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On the existence of solutions for a fractional finite difference inclusion via three points boundary conditions

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Abstract

In this paper, we discussed the existence of solutions for the fractional finite difference inclusion $\Delta^{\nu} x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t))$ via the boundary value conditions $\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0$, $x(\eta) = 0$, and $\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0$, where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$, and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Keywords: fixed point; fractional finite difference inclusion; three points boundary conditions

1 Introduction

There are many works concerned with the existence of solutions for some fractional finite difference equations from different views by using the fixed point theory techniques (see for example, [1–7]). The readers can find more details as regards elementary notions and definitions of fractional finite difference equations in [8–15]. Also, much attention was devoted to the fractional differential inclusions (see for example, [9, 10, 16–24]). To the best of our knowledge, there is no published research work about fractional finite difference inclusions.

In 2011, Goodrich [25] investigated the general discrete fractional boundary problem, namely

$$
\begin{cases}
-\Delta^{\nu} y(t) = f(t + \nu - 1, y(t + \nu - 1)), \\
\alpha y(\nu - 2) - \beta \Delta y(\nu - 2) = 0, \\
\gamma y(\nu + b) - \delta \Delta y(\nu + b) = 0,
\end{cases}
$$

where $t \in [0, b]_{\mathbb{N}_0}$, $\nu \in (1, 2]$, and $\alpha \gamma + \alpha \delta + \beta \gamma \neq 0$ with $\alpha, \beta, \gamma, \delta \geq 0$. In this paper, with this thought and motivation in our minds, we investigate the existence of solution for the fractional finite difference inclusion

$$
\begin{cases}
\Delta^{\nu} x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t)), \\
\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0, \\
x(\eta) = 0, \\
\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0,
\end{cases}
$$

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where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$ and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

2 Preliminaries

As is well known, the Gamma function has some properties as $\Gamma(z + 1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. Define

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined. If $t + 1 - \nu$ is a pole of the gamma function and $t + 1$ is not a pole, then we define $t^{\underline{\nu}} = 0$. One can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu + 1)$ and $t^{\underline{\nu+1}} = (t - \nu)t^{\underline{\nu}}$. We use the notations $\mathbb{N}_a = \{a, a + 1, a + 2, \ldots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a + 1, a + 2, \ldots, b\}$ for all real numbers a and b whenever $b - a$ is a natural number.

Let $\nu > 0$ be such that $m - 1 < \nu \leq m$ for some natural number m. Then the νth fractional sum of f based at a is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} \left(t - \sigma(k)\right)^{\underline{\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+\nu}$. Similarly, we define

$$\Delta_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a}^{t+\nu} \left(t - \sigma(k)\right)^{\underline{-\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+m-\nu}$.

Lemma 2.1 [1] *Let $h : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ be a mapping and $2 < \nu \leq 3$. The general solution of the equation $\Delta_{\nu-3}^{\nu} x(t) = h(t)$ is given by*

$$x(t) = \sum_{i=1}^{3} c_i t^{\underline{\nu-i}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \left(t - \sigma(s)\right)^{\underline{\nu-1}} h(s), \tag{1}$$

where c_1, c_2, c_3 are arbitrary constants.

Since $\Delta t^{\underline{\mu}} = \mu t^{\underline{\mu-1}}$, we have

$$\Delta x(t) = \sum_{i=1}^{3} c_i(\nu - i) t^{\underline{\nu-i-1}} + \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{t-\nu+1} \left(t - \sigma(s)\right)^{\underline{\nu-2}} h(s) \tag{2}$$

for more information see [12].

Let (X, d) be a metric space. Denote by 2^X, $CB(X)$, and $P_{cp}(X)$ the class of all nonempty subsets, the class of all closed and bounded subsets, and the class of all compact subsets of X, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$.

Consider the Hausdorff metric $H_d : 2^X \times 2^X \to [0, \infty)$ by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ a map, and $T : X \to 2^X$ a multifunction.

We say that X obeys the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k. The map T is said to be α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$ [26]. Suppose that Ψ is the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ (for more on this please see [26]).

By using the following fixed point result, we review the existence of solutions for the fractional finite difference inclusion

$$\Delta_{\nu-3}^\nu x(t) \in F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big)$$

via the boundary conditions $\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0$, $\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0$, and $x(\eta) = 0$, where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$, and $F : \mathbb{N}_{\nu-3}^{b+\nu} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 2.2 [26] *Let (X, d) be a complete metric space, $\psi \in \Psi$ a strictly increasing map, $\alpha : X \times X \to [0, \infty)$ a map and $T : X \to CB(X)$ an α-admissible multifunction such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If X obeys the condition (C_α), then T has a fixed point.*

