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On the existence of solutions for a multi-singular pointwise defined fractional q-integro-differential equation

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Abstract

By using the Caputo type and the Riemann–Liouville type fractional q-derivative, we investigate the existence of solutions for a multi-term pointwise defined fractional q-integro-differential equation with some boundary value conditions. In fact, we give some results by considering different conditions and using some classical fixed point techniques and the Lebesgue dominated convergence theorem.

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1 Introduction

It is known that the subject of q-difference equations was introduced by Jackson in 1910 [1]. After that, some researchers studied q-difference equations [2–20]. On the other hand, many modern works on integro-differential equations by using different views and fractional derivatives have been published recently, and young researchers could use the main idea of the works for their works (see, for example, [21–50]).

In 2012, Ahmad *et al.* studied the existence and uniqueness of solutions for the fractional q-difference equation ${}^cD_q^\alpha u(t) = T(t,u(t))$ with boundary conditions $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, where $\alpha \in (1,2]$, α_i, β_i, γ_i, η_i are real numbers for $i = 1,2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$ [6]. In 2013, Zhao *et al.* reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t,u(t)) = 0$ with boundary conditions $u(1) = \mu I_q^\beta u(\eta)$ and $u(0) = 0$ for almost all $t \in (0,1)$, where $q \in (0,1)$, $\alpha \in (1,2]$, $\beta \in (0,2]$, $\eta \in (0,1)$, μ is a positive real number, D_q^α is the q-derivative of Riemann–Liouville and real-valued continuous map u defined on $I \times [0,\infty)$ [15]. In 2014, Ahmad *et al.* investigated the problem

$$ {}^cD_q^\beta\big({}^cD_q^\gamma + \lambda\big)u(t) = pf\big(t,u(t)\big) + kI_q^\xi g\big(t,u(t)\big), $$

with boundary conditions $\alpha_1 u(0) - \beta_1 (t^{(1-\gamma)} D_q u(0))|_{t=0} = \sigma_1 u(\eta_1)$ and $\alpha_2 u(1) + \beta_2 D_q u(1) = \sigma_2 u(\eta_2)$, where $t, q \in [0,1]$, ${}^cD_q^\beta$ is the fractional Caputo q-derivative, $0 < \beta$, $\gamma \le 1$, $I_q^\xi(\cdot)$

denotes the Riemann–Liouville integral with $\xi \in (0,1)$, f and g are given continuous functions, λ and p, k are real constants, $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ and $\eta_i \in (0,1)$ for $i = 1,2$ [5]. In 2017, Wang considered the existence of uniqueness and nonexistence of positive solution for fractional differential equations $D^{\sigma}_{0^+}x(t) + f(t,x(t)) = 0$ for $t \in (0,1)$ under conditions the $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$ and $D^{\alpha}_{0^+}x(1) = \int_0^b \mu(t)D^{\beta}_{0^+}x(t)\,\mathrm{d}t$, where $n-1 < \sigma \le n$, $n \ge 3$, $\alpha \in (0,1)$,

$$\Gamma(\sigma - \alpha)\int_0^b \mu(t)t^{\sigma-\beta-1}\,\mathrm{d}t < \Gamma(\sigma - \beta),$$

$b \in (0,1]$, $D^{\sigma}_{0^+}$, $D^{\alpha}_{0^+}$, $D^{\beta}_{0^+}$ are the standard Riemann–Liouville derivatives, $f : (0,1) \times [0,\infty) \to [0,\infty)$ is continuous and $\mu(t) \in L^1([0,1])$ is nonnegative [51]. Also, in 2018 he investigated the existence and multiplicity of positive solutions for the fractional differential equation $D^{\sigma}_{0^+}x(t) + f(t,x(t)) = 0$ for $t \in (0,1)$ under the conjugate type integral boundary conditions $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$ and $D^{\alpha}_{0^+}x(1) = \int_0^b \mu(t)D^{\beta}_{0^+}x(t)\,\mathrm{d}V(t)$, where $D^{\alpha}_{0^+}$, $D^{\beta}_{0^+}$ are the standard Riemann–Liouville derivatives, $n \ge 3$, $\alpha \in (0,1)$, $0 \le \beta < \sigma - 1$, $b \in (0,1]$, $f(t,x)$ may be singular at $t = 0,1$ and $x = 0$, $\mu(t) \in L^1[0,1] \cap C(0,1)$ is nonnegative, $\int_0^b \mu(t)t^{\sigma-\beta-1}\,\mathrm{d}V(t)$ denotes the Riemann–Stieltjes integral, in which V has bounded variation [52].

In 2019, Samei *et al.* reviewed the existence of solutions for some multi-term q-integro-differential equations with non-separated and initial boundary conditions [12]. Also, Ntouyas *et al.* [10], by applying definitions of the fractional q-derivative of the Caputo type and the fractional q-integral of the Riemann–Liouville type, studied the existence and uniqueness of solutions for multi-term nonlinear fractional q-integro-differential equations under some boundary conditions

$$^cD^{\alpha}_q x(t) = w\bigl(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^cD^{\beta_1}_q x(t), {}^cD^{\beta_2}_q x(t), \ldots, {}^cD^{\beta_n}_q x(t)\bigr).$$

In 2020, Liang *et al.* investigated the existence of solutions for nonlinear problems regular and singular fractional q-differential equation

$$^cD^{\alpha}_q f(t) = w\bigl(t, f(t), f'(t), {}^cD^{\beta}_q f(t)\bigr),$$

with conditions $f(0) = c_1 f(1)$, $f'(0) = c_2\,{}^cD^{\beta}_q f(1)$, and $f^{(k)}(0) = 0$ for $2 \le k \le n-1$, here $n-1 < \alpha < n$ with $n \ge 3$, $\beta, q, c_1 \in (0,1)$, $c_2 \in (0, \Gamma_q(2-\beta))$, function w is an L^{κ}-Carathéodory, $w(t,x_1,x_2,x_3)$ may be singular, and $^cD^{\alpha}_q$ is the fractional Caputo type q-derivative [17]. Also, they discussed the existence of solutions for the fractional q-derivative inclusions

$$^cD^{\alpha}_q x(t) \in F\bigl(t, x(t), x'(t), {}^cD^{\beta}_q x(t)\bigr),$$

$x(0) + x'(0) + {}^cD^{\beta}_q x(0) = \int_0^{\eta_1} x(s)\,\mathrm{d}s$, and $x(1) + x'(1) + {}^cD^{\beta}_q x(1) = \int_0^{\eta_2} x(s)\,\mathrm{d}s$ for any t in I and $q, \eta_1, \eta_2, \beta \in (0,1)$, where F maps $I \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact-valued multifunction and $^cD^{\alpha}_q$ is the fractional Caputo type q-derivative operator of order $\alpha \in (1,2]$, and

$$\Gamma_q(2-\beta)\bigl(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2\bigr) + 2(1-\eta) \ne 0$$

such that $\alpha - \beta > 1$ [14]. Similar results have been presented in other studies [12, 13, 19, 20, 37].

By using the main idea of [41, 42, 53], we are going to investigate the multi-singular fractional q-integro-differential pointwise defined equation

$$D_q^{\alpha} u(t) = \omega\big(t, u(t), u'(t), D_q^{\beta_1} u(t), I_q^{\beta_2} u(t)\big) \tag{1}$$

under two distinct boundary conditions

$$\begin{aligned} &u'(0) = u(a), \qquad u(1) = \int_0^b u(r)\,\mathrm{d}r, \quad \alpha \in [2,3), \\ &u'(0) = u(a), \qquad u(1) = \int_0^b u(r)\,\mathrm{d}r, \quad \alpha \in [3,\infty), \end{aligned} \tag{2}$$

and $u^{(j)}(0) = 0$ for $j = 2, \ldots, [\alpha] - 1$, where $t \in \overline{J} = [0,1]$, $u \in \mathcal{B} = C^1(\overline{J})$, α, β_1, β_2 belong to $[2,\infty)$, $J = (0,1)$, $(1,\infty)$, $a, b \in J$, D_q^{α} is the Caputo fractional q-derivative of order α, and $\omega : \overline{J} \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $\omega(t, \cdot, \cdot, \cdot, \cdot)$ is singular at some points $t \in \overline{J}$.

2 Preliminaries

Here, we recall some basic notion, lemmas, and theorems which are used in the subsequent sections. Let $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ for $n \geq 1$ and $(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [2]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(x-y)_q^{(\alpha)} = x^{\alpha} \prod_{k=0}^{\infty} \big(x - yq^k\big) / \big(x - yq^{\alpha+k}\big).$$

If $y = 0$, then it is clear that $x^{(\alpha)} = x^{\alpha}$ (Algorithm 1). The q-gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$ [1]. Note that $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. The value of q-gamma function is $\Gamma_q(z)$ for input values q and z with counting the number of sentences n in summation by simplifying analysis (see Tables 1–3). For this design, we prepare a pseudo-code description of the technique for estimating q-gamma function of order n, which is shown in Algorithm 2. The q-derivative of function f is defined by $(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x \to 0} (D_q f)(x)$, which is shown in Algorithm 3 [2]. Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^{\alpha} * s$
end if

Output: $(a-b)^{(\alpha)}$

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}$, which is constant, for $x = 9.5, 65, 110, 780$ in Algorithm 2

n	$x = 9.5$	$x = 65$	$x = 110$	$x = 780$
1	2.679786	4432.545834	1,804,225.634753	1.29090809480473E+45
2	2.674552	4423.888518	1,800,701.756560	1.28838678993206E+45
3	2.673899	4422.808467	1,800,262.132108	1.28807224237593E+45
4	2.673818	4422.673494	1,800,207.192468	1.28803293353064E+45
5	2.673808	4422.656623	1,800,200.325222	1.28802802007493E+45
6	2.673806	4422.654514	1,800,199.466820	1.28802740589531E+45
7	2.673806	4422.654250	1,800,199.359519	1.28802732912289E+45
8	2.673806	4422.654217	1,800,199.346107	1.28802731952634E+45
9	2.673806	4422.654213	1,800,199.344430	1.28802731832677E+45
10	2.673806	4422.654213	1,800,199.344221	1.28802731817683E+45
11	2.673806	4422.654212	1,800,199.344195	1.28802731815808E+45
12	2.673806	4422.654212	1,800,199.344191	1.28802731815574E+45
13	2.673806	4422.654212	1,800,199.344191	1.28802731815545E+45
14	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45
15	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45
16	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45
17	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45
18	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45
19	2.673806	4422.654212	1,800,199.344191	1.28802731815541E+45

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 9.5$ of Algorithm 2

n	$q = \frac{1}{8}$	$q = \frac{1}{2}$	$q = \frac{4}{5}$	$q = \frac{8}{9}$
1	2.679786	136.046206	79,062.138227	6,301,918.338883
2	2.674552	119.081545	41,793.335091	2,528,395.395827
3	2.673899	111.658224	26,290.733638	1,232,715.590371
4	2.673818	108.178242	18,589.881264	689,176.848061
5	2.673808	106.492553	14,278.326587	426,538.394173
6	2.673806	105.662861	11,650.586796	285,518.687713
7	2.673806	105.251251	9946.3508930	203,363.796571
⋮	⋮	⋮	⋮	⋮
26	2.673806	104.841780	5522.283831	25,842.863721
27	2.673806	104.841780	5513.202433	25,230.371788
28	2.673806	104.841779	5505.949683	24,699.649904
29	2.673806	104.841779	5500.155385	24,238.446645
⋮	⋮	⋮	⋮	⋮
106	2.673806	104.841779	5477.048235	20,879.606269
107	2.673806	104.841779	5477.048234	20,879.566792
108	2.673806	104.841779	5477.048234	20,879.531702
⋮	⋮	⋮	⋮	⋮
118	2.673806	104.841779	5477.048234	20,879.337427
119	2.673806	104.841779	5477.048234	20,879.327822
120	2.673806	104.841779	5477.048234	20,879.319284

for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [2, 3]. The q-integral of a function f defined on $[0, b]$ is defined by

$$I_q f(x) = \int_0^x f(s)\, \mathrm{d}_q s = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right)$$

for $0 \leq x \leq b$, provided the series is absolutely convergent [2, 3]. The q-derivative of function f is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$, which is shown in

