6}$$

and

$$Lu = \hat{F}(u) \text{ in } \Omega, \quad B_\epsilon u = \hat{\Phi}_\epsilon(u) \text{ on } \partial\Omega, \tag{4.7}$$

have unique solutions $\bar{u}$ *and* $\hat{u}$, *respectively. Moreover,* $\bar{v} \leqq \bar{u} \leqq \hat{u} \leqq \hat{v}$.

Proof. The uniqueness is an immediate consequence of Theorem 2.

By means of a continuity argument it is easily seen that there exists a constant $\sigma > 0$ such that $\bar{w} \equiv \bar{v} + \sigma$ and $\hat{w} \equiv \hat{v} - \sigma$ satisfy

$$L\bar{w} \leqq \bar{F}(\bar{w}) \text{ in } \Omega, \quad B_\epsilon\bar{w} \leqq \bar{\Phi}_\epsilon(\bar{w}) \text{ on } \partial\Omega, \tag{4.8}$$

and

$$L\hat{w} \geqq \hat{F}(\hat{w}) \text{ in } \Omega, \quad B_\epsilon\hat{w} \geqq \hat{\Phi}_\epsilon(\hat{w}) \text{ on } \partial\Omega. \tag{4.9}$$

Indeed,

$$L\bar{w} - \bar{F}(\bar{w}) = L\bar{v} - \bar{F}(\bar{v}) + a\sigma + \bar{F}(\bar{v}) - \bar{F}(\bar{v} + \sigma) \leqq -\delta + a\sigma + \gamma\sigma^{\alpha},$$

and

$$B_\epsilon\bar{w} - \bar{\Phi}_\epsilon(\bar{w}) = B_\epsilon\bar{v} - \bar{\Phi}_\epsilon(\bar{v}) + \beta\sigma + \bar{\Phi}_\epsilon(\bar{v}) - \bar{\Phi}_\epsilon(\bar{v} + \sigma)$$

$$\leqq -\beta(\delta - \sigma) - \epsilon(\delta - \sigma).$$

Hence, for σ sufficiently small, (4.8) is satisfied and a similar argument proves (4.9).

Obviously, $\bar{f}$ is defined on $\bar{\Omega} \times [\bar{w}, \hat{v} + \sigma]$ and satisfies there an inequality of type (4.2) with $m = \gamma\alpha\sigma^{\alpha-1}$. Moreover, by (4.5),

$$L(\hat{v} + \sigma) \geqq L\hat{v} \geqq F(\hat{v}) \geqq \bar{F}(\hat{v}) \geqq \bar{F}(\hat{v} + \sigma) \quad \text{in} \quad \Omega,$$

$$B_\epsilon(\hat{v} + \sigma) \geqq B_\epsilon\hat{v} \geqq \Phi_\epsilon(\hat{v}) \geqq \bar{\Phi}_\epsilon(\hat{v}) \geqq \bar{\Phi}_\epsilon(\hat{v} + \sigma) \quad \text{on} \quad \partial\Omega.$$

Hence, by Theorem 1 and the uniqueness result (applied to the "interval" $[\bar{w}, \hat{v} + \sigma]$), there exists exactly one solution $\bar{u}$ of (4.6) satisfying $\bar{w} \leqq \bar{u} \leqq \hat{v} + \sigma$. A similar consideration shows that there exists exactly one solution $\hat{u}$ of (4.7) with $\bar{v} - \sigma \leqq \hat{u} \leqq \hat{w}$.