3 Main result

In this section, we consider the fractional finite difference inclusion

$$\Delta_{\nu-3}^\nu x(t) \in F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big) \tag{3}$$

via the boundary value conditions $\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0$, $\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0$, and $x(\eta) = 0$, where ξ, β, γ, δ are non-zero numbers, $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$, $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 3.1 *Let $y : \mathbb{N}_0^{b+1} \to \mathbb{R}$ and $2 < \nu < 3$. Then x_0 is a solution for the fractional finite difference equation $\Delta_{\nu-3}^\nu x(t) = y(t)$ via the boundary conditions $\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0$, $x(\eta) = 0$, and $\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0$ if and only if x_0 is a solution of the fractional sum equation $x(t) = \sum_{s=0}^{b+1} G(t, s, \eta) y(s)$, where*

$$
\begin{aligned}
G(t, s, \eta) = {} & \Bigg[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \\
& - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg] \\
& \times (b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} + \Bigg[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}}\Gamma(\nu)} \\
& + \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2 \beta_0 \eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg]\big(\eta - \sigma(s)\big)^{\underline{\nu-1}} \\
& + \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)},
\end{aligned}
$$

whenever $0 \leq s \leq t - \nu \leq b + 1$ *and* $0 \leq s \leq \eta - \nu \leq b + 1$,

$$
G(t,s,\eta) = \left[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right.
$$
$$
\left. - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right]
$$
$$
\times (b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} + \left[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \right.
$$
$$
\left. + \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right]\big(\eta - \sigma(s)\big)^{\underline{\nu-1}},
$$

whenever $0 \leq t - \nu < s \leq \eta - \nu \leq b + 1$,

$$
G(t,s,\eta) = \left[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right.
$$
$$
\left. - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right]
$$
$$
\times (b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} + \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)},
$$

whenever $0 \leq \eta - \nu < s \leq t - \nu \leq b + 1$ *and*

$$
G(t,s,\eta) = \left[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right.
$$
$$
\left. - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right]
$$
$$
\times (b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}},
$$

whenever $0 \leq t - \nu < s \leq b + 1$ *and* $0 \leq \eta - \nu < s \leq b + 1$. *Here,*

$$
\theta = \frac{\eta\beta\nu - \eta\xi - 3\eta\beta - 2\xi + \xi\nu - \beta\nu^2 + 6\beta\nu - 8\beta}{\beta(\nu - 2)},
$$
$$
\mu = \frac{b\xi\delta\nu - 2b\delta\xi + \gamma\xi b^2 + 3b\gamma\xi + \beta b\nu^2\delta + \delta b^2\beta\nu + \beta b\delta\nu - 6\beta\delta b + 3\beta\delta b^2 + 4\xi\delta\nu}{\beta(\nu - 2)}
$$
$$
+ \frac{-8\delta\xi + 4\gamma\xi b + 12\gamma\xi + 4\beta\nu^2\delta + 7\gamma\beta\nu b + 12\gamma\beta\nu + 4\beta\delta\nu - 24\beta\delta + 21\beta\gamma b + 36\beta\gamma}{\beta(\nu - 2)}
$$

and

$$
\beta_0 = \frac{\theta[\delta(\nu - 1) + \gamma(b + 2)](b + 3)(b + 4) + \mu(\eta + 2 - \nu)(\eta + 3 - \nu)}{\theta\mu}.
$$

Proof Let x_0 be a solution for the equation $\Delta_{\nu-3}^{\nu}x(t) = y(t)$ via the boundary conditions $\xi x(\nu - 3) + \beta\Delta x(\nu - 3) = 0$, $x(\eta) = 0$, and $\gamma x(b + \nu) + \delta\Delta x(b + \nu) = 0$. Then by using (2) and Lemma 2.1, we get

$$
x_0(t) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + c_3 t^{\underline{\nu-3}} + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}\big(t - \sigma(s)\big)^{\underline{\nu-1}}y(s)
$$

and

$$\Delta x_0(t) = c_1(\nu - 1)t^{\underline{\nu-2}} + c_2(\nu - 2)t^{\underline{\nu-3}} + c_3(\nu - 3)t^{\underline{\nu-4}}$$

$$+ \frac{1}{\Gamma(\nu - 1)} \sum_{s=0}^{t-\nu+1} \left(t - \sigma(s)\right)^{\underline{\nu-2}} y(s),$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary constants. Now, by using the boundary condition

$$\xi x(\nu - 3) + \beta \Delta x(\nu - 3) = 0,$$

we get $\xi c_3 + \beta[c_2(\nu - 2) + c_3(\nu - 3)] = 0$. Also, by using the condition $x(\eta) = 0$ we obtain

$$c_3 = -(\eta + 2 - \nu)(\eta + 3 - \nu)c_1 - (\eta + 2 - \nu)c_2$$

$$- \frac{1}{\eta^{\underline{\nu-3}}\Gamma(\nu)} \sum_{s=0}^{\eta-\nu} \left(\eta - \sigma(s)\right)^{\underline{\nu-1}} y(s).$$

