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# ON THE EXISTENCE OF PROBABILITY DISTRIBUTIONS WITH GIVEN MARGINALS

Пусть $X = \{0, \dots, n-1\}$ и $\Gamma := \{(x_1, \dots, x_s) \in X^s \colon \sum_{\sigma=1}^{s} x_\sigma = n-1\}$. Дается простая характеризация маргинальных распределений вероятностных законов на $\Gamma$, обладающих тем дополнительным свойством, что они образуют $s$-кортеж убывающих вероятностей на $X$. Эта характеризация имеет интересное применение к асимптотическим спектрам в смысле Штрассена [5], [6]. Некоторые смежные вопросы обсуждаются в приложении.

*Ключевые слова и фразы*: вероятностный закон, маргинальные распределения, асимптотические спектры в смысле Штрассена, необходимые и достаточные условия.

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**1. Introduction and results.** Let $X_1, \dots, X_s$ be nonempty finite sets and let $\Gamma$ be a nonempty subset of $\prod X_\sigma$. For any probability distribution $p$ on $\Gamma$ we denote its marginal distributions on $X_1, \dots, X_s$ by $p^{(1)}, \dots, p^{(s)}$, respectively. We define

$$R := \{\underline{p} = (p_1, \dots, p_s)\colon p_\sigma \text{ are probability distributions of } X_\sigma\},$$
$$M(\Gamma) := \{(p^{(1)}, \dots, p^{(s)})\colon p \text{ is a probability distribution on } \Gamma).$$

Obviously $M(\Gamma)$ and $R$ are polytopes in $E := \prod \mathbf{R}^{X_\sigma}$ and $M(\Gamma) \subset R$. A necessary condition for $\underline{p} \in R$ to belong to $M(\Gamma)$ is for any $A_1 \subset X_1, \dots, A_s \subset X_s$

$$\sum p_\sigma(A_\sigma) \leqslant \max\Big\{\sum I_{A_\sigma}(x_\sigma)\colon (x_1, \dots, x_s) \in \Gamma\Big\}, \tag{1}$$

where $I_A$ denotes the indicator function of $A$. (Notice that the left-hand side equals $\int \sum I_{A_\sigma}\, p(d\underline{x})$, where the $p_\sigma$ are the marginal distributions of $p$.) Dall'Aglio [2] and Kellerer [3] have shown that this condition is sufficient for $s = 2$, but not for $s \geqslant 3$. Nevertheless, by the duality theory for systems of linear inequalities one always has the necessary and sufficient condition that for any real functions $f_\sigma$ on $X_\sigma$

$$\sum \int f_\sigma\, dp_\sigma \leqslant \max\Big\{\sum f_\sigma(x_\sigma)\colon (x_1, \dots, x_s) \in \Gamma\Big\}. \tag{2}$$

(Again the necessity is trivial.) There also exists a fast algorithm for deciding this (the simplex algorithm). However, at least from a theoretical point of view, the above decision problem seems to be typically much harder for $s \geqslant 3$ than for $s = 2$.

In the main part of this paper we will treat the following parametric situation: Take a positive integer $n$ and set $X := \{0, \dots, n-1\}$ and

$$\Gamma := \left\{(x_1, \dots, x_s \in X^s\colon \sum_{\sigma=1}^{s} x_\sigma = n-1\right\}. \tag{3}$$

Thus $\Gamma$ is the simplex with vertices $(n-1, 0, \dots, 0), \dots, (0, \dots, 0, n-1)$ in the discrete hypercube $X^s$. Does $M := M(\Gamma)$ have a simple description?

If we take $f_\sigma(i) := i - (n-1)/s$ for all $\sigma$, then $\pm \sum f_\sigma(x_\sigma) = 0$ on $\Gamma$. Hence by the (trivial) necessity of (2) we have $\sum \int f_\sigma\, dp_\sigma = 0$, i.e.,

$$h(\underline{p}) := \sum_\sigma \sum_i \left(i - \frac{n-1}{s}\right) p_\sigma(i) = 0 \qquad \text{for any} \quad \underline{p} \in M. \tag{4}$$

Call a probability distribution $q$ on $X$ decreasing, when $q(i+1) \leqslant q(i)$ for all $i$. Our main (and perhaps surprising) result is that for $s$-tuples of decreasing probability distributions condition (4) is already sufficient. Define

$$R^> := \{\underline{p} \in R\colon p_\sigma \text{ is decreasing } \forall \sigma\}, \qquad H := \{\underline{p} \in (\mathbf{R}^n)^s\colon h(\underline{p}) = 0\}.$$

$R^>$ is a polytope and $H$ a hyperplane in $(\mathbf{R}^n)^s$. As we have seen above, $M \subset R \cap H$, hence $R^> \cap M \subset R^> \cap H$.

**Theorem 1.** $R^> \cap M = R^> \cap H$.

Thus for systems of decreasing probability distributions a single equation has to be tested to decide on marginality. We note that the restriction to decreasing probability distributions is necessary. In fact $M$ is dramatically different from $R \cap H$ in the sense that $R \cap H$ has at most $sn$ facets (since $R$ is a product of $s$ simplices and therefore has $sn$ facets), whereas $M$ has an exponential number of facets already for $s = 3$, as will be shown in the Appendix.