Finally, by (4.5),

$$L\hat{u} = \hat{F}(\hat{u}) \geqq \bar{F}(\hat{u}) \text{ in } \Omega, \quad B_\epsilon\hat{u} = \hat{\Phi}_\epsilon(\hat{u}) \geqq \bar{\Phi}_\epsilon(\hat{u}) \text{ on } \partial\Omega.$$

Hence

$$L(\hat{u} - \bar{u}) \geqq \bar{F}(\hat{u}) - \bar{F}(\bar{u}) \text{ in } \Omega, \quad B_\epsilon(\hat{u} - \bar{u}) \geqq \bar{\Phi}_\epsilon(\hat{u}) - \bar{\Phi}_\epsilon(\bar{u}) \text{ on } \partial\Omega.$$

Set $\Omega_1 \equiv \{x \in \Omega \mid \hat{u}(x) < \bar{u}(x)\}$ and observe that the monotonicity of $\bar{f}(x, \cdot)$ and $\varphi_1(x, \cdot)$ implies

$$L(\hat{u} - \bar{u}) \geqq 0 \quad \text{in} \quad \Omega_1,$$

$$B_\epsilon(\hat{u} - \bar{u}) \geqq 0 \quad \text{on} \quad \partial\Omega_1 \cap \partial\Omega,$$

$$\hat{u} - \bar{u} = 0 \quad \text{on} \quad \partial\Omega_1 \cap \Omega.$$

Therefore, by the maximum principle, $\bar{u} \leqq \hat{u}$ and the assertion follows. Q.E.D.

Next we show that Lemma 4.1 holds with no additional assumption besides (H).

Lemma 4.2. *Let Hypothesis (H) be satisfied. Then, the boundary value problems*

$$Lu = \bar{F}(u) \text{ in } \Omega, \quad B_\epsilon u = \Phi_\epsilon(u) \text{ on } \partial\Omega, \tag{4.10}$$

and

$$Lu = \hat{F}(u) \text{ in } \Omega, \quad B_\epsilon u = \hat{\Phi}_\epsilon(u) \text{ on } \partial\Omega, \tag{4.11}$$

have unique solutions $\bar{u}$ and $\hat{u}$, respectively. Moreover, $\bar{v} \leqq \bar{u} \leqq \hat{u} \leqq \hat{v}$.

Proof. Denote by e the unique solution of

$$Lu = 1 \text{ in } \Omega, \quad B_\epsilon u = \beta + \epsilon \text{ on } \partial\Omega.$$

Then, the maximum principle implies $e > 0$. For every n, set $\bar{v}_n \equiv \bar{v} - (1/n)e$ and $\hat{v}_n \equiv \hat{v} + (1/n)e$ and define $\bar{f}_n$ and $\hat{f}_n$ on $\bar{\Omega} \times [\bar{v}_n, \hat{v}_n]$ by

$$\bar{f}_n(x, \xi) \equiv f(x, \bar{v}(x)) - \gamma\left(\xi - \bar{v}(x) + \frac{1}{n} e(x)\right)^\alpha,$$

and

$$\hat{f}_n(x, \xi) \equiv f(x, \hat{v}(x)) + \gamma\left(\hat{v}(x) - \xi + \frac{1}{n} e(x)\right)^\alpha.$$

Similarly, on $\partial\Omega \times [\bar{v}_n, \hat{v}_n]$ define $\bar{\varphi}_{1,n}$ and $\hat{\varphi}_{1,n}$ by

$$\bar{\varphi}_{1,n}(x, \xi) \equiv \varphi_1(x, \bar{v}(x)) - \mu\left(\xi - \bar{v}(x) + \frac{1}{n} e(x)\right),$$

and

$$\hat{\varphi}_{1,n}(x, \xi) \equiv \varphi_1(x, \hat{v}(x)) + \mu\left(\hat{v}(x) - \xi + \frac{1}{n} e(x)\right).$$