Moreover, by using the boundary condition $\gamma x(b + \nu) + \delta \Delta x(b + \nu) = 0$, we get

$$c_1\left[\delta(\nu - 1) + \gamma(b + 2)\right](b+\nu)^{\underline{\nu-2}} + c_2\left[\delta(\nu - 2) + \gamma(b + 3)\right](b+\nu)^{\underline{\nu-3}}$$

$$+ c_3\left[\delta(\nu - 3) + \gamma(b + 4)\right](b+\nu)^{\underline{\nu-4}}$$

$$= -\frac{\delta}{\Gamma(\nu - 1)} \sum_{s=0}^{b+1} \left(b + \nu - \sigma(s)\right)^{\underline{\nu-2}} y(s) - \frac{\gamma}{\Gamma(\nu)} \sum_{s=0}^{b} \left(b + \nu - \sigma(s)\right)^{\underline{\nu-1}} y(s).$$

Thus, by using a simple calculation, we get

$$c_1 = -\frac{1}{\beta_0 \theta \eta^{\underline{\nu-3}}\Gamma(\nu)} \sum_{s=0}^{\eta-\nu} \left(\eta - \sigma(s)\right)^{\underline{\nu-1}} y(s)$$

$$- \frac{\gamma + \delta(\nu - 1)}{\beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \sum_{s=0}^{b+1} (b - s + 2)\left(b + \nu - \sigma(s)\right)^{\underline{\nu-2}} y(s),$$

$$c_2 = \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]}{\beta(\nu - 2)\theta^2 \beta_0 \eta^{\underline{\nu-3}}\Gamma(\nu)} \sum_{s=0}^{\eta-\nu} \left(\eta - \sigma(s)\right)^{\underline{\nu-1}} y(s)$$

$$- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]}{\beta(\nu - 2)\theta\beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}$$

$$\times \sum_{s=0}^{b+1} (b - s + 2)\left(b + \nu - \sigma(s)\right)^{\underline{\nu-2}} y(s)$$

and

$$c_3 = \frac{(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0}{\theta^2 \beta_0 \eta^{\underline{\nu-3}}\Gamma(\nu)} \sum_{s=0}^{\eta-\nu} \left(\eta - \sigma(s)\right)^{\underline{\nu-1}} y(s)$$

$$+ \frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]}{\theta\beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \sum_{s=0}^{b+1} (b - s + 2)\left(b + \nu - \sigma(s)\right)^{\underline{\nu-2}} y(s).$$

Hence,

$$
\begin{aligned}
x_0(t) = {} & \left[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right. \\
& \left. - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right] \\
& \times \sum_{s=0}^{b+1} (b - s + 2)(b + \nu - \sigma(s))^{\underline{\nu-2}} y(s) \\
& + \left[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right. \\
& \left. + \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2 \beta_0 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right] \sum_{s=0}^{\eta-\nu} (\eta - \sigma(s))^{\underline{\nu-1}} y(s) \\
& + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} y(s) = \sum_{s=0}^{b+1} G(s, t, \eta) y(s).
\end{aligned}
$$

Now, let x_0 be a solution for the equation $x(t) = \sum_{s=0}^{b+1} G(s, t, \eta) y(s)$. Then we have

$$
\begin{aligned}
x_0(t) = {} & \left[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right. \\
& \left. - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta \beta_0 \mu \Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \right] \\
& \times \sum_{s=0}^{b+1} (b - s + 2)(b + \nu - \sigma(s))^{\underline{\nu-2}} y(s) \\
& + \left[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right. \\
& \left. + \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2 \beta_0 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right] \\
& \times \sum_{s=0}^{\eta-\nu} (\eta - \sigma(s))^{\underline{\nu-1}} y(s) + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} y(s).
\end{aligned}
$$

Since $(\nu - 3)^{\underline{\nu-1}} = (\nu - 3)^{\underline{\nu-2}} = 0$, $(\nu - 3)^{\underline{\nu-3}} = (\nu - 3)^{\underline{\nu-4}} = \Gamma(\nu - 2)$, and

$$
\sum_{s=0}^{-3} (\nu - 3 - \sigma(s))^{\underline{\nu-1}} y(s) = \sum_{s=0}^{-2} (\nu - 3 - \sigma(s))^{\underline{\nu-2}} y(s) = 0,
$$

we get $\xi x_0(\nu - 3) + \beta \Delta x_0(\nu - 3) = 0$. A simple calculation shows us $\gamma x_0(b + \nu) + \delta \Delta x_0(b + \nu) = 0$ and $x_0(\eta) = 0$. On the other hand,

$$
x_0(t) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + c_3 t^{\underline{\nu-3}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{\underline{\nu-1}} y(s)
$$

is a solution for the equation $\Delta_{\nu-3}^{\nu} x(t) = y(t)$ and so $\Delta_{\nu-3}^{\nu} x_0(t) = y(t)$. □