The phenomenon that the number of facets of $M(\Gamma)$ for certain $\Gamma$ grows faster than any polynomial in $n$ occurs even in the case $s = 2$. This is again shown in the Appendix and bases on Proposition 2 which states that if $\Gamma \subset X \times X$ is a partial order with a minimal and a maximal element then the number of concise facets of $M(\Gamma)$ equals the number of order ideals of $X$ minus 2.

In this paper we will prove Theorem 1 for $s = 3$ and refer the reader to [4] for a proof of the general case.

An interesting application of Theorem 1 in the case $s = 3$ concerns asymptotic spectra in the sense of Strassen [5], [6]. These spectra are compact Hausdorff spaces, which control (in a certain asymptotic sense) degenerations as well as the computational complexity of finite dimensional bilinear maps. In [6] a singular 2-simplex $\zeta$ with values in $\Delta$ (the so-called support simplex) is constructed for certain asymptotic spectra $\Delta$, and in [7] it is conjectured that the image of $\zeta$ coincides with $\Delta$. At any rate the support simplex gives *necessary* conditions for the existence of degenerations between the bilinear maps under consideration.

**Corollary 1.** *Let $k$ be an algebraically closed field and let $f$ be the multiplication map of the algebra $k[T]/(T^n)$. Denote the standard 2-simplex by $\Theta$ and the asymptotic spectrum of $f$ by $\Delta \subset \mathbf{R}$. Then the support simplex $\zeta\colon \Theta \to \Delta$ of $f$ is given by*

$$\zeta(\theta) = \mu^{-(n-1)} \prod_{\sigma=1}^{3} \left( \sum_{i=0}^{n-1} \mu^{i/\theta_\sigma} \right)^{\theta_\sigma}, \tag{5}$$

*where $\mu$ is a unique positive solution of*

$$\sum_{\sigma=1}^{3} \frac{\sum_{i=0}^{n-1} i\mu^{i/\theta_\sigma}}{\sum_{i=0}^{n-1} \mu^{i/\theta_\sigma}} = n - 1. \tag{6}$$

By the Chinese Remainder Theorem any algebra $k[T]/(F)$ with $F \in k[T]\backslash\{0\}$ is isomorphic to a direct product of algebras of the type considered in Corollary 1. Therefore our corollary immediately gives the support simplex of the class of all algebras $k[T]/(F)$. (If the above mentioned conjecture is correct, we thus have computed the asymptotic spectrum for this class of bilinear maps. The asymptotic spectrum of a single algebra $k[T]/(F)$ has already been determined in [6].)

My thanks go to Volker Strassen for useful discussion about this matter.

**2. Proof of Theorem 1.** The inclusion «$\subset$» already being known we prove «$\supset$» by induction on $n$, the start $n = 1$ being clear. So assume $n > 1$.

It will suffice to show that the vertices of the polytope $R^{>} \cap H$ belong to $R^{>} \cap M$. Hence we determine the vertices of $R^{>} \cap H$. As $H$ is a hyperplane, any such vertex is the intersection of an edge (a one-dimensional face) of $R^{>}$ with $H$. It is well known and easily seen that the vertices of the polytope of all decreasing probability distributions on $\{0, \dots, n-1\}$ are $r_1, \dots, r_n$, where

$$r_m(i) := \begin{cases} m^{-1} & \text{for } \; i < m, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Now $R^{>}$ is a direct product of three copies of this polytope. An edge of such a product is a product of one edge and two vertices in some order. In our case symmetry allows us to focus on edges of $R^{>}$ of the form $\{(1-\lambda)(r_{m_1}, r_{m_2}, r_{m_3}) + \lambda(r_{m_1}, r_{m_2}, r_{m'_3})\colon \lambda \in [0,1]\}$ (where without loss of generality $m_3 < m'_3$). Hence we will consider vertices of $R^{>} \cap H$ of the form

$$\underline{p} = (1-\lambda)(r_{m_1}, r_{m_2}, r_{m_3}) + \lambda(r_{m_1}, r_{m_2}, r_{m'_3}), \qquad m_3 < m'_3, \tag{8}$$

where $\lambda$ is such that $\underline{p} \in H$. Now $h(r_{m_1}, r_{m_2}, r_{m_3}) = -(n-1) + \sum_\sigma \sum_i i r_{m_\sigma}(i) = -(n-1) + \sum_\sigma (m_\sigma - 1)/2$. Therefore the condition $\underline{p} \in H$ gives $0 = (1-\lambda)\, h(r_{m_1}, r_{m_2}, r_{m_3}) + \lambda h(r_{m_1}, r_{m_2}, r_{m'_3}) = -(n-1) + \frac{1}{2}((m_1 - 1) + (m_2 - 1) + (1-\lambda)(m_3 - 1) + \lambda(m'_3 - 1)) = -(n + \frac{1}{2}) + \frac{1}{2}(m_1 + m_2 + m_3 + \lambda(m'_3 - m_3))$, hence