Finally, set $\bar{\varphi}_{0,n} \equiv \hat{\varphi}_{0,n} \equiv \varphi_0$. Then (H) implies

$$L\bar{v}_n - \bar{F}_n(\bar{v}_n) = L\bar{v} - \frac{1}{n} - F(\bar{v}) \leqq -\frac{1}{n} \quad \text{in} \quad \Omega,$$

$$B_\epsilon\bar{v}_n - \bar{\Phi}_{\epsilon,n}(\bar{v}_n) = B_\epsilon\bar{v} - \frac{1}{n}(\beta + \epsilon) - \Phi_\epsilon(\bar{v}) \leqq -\frac{1}{n}(\beta + \epsilon) \quad \text{on} \quad \partial\Omega,$$

and, analogously,

$$L\hat{v}_n - \hat{F}_n(\hat{v}_n) \geqq \frac{1}{n} \quad \text{in} \quad \Omega, \qquad B_\epsilon\hat{v}_n - \hat{\Phi}_{\epsilon,n}(\hat{v}_n) \geqq \frac{1}{n}(\beta + \epsilon) \quad \text{on} \quad \partial\Omega.$$

Hence, according to Lemma 4.1 (with $\delta = 1/n$), for every n, there exists exactly one solution $\bar{u}_n$ and $\hat{u}_n$ of

$$Lu = \bar{F}_n(u) \text{ in } \Omega, \quad B_\epsilon u = \bar{\Phi}_{\epsilon,n}(u) \text{ on } \partial\Omega,$$

and

$$Lu = \hat{F}_n(u) \text{ in } \Omega, \quad B_\epsilon u = \hat{\Phi}_{\epsilon,n}(u) \text{ on } \partial\Omega,$$

respectively. Moreover, $\bar{v}_n \leqq \bar{u}_n \leqq \hat{u}_n \leqq \hat{v}_n$. On $\bar{\Omega} \times [\bar{v}_{n-1}, \hat{v}_{n-1}]$, we have $\bar{f}_n \geqq \bar{f}_{n-1}$ and, on $\partial\Omega \times [\bar{v}_{n-1}, \hat{v}_{n-1}]$, we have $\bar{\varphi}_{1,n} \geqq \bar{\varphi}_{1,n-1}$. Hence,

$$L\bar{u}_n = \bar{F}_n(\bar{u}_n) \geqq \bar{F}_{n-1}(u_n), \quad B_\epsilon\bar{u}_n = \bar{\Phi}_{\epsilon,n}(\bar{u}_n) \geqq \bar{\Phi}_{\epsilon,n-1}(\bar{u}_n),$$

and therefore

$$L(\bar{u}_n - \bar{u}_{n-1}) \geqq \bar{F}_{n-1}(u_n) - \bar{F}_{n-1}(u_{n-1}),$$

$$B_\epsilon(\bar{u}_n - \bar{u}_{n-1}) \geqq \bar{\Phi}_{\epsilon,n-1}(\bar{u}_n) - \bar{\Phi}_{\epsilon,n-1}(\bar{u}_{n-1}),$$

and, by the monotonicity of $\bar{f}_n(x, \cdot)$ and $\bar{\varphi}_{1,n}(x, \cdot)$ and by the maximum principle, one proves, similarly as in the proof of Theorem 2, that $\bar{u}_{n-1} \leqq \bar{u}_n$. Analogously, $\hat{u}_n \leqq \hat{u}_{n-1}$ and, hence,

$$\bar{v}_1 \leqq \bar{u}_1 \leqq \bar{u}_2 \leqq \cdots \leqq \bar{u}_n \leqq \cdots \leqq \hat{u}_n \leqq \hat{u}_{n-1} \leqq \cdots \leqq \hat{u}_1 \leqq \hat{v}_1 .$$

Therefore, the sequence $\{\bar{u}_n\}$ converges pointwise to a function $\bar{u}$ on $\bar{\Omega}$ and $\{\hat{u}_n\}$ converges pointwise to a function $\hat{u}$ on $\bar{\Omega}$ such that