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 110$ of Algorithm 2

n	$q = \frac{1}{8}$	$q = \frac{1}{2}$	$q = \frac{4}{5}$	$q = \frac{8}{9}$
1	1,804,225.634753	2.43388915243820E+32	1.10933564801075E+75	2.3996994906237E+102
2	1,800,701.756560	2.12965300838343E+32	5.41355796236824E+74	7.1431517307455E+101
3	1,800,262.132108	1.99654969535946E+32	3.19616462101800E+74	2.6837217226512E+101
4	1,800,207.192468	1.93415751737948E+32	2.14884539802207E+74	1.1944485864825E+101
5	1,800,200.325222	1.90393630617042E+32	1.58553847001434E+74	6.0526350536381E+100
6	1,800,199.466820	1.88906180377847E+32	1.25302695267477E+74	3.3987862057282E+100
7	1,800,199.359519	1.88168265610746E+32	1.04280391429109E+74	2.0741306563269E+100
8	1,800,199.346107	1.87800749466975E+32	9.02841142168746E+73	1.3555712905453E+100
9	1,800,199.344430	1.87617350297573E+32	8.05899312693661E+73	9.38129101307050E+99
10	1,800,199.344221	1.87525740263248E+32	7.36673088857628E+73	6.81335603265770E+99
11	1,800,199.344195	1.87479957611817E+32	6.86049299667128E+73	5.15556440821410E+99
12	1,800,199.344191	1.87457071874804E+32	6.48333340557523E+73	4.04051908444650E+99
⋮	⋮	⋮	⋮	⋮
48	1,800,199.344191	1.87434189862553E+32	5.18960499065178E+73	6.66324790738213E+98
⋮	⋮	⋮	⋮	⋮
90	1,800,199.344191	1.87434189862553E+32	5.18923469131315E+73	6.50025876524830E+98
91	1,800,199.344191	1.87434189862553E+32	5.18923468501255E+73	6.50013085733126E+98
92	1,800,199.344191	1.87434189862553E+32	5.18923467997207E+73	6.50001716364224E+98
93	1,800,199.344191	1.87434189862553E+32	5.18923467593968E+73	6.49991610435300E+98
⋮	⋮	⋮	⋮	⋮
118	1,800,199.344191	1.87434189862553E+32	5.18923465987107E+73	6.49915022957670E+98
119	1,800,199.344191	1.87434189862553E+32	5.18923465985889E+73	6.49914550293450E+98
120	1,800,199.344191	1.87434189862553E+32	5.18923465984914E+73	6.49914130147782E+98

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: $n, q \in (0,1), x \in \mathbb{R}\backslash\{0,-1,2,\ldots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1), f(x), x$

syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q*z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q*x))/((1-q)x)$
end if

Output: $(D_q f)(x)$

Algorithm 3 [2, 3]. If $a \in [0,b]$, then

$$\int_a^b f(u)\,\mathrm{d}_q u = (1-q)\sum_{k=0}^{\infty} q^k\bigl[bf\bigl(bq^k\bigr) - af\bigl(aq^k\bigr)\bigr],$$

whenever the series exists [2, 3]. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1}h))(x)$ for $n \geq 1$ and $g \in C([0,b])$ [2, 3]. It has been proved that $(D_q(I_q f))(x) = f(x)$

Algorithm 4 The proposed method for calculated $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x
1: $s \leftarrow 0$
2: **for** $i = 0$ to n **do**
3: $pf \leftarrow (1-q^{i+1})^{\alpha-1}$
4: $s \leftarrow s + pf * q^i * f(x * q^i)$
5: **end for**
6: $g \leftarrow (x^{\alpha} * (1-q) * s)/(\Gamma_q(x))$
Output: $(I_q^{\alpha} f)(x)$

and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [2, 3]. The fractional Riemann–Liouville type q-integral of the function f on J for $\alpha \geq 0$ is defined by $(I_q^0 f)(t) = f(t)$ and

$$\left(I_q^{\alpha} f\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s)\, \mathrm{d}_q s$$

for $t \in J$ and $\alpha > 0$ [9]. Also, the Caputo fractional q-derivative of a function f is defined by

$$\begin{aligned} \left({}^c D_q^{\alpha} f\right)(t) &= \left(I_q^{[\alpha]-\alpha}\left(D_q^{[\alpha]} f\right)\right)(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t-qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f\right)(s)\, \mathrm{d}_q s, \end{aligned} \tag{3}$$

where $t \in J$ and $\alpha > 0$ ([9]). It has been proved that $(I_q^{\beta}(I_q^{\alpha} f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^{\alpha}(I_q^{\alpha} f))(x) = f(x)$, where $\alpha, \beta \geq 0$ ([9]). By using Algorithm 2, we can calculate $(I_q^{\alpha} f)(x)$, which is shown in Algorithm 4.

We say f is multi-singular when it is singular at more than one point t. Also, we say that $D_q^{\alpha} u(t) + h(t) = 0$ is a pointwise defined equation on $\overline{J}$ if there exists a set $E \subset \overline{J}$ such that the measure of E^c is zero and the equation holds on E. In this paper, we use $\|\cdot\|_1$, $\|\cdot\|$ and $\|w\|_* = \max\{\|w\|, \|w'\|\}$ as the norm of $\overline{\mathcal{L}} = L^1(\overline{J})$, the sup norm $\overline{\mathcal{A}} = C(\overline{J})$, and the norm of $\overline{\mathcal{B}} = C^1(\overline{J})$, respectively. Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ [54]. One can check that $\psi(t) < t$ for all $t > 0$ [54]. Let $T : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [55]. Let $(\mathcal{X}, \rho)$ be a complete metric space, $\psi \in \Psi$, and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a map. A self-map $T : \mathcal{X} \to \mathcal{X}$ is called an α-ψ-contraction whenever $\alpha(s,t)\rho(Ts, Tt) \leq \psi(\rho(s,t))$ for all $s, t \in \mathcal{X}$ [55]. We need the following results.

Lemma 1 ([56]) *Suppose that* $0 < n-1 \leq \alpha < n$ *and* $u \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$. *Then* $I_q^{\alpha} D_q^{\alpha} u(t) = u(t) + \sum_{i=0}^{n-1} c_i t^i$ *for some constants* $c_i \in \mathbb{R}$.

Lemma 2 ([55]) *Let* $(\mathcal{X}, \rho)$ *be a complete metric space,* $\psi \in \Psi$, $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ *be a map, and* $T : \mathcal{X} \to \mathcal{X}$ *be an* α*-admissible* α-ψ*-contraction. Then* T *has a fixed point whenever* T *is continuous and there exists* $x_0 \in \mathcal{X}$ *such that* $\alpha(x_0, Tx_0) \geq 1$.

3 Main results

First, we state and prove the following key results.

Lemma 3 *Let* $\alpha \geq 2$, $a, b \in J$, *and* $v_0 \in L^1(\overline{J})$. *Then* $v(t) = \int_0^1 G_q(t,s)v_0(s)\,\mathrm{d}s$ *is a solution for the pointwise defined problem* $D_q^\alpha u(t) + v_0(t) = 0$ *with boundary conditions* (2), *where*

$$G_q(t,s) = G_q^0(t,s) + \frac{1}{1-b}\int_0^b G_q^0(t,s)\,\mathrm{d}t \tag{4}$$

and

$$G_q^0(t,s) = \begin{cases} \frac{1}{\Gamma_q(\alpha)}[(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \\ \quad + (1-t)(a-qs)^{(\alpha-1)}], & 0 \leq s \leq t \leq 1, s \leq a, \\ \frac{(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq a \leq s \leq t \leq 1, \\ \frac{(1-qs)^{(\alpha-1)} + (1-t)(a-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq a \leq 1, \\ \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq 1, a \leq s. \end{cases}$$

Proof Let $E \subset \overline{J}$ be such that the equation $D_q^\alpha u(t) + v_0(t) = 0$ holds for all $t \in E$ and the measure of E^c is zero. Choose $v \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$ such that $v = v_0$ on E. If $v_0 \in \overline{\mathcal{A}}$ is a solution for the pointwise defined problem, then we put $v(t) = -D_q^\alpha u_0(t)$ for all $t \in \overline{J}$. Note that $v \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$ and $v = v_0|_E$. Also, we have

$$\begin{aligned} I_q^\alpha\big(v_0(t)\big) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s \\ &= \frac{1}{\Gamma_q(\alpha)}\bigg[\int_{[0,t]\cap E} (t-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s \\ &\quad + \int_{[0,t]\cap E^c} (t-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s\bigg] \\ &= \frac{1}{\Gamma_q(\alpha)}\int_{[0,t]\cap E} (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \\ &= \frac{1}{\Gamma_q(\alpha)}\bigg[\int_{[0,t]\cap E} (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \\ &\quad + \int_{[0,t]\cap E^c} (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s\bigg] \\ &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s = I_q^\alpha\big(v(t)\big) \end{aligned}$$

for each $t \in E$. Let $t \in E^c \backslash \{0\}$. Choose $\{t_n\}$ in E such that $t_n \to t^-$. Hence,

$$\begin{aligned} I_q^\alpha\big(v_0(t)\big) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s \\ &= \lim_{n\to\infty}\frac{1}{\Gamma_q(\alpha)}\int_0^{t_n} (t_n-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s = \lim_{n\to\infty} I_q^\alpha\big(v_0(t_n)\big) \\ &= \lim_{n\to\infty} I_q^\alpha\big(v(t_n)\big) = \lim_{n\to\infty}\frac{1}{\Gamma_q(\alpha)}\int_0^{t_n} (t_n-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \\ &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v_0(s)\,\mathrm{d}_q s = I_q^\alpha\big(v(t)\big). \end{aligned}$$

If $t = 0 \in E^c$, then $I_q^\alpha(\nu_0(t)) = I_q^\alpha(\nu(t)) = 0$, and so $I_q^\alpha(\nu_0(t)) = I_q^\alpha(w(t))$ for all $t \in \overline{J}$. Thus, $I_q^\alpha(D_q^\alpha u(t)) = I_q^\alpha(-\nu_0(t))$ for each $t \in \overline{J}$ whenever $D_q^\alpha u(t) + \nu_0(t) = 0$ for $t \in E$. Hence, $I_q^\alpha(D^\alpha u(t)) = I_q^\alpha(-\nu(t))$ on $\overline{J}$. By employing the boundary conditions and Lemma 1, we can conclude that

$$u(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \nu(s) + c_0 + c_1 t.$$

Since $u'(0) = u(a)$, $c_1 = -I_q^\alpha \nu(a)$, and so $u(t) = -I_q^\alpha \nu(t) + c_0 - I_q^\alpha \nu(a)$. Hence,

$$\int_0^b u(s)\,\mathrm{d}s = u(1) = -I_q^\alpha \nu(1) + c_0 - I_q^\alpha \nu(a).$$