$$\lambda = \frac{(2n+1) - m_1 - m_2 - m_3}{m'_3 - m_3}. \tag{9}$$

In order that $0 \leqslant \lambda \leqslant 1$ the following inequalities have to be satisfied:

$$m_1 + m_2 + m_3 \leqslant 2n + 1 \leqslant m_1 + m_2 + m'_3. \tag{10}$$

Now we take a vertex $\underline{p}$ of $R^{>} \cap H$. In order to apply the induction hypothesis we choose $k \in \{1, \ldots, n-1\}$ and imbed the cube $\{0, \ldots, n-k-1\}^3$ into the cube $\{0, \ldots, n-1\}^3$ by means of the map $\varphi := \varphi_1 \times \varphi_2 \times \varphi_3$, where $\varphi_1, \varphi_2$ are the natural imbeddings of $\{0, \ldots, n-k-1\}$ into $\{0, \ldots, n-1\}$ and $\varphi_3$ is defined by $\varphi_3(i) := i+k$. This imbedding $\varphi$ has the property that $\varphi^{-1}(\Gamma_n) = \Gamma_{n-k}$ (with obvious notation; similar notation will be used for $R, M$, and $R^{>}$).

**Lemma 1.** *Under the induction hypothesis the following is true: If* $1 \leqslant k \leqslant n-1$ *and* $\underline{u} \in R \cap H$ *is such that* $u_1(n-1-i) = u_2(n-1-i) = u_3(i) = 0$ *for* $0 \leqslant i \leqslant k-1$ *and* $u_1, u_2$ *are decreasing, whereas* $u_3$ *is decreasing on* $\{k, \ldots, n-1\}$, *then* $\underline{u} \in M$.

P r o o f. Define $\underline{\varphi}$: $R_{n-k} \to R_n$ by $\underline{\varphi}(\underline{q}) := (\varphi_1(q_1), \varphi_2(q_2), \varphi_3(q_3))$. Then we have for any probability distribution $q$ on $\Gamma_{n-k}$ that the marginal triple of $\varphi(q)$ is the image under $\underline{\varphi}$ of the marginal triple of $q$. Hence $\underline{\varphi}$ maps $M_{n-k}$ into $M_n$. Moreover

$$h_{n-k}(\underline{q}) = -(n-k-1) + \sum_{\sigma} \sum_{i=0}^{n-k-1} i q_\sigma(i) = -(n-k-1) + \sum_{i=0}^{n-1} i(\varphi_1 q_1)(i) + \sum_{i=0}^{n-1} i(\varphi_2 q_2)(i) + \sum_{i=0}^{n-1} (i-k)(\varphi_3 q_3)(i) = h_n(\underline{\varphi}\underline{q}).$$

Now let $\underline{u} \in R_n \cap H_n$ satisfy the assumptions of the lemma. It is clear that there exists a (unique) $\underline{q} \in R^{>}_{n-k}$ such that $\underline{u} = \underline{\varphi}\,\underline{q}$. The above computation shows that $\underline{q}$ is even in $R^{>}_{n-k} \cap H_{n-k}$. By induction hypothesis $\underline{q} \in M_{n-k}$. Hence $\underline{u}$ is in $M_n$. Lemma 1 is proved.

It is clear that $\underline{p}$ does not satisfy the assumptions of the lemma, since $p_3$ is decreasing. Therefore we write $\underline{p}$ as a convex combination $\underline{p} = (1-\alpha)\,\underline{u} + \alpha \underline{v}$, where $\underline{u}, \underline{v} \in R \cap H$ are such that either both $\underline{u}$ and $\underline{v}$ satisfy the assumptions of the lemma (up to permutation of the three coordinates), or $\underline{u}$ does while $\underline{v}$ may be treated by the following lemma (or a permuted version of it).

**Lemma 2.** *Suppose* $\underline{v} \in R$ *is such that* $v_3$ *is the point mass at* 0 *(i.e.,* $v_3(0) = 1$*). Then* $\underline{v} \in M$ *if and only if* $v_2(i) = v_1(n-1-i)$ *for all* $i$.

P r o o f. If $v_3$ is concentrated on 0, any probability distribution $v$ on $\Gamma$ with marginal triple $\underline{v}$ is supported by $\{(n-1-i, i, 0)\colon 0 \leqslant i \leqslant n-1\}$. Hence $v_2(i) = v(n-1-i, i, 0) = v_1(n-1-i)$. Conversely, the probability distribution $v$ on $\Gamma$ defined by $v(n-1-i, i, 0) := v_1(n-1-i) = v_2(i)$ has marginal triple $\underline{v}$. Lemma 2 is proved.

We continue with the proof of the theorem by a case by case analysis. Remember that $p$ is of the form (8) with (9) and (10).

C a s e 1: $p_1(n-1) > 0, p_2(n-1) > 0$. By (8) this means that $m_1 = m_2 = n$, hence by the first inequality of (10) we have $m_3 = 1$. From (9) we now get $\lambda = 0$, which in view of (8) says that $p_3$ is the point mass at 0. (The converse is also true: If $p_3$ is the point mass at 0, the assumption $m_3 < m_3'$ together with (8) gives $\lambda = 0$ and $m_3 = 1$, therefore by (9) one gets $m_1 = m_2 = n$.) Now we apply Lemma 2.