A function $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ is a solution of the problem (3) whenever it satisfies the boundary conditions and there exists a function $y : \mathbb{N}_0^{b+1} \to \mathbb{R}$ such that

$$y(t) \in F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big)$$

for all $t \in \mathbb{N}_0^{b+1}$ and

$$
\begin{aligned}
x(t) = {} & \left[\frac{[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-1}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \\
& \left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right] \\
& \times \sum_{s=0}^{b+1} (b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} y(s) \\
& + \left[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right. \\
& \left. + \frac{[-\xi + \beta(\nu-3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right] \\
& \times \sum_{s=0}^{\eta-\nu} \big(\eta - \sigma(s)\big)^{\underline{\nu-1}} y(s) + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} y(s).
\end{aligned}
$$

Let $\mathcal{X}$ be the set of all functions $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|x(t)\big| + \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|\Delta x(t)\big| + \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|\Delta^2 x(t)\big|.$$

We show that $(\mathcal{X}, \|\cdot\|)$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$ and $\epsilon > 0$ be given. Choose a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $m, n > N$. This implies that $\max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} |x_n(t) - x_m(t)| < \epsilon$, $\max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} |\Delta x_n(t) - \Delta x_m(t)| < \epsilon$ and

$$\max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|\Delta^2 x_n(t) - \Delta^2 x_m(t)\big| < \epsilon.$$

Choose $x(t), z(t), w(t) \in \mathbb{R}$ such that $x_n(t) \to x(t)$, $\Delta x_n(t) \to z(t)$, and $\Delta^2 x_n(t) \to w(t)$ for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Note that $\Delta x_n(t) = x_n(t+1) - x_n(t)$ and so $\Delta x(t) = x(t+1) - x(t) = z(t)$. Similarly, we get $\Delta^2 x(t) = w(t)$. This implies that $|x_n(t) - x(t)| < \frac{\epsilon}{3}$, $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{3}$, and $|\Delta^2 x_n(t) - \Delta^2 x(t)| < \frac{\epsilon}{3}$ for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and $n > M$ for some natural number M. Thus,

$$\|x_n - x\| = \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|x_n(t) - x(t)\big| + \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|\Delta x_n(t) - \Delta x(t)\big| + \max_{t \in \mathbb{N}_{\nu-3}^{b+\nu+1}} \big|\Delta^2 x(t) - \Delta^2 x(t)\big| < \epsilon.$$

Hence, $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Let $x \in \mathcal{X}$. Define the set of selections of F by

$$S_{F,x} = \big\{ y : \mathbb{N}_0^{b+1} \to \mathbb{R} \mid y(t) \in F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big) \text{ for all } t \in \mathbb{N}_0^{b+1} \big\}.$$

Since $F(t, x(t), \Delta x(t), \Delta^2 x(t)) \neq \emptyset$, the selection principle implies that $S_{F,x}$ is nonempty.

Theorem 3.2 *Suppose that $\psi \in \Psi$ and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{\mathrm{cp}}(\mathbb{R})$ is a multifunction such that*

$$
H_d\big(F(t,x_1,x_2,x_3) - F(t,z_1,z_2,z_3)\big) \le \psi\big(|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|\big)
$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and $x_1,x_2,x_3,z_1,z_2,z_3 \in \mathbb{R}$. Then the boundary value inclusion (3) has a solution.

Proof Choose $y \in S_{F,x}$ and put $h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s)$ for all $t \in \mathbb{N}_{\nu-3}^{\nu+b+1}$. Then $h \in \mathcal{X}$ and so the set

$$
\left\{ h \in \mathcal{X} : \text{there exists } y \in S_{F,x} \text{ such that } h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s) \text{ for all } t \in \mathbb{N}_{\nu-3}^{b+\nu+1} \right\}
$$

is nonempty. Now define $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$
\mathcal{F}(x) = \left\{ h \in \mathcal{X} : \text{there exists } y \in S_{F,x} \text{ such that } h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s) \right.
$$
$$
\left. \text{for all } t \in \mathbb{N}_{\nu-3}^{b+\nu+1} \right\}.
$$

We show that the multifunction $\mathcal{F}$ has a fixed point. First, we show that $\mathcal{F}(x)$ is closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\ge 1}$ be a sequence in $\mathcal{F}(x)$ with $u_n \to u$. For each n, choose $y_n \in S_{F,x}$ such that