$$\bar{v} \leqq \bar{u} \leqq \hat{u} \leqq \hat{v}.$$

For every n, we define g_n on $\bar{\Omega} \times [\bar{v}_1, \hat{v}_1]^2$ by

$$g_n(x, \xi, \eta) \equiv \bar{f}_n(x, \xi),$$

and ψ_n on $\partial\Omega \times [\bar{v}_1, \hat{v}_1]^2$ by

$$\psi_n(x, \xi, \eta) \equiv \bar{\varphi}_{1,n}(x, \xi).$$

Then, $g_n \to \bar{f}$ and $\psi_n \to \bar{\varphi}_1$ pointwise and $\{g_n\}$ and $\{\psi_n\}$ are bounded in $C^\alpha(\bar{\Omega} \times [\bar{v}_1, \hat{v}_1]^2)$ and $C^{1+\alpha}(\partial\Omega \times [\bar{v}_1, \hat{v}_1]^2)$, respectively. Hence, by Proposition 3.3, $\bar{u} \in C^{2+\alpha}(\bar{\Omega})$ and it is a solution of (4.10) which, by the monotonicity of $\bar{f}(x, \cdot)$ and $\bar{\varphi}_1(x, \cdot)$, is unique. In the same way it is shown that $\hat{u}$ is the unique solution of (4.11). Q.E.D.

Now we are ready for the proof of our main existence theorem. This theorem is proved by means of a non-linear iteration argument which in some sense is similar to the proof of Theorem 1. But instead of using a hypothesis of type (4.2) we use the fact that a Hölder-continuous function $h(\xi)$ can be majorized by "parabolas" of the form $h(\xi_0) \pm \gamma\, |\xi - \xi_0|^\alpha$.

Theorem 3. *Let Hypothesis (H) be satisfied. Then, there exists at least one solution of* (4.1) *such that $\bar{v} \leqq u \leqq \hat{v}$. Moreover, there exists a maximal solution $\hat{u}$ with $\bar{v} \leqq \hat{u} \leqq \hat{v}$ and a minimal solution $\bar{u}$ with $\bar{v} \leqq \bar{u} \leqq \hat{v}$ in the sense that, for every solution u with $\bar{v} \leqq u \leqq \hat{v}$,*

$$\bar{v} \leqq \bar{u} \leqq u \leqq \hat{u} \leqq \hat{v}.$$

Proof. For every pair $\bar{w}, \hat{w} \in C^{2+\alpha}(\bar{\Omega})$ satisfying $\bar{v} \leqq \bar{w} \leqq \hat{w} \leqq \hat{v}$ we define $\bar{f}(\cdot\,;\bar{w}) \in C^\alpha(\bar{\Omega} \times [\bar{w}, \hat{v}])$ and $\hat{f}(\cdot\,;\hat{w}) \in C^\alpha(\bar{\Omega} \times [\bar{v}, \hat{w}])$ by

$$\bar{f}(x, \xi; \bar{w}) \equiv f(x, \bar{w}(x)) - \gamma(\xi - \bar{w}(x))^\alpha,$$

and

$$\hat{f}(x, \xi; \hat{w}) \equiv f(x, \hat{w}(x)) + \gamma(\hat{w}(x) - \xi)^\alpha.$$

Similarly, we define $\bar{\varphi}_1(\cdot\,;\bar{w}) \in C^{1+\alpha}(\partial\Omega \times [\bar{w}, \hat{v}])$ and $\hat{\varphi}_1(\cdot\,;\hat{w}) \in C^{1+\alpha}(\partial\Omega \times [\bar{v}, \hat{w}])$ by

$$\bar{\varphi}_1(x, \xi; \bar{w}) \equiv \varphi_1(x, \bar{w}(x)) - \mu(\xi - \bar{w}(x)),$$

and

$$\hat{\varphi}_1(x, \xi; \hat{w}) \equiv \varphi_1(x, \hat{w}(x)) + \mu(\hat{w}(x) - \xi),$$

respectively. Here, γ and μ have the same meaning as in the definitions preceding Lemma 4.1.