So $c_0 = \int_0^b u(s)\,\mathrm{d}s + I_q^\alpha \nu(1) + I_q^\alpha \nu(a)$. Thus,

$$u(t) = -I_q^\alpha \nu(t) - t I_q^\alpha \nu(a) + \int_0^b u(s)\,\mathrm{d}_q s + I_q^\alpha \nu(1) + I_q^\alpha \nu(a).$$

Put $h(t) = -I_q^\alpha \nu(t) + (1-t) I_q^\alpha \nu(a) + I_q^\alpha \nu(1)$. Then we get

$$u(t) = h(t) + \int_0^b u(s)\,\mathrm{d}s. \tag{5}$$

We consider two cases. If $t \geq a$, then

$$\begin{aligned} h(t) = {} & -\frac{1}{\Gamma_q(\alpha)} \biggl[\int_0^a (t-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \\ & + \int_a^t (t-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \biggr] \\ & - \frac{t}{\Gamma_q(\alpha)} \int_0^a (a-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \biggl[\int_0^a (1-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \\ & + \int_a^t (1-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s + \int_t^1 (1-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \biggr] \\ & + \frac{1}{\Gamma_q(\alpha)} \int_0^a (a-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s \\ = {} & \frac{1}{\Gamma_q(\alpha)} \int_0^a \bigl[(1-sq)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \\ & + (1-t)(a-qs)^{(\alpha-1)} \bigr] \nu(s)\,\mathrm{d}_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_a^t \bigl[(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \bigr] \nu(s)\,\mathrm{d}_q s \\ & + \frac{1}{\Gamma_q(\alpha)} \int_t^1 (1-qs)^{(\alpha-1)} \nu(s)\,\mathrm{d}_q s. \end{aligned} \tag{6}$$

If $t \leq a$, then we have

$$\begin{aligned}
h(t) = &-\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \\
&- \frac{t}{\Gamma_q(\alpha)} \left[\int_0^t (a-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right. \\
&\left. + \int_t^a (a-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right] \\
&+ \frac{1}{\Gamma_q(\alpha)} \left[\int_0^t (1-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right. \\
&\left. + \int_t^a (1-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s + \int_a^1 (1-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right] \\
&+ \frac{1}{\Gamma_q(\alpha)} \left[\int_0^t (a-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right. \\
&\left. + \int_t^a (a-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s \right] \\
= &\frac{1}{\Gamma_q(\alpha)} \int_0^t \left[(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \right. \\
&\left. + (1-t)(a-qs)^{(\alpha-1)} + \right] v(s)\,\mathrm{d}_q s \\
&+ \frac{1}{\Gamma_q(\alpha)} \int_t^a \left[(1-qs)^{(\alpha-1)} - t(a-qs)^{(\alpha-1)} \right. \\
&\left. + (a-qs)^{(\alpha-1)} \right] v(s)\,\mathrm{d}_q s \\
&+ \frac{1}{\Gamma_q(\alpha)} \int_a^1 (1-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s.
\end{aligned} \tag{7}$$

Thus, equations (6) and (7) imply that $h(t) = \int_0^1 G_q^0(t,s)v(s)\,\mathrm{d}_q s$, and by entering $h(t)$ in equation (5), we see that

$$u(t) = \int_0^1 G_q^0(t,s)v(s)\,\mathrm{d}_q s + \int_0^b u(s)\,\mathrm{d}s.$$

This implies that

$$\begin{aligned}
\int_0^b u(t)\,\mathrm{d}t &= \int_0^b \int_0^1 G_q^0(t,s)v(s)\,\mathrm{d}_q s\,\mathrm{d}t + \int_0^b \int_0^b u(s)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^1 \left[\int_0^b G_q^0(t,s)\,\mathrm{d}t \right] v(s)\,\mathrm{d}_q s + b\int_0^b u(s)\,\mathrm{d}s.
\end{aligned}$$

Thus, $(1-b)\int_0^b u(t)\,\mathrm{d}t = \int_0^1 [\int_0^b G_q^0(t,s)\,\mathrm{d}t] v(s)\,\mathrm{d}_q s$, and so

$$\int_0^b u(t)\,\mathrm{d}t = \int_0^1 \frac{1}{1-b} \left[\int_0^b G_q^0(t,s)\,\mathrm{d}t \right] v(s)\,\mathrm{d}_q s.$$

Hence,

$$\begin{aligned} u(t) &= \int_0^1 G_q^0(t,s)v(s)\,\mathrm{d}_qs + \int_0^1 \frac{1}{1-b}\biggl[\int_0^b G_q^0(t,s)\,\mathrm{d}t\biggr]v(s)\,\mathrm{d}_qs \\ &= \int_0^1 \biggl[G_q^0(t,s) + \frac{1}{1-b}\int_0^b G_q^0(t,s)\,\mathrm{d}t\biggr]v(s)\,\mathrm{d}_qs \\ &= \int_0^1 G_q(t,s)v(s)\,\mathrm{d}_qs = \int_0^1 G_q(t,s)v_0(s)\,\mathrm{d}_qs. \end{aligned}$$

This completes the proof. □

Lemma 4 *Let $G_q(t,s)$ be given in Lemma 3. Then*

$$0 \le G_q(t,s) \le A_1(\alpha,b)(1-qs)^{(\alpha-1)},$$

$|\frac{\partial G_q}{\partial t}(t,s)| \le A_2(\alpha,b)(1-qs)^{(\alpha-1)}$, *where*

$$\begin{aligned} A_1(\alpha,b) &= \frac{3}{(1-b)\Gamma_q(\alpha)}, \\ A_2(\alpha,b) &= \frac{2}{(1-b)\Gamma_q(\alpha-1)}, \end{aligned}$$

and finally

$$0 \le \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\biggl[-t + \frac{2-b^2}{2(1-b)}\biggr] \le G_q(t,s) \tag{8}$$

for $t,s \in \overline{J}$.

Proof We consider some cases. If $0 \le s \le t \le 1$ and $s \le a$, then $(a-qs)^{\alpha-1} \ge t(a-qs)^{(\alpha-1)}$ and $(1-qs)^{(\alpha-1)} \ge (t-qs)^{(\alpha-1)}$. Hence,

$$(1-qs)^{(\alpha-1)} + (1-t)(a-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \ge 0$$

and so $G_q^0(t,s) \ge 0$. Thus, $G_q(t,s) \ge 0$. In other cases, the proof is easy. One can see that $G_q^0(t,s) \le 3(1-qs)^{(\alpha-1)}$ for each $t,s \in \overline{J}$, and so

$$\begin{aligned} G_q(t,s) &\le 3(1-qs)^{(\alpha-1)} + \frac{1}{(1-b)\Gamma_q(\alpha)}\int_0^b 3(1-qs)^{(\alpha-1)}\,\mathrm{d}t \\ &= 3(1-qs)^{(\alpha-1)} + \frac{3b(1-qs)^{(\alpha-1)}}{(1-b)\Gamma_q(\alpha)} \\ &= \frac{3(1-qs)^{(\alpha-1)}}{(1-b)\Gamma_q(\alpha)} = A_1(\alpha,b)(1-qs)^{(\alpha-1)}. \end{aligned}$$

From q-Green function $G_q^0(t,s)$ be given in Lemma 3, since

$$\frac{\partial G_q^0(t,s)}{\partial t} = \begin{cases} \frac{-(t-qs)^{(\alpha-2)}-(a-qs)^{(\alpha-1)}}{\Gamma_q(\alpha-1)}, & 0 \leq s \leq t \leq 1, s \leq a, \\ \frac{-(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}, & 0 \leq s \leq a \leq t \leq 1, \\ \frac{-(a-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq a \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, a \leq s, \end{cases}$$

we have

$$\left|\frac{\partial G_q^0(t,s)}{\partial t}\right| \leq \frac{(t-qs)^{(\alpha-1)} + (a-qs)^{(\alpha-1)}}{\Gamma_q(\alpha-1)} \leq \frac{2(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha-1)},$$

and so

$$\begin{aligned} \left|\frac{\partial G_q(t,s)}{\partial t}\right| &\leq \frac{2(1-qs)^{\alpha-1}}{\Gamma_q(\alpha-1)} + \frac{2b(1-qs)^{(\alpha-1)}}{1-b} \\ &= \frac{2(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha-1)}\left[1 + \frac{b}{1-b}\right] \\ &= \frac{2(1-qs)^{(\alpha-1)}}{(1-b)\Gamma_q(\alpha-1)} = A_2(\alpha,b)(1-qs)^{(\alpha-1)}. \end{aligned}$$

If $0 < s < t < 1$ and $s \leq a$, then $t - st > 0$, and so $t(1-qs) - s + t > 0$. Hence, $s - t < (1-qs)t$ and $t(1-qs) > t - qs$. Since $t < 1$ and $\alpha \geq 2$,

$$\left(\frac{1-qs}{t-qs}\right)^{(\alpha-1)} > \left(\frac{1}{t}\right)^{(\alpha-1)} > \frac{1}{t},$$

and so

$$\begin{aligned} &(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} + (1-t)(a-qs)^{(\alpha-1)} \\ &\quad > (1-qs)^{(\alpha-1)} - t(1-qs)^{(\alpha-1)} + (1-t)(a-qs)^{(\alpha-1)} \\ &\quad = (1-t)\bigl((1-qs)^{(\alpha-1)} + (a-qs)^{(\alpha-1)}\bigr) \\ &\quad \geq (1-t)(1-qs)^{(\alpha-1)}. \end{aligned}$$

Thus, $G_q^0(t,s) > (1-t)(1-qs)^{(\alpha-1)}$. If $0 < s \leq a < t < 1$, then

$$\begin{aligned} -(t-qs)^{(\alpha-1)} + (1-qs)^{(\alpha-1)} &> -t(1-qs)^{(\alpha-1)} + (1-qs)^{(\alpha-1)} \\ &= (1-t)(1-qs)^{(\alpha-1)}, \end{aligned}$$

and so $G_q^0(t,s) > \frac{(1-t)(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}$. Hence,

$$\begin{aligned} G_q(t,s) &\geq \frac{1}{\Gamma_q(\alpha)}\left[(1-t)(1-qs)^{(\alpha-1)} + \frac{1}{1-b}\int_0^b (1-t)(1-qs)^{(\alpha-1)}\,\mathrm{d}t\right] \\ &= \frac{1}{\Gamma_q(\alpha)}\left[(1-t)(1-qs)^{(\alpha-1)} + \frac{(1-qs)^{(\alpha-1)}}{1-b}\left(b - \frac{b^2}{2}\right)\right] \end{aligned}$$

$$= \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[-t + \frac{2-b^2}{2(1-b)}\right] \geq 0,$$

and so inequality (8) holds. □

Consider the self-map $T : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ defined by

$$T_u(t) = \int_0^1 G_q(t,s)\omega\big(s, u(s), u'(s), D_q^{\beta_1} u(s), I_q^{\beta} u(s)\big)\,\mathrm{d}s, \tag{9}$$

where $G_q(t,s)$ is the q-Green function in Lemma 3. By applying Lemma 3, one can easily see that the fractional q-integro-differential equation (1) has a solution if and only if T has a fixed point.

Here, we provide our first result about the existence of solutions for problem (1).