C a s e 2: $p_1(n-1) = p_2(n-1) = 0$. Here we use a decomposition $\underline{p} = (1-\alpha)\,\underline{u} + \alpha \underline{v}$, where $u_3(0) = 0$ and $v_3(0) = 1$ (i.e., $v_3$ is the point mass at 0). We will treat $\underline{u}$ by Lemma 1 and $\underline{v}$ by Lemma 2. Evaluating the equation $p_3 = (1-\alpha)\,u_3 + \alpha v_3$ at 0 we obtain $\alpha = p_3(0) > 0$ ($p_3$ is decreasing). Since $p_3(0) = 1$ is equivalent to Case 1 (as we have seen above), we conclude that $0 < \alpha < 1$. As a consequence $\underline{u}$ is determined by $\underline{v}$, i.e., by $v_1$ and $v_2$. By Lemma 2 the condition $\underline{v} \in M$ means that $v_2(i) = v_1(n-1-i)$ for all $i$. Using this as a definition of $v_2$ in terms of $v_1$ it remains to define $v_1$. The requirement that $u_1$ and $u_2$ be decreasing probability distributions actually dictates that $v_1$ is the rectangular distribution $r_{n-m_2, m_1}$, where

$$r_{l,m}(i) := \begin{cases} (m-l)^{-1} & \text{if } \; l \leqslant i < m, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

for $0 \leqslant l < m \leqslant n$. (By (10) we have $n+1 \leqslant m_1 + m_2 + m_3' - n \leqslant m_1 + m_2$, hence $n - m_2 < m_1$, hence $r_{n-m_2, m_1}$ is well defined.) By the above definitions $\underline{v}$ is in $M$. Since $M \subset H$ this implies that $\underline{u}$ is in $H$. It remains to show that $\underline{u}$ is in $R$, i.e., that $u_1$ and $u_2$

are decreasing probability distributions. (Then we may apply Lemma 1 with $k = 1$ to $\underline{u}$ and finish Case 2.) By symmetry it suffices to treat $u_1$. Since $p_1 = r_{m_1}$ and $v_1 = r_{n-m_2,m_1}$, we obtain

$$u_1 = \frac{r_{m_1} - \alpha r_{n-m_2,m_1}}{1-\alpha}. \tag{12}$$

We only have to show that $u_1$ is nonnegative, i.e., that $\alpha \leqslant (m_1 + m_2 - n)/m_1$. (Then automatically $u_1$ is decreasing.) Now $\alpha = p_3(0)$. Hence by (8) and (9) we obtain

$$\begin{aligned} \alpha &= (1-\lambda)\, r_{m_3}(0) + \lambda\, r_{m_3'}(0) = \frac{m_3' - \lambda(m_3' - m_3)}{m_3' m_3} \\ &= \frac{1}{m_3'}\left(1 + \frac{m_3' - (2n+1) + m_1 + m_2}{m_3}\right). \end{aligned}$$

The numerator in the last expression is nonnegative by (10), hence by replacing $m_3$ by its minimal value 1 we obtain $\alpha \leqslant 1 + (-2n + m_1 + m_2)/m_3'$. Obviously, $-2n + m_1 + m_2 \leqslant 0$, hence by replacing $m_3'$ by its maximal value $n$ we obtain $\alpha \leqslant (m_1 + m_2 - n)/n$. Since $n \geqslant m_1$ and $\alpha$ is positive, this implies the desired inequality and therefore finishes Case 2.

Before we treat the remaining two cases we argue that we may assume $0 < \lambda < 1$ (and hence strict inequalities in (10)). Indeed, when $\lambda$ is either 0 or 1, the triple $\underline{p}$ consists of three rectangular distributions of the form (7). By symmetry we may assume that either $p_1(n-1) > 0$, $p_2(n-1) > 0$, or $p_1(n-1) = p_2(n-1) = 0$. These cases have already been handled in Case 1 and Case 2, respectively.

Without loss of generality we will also assume that $m_1 \leqslant m_2$.

Case 3: $p_1(n-1) = 0, p_2(n-1) > 0, p_3(n-1) = 0$. This will be treated in a similar way as Case 2, using a decomposition $\underline{p} = (1-\alpha)\,\underline{u} + \alpha\underline{v}$, where $u_2(0) = 0$ and $v_2(0) = 1$. We apply a permuted version of Lemma 1 to $\underline{u}$ and of Lemma 2 to $\underline{v}$. Evaluating the equation $p_2 = (1-\alpha)\, u_2 + \alpha v_2$ at 0 we obtain $\alpha = p_2(0) = 1/n$. As in Case 2 it suffices to define $v_1$. We set $v_1 := r_{n-m_3',m_1}$. By similar arguments as in Case 2 one proves that $r_{n-m_3',m_1}$ is well defined and that it suffices to show $u_1(m_1 - 1) \geqslant 0$ and $u_3(m_3' - 1) \geqslant 0$. This is equivalent to $n(m_1 + m_3' - n) \geqslant m_1$ and $\lambda n(m_1 + m_3' - n) \geqslant m_3'$, where $\lambda$ is given by (9). The first inequality is clear, since $m_1 + m_3' - n \geqslant 1$ by the well definedness of $r_{n-m_3',m_1}$. Instead of the second inequality we will prove the stronger inequality $\lambda(m_1 + m_3' - n) \geqslant 1$. In view of $m_2 = n$ this amounts to $(n - m_1 - m_3 + 1)(m_1 + m_3' - n) \geqslant m_3' - m_3$. Since $\lambda > 0$ the first factor on the left-hand side is $\geqslant 1$. We already know that the second factor is $\geqslant 1$. Therefore their product is greater or equal than their sum $m_3' - m_3 + 1$. This finishes Case 3.