$$
u_n(t) = \left[\frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\nu-3} - \theta[\gamma + \delta(\nu-1)]t^{\nu-1}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\nu-4}} \right.
$$
$$
\left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\nu-2}}{\beta(\nu-2)\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\nu-4}} \right]
$$
$$
\times \sum_{s=0}^{b+1} (b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} y_n(s)
$$
$$
+ \left[\frac{[(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\nu-3} - \theta t^{\nu-1}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right.
$$
$$
\left. + \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\nu-2}}{\beta(\nu-2)\theta^2 \beta_0 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right]
$$
$$
\times \sum_{s=0}^{\eta-\nu} \big(\eta - \sigma(s)\big)^{\underline{\nu-1}} y_n(s) + \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} y_n(s)
$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Since F has compact values, $\{y_n\}_{n\ge 1}$ has a subsequence which converges to some $y \in S_{F,x}$. We denote this subsequence again by $\{y_n\}_{n\ge 1}$. So

$$
u_n(t) \to u(t)
$$
$$
= \left[\frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\nu-3} - \theta[\gamma + \delta(\nu-1)]t^{\nu-1}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\nu-4}} \right.
$$

$$
- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg]
$$

$$
\times \sum_{s=0}^{b+1}(b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}}y(s)
$$

$$
+ \Bigg[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg]
$$

$$
\times \sum_{s=0}^{\eta-\nu}\big(\eta - \sigma(s)\big)^{\underline{\nu-1}}y(s) + \sum_{s=0}^{t-\nu}\frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y(s)
$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. This implies that $u \in \mathcal{F}(x)$. Thus, the multifunction $\mathcal{F}$ has closed values. Since F is a compact multifunction, it is easy to check that $\mathcal{F}(x)$ is bounded set in $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x, z \in \mathcal{X}$, $h_1 \in \mathcal{F}(x)$, and $h_2 \in \mathcal{F}(z)$. Choose $y_1 \in S_{F,x}$ and $y_2 \in S_{F,z}$ such that

$$
h_1(t) = \Bigg[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}}
$$

$$
- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg]
$$

$$
\times \sum_{s=0}^{b+1}(b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}}y_1(s)
$$

$$
+ \Bigg[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg]
$$

$$
\times \sum_{s=0}^{\eta-\nu}\big(\eta - \sigma(s)\big)^{\underline{\nu-1}}y_1(s) + \sum_{s=0}^{t-\nu}\frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y_1(s)
$$

and

$$
h_2(t) = \Bigg[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}}
$$

$$
- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg]
$$

$$
\times \sum_{s=0}^{b+1}(b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}}y_2(s)
$$

$$
+ \Bigg[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg]
$$

$$
\times \sum_{s=0}^{\eta-\nu}\big(\eta - \sigma(s)\big)^{\underline{\nu-1}}y_2(s) + \sum_{s=0}^{t-\nu}\frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y_2(s)
$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Since

$$H_d\big(F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big) - F\big(t, z(t), \Delta z(t), \Delta^2 z(t)\big)\big)$$
$$\leq \psi\big(\big|x(t) - z(t)\big| + \big|\Delta x(t) - \Delta z(t)\big| + \big|\Delta^2 x(t) - \Delta^2 z(t)\big|\big)$$

for all $x, z \in \mathcal{X}$ and $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$, we get

$$\big|y_1(t) - y_2(t)\big| \leq \psi\big(\big|x(t) - z(t)\big| + \big|\Delta x(t) - \Delta z(t)\big| + \big|\Delta^2 x(t) - \Delta^2 z(t)\big|\big).$$

Now, put

$$
\begin{aligned}
G_1 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} & \left\{ \left| \frac{[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-1}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \right. \\
& \left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right| \\
& \times \sum_{s=0}^{b+1}(b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} + \left| \frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right. \\
& \left. + \frac{[-\xi + \beta(\nu-3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2 \beta_0 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right| \\
& \left. \times \sum_{s=0}^{\eta-\nu}\big(\eta - \sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \right\},
\end{aligned}
$$

$$
\begin{aligned}
G_2 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} & \left\{ \left| \frac{(\nu-3)[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-4}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \right. \\
& - \frac{(\nu-1)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-2}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
& \left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}}}{\beta \theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right| \\
& \times \sum_{s=0}^{b+1}(b - s + 2)\big(b + \nu - \sigma(s)\big)^{\underline{\nu-2}} \\
& + \left| \frac{(\nu-3)[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0 \theta^2 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right. \\
& \left. + \frac{[-\xi + \beta(\nu-3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta \beta_0]t^{\underline{\nu-3}}}{\beta \theta^2 \beta_0 \eta^{\underline{\nu-3}} \Gamma(\nu)} \right| \\
& \left. \times \sum_{s=0}^{\eta-\nu}\big(\eta - \sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+1} \frac{(t - \sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
G_3 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} & \left\{ \left| \frac{(\nu-3)(\nu-4)[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-5}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \right. \\
& - \frac{(\nu-1)(\nu-2)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-3}}}{\theta \beta_0 \mu \Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}
\end{aligned}
$$

$$
- \frac{(\nu - 3)[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\nu-4}}{\beta\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg|
$$

$$
\times \sum_{s=0}^{b+1} (b - s + 2)(b + \nu - \sigma(s))^{\underline{\nu-2}}
$$

$$
+ \Bigg| \frac{(\nu - 3)(\nu - 4)[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-5} - \theta(\nu - 1)(\nu - 2)t^{\nu-3}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{(\nu - 3)[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-4}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg|
$$

$$
\times \sum_{s=0}^{\eta-\nu}(\eta - \sigma(s))^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+2} \frac{(t - \sigma(s))^{\underline{\nu-3}}}{\Gamma(\nu - 2)} \Bigg\}.
$$