Theorem 5 *Assume that the map T is defined by equation (9) and $\omega : \overline{J} \times \overline{\mathcal{A}}^4 \to \mathbb{R}$ is a singular function at some points $t \in \overline{J}$, $\mu_1, \ldots, \mu_4 \in \overline{\mathcal{L}}$ are some nonnegative real-valued maps. Then fractional differential pointwise defined equation (1) under boundary conditions (2) has a solution whenever the following assumptions hold:*

(1) *The function ω satisfies the contraction condition*

$$\big|\omega(t, u_1, \ldots, u_4) - \omega(t, v_1, \ldots, v_4)\big| \leq \sum_{i=1}^{4} \mu_i(t)\|u_i - v_i\|$$

for all $u_1, \ldots, u_4, v_1, \ldots, v_4 \in \overline{\mathcal{B}}$ and $t \in \overline{J}$.

(2) *There exist a natural number k_0, some functions $\gamma_1, \ldots, \gamma_{k_0} \in \overline{\mathcal{L}}$, $\Theta_1, \ldots, \Theta_{k_0} : \mathbb{R}^4 \to [0, \infty)$, nonnegative maps $\gamma_1, \ldots, \gamma_{k_0}$, and nonnegative and nondecreasing maps in their all components $\Theta_1, \ldots, \Theta_{k_0}$ such that*

$$\big|\omega(t, u_1, \ldots, u_4)\big| \leq \sum_{i=1}^{k_0} \gamma_i(t)\Theta_i(u_1, \ldots, u_4)$$

for all $(u_1, \ldots, u_4) \in \overline{\mathcal{B}}^4$ and $t \in \overline{J}$ and $\lim_{w\to\infty} \frac{\Theta_i(w,w,w,w)}{w} = \eta_0$, where η_0 is a nonnegative real number with

$$0 \leq \eta_0 \leq \frac{m_0}{M(\alpha,b)\sum_{i=1}^{k_0}\|\gamma_i\| + \delta_0}$$

for some $\delta_0 > 0$, $M(\alpha,b) = \max\{A_1(\alpha,b), A_2(\alpha,b)\}$, and

$$m_0 = \min\big\{1, \Gamma_q(2-\beta_1), \Gamma_q(\beta_2+1)\big\}.$$

(3) *We have*

$$\tau(\alpha,b) = \left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right] M(\alpha,b) < 1,$$

where $\hat{\mu}_i = \int_0^1 (1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_q s = \Gamma_q(\alpha) I_q^{\alpha}\mu_i(1)$.

Proof Let $u_1, u_2 \in \overline{\mathcal{B}}$ and t belong to $\overline{J}$. Then we obtain

$$\begin{aligned}
\left|T_{u_1}(t) - T_{u_2}(t)\right| &\leq \int_0^1 G_q(t,s)\left|\omega\left(u_1(s), u_1'(s), D_q^{\beta_1}u_1(s), I_q^{\beta_2}u_1(s)\right)\right. \\
&\quad \left. - \omega\left(u_2(s), u_2'(s), D_q^{\beta_1}u_2(s), I_q^{\beta_2}u_2(s)\right)\right| \mathrm{d}_q s \\
&\leq \int_0^1 A_1(\alpha,b)(1-qs)^{(\alpha-1)}\left(\mu_1(s)\|u_1 - u_2\| + \mu_2(s)\left\|u_1' - u_2'\right\|\right. \\
&\quad \left. + \mu_3(s)\left\|D_q^{\beta_1}u_1 - D_q^{\beta_1}u_2\right\| - \mu_4(s)\left\|I_q^{\beta_2}u_1 - I_q^{\beta_2}u_2\right\|\right) \mathrm{d}_q s.
\end{aligned}$$

Since

$$\begin{aligned}
\left|I_q^{\beta_2}u(t)\right| &\leq \frac{1}{\Gamma_q(\beta_2)}\int_0^t (t-qs)^{(\beta_2)}\left|u(s)\right| \mathrm{d}_q s \\
&\leq \frac{\|u\|}{\Gamma_q(\beta_2)}\left(\frac{1}{\beta_2}\left[(t-qs)^{(\beta_2)}|_0^t\right]\right) = \frac{\|u\|}{\Gamma_q(\beta_2+1)}t^{\beta_2},
\end{aligned}$$

$\Gamma_q(\beta_2+1)\|I_q^{\beta_2}u\| \leq \|u\|$. Hence,

$$\Gamma_q(\beta_2+1)\left\|I_q^{\beta_2}u_1 - I_q^{\beta_2}u_2\right\| = \Gamma_q(\beta_2+1)\left\|I_q^{\beta_2}(u_1-u_2)\right\| \leq \|u_1 - u_2\|.$$

Similarly, one can conclude that $\Gamma_q(2-\beta_1)\|D_q^{\beta_1}u_1 - D_q^{\beta_1}u_2\| \leq \|u_1 - u_2\|$ for each $u_1, u_2 \in \overline{\mathcal{B}}$. Therefore

$$\begin{aligned}
\left|T_{u_1}(t) - T_{u_2}(t)\right| &\leq A_1(\alpha,b)\int_0^1 \left[\mu_1(s)\|u_1 - u_2\| + \mu_2(s)\left\|u_1' - u_2'\right\|\right. \\
&\quad \left. + \mu_3(s)\frac{\|u_1' - u_2'\|}{\Gamma_q(2-\beta_1)} + \mu_4(s)\frac{\|u_1 - u_2\|}{\Gamma_q(\beta_2+1)}\right](1-qs)^{(\alpha-1)} \mathrm{d}_q s \\
&= A_1(\alpha,b)\int_0^1 \left[\left(\mu_1(s) + \frac{\mu_4(s)}{\Gamma_q(\beta_2+1)}\right)(1-qs)^{(\alpha-1)}\|u_1 - u_2\|\right. \\
&\quad \left. + \left(\mu_2(s) + \frac{\mu_3(s)}{\Gamma_q(2-\beta_1)}\right)(1-qs)^{(\alpha-1)}\left\|u_1' - u_2'\right\|\right] \mathrm{d}_q s \\
&\leq A_1(\alpha,b)\|u_1 - u_2\|_* \int_0^1 \left[\mu_1(s) + \mu_2(s)\right. \\
&\quad \left. + \frac{\mu_3(s)}{\Gamma_q(2-\beta_1)} + \frac{\mu_4(s)}{\Gamma_q(\beta_2+1)}\right](1-qs)^{(\alpha-1)} \mathrm{d}_q s \\
&= A_1(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u_1 - u_2\|_*. \qquad (10)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\left|T_{u_1}'(t) - T_{u_2}'(t)\right| &\leq \int_0^1 \left|\frac{\partial G_q(t,s)}{\partial t}\right| |\omega\left(s, u_1(s), u_1(s), D_q^{\beta_1}u_1(s), I_q^{\beta_2}u_1(s)\right) \\
&\quad - \omega\left(s, u_2(s), u_2(s), D_q^{\beta_1}u_2(s), I_q^{\beta_2}u_2(s)\right)| \mathrm{d}_q s \\
&\leq \int_0^1 A_2(\alpha,b)(1-qs)^{(\alpha-1)}\left[\mu_1(s)\|u_1 - u_2\| + \mu_2(s)\left\|u_1' - u_2'\right\|\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \mu_3(s)\big\|D_q^{\beta_1}u_1 - D_q^{\beta_1}u_2\big\| + \mu_1(s)\big\|I_q^{\beta_2}u_1 - I_q^{\beta_2}u_2\big\|\big]\,\mathrm{d}_qs \\
&\le A_2(\alpha,b)\int_0^1\bigg[\mu_1(s)\|u_1-u_2\| + \mu_2(s)\big\|u_1'-u_2'\big\| \\
&+ \mu_3(s)\frac{\|u_1'-u_2'\|}{\Gamma_q(2-\beta_1)} + \mu_4(s)\frac{\|u_1-u_2\|}{\Gamma_q(\beta_2+1)}\bigg](1-qs)^{(\alpha-1)}\,\mathrm{d}_qs \\
&= A_2(\alpha,b)\bigg[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\bigg]\|u_1-u_2\|_*. \qquad (11)
\end{aligned}
$$

By using (10) and (11), we obtain

$$
\begin{aligned}
\|T_{u_1} - T_{u_2}\|_* &= \max\big\{\|T_{u_1} - T_{u_2}\|, \big\|T_{u_1}' - T_{u_2}'\big\|\big\} \\
&\le M(\alpha,b)\bigg[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\bigg]\|u_1-u_2\|_*.
\end{aligned}
$$

Hence, $\|T_{u_1} - T_{u_2}\|_* \to 0$ as $\|u_1-u_2\|_* \to 0$, and so T is continuous. Since $\eta_0 M(\alpha,b) \times \sum_{i=1}^{k_0}\|\gamma_i\| < m_0$, we can choose $\varepsilon_0 > 0$ such that

$$
(\eta_0+\varepsilon_0)M(\alpha,b)\sum_{i=1}^{k_0}\|\gamma_i\| < m_0.
$$

Since $\frac{\Theta(w,w,w,w)}{w} \to \eta_0$ as $w\to\infty$, there exists $r = \Delta(\varepsilon_0) > 0$ such that $\frac{\Theta(w,w,w,w)}{w} < \eta_0 + \varepsilon_0$ for all $w \ge \Delta(\varepsilon_0)$. So

$$
\Theta(w,w,w,w) < (\eta_0+\varepsilon_0)w \qquad (12)
$$

for $w \ge \Delta(\varepsilon_0)$. Put $B_r = \{u \in \overline{\mathcal{B}} : \|u\|_* < r\}$ and define $\alpha : \overline{\mathcal{B}}^2 \to [0,\infty)$ by $\alpha(u,v) = 1$ whenever $u,v \in B_r$ and $\alpha(u,v) = 0$ otherwise. If $\alpha(u,v) \ge 1$, then $\|u\|_*$ and $\|v\|_*$ are less than r. Let $t \in \overline{J}$. Then we obtain

$$
\begin{aligned}
\big|T_u(t)\big| &\le \int_0^1 G_q(t,s)\big|\omega\big(s,u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\big|\,\mathrm{d}_qs \\
&\le A_1(\alpha,b)\int_0^1(1-qs)^{(\alpha-1)} \\
&\quad\times\sum_{i=1}^{k_0}\gamma_i(s)\Theta_i\big(u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\,\mathrm{d}_qs \\
&\le A_1(\alpha,b)\sum_{i=1}^{k_0}\int_0^1(1-qs)^{(\alpha-1)}\gamma_i(s) \\
&\quad\times\Theta_i\bigg(\|u\|,\big\|u_1'\big\|,\frac{\|u'\|}{\Gamma_q(2-\beta_1)},\frac{\|u\|}{\Gamma_q(\beta_2+1)}\bigg)\,\mathrm{d}_qs \\
&\le A_1(\alpha,b)\sum_{i=1}^{k_0}\Theta_i\bigg(r,r,\frac{r}{\Gamma_q(2-\beta_1)},\frac{r}{\Gamma_q(\beta_2+1)}\bigg)
\end{aligned}
$$

$$\begin{aligned}
&\times \int_0^1 \gamma_i(s) \sup(1-qs)^{(\alpha-1)}\, \mathrm{d}_q s \\
&\le A_1(\alpha,b) \sum_{i=1}^{k_0} \Theta_i\left(\frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}\right) \|\gamma_i\|_1,
\end{aligned}$$

where

$$m_0 = \min\bigl\{1, \Gamma_q(\beta_2+1), \Gamma_q(2-\beta_1)\bigr\} = \min\bigl\{\Gamma_q(\beta_2+1), \Gamma_q(2-\beta_1)\bigr\}.$$