Case 4: $p_1(n-1) = 0, p_2(n-1) > 0, p_3(n-1) > 0$. Here we write $\underline{p} = (1-\alpha)\,\underline{u} + \alpha\underline{v}$ in such a way that Lemma 1 applies to $u$ and a permuted version of Lemma 1 applies to $v$. Explicitly we define

$$\begin{aligned} \alpha &:= \frac{m_1 + m_3 - 1}{n} && \text{(our assumption } \lambda > 0 \text{ implies } 0 < \alpha < 1\text{)}, \\ u_1 &:= r_{m_1}, && v_1 := r_{m_1}, \\ u_2 &:= r_{n-m_1-m_3+1}, && v_2 := r_{n-m_1-m_3+1,n}, \\ u_3 &:= r_{m_3,n}, && v_3 := r_{m_3}. \end{aligned}$$

It is easily checked that $\underline{u}$ and $\underline{v}$ are well defined and lie in $R \cap H$. Now Lemma 1 applies to $\underline{u}$ with $k := m_3$ and to $\underline{v}$ with $k := n - m_1 - m_3 + 1$.

Since we have assumed $m_1 \leqslant m_2$, the four cases above exhaust all possibilities and the theorem is proved.

**3. Appendix.** First we are going to show that for $s = 3$ and $\Gamma$ given by (3) the polytope $M = M(\Gamma)$ has at least $2^{n-2} - 1$ facets. Then we will consider the case $s = 2$ and construct subsets $\Gamma$ of $X \times X$ such that $M(\Gamma)$ has exactly $2^{n-1}$ facets.

**3.1.** We begin with a general discussion of the faces of $M(\Gamma) \subset E = \prod \mathbf{R}^{X_\sigma}$ for arbitrary $\Gamma \subset \prod X_\sigma$.

First note that the set of vertices of $M(\Gamma)$ is

$$\{I_{\underline{x}} := (I_{x_1}, \ldots, I_{x_s}) \colon \underline{x} \in \Gamma\},$$

where $I_x$ for $x \in X_\sigma$ denotes the indicator of $\{x\}$. Indeed, $M(\Gamma)$ is obviously generated by this set and $(1 - I_{x_1}, \dots, 1 - I_{x_s}) \in E^*$ vanishes on $I_{\underline{x}}$, while it is positive on all other $I_{\underline{y}}$.

Any face $S$ of $M(\Gamma)$ is generated by the vertices it contains, i.e., by $\{I_{\underline{x}}: \underline{x} \in \Gamma_S\}$, where $\Gamma_S$ is a subset of $\Gamma$, uniquely determined by $S$. Since $M(\Gamma) \subset R$ lives in an affine subspace of $E$ not containing 0, the face $S$ is cut out by some $\underline{f} \in E^*$, so that $\underline{f} = 0$ on $S$, while $\underline{f} > 0$ on $M(\Gamma)\backslash S$. Then we have

$$\Gamma_S = \left\{\underline{x}: \sum_\sigma f_\sigma(x_\sigma) = 0\right\}.$$

If we call a subset of $\Gamma$ closed, when it is of the form $\Gamma_S$ for some face $S$ of $M(\Gamma)$, we have a bijective correspondence $S \mapsto \Gamma_S$ between faces of $M(\Gamma)$ and closed subsets of $\Gamma$. Under this bijection the facets of $M(\Gamma)$ correspond to the maximal proper closed subsets of $\Gamma$.

Let $\pi_\sigma$ denote the projection from $\Gamma$ to $X_\sigma$, i.e., $\pi_\sigma(\underline{x}) = x_\sigma$ for $\underline{x} \in \Gamma$, and call $\Gamma$ concise when all $\pi_\sigma$ are surjective. We say that a face $S$ of $M(\Gamma)$ is concise, when $\Gamma_S$ is concise.

It is easy to see that a nonconcise facet is generated by $\pi_\sigma^{-1}(X\backslash\{x_\sigma\})$ for some $\sigma \in \{1, \dots, s\}$, $x_\sigma \in X_\sigma$. Thus the number of nonconcise facets is bounded by the sum of the sizes of the $X_\sigma$. Since our aim is to exhibit more than polynomial growth of the number of facets in certain situations, we will focus on concise facets.