Consider the sequences $\{\bar{u}_n\}$, $\{\hat{u}_n\} \subset C^{2+\alpha}(\bar{\Omega})$ defined by

$$\bar{u}_0 = \bar{v},$$

$$L\bar{u}_n = \bar{F}(\bar{u}_n\,;\bar{u}_{n-1}), \qquad B_\epsilon \bar{u}_n = \epsilon\bar{\Phi}_1(\bar{u}_n\,;\bar{u}_{n-1}) + (1-\epsilon)\varphi_0\,,$$

and

$$\hat{u}_0 = \hat{v},$$

$$L\hat{u}_n = \hat{F}(\hat{u}_n\,;\hat{u}_{n-1}), \qquad B_\epsilon \hat{u}_n = \epsilon\hat{\Phi}_1(\hat{u}_n\,;\hat{u}_{n-1}) + (1-\epsilon)\varphi_0\,.$$

Then, by means of Lemma 4.2, with $\bar{v}$ replaced by $\bar{u}_{n-1}$ and $\hat{v}_n$ replaced by $\hat{u}_{n-1}$, it follows easily that these sequences are well-defined and satisfy the inequalities.

$$\bar{v} \leqq \bar{u}_1 \leqq \cdots \leqq \bar{u}_n \leqq \cdots \leqq \hat{u}_n \leqq \cdots \leqq \hat{u}_1 \leqq \hat{v}.$$

Hence, $\{\bar{u}_n\}$ and $\{\hat{u}_n\}$ converge pointwise on $\bar{\Omega}$ to functions $\bar{u}$ and $\hat{u}$, respectively, such that $\bar{v} \leqq \bar{u} \leqq \hat{u} \leqq \hat{v}$.

For every n, define $g_n \in C^\alpha(\bar{\Omega} \times [\bar{v}, \hat{v}]^2)$ by

$$g_n(x, \xi, \eta) \equiv \begin{cases} f(x, \xi) & \text{if } \bar{v}(x) \leqq \xi \leqq \eta \leqq \hat{v}(x), \\ f(x, \eta) - \gamma(\xi - \eta)^\alpha & \text{if } \bar{v}(x) \leqq \eta \leqq \xi \leqq \hat{v}(x), \end{cases}$$

and, similarly, define $\psi_n \in C^{1+\alpha}(\partial\Omega \times [\bar{v}, \hat{v}]^2)$ by

$$\psi_n(x, \xi, \eta) \equiv \varphi_1(x, \eta) - \mu(\xi - \eta).$$

With these definitions, the sequence $\{\bar{u}_n\}$ satisfies

$$L\bar{u}_n = G_n(\bar{u}_n\,, \bar{u}_{n-1}) \text{ in } \Omega,$$

$$B_\epsilon \bar{u}_n = \epsilon\Psi_n(u_n\,, u_{n-1}) + (1-\epsilon)\varphi_0 \text{ on } \partial\Omega.$$

Hence Proposition 3.3 can be applied to prove that $\bar{u} \in C^{2+\alpha}(\bar{\Omega})$ and satisfies

$$L\bar{u} = F(\bar{u}) \quad \text{in} \quad \Omega,$$

$$B_\epsilon \bar{u} = \Phi_\epsilon(\bar{u}) \quad \text{on} \quad \partial\Omega.$$

In the same way it follows that $\hat{u}$ is a solution of (4.1). The fact that $\bar{u}$ is the minimal and $\hat{u}$ is the maximal solution of (4.1) follows as in the proof of Theorem 1. Q.E.D.

Remark 4.3. It is immediately seen that all the sequences which appear throughout the preceding proofs are well-defined if the coefficients of L satisfy the weaker continuity hypotheses a_{ik}, a_i, $a \in C^\alpha(\bar{\Omega})$. The only occasion where the stronger assumptions $a_{ik} \in C^{2+\alpha}(\bar{\Omega})$, $a_i \in C^{1+\alpha}(\bar{\Omega})$ have explicitly been used was in the proof of Lemma 3.2, namely in order to guarantee that the adjoint operator L^* has Hölder-continuous coefficients. But (see Remark 3.4), Lemma 3.2 is only used in case the boundary conditions are not independent of the solution u. Hence, *if the boundary conditions are independent of* u (*i.e.* $\varphi_1(x, \xi) =$

$\varphi_1(x)$ or in case of the first boundary value problem $\epsilon = 0$) *then it suffices to assume that* a_{ik} , a_i , $a \in C^\alpha(\bar{\Omega})$.