Since $r > m_0 r$, by using (12) we obtain

$$\Theta_i\left(\frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}\right) < (\eta_0 + \varepsilon_0)\frac{r}{m_0},$$

and so

$$\begin{aligned}
\bigl|T_u(t)\bigr| &\le A_1(\alpha,b) \sum_{i=1}^{k_0} \frac{r}{m_0}(\eta_0 + \varepsilon_0)\|\gamma_i\|_1 \\
&= (\eta_0 + \varepsilon_0)\left[\frac{A_1(\alpha,b)\sum_{i=1}^{k_0}\|\gamma_i\|_1}{m_0}\right] r < r.
\end{aligned}$$

Hence, $\|T_u\| \le r$. Also, one can conclude that

$$\begin{aligned}
\bigl|T'_u(t)\bigr| &\le \int_0^1 \left|\frac{\partial G_q(t,s)}{\partial t}\right| \bigl|\omega\bigl(s, u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\bigr)\bigr|\, \mathrm{d}_q s \\
&\le A_2(\alpha,b) \int_0^1 (1-qs)^{(\alpha-1)} \\
&\quad \times \sum_{i=1}^{k_0} \gamma_i(s)\Theta_i\bigl(u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\bigr)\, \mathrm{d}_q s \\
&\le A_2(\alpha,b)\left[\sum_{i=1}^{k_0} \Theta_i\left(r, r, \frac{r}{\Gamma_q(2-\beta_1)}, \frac{r}{\Gamma_q(\beta_2+1)}\right)\right] \\
&\quad \times \int_0^1 (1-qs)^{(\alpha-1)}\gamma_i(s)\, \mathrm{d}_q s \\
&\le A_2(\alpha,b) \sum_{i=1}^{k_0} \Theta_i\left(\frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}, \frac{r}{m_0}\right) \int_0^1 \|\gamma_i\|_1 \\
&\le (\eta_0 + \varepsilon_0)\left[\frac{A_2(\alpha,b)\sum_{i=1}^{k_0}\|\gamma_i\|_1}{m_0}\right] r < r
\end{aligned}$$

and $\|T_u\|_* = \max\{\|T_u\|, \|T'_u\|_*\} \le r$. This implies that T_u and so $T_v \in B_r$, that is, $\alpha(T_u, T_v) \ge 1$. Thus, T is α-admissible. Since $B_r \neq \emptyset$, there exists $u_0 \in B_r$ such that $T_{u_0} \in B_r$. Hence, $\alpha(u_0, T_{u_0}) \ge 1$. Put $\psi(t) = \tau(\alpha,b)t$ for each $t \in [0,\infty)$, here $\tau(\alpha,b) < 1$. Since

$$\sum_{n=1}^{\infty} \psi^n(t) = \sum_{n=1}^{\infty} \tau(\alpha,b)^n t = \left(\frac{\tau(\alpha,b)}{1-\tau(\alpha,b)}\right) t < \infty$$

and $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing, we get $\psi \in \Psi$. Note that

$$\begin{aligned}\left|T_u(t) - T_v(t)\right| &\leq \int_0^1 G_q(t,s)\big|\omega\big(s,u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\\ &\quad - \omega\big(s,v(s),v'(s),D_q^{\beta_1}v(s),I_q^{\beta_2}v(s)\big)\big|\,\mathrm{d}_q s\\ &\leq A_1(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u-v\|_*,\end{aligned}$$

and so

$$\|T_u - T_v\| \leq A_1(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u-v\|_*. \tag{13}$$

Also,

$$\begin{aligned}\left|T'_u(t) - T'_v(t)\right| &\leq \int_0^1 \left|\frac{\partial G_q(t,s)}{\partial t}\right|\big|\omega\big(s,u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\\ &\quad - \omega\big(s,v(s),v'(s),D_q^{\beta_1}v(s),I_q^{\beta_2}v(s)\big)\big|\,\mathrm{d}_q s\\ &\leq A_2(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u-v\|_*.\end{aligned}$$

This implies

$$\left\|T'_u - T'_v\right\| \leq A_2(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u-v\|_*. \tag{14}$$

Thus, equations (13) and (14) imply that

$$\begin{aligned}\|T_u - T_v\|_* &\leq M(\alpha,b)\left[\hat{\mu}_1 + \hat{\mu}_2 + \frac{\hat{\mu}_3}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}_4}{\Gamma_q(\beta_2+1)}\right]\|u-v\|_*\\ &= \tau(\alpha,b)\|u-v\|_* = \psi\big(\|u-v\|_*\big)\end{aligned}$$

for u and v in B_r. Hence, $\alpha(u,v)\|T_u - T_v\|_* \leq \psi(\rho(u,v))$ for each $u,v \in \overline{\mathcal{B}}$. By using Lemma 2, T has a fixed point, which is a solution for problem (1). □

Theorem 6 *Let ω be a real-valued function on $\overline{J} \times \overline{\mathcal{A}}^4$. Then the pointwise defined problem (1) with boundary conditions (2) has a solution whenever the following assumptions hold:*

(1) *There exist natural numbers k_1, some maps $\Theta_1,\dots,\Theta_{k_1} : \mathbb{R}^4 \to \mathbb{R}$ which are nondecreasing in their all components, $\Theta_i(w,w,w,w) \geq 0$ for all $w \geq 0$ and $\frac{\Theta_i(w,w,w,w)}{w} \to \eta_i$ as $w \to 0^+$ for some $\eta_i \in [0,1)$ $(i = 1,\dots,k_1)$, and there are some nonnegative real-valued functions $\mu_1,\dots,\mu_{k_1} : \overline{J} \to [0,\infty)$ such that*

$$\begin{aligned}&\big|\omega(t,u_1,u_2,u_3,u_4) - \omega(t,v_1,v_2,v_3,v_4)\big|\\ &\quad\leq \sum_{i=1}^{k_1}\mu_i(t)\Theta_i(u_1-v_1,u_2-v_2,u_3-v_3,u_4-v_4)\end{aligned}$$

for all $u_1,\dots,u_4,v_1,\dots,v_4 \in \overline{\mathcal{B}}$ with $u_j \geq v_j \geq 0$ $(j = 1,\dots,4)$ and $t \in \overline{J}$.

(2) *If* $M(\alpha,b)=\max\{A_1(\alpha,b),A_2(\alpha,b)\}$, *then*

$$M(\alpha,b)\sum_{i=1}^{k_1}\big\|(1-qt)^{(\alpha-1)}\mu_i\big\|_1\le 1.$$

Proof Since $\lim_{w\to 0^+}\frac{\Theta_i(w,w,w,w)}{w}=\eta_i<1$ for $i=1,\dots,k_1$, for each $\varepsilon_i>0$ there exist $\delta_i=\delta(\varepsilon_i)>0$ such that $\frac{w}{m_0}\in(0,\delta_i)$ implies

$$\Theta_i\left(\frac{w}{m_0},\dots,\frac{w}{m_0}\right)\le(\eta_i+\varepsilon_i)\frac{w}{m_0},$$

where $m_0=\min\{\Gamma_q(2-\beta_1),\Gamma_q(\beta_2+1)\}$. Let ε_i^0 be such that $\eta_i+\varepsilon_i^0<1$ and $\delta_i^0=\delta(\varepsilon_i^0)$. Put $\eta=\max\{\eta_1,\dots,\eta_{k_1}\}$, $\varepsilon_0=\min\{\varepsilon_1^0,\dots,\varepsilon_{k_1}^0\}$, and $\delta=\min\{\delta_1^0,\dots,\delta_{k_1}^0,\varepsilon_0\}$. Thus, $\eta+\varepsilon_0<1$ and

$$\Theta_i\left(\frac{w}{m_0},\dots,\frac{w}{m_0}\right)<(\eta_i+\varepsilon_0)\frac{w}{m_0}$$

for $\frac{w}{m_0}\in(0,\delta)$ and $1\le i\le k_1$. Also,

$$\Theta_i\left(\frac{\delta}{m_0},\dots,\frac{\delta}{m_0}\right)<(\eta+\varepsilon_0)\frac{\delta}{m_0}m_0=(\eta+\varepsilon_0)\delta\le(\eta+\varepsilon_0)\varepsilon_0.$$

Now, we define the map $\alpha:\overline{\mathcal{B}}\times\overline{\mathcal{B}}\to[0,\infty)$ by $\alpha(u,v)=1$ whenever $\|u-v\|_*\le\delta$ and $\alpha(u,v)=0$ otherwise. If $\alpha(u,v)\ge 1$, then $\|u-v\|_*\le\delta$, and so

$$\begin{aligned}
\big|T_u(t)-T_v(t)\big| &\le\int_0^1 G_q(t,s)\big|\omega\big(s,u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\\
&\quad-\omega\big(s,v(s),v'(s),D_q^{\beta_1}v(s),I_q^{\beta_2}v(s)\big)\big|\,\mathrm{d}_qs\\
&\le\int_0^1 G_q(t,s)\sum_{i=1}^{k_1}\mu_i(s)\\
&\quad\times\big|\Theta_i\big((u-v)(s),(u-v)'(s),D_q^{\beta_1}(u-v)(s),I_q^{\beta_2}(u-v)(s)\big)\big|\,\mathrm{d}_qs\\
&\le A_1(\alpha,b)\int_0^1(1-qs)^{(\alpha-1)}\sum_{i=1}^{k_1}\mu_i(s)\\
&\quad\times\big|\Theta_i\big(\|u-v\|,\big\|(u-v)'\big\|,\big\|D_q^{\beta_1}(u-v)\big\|,\big\|I_q^{\beta_2}(u-v)\big\|\big)\big|\,\mathrm{d}_qs\\
&\le A_1(\alpha,b)\sum_{i=1}^{k_1}\left(\int_0^1(1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_qs\right)\\
&\quad\times\Theta_i\left(\delta,\delta,\frac{\delta}{\Gamma_q(2-\beta_1)},\frac{\delta}{\Gamma_q(\beta_2+1)}\right)\\
&\le A_1(\alpha,b)\sum_{i=1}^{k_1}\left(\int_0^1(1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_qs\right)\\
&\quad\times\Theta_i\left(\frac{\delta}{m_0},\frac{\delta}{m_0},\frac{\delta}{m_0},\frac{\delta}{m_0}\right)
\end{aligned}$$

$$\le A_1(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1(\eta+\varepsilon_0)\delta$$

$$\le A_1(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1\delta\le\delta. \tag{15}$$

Hence, $\|T_u - T_v\|_* \le \delta$, which implies $\alpha(T_u, T_v) = 1$. If

$$\lambda = M(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1(\eta+\varepsilon_0),$$

then by using the assumption we get $\lambda < 1$. If $\psi(t) = \lambda t$, then $\psi \in \Psi$. If $\|u - v\| \le \delta$, then

$$\|T_u - T_v\| \le A_1(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1(\eta+\varepsilon_0)\|u-v\|_* \le \lambda\|u-v\|_* \tag{16}$$

and

$$\left\|T'_u - T'_v\right\| \le A_2(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1(\eta+\varepsilon_0)\|u-v\|_* \le \lambda\|u-v\|_*. \tag{17}$$