Then we have

$$
\big|h_1(t) - h_2(t)\big|
$$

$$
= \Bigg| \Bigg[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\nu-3} - \theta[\gamma + \delta(\nu - 1)]t^{\nu-1}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}}
$$

$$
- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\nu-2}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg]
$$

$$
\times \sum_{s=0}^{b+1} (b - s + 2)(b + \nu - \sigma(s))^{\underline{\nu-2}}(y_1 - y_2)(s)
$$

$$
+ \Bigg[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-3} - \theta t^{\nu-1}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-2}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg]
$$

$$
\times \sum_{s=0}^{\eta-\nu}(\eta - \sigma(s))^{\underline{\nu-1}}(y_1 - y_2)(s) + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}(y_1 - y_2)(s) \Bigg|
$$

$$
\leq \Bigg| \frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\nu-3} - \theta[\gamma + \delta(\nu - 1)]t^{\nu-1}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}}
$$

$$
- \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\nu-2}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \Bigg|
$$

$$
\times \sum_{s=0}^{b+1} (b - s + 2)(b + \nu - \sigma(s))^{\underline{\nu-2}}\big|y_1(s) - y_2(s)\big|
$$

$$
+ \Bigg| \frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-3} - \theta t^{\nu-1}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}
$$

$$
+ \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\nu-2}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg|
$$

$$
\times \sum_{s=0}^{\eta-\nu}(\eta - \sigma(s))^{\underline{\nu-1}}\big|y_1(s) - y_2(s)\big| + \sum_{s=0}^{t-\nu} \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\big|y_1(s) - y_2(s)\big|
$$

$$
\leq \max_{t \in \mathbb{N}_0^{b+1}} \big|y_1(t) - y_2(t)\big|
$$

$$\times \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \left\{ \left| \frac{[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \right.$$

$$\left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right|$$

$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} + \left| \frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \right.$$

$$\left. + \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right|$$

$$\left. \times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \right\}$$

$$\leq \psi\big(\big|x(t)-z(t)\big| + \big|\Delta x(t) - \Delta z(t)\big| + \big|\Delta^2 x(t) - \Delta^2 z(t)\big|\big) \times G_1.$$

Since

$$\Delta h_1(t) = \left[\frac{(\nu-3)[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-4}} - (\nu-1)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right.$$

$$\left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right]$$

$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y_1(s)$$

$$+ \left[\frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \right.$$

$$\left. + \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right]$$

$$\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y_1(s) + \sum_{s=0}^{t-\nu+1}\frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)}y_1(s),$$

we get

$$\big|\Delta h_1(t) - \Delta h_2(t)\big|$$

$$\leq \left| \frac{(\nu-3)[\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-4}} - (\nu-1)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right.$$

$$\left. - \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right|$$

$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}\big|y_1(s) - y_2(s)\big|$$

$$+ \left| \frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \right.$$

$$\left. + \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right|$$

$$\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}\big|y_1(s) - y_2(s)\big| + \sum_{s=0}^{t-\nu+1}\frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)}\big|y_1(s) - y_2(s)\big|$$

$$\leq \max_{t \in \mathbb{N}_0^{b+1}} \left| y_1(t) - y_2(t) \right|$$

$$\times \max_{t \in \mathbb{N}_{v-3}^{b+1+v}} \left\{ \left| \frac{(v-3)[\gamma + \delta(v-1)][(\eta + 2 - v)(\eta + 3 - v)]t^{\underline{v-4}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}} \right.\right.$$

$$- \frac{(v-1)\theta[\gamma + \delta(v-1)]t^{\underline{v-2}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}}$$

$$\left. - \frac{[\xi - \beta(v-3)][\gamma + \delta(v-1)][(\eta + 2 - v)(\eta + 3 - v)]t^{\underline{v-3}}}{\beta \theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}} \right|$$

$$\times \sum_{s=0}^{b+1} (b - s + 2)\left(b + v - \sigma(s)\right)^{\underline{v-2}}$$

$$+ \left| \frac{(v-3)[(\eta + 2 - v)(\eta + 3 - v) - \theta \beta_0]t^{\underline{v-4}} - \theta(v-1)t^{\underline{v-2}}}{\beta_0 \theta^2 \eta^{\underline{v-3}} \Gamma(v)} \right.$$

$$\left. + \frac{[-\xi + \beta(v-3)][(\eta + 2 - v)(\eta + 3 - v) - \theta \beta_0]t^{\underline{v-3}}}{\beta \theta^2 \beta_0 \eta^{\underline{v-3}} \Gamma(v)} \right|$$

$$\left. \times \sum_{s=0}^{\eta - v} \left(\eta - \sigma(s)\right)^{\underline{v-1}} + \sum_{s=0}^{t-v+1} \frac{(t - \sigma(s))^{\underline{v-2}}}{\Gamma(v-1)} \right\}$$

$$\leq \psi \left(\left| x(t) - z(t) \right| + \left| \Delta x(t) - \Delta z(t) \right| + \left| \Delta^2 x(t) - \Delta^2 z(t) \right| \right) \times G_2.$$