**3.2.** Here we specialize to the case $s = 3$, $X = \{0, \dots, n-1\}$, $\Gamma := \{\underline{x} \in X^3: \sum_\sigma x_\sigma = n-1\}$. The projections $\pi_\sigma$ are obviously surjective. We are going to prove that $M = M(\Gamma)$ has at least $2^{n-2} - 1$ concise facets.

For $\varnothing \neq A \subset \{0, \dots, n-3\}$ we define the subset $\overline{A}$ of $\Gamma$ by

$$\begin{aligned}\overline{A} := &\{(i, n-3-i, 2): i \in A\} \bigcup \{(i, n-2-i, 1): 0 \leqslant i < n-2\} \\ &\bigcup \{(i, n-1-i, 0): i-1 \notin A\}.\end{aligned}$$

Thus, when $\Gamma$ is considered as a discrete triangle, $\overline{A}$ is concentrated on the three lowest rows of $\Gamma$ in the following way: In row 2 one has $A$, row 1 is filled completely, and row 0 carries the complement of the vertical copy of row 2.

**Proposition 1.** *There exists exactly one concise facet $S(A)$ of $M$ such that $\overline{A} \subset \Gamma_{S(A)}$, and the map $A \mapsto S(A)$ is injective. Consequently, $M$ has at least $2^{n-2} - 1$ concise facets.*

P r o o f. Suppose $\underline{f} \in E^*$ cuts out a concise facet $S$ of $M$ such that $\overline{A} \subset \Gamma_S$. In particular, for all $\underline{x} \in \overline{A}$ we have $\sum_\sigma f_\sigma(x_\sigma) = 0$ and for all $\underline{x} \in \Gamma$ we have $\sum_\sigma f_\sigma(x_\sigma) \geqslant 0$.

The conciseness of $S$ means that $\Gamma_S$ has surjective projections. Since $\overline{A} \subset \Gamma_S$, this is automatically true for $\pi_1$ and $\pi_2$. Hence $S$ is concise if and only if

$$f_3(k) = -\min_{i,j:\, i+j+k=n-1} f_1(i) + f_2(j) \qquad \text{for} \quad k \geqslant 3. \tag{13}$$

We may adjust $\underline{f}$ by adding a linear combination of $\underline{h}, (0,1,-1), (1,-1,0) \in E^*$ (where $\underline{h}$ has been defined in the introduction and 1 denotes the constant function 1 on $X$), since these linear forms vanish on $M$. Therefore we may assume that $f_3(0) = f_3(2)$, $f_3(1) = 0$ and $f_1(0) = 0$, and by scaling $f_3(0) \in \{\pm 1, 0\}$.

(row 1) For all $0 \leqslant i \leqslant n-2$ we have $(i, n-2-i, 1) \in \overline{A}$, therefore $f_1(i) + f_2(n-2-i) = 0$.

(row 2) For $1 \leqslant i \leqslant n-2$ we have $(i-1, n-2-i, 2) \in \Gamma$, hence from what we already know we get $0 \leqslant f_1(i-1) + f_2(n-2-i) + f_3(2) = f_1(i-1) - f_1(i) + f_3(0)$, i.e., $f_1(i) \leqslant f_1(i-1) + f_3(0)$.

Changing $i$ to $i+1$ we obtain $f_1(i+1) \geqslant f_1(i) - f_3(0)$ for $0 \leqslant i \leqslant n-3$. Moreover, $f_1(i+1) = f_1(i) + f_3(0)$ when $i \in A$.

(row 0) For $0 \leqslant i \leqslant n-2$ we have $(i+1, n-2-i, 0) \in \Gamma$, hence $0 \leqslant f_1(i+1) + f_2(n-2-i) + f_3(0) = f_1(i+1) - f_1(i) + f_3(0)$, i.e., $f_1(i+1) \geqslant f_1(i) - f_3(0)$. Moreover, $f_1(i+1) = f_1(i) - f_3(0)$ when $i \notin A$.

Take $i \in A$. Then (row 2) and (row 0) give $f_3(0) = f_1(i+1) - f_1(i) \geqslant -f_3(0)$, hence $f_3 \geqslant 0$ and therefore $f_3(0) \in \{0, 1\}$.

Suppose $f_3(0) = 1$. Since $f_1(0) = 0$, (row 2) and (row 0) determine $f_1$ completely, (row 1) then fixes $f_2(i)$ for $i \leqslant n-2$. But note that $(0, n-1, 0) \in \overline{A}$, thus by what we already know we have $f_2(n-1) = -1$. Hence $f_2$ and by (13) also $f_3$ are determined completely. It is clear (essentially again by (13)) that $\underline{f}$ cuts out a concise face of $M$.

A similar reasoning shows that $f_3(0) = 0$ leads to $\underline{f} = 0$, which is impossible.

Thus we have shown that there is exactly one concise face $S(A)$ of $M$ such that $\overline{A} \subset \Gamma_s$. This face is necessarily a facet. Finally $S(A)$ together with the normalisation $f_3 = 1$ determines $\underline{f}$, which by (row 2) and (row 0) determines $A$.