Thus, from inequality (16), we have $\|T_u - T_v\|_* \le \lambda\|u - v\|_* = \psi(\|u - v\|_*)$ and so $\alpha(u,v)\|T_u - T_v\|_* \le \psi(\|u-v\|_*)$ for each $u, v \in \overline{\mathcal{B}}$. Let $\varepsilon > 0$ be given. Since $\Theta_i(\frac{w}{m_0},\dots,\frac{w}{m_0}) \to 0$ as $w \to 0^+$, for each $i = 1,\dots,k_1$, there exists $\delta_i > 0$ such that $\Theta_i(\frac{w}{m_0},\dots,\frac{w}{m_0}) < \frac{\varepsilon}{\lambda'}$ for all $0 < w \le \delta_i$, where

$$\lambda' = \left[M(\alpha,b)\sum_{i=1}^{k_1}\left\|(1-qt)^{(\alpha-1)}\mu_i\right\|_1\right]+1.$$

If $\delta = \min\{\delta_i : 1 \le i \le k_1\}$, then $\Theta_i(\frac{w}{m_0},\dots,\frac{w}{m_0}) < \frac{\varepsilon}{\lambda'}$ for all $w \in (0,\delta]$ and $i = 1,\dots,k_1$. If $u_n \to u$, then there exists a natural number n_0 such that $\|u_n - u\|_* < \delta$ for all $n \ge n_0$. Hence,

$$\begin{aligned}
&\left|T_{u_n}(t) - T_u(t)\right| \\
&\quad\le \int_0^1 G_q(t,s)\left|\omega\left(s,u_n(s),u_n'(s),D_q^{\beta_1}u_n(s),I_q^{\beta_2}u_n(s)\right)\right. \\
&\qquad\left.-\,\omega\left(s,u(s),u'(s),D_q^{\beta_2}u(s),I_q^{\beta_2}u(s)\right)\right|\,\mathrm{d}s \\
&\quad\le \int_0^1 G_q(t,s)\sum_{i=1}^{k_1}\mu_i(s) \\
&\qquad\times\Theta_i\left(\|u_n-u\|,\left\|(u_n-u)'\right\|,\left\|D_q^{\beta_1}(u_n-u)\right\|,\left\|I_q^{\beta_2}(u_n-u)\right\|\right)\mathrm{d}s \\
&\quad\le A_1(\alpha,b)\sum_{i=1}^{k_1}\left[\Theta_i\left(\delta,\delta,\frac{\delta}{\Gamma_q(2-\beta_1)},\frac{\delta}{\Gamma_q(\beta_2+1)}\right)\right. \\
&\qquad\left.\times\int_0^1(1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_qs\right]
\end{aligned}$$

$$\begin{aligned}
&\leq A_1(\alpha,b)\sum_{i=1}^{k_1}\biggl[\Theta_i\biggl(\frac{\delta}{m_0},\frac{\delta}{m_0},\frac{\delta}{m_0},\frac{\delta}{m_0}\biggr)\\
&\quad\times\int_0^1(1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_qs\biggr]\\
&\leq A_1(\alpha,b)\frac{\varepsilon}{k'}\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|_1<\varepsilon
\end{aligned}\tag{18}$$

for $t\in\overline{J}$ and $n\geq n_0$. By repeating the similar method, we obtain

$$\begin{aligned}
\bigl|T'_{u_n}(t)-T'_u(t)\bigr|\leq{}&\int_0^1\left|\frac{\partial G_q(t,s)}{\partial t}\right|\bigl|\omega\bigl(s,u_n(s),u'_n(s),D_q^{\beta_1}u_n(s),I_q^{\beta_2}u_n(s)\bigr)\\
&-\omega\bigl(s,u(s),u'(s),D_q^{\beta}u(s),I_q^{\beta_2}u(s)\bigr)\bigr|\,\mathrm{d}_qs\\
\leq{}&A_2(\alpha,b)\frac{\varepsilon}{k'}\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|_1<\varepsilon
\end{aligned}\tag{19}$$

for all t in $\overline{J}$ and $n\geq n_0$. Now, (18) and (19) imply that $\|T_{u_n}-T_u\|\leq\varepsilon$ and $\|T'_{u_n}-T'_u\|\leq\varepsilon$, respectively, for $n\geq n_0$. Thus,

$$\|T_{u_n}-T_u\|_*=\max\bigl\{\|T_{u_n}-T_u\|,\bigl\|T'_{u_n}-T'_u\bigr\|\bigr\}\leq\varepsilon$$

for $n\geq n_0$, and so $T_{u_n}\to T_u$ as $u_n\to u$. Indeed, T is continuous. Now, we show that there exists $u_0\in\overline{\mathcal{B}}$ such that $\alpha(u_0,T_{u_0})=1$. In this way, we have to show that $\|T_{u_0}-u_0\|\leq\delta$ for some $u_0\in\overline{\mathcal{B}}$. Let $r_0>0$ be a fixed real number. Since $\Theta_i(w,w,w,w)\to 0$ as $w\to 0^+$, for each $\varepsilon>0$, there exists $n=n(\varepsilon)$ such $0<w\leq\frac{r_0}{n}$ implies $\Theta_i(w,w,w,w)\leq\varepsilon$ for all $1\leq i\leq k_1$. Hence, $\Theta_i(\frac{r_0}{n},\dots,\frac{r_0}{n})\leq\varepsilon$. Put

$$M=\max\biggl\{\frac{1}{\Gamma_q(2-\beta_1)},\frac{1}{\Gamma_q(\beta_2+1)},1\biggr\}=\max\biggl\{\frac{1}{\Gamma_q(2-\beta_1)},\frac{1}{\Gamma_q(\beta_2+1)}\biggr\}$$

and choose ε_M such that

$$\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|_1\varepsilon_MM(\alpha,b)<\delta.$$

Take $n_1=n(\varepsilon_M)$ and choose a natural number n_2 such that

$$\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|_1\varepsilon_MM(\alpha,b)\leq\delta-\frac{1}{n_2}.$$

If $n_0=\max\{n_1,n_2\}$, then $\Theta_i(\frac{M}{n_0},\dots,\frac{M}{n_0})\leq\varepsilon_M$ for $1=1,\dots,k_1$. Define

$$u_0(t)=\begin{cases}0, & t\leq\frac{1}{n_0+1},\\ \frac{6n_0^2}{6n_0^2+5n_0+2}\\ \quad\times\bigl[\frac{t^3}{3}-\frac{2n_0+1}{2n_0(n_0+1)}t^2+\frac{t}{n_0(n_0+1)}\bigr]+\frac{1}{n_0+2}, & \frac{1}{n_0+1}<t<\frac{1}{n_0},\\ \frac{1}{n_0}, & \frac{1}{n_0}\leq t.\end{cases}$$

One can easily see that $u_0 \in \overline{\mathcal{A}}$ and $u(t) \in [0, \frac{1}{n_0}]$, $u_0(\frac{1}{n_0+1}) = 0$ and

$$u_0'(t) = \begin{cases} 0, & t \leq \frac{1}{n_0+1}, \\ \frac{6n_0^2}{6n_0^2+5n_0+2}[t^2 - \frac{2n_0+1}{n_0(n_0+1)}t + \frac{1}{n_0(n_0+1)}], & \frac{1}{n_0+1} < t < \frac{1}{n_0}, \\ 0, & \frac{1}{n_0} \leq t. \end{cases}$$

Hence, u_0' belongs to $\overline{\mathcal{A}}$ and $u_0'(\frac{1}{n_0+1}) = u_0'(\frac{1}{n_0}) = 0$. Thus, $u_0 \in \overline{\mathcal{B}}$. Also, we have

$$\begin{aligned} u_0'(t) &\leq \frac{6n_0^2}{6n_0^2 + 5n_0 + 2}\left(\frac{1}{n_0^2} + \frac{1}{n_0^4(n_0+1)} - \frac{2n_0+1}{2n_0(n_0+1)^2}\right) \\ &\leq 1 \times \frac{1}{n_0} \times \left(\frac{1}{n_0} + \frac{1}{n_0^3(n_0+1)} - \frac{2n_0}{(n_0+1)^2}\right) \leq \frac{1}{n_0}, \end{aligned}$$

and so $n_0\|u_{n_0}\|_* \leq 1$. This implies that

$$\begin{aligned} &\bigl|T_{u_0}(t) - u_0(t)\bigr| \\ &\quad= \left|\int_0^1 G_q(t,s)\omega\bigl(s, u_0(s), u_0'(s), D_q^{\beta_1}u_0(s), I_q^{\beta_2}u_0(s)\bigr)\,\mathrm{d}_qs - u_0(s)\right| \\ &\quad\leq \int_0^1 G_q(t,s)\bigl|\omega\bigl(s, u_0(s), u_0'(s), D_q^{\beta}u_0(s), I_q^{\beta_2}u_0(s)\bigr)\,\mathrm{d}_qs\bigr| + \frac{1}{n_0} \\ &\quad\leq A_1(\alpha,b)\int_0^1 (1-qs)^{(\alpha-1)} \\ &\qquad\times\left[\sum_{i=1}^{k_1}\mu_i(s)\Theta_i\left(\|u_0\|, \|u_0'\|, \frac{\|u_0'\|}{\Gamma_q(2-\beta_1)}, \frac{\|u_0\|}{\Gamma_q(\beta_2+1)}\right)\right]\mathrm{d}_qs + \frac{1}{n_0} \\ &\quad\leq A_1(\alpha,b)\sum_{i=1}^{k_1}\left[\left(\left(\int_0^1 (1-qs)^{(\alpha-1)}\mu_i(s)\,\mathrm{d}_qs\right)\right.\right. \\ &\qquad\left.\times\Theta_i\left(\frac{r_0}{n_0}, \frac{r_0}{n_0}, \frac{r_0}{n_0}, \frac{r_0}{n_0}\right)\right] + \frac{1}{n_0} \\ &\quad\leq A_1(\alpha,b)\left[\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|\right]\varepsilon_M + \frac{1}{n_0} \leq \delta \end{aligned} \tag{20}$$

for $t \in \overline{J}$, and so $\|T_{u_0} - u_0\| \leq \delta$. By using a similar method, we get

$$\begin{aligned} \bigl|\bigl(T_{u_0}(t) - u_0(t)\bigr)'\bigr| &= \bigl|T_{u_0}'(t) - u_0'(t)\bigr| \\ &\leq \int_0^1\left|\frac{\partial G_q}{\partial t}(t,s)\right|\bigl|\omega\bigl(s, u_0(s), u_0'(s), D_q^{\beta_1}u_0(s), I_q^{\beta_2}u_0(s)\bigr)\,\mathrm{d}_qs\bigr| + \frac{1}{n_0} \\ &\leq A_2(\alpha,b)\left[\sum_{i=1}^{k_1}\bigl\|(1-qt)^{(\alpha-1)}\mu_i\bigr\|\right]\varepsilon_M + \frac{1}{n_0} \leq \delta, \end{aligned} \tag{21}$$

and so $\|(T_{u_0} - u_0)'\| \leq \delta$. Hence, from equations (20) and (21), we obtain $\|T_{u_0} - u_0\|_* = \max\{\|T_{u_0} - u_0\|, \|(T_{u_0} - u_0)'\|\} \leq \delta$. Thus, $\alpha(u_0, T_{u_0}) = 1$. Now, by using Lemma 2, the map T has a fixed point, which is a solution for the multi-singular fractional q-problem (1). □

Now, we present our final result.