Also, we have

$$\left| \Delta^2 h_1(t) - \Delta^2 h_2(t) \right|$$

$$\leq \left| \frac{(v-3)(v-4)[\gamma + \delta(v-1)][(\eta + 2 - v)(\eta + 3 - v)]t^{\underline{v-5}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}} \right.$$

$$- \frac{(v-1)(v-2)\theta[\gamma + \delta(v-1)]t^{\underline{v-3}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}}$$

$$\left. - \frac{(v-3)[\xi - \beta(v-3)][\gamma + \delta(v-1)][(\eta + 2 - v)(\eta + 3 - v)]t^{\underline{v-4}}}{\beta \theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}} \right|$$

$$\times \sum_{s=0}^{b+1} (b - s + 2)\left(b + v - \sigma(s)\right)^{\underline{v-2}} \left| y_1(s) - y_2(s) \right|$$

$$+ \left| \frac{(v-3)(v-4)[(\eta + 2 - v)(\eta + 3 - v) - \theta \beta_0]t^{\underline{v-5}} - \theta(v-1)(v-2)t^{\underline{v-3}}}{\beta_0 \theta^2 \eta^{\underline{v-3}} \Gamma(v)} \right.$$

$$\left. + \frac{(v-3)[-\xi + \beta(v-3)][(\eta + 2 - v)(\eta + 3 - v) - \theta \beta_0]t^{\underline{v-4}}}{\beta \theta^2 \beta_0 \eta^{\underline{v-3}} \Gamma(v)} \right|$$

$$\times \sum_{s=0}^{\eta - v} \left(\eta - \sigma(s)\right)^{\underline{v-1}} \left| y_1(s) - y_2(s) \right| + \sum_{s=0}^{t-v+2} \frac{(t - \sigma(s))^{\underline{v-3}}}{\Gamma(v-2)} \left| y_1(s) - y_2(s) \right|$$

$$\leq \max_{t \in \mathbb{N}_0^{b+1}} \left| y_1(t) - y_2(t) \right|$$

$$\times \max_{t \in \mathbb{N}_{v-3}^{b+1+v}} \left\{ \left| \frac{(v-3)(v-4)[\gamma + \delta(v-1)][(\eta + 2 - v)(\eta + 3 - v)]t^{\underline{v-5}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}} \right.\right.$$

$$- \frac{(v-1)(v-2)\theta[\gamma + \delta(v-1)]t^{\underline{v-3}}}{\theta \beta_0 \mu \Gamma(v)(b+v)^{\underline{v-4}}}$$

$$- \frac{(v-3)[\xi - \beta(v-3)][\gamma + \delta(v-1)][(\eta+2-v)(\eta+3-v)]t^{\underline{v-4}}}{\beta\theta\beta_0\mu\Gamma(v)(b+v)^{\underline{v-4}}} \Bigg|$$

$$\times \sum_{s=0}^{b+1}(b-s+2)(b+v-\sigma(s))^{\underline{v-2}}$$

$$+ \Bigg| \frac{(v-3)(v-4)[(\eta+2-v)(\eta+3-v)-\theta\beta_0]t^{\underline{v-5}} - \theta(v-1)(v-2)t^{\underline{v-3}}}{\beta_0\theta^2\eta^{\underline{v-3}}\Gamma(v)}$$

$$+ \frac{(v-3)[-\xi + \beta(v-3)][(\eta+2-v)(\eta+3-v)-\theta\beta_0]t^{\underline{v-4}}}{\beta\theta^2\beta_0\eta^{\underline{v-3}}\Gamma(v)} \Bigg|$$

$$\times \sum_{s=0}^{\eta-v}(\eta-\sigma(s))^{\underline{v-1}} + \sum_{s=0}^{t-v+2}\frac{(t-\sigma(s))^{\underline{v-3}}}{\Gamma(v-2)} \Bigg\}$$

$$\leq \psi\big(|x(t)-z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)|\big) \times G_3.$$