**3.3.** Now we consider the case $s = 2$. We begin with an arbitrary $\Gamma \subset X_1 \times X_2$. Without loss of generality we may assume that $X_1 \cap X_2 = \varnothing$. We interpret $\Gamma \subset X_1 \times X_2$ as the set of edges of a bipartite graph on $X_1 \dot\cup X_2$, which we denote by $G(\Gamma)$. Observe that $\Gamma$ is automatically concise when $G(\Gamma)$ is connected.

**Lemma 3.** *Suppose $G(\Gamma)$ has $k$ connected components. Then*

$$\dim M(\Gamma) = |X_1| + |X_2| - k - 1.$$

P r o o f. Since $0 \notin M(\Gamma)$, we have to prove

$$\dim \operatorname{lin} M(\Gamma) = |X_1| + |X_2| - k. \tag{14}$$

Without loss of generality we may assume that $\Gamma$ is concise. Since $G(\Gamma)$ has $k$ connected components, we have decompositions $X_\sigma = \dot\bigcup_{1\leqslant j\leqslant k} X_{\sigma j}$ into nonempty subsets such that $\Gamma \subset \cup(X_{1j} \times X_{2j})$ and $(X_{1j} \dot\cup X_{2j}, (X_{1j} \times X_{2j}) \cap \Gamma)$ is connected. Since both sides of (14) behave additively with respect to such a decomposition, it suffices to prove (14) for connected $\Gamma$, i.e., to show $\operatorname{codim}(\operatorname{lin} M(\Gamma)) = 1$ for connected $\Gamma$. Clearly, the linear form $(-1, 1)$ vanishes on $\operatorname{lin} M(\Gamma)$. On the other hand, for any linear form $(f_1, f_2)$ vanishing on $M(\Gamma)$ we have a decomposition

$$\Gamma = \bigcup_{\lambda \in \operatorname{im} f_1} (f_1^{-1}(\lambda) \times f_2(f_1^{-1}(-\lambda)) \cap \Gamma.$$

(Indeed, for $(x_1, x_2) \in \Gamma$ let $f_1(x_1) = \lambda$. Then $f_2(x_2) = -\lambda$.) Since $\Gamma$ is connected, this decomposition has only one term, so that $f_1$ and $f_2$ are constant and $f_2 = -f_1$.

**Lemma 4.** *Suppose that $\Gamma$ is connected. Then the number of concise facets of $M(\Gamma)$ equals the number of pairs of nonempty subsets $A_1 \subset X_1$, $A_2 \subset X_2$ such that $(A_1 \times A_2) \cap \Gamma = \varnothing$ and $(A_1 \dot\cup A_2^c, (A_1 \times A_2^c) \cap \Gamma)$ and $(A_1^c \dot\cup A_2, (A_1^c \times A_2) \cap \Gamma)$ are connected.*

P r o o f. We construct a bijection between the two sets corresponding to the considered numbers.

First suppose $S$ is a concise facet of $M(\Gamma)$. Then by Lemma 3 we have $\dim S = \dim M(\Gamma) - 1 = |X_1| + |X_2| - 3$, which implies (again by Lemma 3) that $G(\Gamma_S)$ has two connected components. Hence we have decompositions $X_\sigma = A_\sigma \dot\cup A_\sigma^c$ such that $\Gamma_S \subset (A_1 \times A_2^c) \dot\cup (A_1^c \times A_2)$ and $(A_1 \dot\cup A_2^c, (A_1 \times A_2^c) \cap \Gamma)$ as well as $(A_1^c \dot\cup A_2, (A_1^c \times A_2) \cap \Gamma)$ are connected and consist of at least one point.

If $A_1 = \varnothing$, then by the conciseness of $S$ we have $X_2 = \pi_2(\Gamma_S) \subset \pi_2(X_1 \times A_2) = A_2$, hence $A_1 \dot\cup A_2^c = \varnothing$. So $A_1 \neq \varnothing$. Analogously we get $A_2 \neq \varnothing$.

Now the linear form $(I_{A_1}, I_{A_2} - 1)$ vanishes on $S$. Since $S$ is a facet the only linear forms vanishing on $S$ are linear combinations of $(1, -1)$ and $(I_{A_1}, I_{A_2} - 1)$ and hence $S$ is cut out by either $(I_{A_1}, I_{A_2} - 1)$ or $(-I_{A_1}, 1 - I_{A_2})$. Therefore either $(A_1^c \times A_2^c) \cap \Gamma = \varnothing$ or $(A_1 \times A_2) \cap \Gamma = \varnothing$. These two cases differ only in notation.

Conversely suppose $A_1, A_2$ are nonempty subsets of $X_1, X_2$, respectively, such that $(A_1 \times A_2) \cap \Gamma = \varnothing$ and $(A_1 \times A_2^c) \cap \Gamma$ and $(A_1^c \times A_2) \cap \Gamma$ are connected. Since $(A_1 \times A_2) \cap \Gamma = \varnothing$ the linear form $(-I_{A_1}, 1 - I_{A_2})$ is $\geqslant 0$ on $M(\Gamma)$ and cuts out $\Gamma_S := ((A_1 \times A_2^c) \cap \Gamma) \cup ((A_1^c \times A_2) \cap \Gamma)$ of $\Gamma$. Since $(A_1 \times A_2^c) \cap \Gamma$ and $(A_1^c \times A_2) \cap \Gamma$ are connected we have on the one hand from Lemma 3 that $\dim M(\Gamma_S) = \dim M(\Gamma) - 1$, therefore $\Gamma_S$ is a facet of $\Gamma$. On the other hand we get $\pi_1(\Gamma_S) = A_1^c \cup A_1 = X_1$ and analogously $\pi_2(\Gamma_S) = X_2$. So $\Gamma_S$ and the corresponding face $S$ are concise facets.