Theorem 7 *Assume that $\omega : \overline{J} \times \overline{\mathcal{B}}^4 \to [0,\infty]$ is such that $\omega(t,u_1,u_2,u_3,u_4) < \infty$ for all $u_1,u_2,u_3,u_4 \in \overline{\mathcal{B}}$ and $t \in E$, where E^c is a null subset of $\overline{J}$, that is, the measure of E^c is zero, the map $\omega(t,u_1,u_2,u_3,u_4)$ is continuous with respect to the components u_1, u_2, u_3, and u_4 for all $t \in E$. Then the pointwise defined problem (1) with boundary conditions (2) has a solution whenever the following assumptions hold*:

(1) *There exist a natural number $n_1 \geq 1$ and some maps $\mu_1,\dots,\mu_{n_1} : \overline{J} \to [0,\infty)$ such that $\mu_1,\dots,\mu_{n_1} \in \overline{\mathcal{L}}$, the maps $F_1,\dots,F_{n_1} : \mathbb{R}^4 \to [0,\infty)$ and $\Omega : \mathbb{R}^4 \to [0,\infty)$ so that*

$$\|\Omega\|_1^* = \sup_{x\in\overline{\mathcal{L}}} \int_0^1 \Omega\big(u(t),u(t),u(t),u(t)\big)\,\mathrm{d}t < \infty,$$

$$\|F_i\|_\infty = \sup_{w\in\mathbb{R}} \big\{F_i(w,w,w,w)\big\} < \infty$$

for $i = 1,\dots,n_1$ and

$$\big|\omega(t,u_1,u_2,u_3,u_4)\big| \leq \sum_{i=1}^{n_1} \mu_i(t)F_i(u_1,u_2,u_3,u_4) + \Omega(u_1,u_2,u_3,u_4)$$

for $u_1,\dots,u_4 \in \overline{\mathcal{B}}$ and $t \in \overline{J}$.

(2) *There exist some maps $\psi : \mathbb{R}^4 \to [0,\infty)$ and $h : \overline{J} \to [0,\infty)$ such that $\|h\|_1^L = \Gamma_q(\alpha)I_q^\alpha h(1) < \infty$ and $h(t)\psi(u_1,\dots,u_4) \leq \omega(t,u_1,\dots,u_4)$ for $u_1,\dots,u_4 \in \overline{\mathcal{B}}$ and $t \in \overline{J}$.*

(3) *There exist $\gamma_1,\gamma_2,\gamma_3,\gamma_4 \in \overline{\mathcal{L}}$ and $\phi : [0,\infty) \to [0,\infty)$ such that*

$$M(\alpha,b)\sum_{i=1}^{4} \|\gamma_i\|_1 < 1,$$

$\phi_{m_0} \in \Psi$ and

$$\big|\omega(t,u_1,\dots,u_4) - w(t,v_1,\dots,v_4)\big| \leq \sum_{i=1}^{4} \gamma_i(t)\phi\big(\|u_i - v_i\|\big)$$

for all $(u_1,\dots,u_4)$ and $(v_1,\dots,v_4) \in \overline{\mathcal{B}}^4$ with $\|u_i\|, \|v_i\| \in [\delta_1,\delta_2]$, where $\phi_\lambda(z) := \phi(\frac{z}{\lambda})$ for all $\lambda \in (0,\infty)$,

$$\|\psi\|_m := \min\big\{\psi(u_1,\dots,u_4) : (u_1,\dots,u_4) \in \mathbb{R}^4\big\},$$

$2\delta_1\Gamma_q(\alpha)(1-\alpha) \leq \|\psi\|_m\|h\|_1^L(4-\alpha^2-2\alpha)$ and

$$\delta_2 \geq M(\alpha,b)\left(\sum_{i=1}^{n_1} \|F_i\|_\infty\|\mu_i\|_1 + \|\Omega\|_1^*\right).$$

Proof Let $\{u_n\}$ be a sequence such that $\|u_n - u\|_* \to 0$. Then $u_n \to u$ and $u'_n \to u'$. By using the inequalities $\Gamma_q(2-\beta_1)\|D_q^{\beta_1}(u_n - u)\| \le \|(u_n - u)'\|$ and

$$\Gamma_q(\beta_2 + 1)\big\|I_q^{\beta_2}(u_n - u)\big\| \le \|u_n - u\|,$$

we get $D_q^{\beta}u_n \to D_q^{\beta}u$ and $I_q^{\beta_2}u_n \to I_q^{\beta_2}u$. Since $\omega(t, u_1, \dots, u_4)$ is continuous with respect to $u_1, \dots, u_4$ for all $t \in E$, we can conclude that

$$\omega\big(t, u_n, u'_n, D_q^{\beta_1}u_n, I_q^{\beta_2}u_n\big) \to \omega\big(t, u, u', D_q^{\beta_1}u, I_q^{\beta_2}u\big)$$

for $t \in E$. Let $u \in \overline{\mathcal{B}}$ be given and $t \in \overline{J}$. Then we have

$$\begin{aligned} \big|T_u(t)\big| \le A_1(\alpha, b)\Bigg[\int_0^1 (1-qs)^{\alpha-1}\Bigg(\Bigg[\sum_{i=1}^{n_1}\mu_i(s)F_i\big(u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\big)\Bigg] \\ + \Omega\big(u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\big)\Bigg)\,\mathrm{d}_q s\Bigg]. \end{aligned}$$

If $u_M(s) := \max\{u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\}$, then $u_M \in \overline{\mathcal{A}}$, and so

$$\begin{aligned} \big|T_u(t)\big| &\le A_1(\alpha, b)\Bigg[\sum_{i=1}^{n_1} F_i\Bigg(\|u\|, \big\|u'\big\|, \frac{\|u'\|}{\Gamma_q(2-\beta_1)}, \frac{\|u\|}{\Gamma_q(\beta_2+1)}\Bigg)\Bigg]\int_0^1 \mu_i(s)\,\mathrm{d}s \\ &\quad + \int_0^1 \Omega\big(u_M(s), u_M(s), u_M(s), u_M(s)\big)\,\mathrm{d}s \\ &\le A_1(\alpha, b)\Bigg[\sum_{i=1}^{n_1}\|F_i\|_\infty\|\mu_i\| + \|\Omega\|_1^*\Bigg]. \end{aligned}$$

Similarly, one can see that $|T'_u(t)| \le A_2(\alpha, b)[\sum_{i=1}^{n_1}\|F_i\|_\infty\|\mu_i\| + \|\Omega\|_1^*]$. Thus,

$$\|T_u\|_* \le M(\alpha, b)\Bigg[\sum_{i=1}^{n_1}\|F_i\|_\infty\|\mu_i\| + \|\Omega\|_1^*\Bigg] < \infty \tag{22}$$

for all $u \in \overline{\mathcal{B}}$. By using the Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned} T_{u_n}(t) &= \int_0^1 G_q(t,s)\omega\big(s, u_n(s), u'_n(s), D_q^{\beta_1}u_n(s), I_q^{\beta_2}u_n(s)\big)\,\mathrm{d}_q s \\ &\to \int_0^1 G_q(t,s)\omega\big(s, u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\big)\,\mathrm{d}_q s = T_u(t) \end{aligned}$$

for t belonging to $\overline{J}$, and so the self-map T on $\overline{\mathcal{B}}$ is continuous. Define the map $\alpha : \overline{\mathcal{B}}^2 \to [0,\infty)$ by $\alpha(u, v) = 1$ whenever $\|u\|_*, \|v\|_* \in [\delta_1, \delta_2]$, $\alpha(u, v) = 0$, otherwise. If $\alpha(u, v) \ge 1$, then $\|u\|_*, \|v\|_* \in [\delta_1, \delta_2]$, and so

$$\begin{aligned} \big|T_u(t)\big| &= \Bigg|\int_0^1 G_q(t,s)\omega\big(s, u(s), u'(s), D_q^{\beta_1}u(s), I_q^{\beta_2}u(s)\big)\,\mathrm{d}_q s\Bigg| \\ &\ge \int_0^1 \frac{(1-qs)^{b-1}}{\Gamma_q(\alpha)}\bigg[-t + \frac{2-b^2}{2(1-b)}\bigg]h(s) \end{aligned}$$

$$
\begin{aligned}
&\times \psi\big[u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big]\,\mathrm{d}_qs\\
&\geq \|\psi\|_m\bigg[\frac{-t}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{\alpha-1}h(s)\,\mathrm{d}_qs\\
&\quad+\frac{2-b^2}{2(1-b)}\int_0^1(1-qs)^{(b-1)}h(s)\,\mathrm{d}_qs\bigg]\\
&\geq \|\psi\|_m\|h\|_1^L\bigg[\frac{1}{\Gamma_q(\alpha)}+\frac{2-b^2}{2(1-b)}\bigg]\\
&= \|\psi\|_m\|h\|_1^L\bigg[\frac{4-b^2-2b}{2\Gamma_q(\alpha)(1-b)}\bigg]
\end{aligned}
$$

for $t\in\bar{J}$. Thus, $2\Gamma_q(\alpha)(1-b)\|T_u\|\geq\|\psi\|_m\|h\|_1^L(4-b^2-2b)$, and so

$$
\|T_u\|_*:=\max\big\{\|T_u\|,\big\|T_u'\big\|\big\}\geq\frac{\|\psi\|_m\|h\|_1^L(4-b^2-2b)}{2\Gamma_q(\alpha)(1-b)}\geq\delta_1.
$$

By using (22), we obtain

$$
\|T_x\|_*\leq M(\alpha,b)\Bigg(\sum_{i=1}^{n_1}\|F_i\|_\infty\|\mu_i\|_1+\|\Omega\|_1^*\Bigg)\leq\delta_2,
$$

and so $\alpha(T_u,F_v)\geq 1$. If $u_0\in[\delta_1,\delta_2]$, then it is easy to check that $\alpha(T_{u_0},u_0)\geq 1$. Let $u,v\in[\delta_1,\delta_2]$. Then

$$
\begin{aligned}
&\alpha(u,v)\big|T_u(t)-T_v(t)\big|\\
&\quad\leq\int_0^1 G_q(t,s)\big|\omega\big(s,u(s),u'(s),D_q^{\beta_1}u(s),I_q^{\beta_2}u(s)\big)\\
&\qquad-\omega\big(s,v(s),v'(s),D_q^{\beta_1}v(s),I_q^{\beta_2}v(s)\big)\big|\,\mathrm{d}_qs\\
&\quad\leq A_1(\alpha,b)\int_0^1(1-qs)^{(\alpha-1)}\big(\gamma_1(s)\phi\big(|u-v|\big)+\gamma_2(s)\phi\big(\big|u'-v'\big|\big)\\
&\qquad+\gamma_3(s)\phi\big(\big|D_q^{\beta_1}u-D_q^{\beta_1}v\big|\big)+\gamma_4(s)\phi\big(\big|I_q^{\beta_2}u-I_q^{\beta_2}v\big|\big)\big)\,\mathrm{d}_qs\\
&\quad\leq A_1(\alpha,b)\int_0^1(1-qs)^{(\alpha-1)}\bigg(\gamma_1(s)\phi\big(\|u-v\|\big)+\gamma_2(s)\phi\big(\big\|u'-v'\big\|\big)\\
&\qquad+\gamma_3(s)\phi\bigg(\frac{\|u'-v'\|}{\Gamma_q(2-\beta_1)}\bigg)+\gamma_4(s)\phi\bigg(\frac{\|u-v\|}{\Gamma_q(\beta_2+1)}\bigg)\bigg)\,\mathrm{d}_qs\\
&\quad\leq A_1(\alpha,b)\int_0^1(1-qs)^{(\alpha-1)}\phi\bigg(\frac{\|u-v\|_*}{m_0}\bigg)\Bigg[\sum_{i=1}^4\gamma_i(s)\Bigg]\,\mathrm{d}_qs\\
&\quad\leq A_1(\alpha,b)\phi\bigg(\frac{\|u-v\|_*}{m_0}\bigg)\sum_{i=1}^4\int_0^1(1-qs)^{(\alpha-1)}\gamma_i(s)\,\mathrm{d}_qs\\
&\quad\leq\sum_{i=1}^4\|\gamma_i\|_1\phi\bigg(\frac{\|u-v\|_*}{m_0}\bigg)\leq\phi_{\gamma_0}\big(\|u-v\|_*\big).
\end{aligned}
\tag{23}
$$