Hence, we obtain

$$\|h_1 - h_2\| = \max_{t \in \mathbb{N}_{v-3}^{b+1+v}}|h_1(t) - h_2(t)| + \max_{t \in \mathbb{N}_{v-3}^{b+1+v}}|\Delta h_1(t) - \Delta h_2(t)|$$

$$+ \max_{t \in \mathbb{N}_{v-3}^{b+1+v}}|\Delta^2 h_1(t) - \Delta^2 h_2(t)|$$

$$\leq \psi\big(|x(t)-z(t)| + |\Delta x(t) - \Delta z(t)| + |\Delta^2 x(t) - \Delta^2 z(t)|\big)(G_1 + G_2 + G_3)$$

$$\leq (G_1 + G_2 + G_3)\psi\big(\|x - z\|\big)$$

for all $x, z \in \mathcal{X}$, $h_1 \in \mathcal{F}(x)$, and $h_2 \in \mathcal{F}(z)$. So $H_d(\mathcal{F}(x), \mathcal{F}(z)) \leq (G_1 + G_2 + G_3)\psi(\|x - z\|)$ for all $x, z \in \mathcal{X}$.

Define the function α on $\mathcal{X} \times \mathcal{X}$ by $\alpha(x, z) = 1$ whenever $G_1 + G_2 + G_3 < 1$ and $\alpha(x, z) = \frac{1}{G_1+G_2+G_3}$ otherwise. Thus,

$$\alpha(x, z)H_d\big(\mathcal{F}(x), \mathcal{F}(z)\big) \leq \psi\big(\|x - z\|\big)$$

for all $x, z \in \mathcal{X}$. Let $\{x_n\}$ be a sequence in $\mathcal{X}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$. Then it is easy to check that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k. This implies that $\mathcal{X}$ obeys the condition (C_α). If $x \in \mathcal{X}$ and $y \in \mathcal{F}(x)$ with $\alpha(x, y) \geq 1$, then it is easy to see that $\alpha(y, z) \geq 1$ for all $z \in \mathcal{F}(y)$. Thus, $\mathcal{F}$ is an α-admissible α-ψ-contractive multifunction. Hence by using Theorem 2.2, there exists $x^* \in \mathcal{X}$ such that $x^* \in \mathcal{F}(x^*)$. One can check that x^* is a solution for the problem (3). □

Example 3.1 Consider the fractional finite difference inclusion

$$\Delta_{-0.5}^{2.5}x(t) \in \left[1, e^{t^2} + 2 + \frac{\sin x(t)}{e^{2|t|}} + \sinh^2 t + \frac{|\Delta x(t)|}{4|t|} + \frac{3}{6t^2 - 1} + \frac{|\Delta^2 x(t)|}{\cosh|3t|}\right] \tag{4}$$

via the boundary value conditions $\xi x(-0.5) + \beta \Delta x(-0.5) = 0$, $\gamma x(6.5) + \delta \Delta x(6.5) = 0$, and $x(3.5) = 0$, where $\xi, \beta, \gamma, \delta$ are non-zero numbers. In fact, this problem is a special case of the problem (3), where $v = 2.5$, $\eta = 3.5$, $b = 4$, and

$$F(t, x_1, x_2, x_3) = \left[1, e^{t^2} + 2 + \frac{\sin x_1}{e^{2|t|}} + \sinh^2 t + \frac{|x_2|}{4|t|} + \frac{3}{6t^2 - 1} + \frac{|x_3|}{\cosh|3t|}\right].$$

Note that $e^{t^2} + 2 + \frac{\sin x_1}{e^{2|t|}} + \sinh^2 t + \frac{|x_2|}{4|t|} + \frac{3}{6t^2 - 1} + \frac{|x_3|}{\cosh|3t|} > 1$ for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and $x_1, x_2, x_3 \in \mathbb{R}$. Also, $e^{2|t|} \geq 2$, $4|t| \geq 2$, and $\cosh|3t| \geq 2$ for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and F is a compact valued multi-function on $\mathbb{N}_{-0.5}^{7.5} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Now, define $\psi \in \Psi$ by $\psi(z) = \frac{z}{2}$ for all $z \geq 0$. Since

$$
\begin{aligned}
& H_d\big(F(t, x_1, x_2, x_3), F(t, z_1, z_2, z_3)\big) \\
& \leq \left| \frac{\sin x_1}{e^{2|t|}} - \frac{x_2}{4|t|} + \frac{x_3}{\cosh|3t|} - \frac{\sin z_1}{e^{2|t|}} + \frac{z_2}{4|t|} - \frac{z_3}{\cosh|3t|} \right| \\
& \leq \frac{|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|}{2} \\
& = \psi\big(|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|\big)
\end{aligned}
$$

for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and $x_1, x_2, x_3, z_1, z_2, z_3 \in \mathbb{R}$, by using Theorem 3.2 the problem (4) has at least one solution.

4 Conclusions

In this manuscript, based on a fixed point theorem, we provided the existence result for a fractional finite difference inclusion in the presence of the general boundary conditions. An example illustrates our result.

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