It is easy to see that both defined maps are inverse to each other and give the bijection mentioned above.

**Proposition 2.** *Suppose that $\Gamma \subset X \times X$ is a partial order with a minimal and a maximal elements. Then the number of concise facets of $M(\Gamma)$ coincides with the number of order ideals of $X$ not equal to $\varnothing$ or $X$.*

P r o o f. Since $\Gamma$ has a minimal and a maximal elements, it is clear that $\Gamma$ is connected. Now by Lemma 4 the number of concise facets equals the number of pairs $A_1, A_2$ of nonempty subsets of $X$ such that $(A_1 \times A_2) \cap \Gamma = \varnothing$ and $(A_1 \dot\cup A_2^c, (A_1 \times A_2^c) \cap \Gamma)$ and $(A_1^c \dot\cup A_2, (A_1^c \times A_2) \cap \Gamma)$ are connected (in the following called property (A)).

Now it suffices to prove that the pair $A_1, A_2$ has property (A) if and only if $A_2$ is an order ideal not equal to $\varnothing$ or $X$ and $A_1 = A_2^c$ (property (B)).

Suppose first that $A_1, A_2$ is a pair with property (A). $(A_1 \times A_2) \cap \Gamma) = \varnothing$ implies that for each $x_1 \in A_1$, $x_2 \in A_2$ we have $x_1 \not\leqslant x_2$, in particular $A_1 \cap A_2 = \varnothing$. Next we assert $A_1 \cup A_2 = X$. Suppose $x \in A_1^c \cap A_2^c$. Since $(A_1 \dot\cup A_2^c, (A_1 \times A_2^c) \cap \Gamma)$ is connected, there exists $x_1 \in A_1$ with $x_1 \leqslant x$. Since $(A_1^c \dot\cup A_2, (A_1^c \times A_2) \cap \Gamma)$ is connected there exists $x_2 \in A_2$ with $x \leqslant x_2$. But now $x_1 \leqslant x_2$, hence $(x_1, x_2) \in (A_1 \times A_2) \cap \Gamma$ which is impossible. Hence $X = A_1 \dot\cup A_2$. This implies immediately that $A_2$ is an order ideal: For each $x_2 \in A_2$ and $x_1 \leqslant x_2$ we have $x_1 \in A_1^c = A_2$. So $A_1, A_2$ has property (B).

Conversely, suppose $A_1, A_2$ is a pair with property (B). Since $A_2$ is nonempty the minimal element of $X$ is in $A_2 = A_1^c$, therefore $(A_1^c \dot\cup A_2, (A_1^c \times A_2) \cap \Gamma)$ is connected. Since $A_2 \neq X$ the maximal element of $X$ is not in $A_2$ and therefore $(A_1 \dot\cup A_2^c, (A_1 \times A_2^c) \cap \Gamma)$ is connected.

Applying Proposition 2 to $\{0, \dots, n-1\}$ with the natural order $\Gamma_1$, we see that $M(\Gamma_1)$ has (only) $n-2$ concise facets.

But already for $X = \{0, \dots, n-1\}^2$ with the product order $\Gamma_2 = \Gamma_1 \times \Gamma_1$ we get $\binom{2n}{n}$ order ideals. (Represent an order ideal by a staircase, i.e., a weakly decreasing function from $\{0, \dots, n-1\}$ to $\{0, \dots, n\}$.) Hence $M(\Gamma_2) = M(\Gamma_1) \otimes M(\Gamma_1)$ has $\binom{2n}{n} - 2$ facets. In particular, we obtain the somewhat surprising result that a tensor product of two polytopes, each with a number of facets linear in the number of vertices, may have an almost exponential number of facets. For the number of order ideals of higher powers of $\{0, \dots, n-1\}$ consult [1].

A truly exponential number of facets is exhibited in the case of $X := \{0, \dots, n-1\}$ with the partial order $\Gamma$ generated by $\{(0,1), \dots, (0, n-2), (1, n-1), \dots, (n-2, n-1)\}$. Here we have $2^{n-2}$ order ideals apart from $\varnothing$ and $X$ namely all $\{0\} \cup A$ with $A \subset \{1, \dots, n-2\}$.

If we define $X_1 := \{0, \dots, n-2\}$ and $X_2 := \{1, \dots, n-1\}$, then $M(\Gamma|_{X_1 \times X_2})$ has the same number of facets as $\Gamma$, but $X_1, X_2$ have only size $n-1$. This example is easily seen to give the maximum number of facets of $M(\Gamma)$ for any $\Gamma \subset Y_1 \times Y_2$ with $|Y_\sigma| \leqslant n-1$.

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