5. Proof of the Remaining Theorems. With $\bar{v} \equiv 0$ and $\hat{v} \equiv v$ it is obvious that the sufficient part of Theorem A is a consequence of Theorem 3. That the stated condition is also necessary is trivial.

Proof of Theorem B. Denote by v the unique solution of the linear boundary value problem

$$Lu = g \text{ in } \Omega,$$
$$B_\epsilon u = \epsilon\psi + (1 - \epsilon)\varphi_0 \text{ on } \partial\Omega.$$

The assumptions of Theorem B imply $g \geqq 0$ and $\psi \geqq 0$ hence, by the maximum principle, $v \geqq 0$. Moreover,

$$Lv = g \geqq F(v) \text{ in } \Omega,$$
$$B_\epsilon v = \epsilon\psi + (1 - \epsilon)\varphi_0 \geqq \Phi_\epsilon(v) \text{ on } \partial\Omega.$$

Hence Theorem A is applicable and the proof follows. Q.E.D.

Proof of Theorem C. Since, for all $u \geqq 0$, $F(u) \geqq 0$ and $\Phi_\epsilon(u) \geqq 0$, it is an immediate consequence of the maximum principle that no $\lambda < 0$ belongs to the spectrum of (2.3).

Let $\lambda_1 > 0$ belong to the spectrum of (2.3) and let u_1 denote a corresponding "eigenfunction", *i.e.* $u_1 \geqq 0$ and

$$Lu_1 = \lambda_1 F(u_1) \text{ in } \Omega, \quad B_\epsilon u_1 = \lambda_1 \Phi_\epsilon(u_1) \text{ on } \partial\Omega.$$

Hence, for every $\lambda \in [0, \lambda_1]$,

$$Lu_1 \geqq \lambda F(u_1) \quad \text{in} \quad \Omega,$$
$$B_\epsilon u_1 \geqq \lambda \Phi_\epsilon(u_1) \quad \text{on} \quad \partial\Omega,$$

and Theorem A shows that λ belongs to the spectrum too. Therefore, the spectrum is an interval with 0 as left end-point. In order to show that this interval is non-degenerate denote by v_0 the unique solution of

$$Lu = 1 \text{ in } \Omega, \quad B_\epsilon u = 1 \text{ on } \partial\Omega.$$

By the maximum principle, $v_0 > 0$. Evidently, for $\lambda_0 > 0$ small enough, $\lambda_0 F(v_0) \leqq 1$ and $\lambda_0 \Phi_\epsilon(v_0) \leqq 1$. Hence,

$$Lv_0 \geqq \lambda_0 F(v_0) \text{ in } \Omega, \quad B_\epsilon v_0 \geqq \lambda_0 \Phi_\epsilon(v_0) \text{ on } \partial\Omega$$

and Theorem A implies that λ_0 belongs to the spectrum. Finally, since $\lambda = 0$ belongs to the spectrum the theorem follows. Q.E.D.

Proof of Theorem D. Assume $\lambda \geqq \lambda_0(r)$ and let $v \geqq 0$ be a solution of (2.5). Hence, by (2.4), $v > 0$ and

$$Lv = \lambda F(v) \geqq \lambda f_0 + \lambda r v \text{ in } \Omega,$$
$$B_\epsilon v = 0 \text{ on } \partial\Omega.$$

Therefore, by Theorem A, there exists at least one non-negative solution u of

$$Lu = \lambda f_0 + \lambda r u \text{ in } \Omega, \quad B_\epsilon u = 0 \text{ on } \partial\Omega. \tag{5.1}$$

Hence,

$$Lu - \lambda r u > 0 \text{ in } \Omega,$$
$$B_\epsilon u = 0 \text{ on } \partial\Omega,$$

which implies, in particular, that $u > 0$. Now denote by w_0 a non-negative eigenfunction belonging to $\lambda_0(r)$. It is well-known that $w_0(x) > 0$ for all $x \in \Omega$.

By integrating the expression $wLu - uLw$ over Ω and using relation (5.1) we obtain

$$(\lambda_0(r) - \lambda) \int_\Omega ruw\, dx = \lambda \int_\Omega f_0 w\, dx.$$

Since both integrals are positive, this implies $\lambda < \lambda_0(r)$ which contradicts our assumption. This proves the theorem.

Proof of Theorem E. The inequality (2.7) implies $g \geqq 0$ and hence, since $h \leqq a$, the boundary value problem

$$(L - h)u - \lambda r u = g \text{ in } \Omega,$$
$$B_\epsilon u = 0 \text{ on } \partial\Omega,$$

has, for every $\lambda < \mu_0(h)$, a unique solution v which satisfies $v \geqq 0$ by the weak "Positivity Lemma" of Keller and Cohen (*e.g.* [3, p. 308]). But

$$Lv = g + hv + \lambda r v \geqq F(v, \lambda) + \lambda r v \text{ in } \Omega,$$
$$B_\epsilon v \geqq 0 \text{ on } \partial\Omega,$$

and Theorem A applies to the problem

$$Lu = F(u, \lambda) + \lambda r u \text{ in } \Omega,$$
$$B_\epsilon u = 0 \text{ on } \partial\Omega,$$

which proves the theorem.

References

1. S. Agmon, A. Douglis, & L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I.*, Comm. Pure Appl. Math. **XII** (1959), 623–727.
2. D. S., Cohen, *Positive solutions of a class of nonlinear eigenvalue problems*, J. Math. Mech. **17** (1967), 209–216.

3. ———, *Positive solutions of non-linear eigenvalue problems: applications to non-linear reactor dynamics*, Arch. Rat. Mech. Anal. **26** (1967), 305–315.
4. ——— & T. W. LAETSCH, *Non-linear boundary value problems suggested by chemical reactor theory*, J. Diff. Equ. **7** (1970), 217–226.
5. E. GAGLIARDO, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova **XXVII** (1957), 284–305.
6. H. B. KELLER, *Nonexistence and uniqueness of positive solutions of nonlinear eigenvalue problems*, Bull. Amer. Math. Soc. **74** (1968), 887–891.
7. ———, *Positive solutions of some non-linear eigenvalue problems*, J. Math. Mech. **19** (1969), 279–295.
8. ———, *Elliptic boundary value problems suggested by non-linear diffusion processes*, Arch. Rat. Mech Anal. **35** (1969), 363–381.
9. ——— & D. S. COHEN, *Some positone problems suggested by non-linear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.
10. O. A. LADYZHENSKAYA & N. N. URAL'TSEVA, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.
11. C. MIRANDA, *Partial differential equations of elliptic type*, Springer-Verlag, New York, 1970.
12. C. B. MORREY, *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966.
13. M. H. PROTTER & H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967.
14. M. SCHECHTER, *On L^p estimates and regularity, I.*, Amer. J. Math. **LXXXV** (1963), 1–13.
15. ———, *Coerciveness in L^p*, Trans. Amer. Math. Soc. **107** (1963), 10–29.
16. L. F. SHAMPINE, *Some non-linear eigenvalue problems*, J. Math. Mech. **17** (1968), 1065–1072.
17. ——— & G. M. WING, *Existence and uniqueness of solutions of a class of non-linear elliptic boundary value problems*, J. Math. Mech. **19** (1970), 971–979.
18. R. B. SIMPSON & D. S. COHEN, *Positive solutions of non-linear elliptic eigenvalue problems*, J. Math. Mech. **19** (1970), 895–910.

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