Algorithm 5 The proposed method for calculated $\int_a^b f(r)\,\mathrm{d}_q r$

Input: $q \in (0,1)$, α, n, $f(x)$, a, b

```
1: s ← 0
2: for i = 0 : n do
3:     s ← s + q^i * (b * f(b * q^i) − a * f(a * q^i))
4: end for
5: g ← (1 − q) * s
```

Output: $\int_a^b f(r)\,\mathrm{d}_q r$

Similarly, we conclude that

$$\alpha(u,v)\left\| T'_u - T'_v \right\| \le \phi_{m_0}\left(\|u-v\|_*\right). \tag{24}$$

Now, (23) and (24) imply that $\alpha(u,v)\| T_u - T_v\|$ and $\alpha(u,v)\| T'_u - T'_v\|_*$ are less than or equal to $\phi_{m_0}(\|u-v\|_*)$ for all $u,v \in \overline{\mathcal{B}}$. By using Theorem 2, the self-map T has a fixed point, which is a solution for problem (1). □

4 Examples and algorithms for the problem

Here, we provide some examples to illustrate our main results. In this way, we give a computational technique for checking problem (1). We need to present that a simplified analysis could be executed on values of the q-gamma function. To this aim, we consider a pseudo-code description of the method for calculation of the q-gamma function of order n in Algorithms 2, 3, 4, and 5 (for more details, see the link https://en.wikipedia.org/wiki/Q-gamma_function).

Example 1 Consider the following pointwise defined problem, similar to (1):

$$D_q^{\frac{7}{2}} u(t) + \frac{1}{96\sqrt{\pi}[g(t)]^{\frac{1}{3}}}\left(\|u\| + \left\|u'\right\| + \left\|D_q^{\frac{2}{3}} u\right\| + \left\|I_q^{\frac{1}{2}} u\right\|\right) = 0 \tag{25}$$

with boundary conditions $u'(0) = u(\frac{5}{6})$, $u(1) = \int_0^{\frac{1}{2}} u(s)\,\mathrm{d}s$, and $u''(0) = 0$, where $g(t) = 0$ whenever $t \in \overline{J} \cap Q$, $g(t) = t$ whenever $t \in J \cap Q^c$. Let $\alpha = \frac{7}{2}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{1}{3}$, $a = \frac{5}{6}$, and $b = \frac{1}{2}$. Then we have

$$\begin{aligned} M(\alpha,b) &= \max\left\{A_1(\alpha,b), A_2(\alpha,b)\right\} \\ &= \max\left\{\frac{3}{(1-b)\Gamma_q(\alpha)}, \frac{2}{(1-b)\Gamma_q(\alpha-1)}\right\} \\ &= \max\left\{\frac{3}{(1-\frac{1}{2})\Gamma_q(\frac{7}{2})}, \frac{2}{(1-\frac{1}{2})\Gamma_q(\frac{5}{2})}\right\} \end{aligned} \tag{26}$$

and

$$m_0 = \min\left\{\Gamma_q(2-\beta_1), \Gamma_q(\beta_2+1)\right\} = \min\left\{\Gamma_q\left(\frac{4}{3}\right), \Gamma_q\left(\frac{3}{2}\right)\right\}. \tag{27}$$

Table 4 Some numerical results of $M(\boldsymbol{\alpha}, b)$ in equation (26) in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$

n	$q=\frac{1}{7}$			$q=\frac{1}{2}$			$q=\frac{8}{9}$		
	$A_1(\boldsymbol{\alpha},b)$	$A_2(\boldsymbol{\alpha},b)$	$M(\boldsymbol{\alpha},b)$	$A_1(\boldsymbol{\alpha},b)$	$A_2(\boldsymbol{\alpha},b)$	$M(\boldsymbol{\alpha},b)$	$A_1(\boldsymbol{\alpha},b)$	$A_2(\boldsymbol{\alpha},b)$	$M(\boldsymbol{\alpha},b)$
1	4.8544	3.7471	4.8544	2.4645	2.8302	2.8302	0.1472	0.5474	0.5474
2	4.8685	3.7575	4.8685	2.7543	3.0915	3.0915	0.2357	0.7566	0.7566
3	4.8705	3.759	4.8705	2.9055	3.2248	3.2248	0.3356	0.9602	0.9602
4	4.8708	3.7592	4.8708	2.9826	3.292	3.292	0.4424	1.1541	1.1541
5	4.8709	3.7592	4.8709	3.0216	3.3258	3.3258	0.5522	1.336	1.336
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	4.8709	3.7592	4.8709	3.0608	3.3596	3.3596	1.4427	2.4896	2.4896
16	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	1.5002	2.5526	2.5526
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
80	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	2.0086	3.0741	3.0741
91	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	2.0088	3.0743	3.0743
92	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	2.0088	3.0744	3.0744
93	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	2.0089	3.0744	3.0744
94	4.8709	3.7592	4.8709	3.0608	3.3597	3.3597	2.0089	3.0744	3.0744

Table 5 Some numerical results of m_0 in equation (27) in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$

n	$q=\frac{1}{7}$			$q=\frac{1}{2}$			$q=\frac{8}{9}$		
	$\Gamma_q(\beta_2+1)$	$\Gamma_q(2-\beta_1)$	m_0	$\Gamma_q(\beta_2+1)$	$\Gamma_q(2-\beta_1)$	m_0	$\Gamma_q(\beta_2+1)$	$\Gamma_q(2-\beta_1)$	m_0
1	0.9661	0.9656	0.9656	0.9965	0.9772	0.9772	1.6936	1.389	1.389
2	0.9644	0.9642	0.9642	0.9565	0.9493	0.9493	1.4923	1.2733	1.2733
3	0.9641	0.964	0.964	0.9382	0.9364	0.9364	1.3628	1.1967	1.1967
4	0.9641	0.964	0.964	0.9294	0.9302	0.9294	1.2721	1.1419	1.1419
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.9641	0.964	0.964	0.921	0.9243	0.921	1.031	0.9905	0.9905
11	0.9641	0.964	0.964	0.9209	0.9242	0.9209	1.0124	0.9783	0.9783
12	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.9967	0.9681	0.9681
13	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.9833	0.9593	0.9593
14	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.9719	0.9517	0.9517
15	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.962	0.9452	0.9452
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
67	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8928	0.8988	0.8928
68	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8928	0.8988	0.8928
69	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8928	0.8988	0.8928
70	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8927	0.8988	0.8927
71	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8927	0.8988	0.8927
72	0.9641	0.964	0.964	0.9209	0.9242	0.9209	0.8927	0.8988	0.8927

Tables 4 and 5 show the values of $M(\alpha, b)$ in equation (26) and m_0 in equation (27), respectively. Put

$$\omega\big(t, u(t), u'(t), D_q^{\beta_1} u(t), I_q^{\beta_2} u(t)\big) := \frac{1}{96\sqrt{\pi}[g(t)]^{\frac{1}{3}}} \big(\|u\| + \|u'\| + \|D_q^{\frac{2}{3}} u\| + \|I_q^{\frac{1}{2}} u\|\big),$$

$\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu_4(t) = \mu(t) = \frac{1}{96\sqrt{\pi} t^{\frac{1}{3}}}$, $\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = \gamma_4(t) = \gamma(t) = \frac{1}{96\sqrt{\pi} t^{\frac{1}{3}}}$ and $\Theta_i(u_1, u_2, u_3, u_4) := \|u_i\|$ for $i = 1, \ldots, 4$. Then $\|\mu\|_1 = \|\gamma\|_1 = \frac{1}{96\sqrt{\pi}(1-\frac{1}{3})} = \frac{1}{64\sqrt{\pi}}$,

$$\big|\omega(t, u_1, u_2, u_3, u_4) - \omega(t, v_1, v_2, v_3, v_4)\big|$$

$$
\begin{aligned}
&= \frac{1}{96\sqrt{\pi}(g(t))^{\frac{1}{3}}} \sum_{i=1}^{4} \|u_i\| - \|v_i\| \\
&\leq \frac{1}{96\sqrt{\pi}(g(t))^{\frac{1}{3}}} \sum_{i=1}^{4} \|u_i - v_i\| \\
&= \frac{1}{96\sqrt{\pi}\, t^{\frac{1}{3}}} \sum_{i=1}^{4} \|u_i - v_i\| \\
&= \sum_{i=1}^{4} \mu(t)\|u_i - v_i\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\omega(t,u_1,u_2,u_3,u_4)\right| &= \frac{1}{96\sqrt{\pi}(g(t))^{\frac{1}{3}}} \sum_{i=1}^{4} \|u_i\| \\
&= \frac{1}{96\sqrt{\pi}\, t^{\frac{1}{3}}} \sum_{i=1}^{4} \|u_i\| \\
&= \sum_{i=1}^{4} \gamma(t)\Theta_i(u_1,u_2,u_3,u_4)
\end{aligned}
$$

for all (u_1,u_2,u_3,u_4) and $(v_1,v_2,v_3,v_4) \in \overline{\mathcal{B}}^4$ and $t \in \overline{J}$. On the other hand, we have $\lim_{w\to\infty} \frac{\Theta_i(w,w,w,w)}{w} = 1$, and by equations (26) and (27), we get

$$
\Lambda = \frac{m_0}{M(\alpha,b)\sum_{i=1}^{4}\|\gamma_i(t)\|_1} = \frac{m_0}{M(\alpha,b)\sum_{i=1}^{4}\|\gamma(t)\|_1} > 1. \tag{28}
$$

Table 6 shows the values of Λ. Choose $\delta_0 > 0$ such that $m_0 \geq M(\alpha,b)\sum_{i=1}^{4}\|\gamma\|_1 + \delta_0$. Since

$$
\hat{\mu} = \int_0^1 (1-qs)^{(\frac{7}{2}-1)}\mu(s)\,\mathrm{d}_q s \leq \int_0^1 \mu(s)\,\mathrm{d}s = \|\mu\|_1 = \frac{1}{64\sqrt{\pi}},
$$

we obtain

$$
\begin{aligned}
\tau(\alpha,b) &= \left[\hat{\mu} + \hat{\mu} + \frac{\hat{\mu}}{\Gamma_q(2-\beta_1)} + \frac{\hat{\mu}}{\Gamma_q(\beta_2+1)}\right] M(\alpha,b) \\
&\leq \frac{1}{64\sqrt{\pi}}\left[1 + 1 + \frac{1}{\Gamma_q(\frac{4}{3})} + \frac{1}{\Gamma_q(\frac{3}{2})}\right] M(\alpha,b) < 1.
\end{aligned} \tag{29}
$$

Table 7 shows the values of $\tau(\alpha,b)$. Now, by using Theorem 5, the pointwise defined problem (25) has a solution.

Example 2 Consider the fractional q-integro-differential equation

$$
D_q^{\frac{5}{2}} u(t) + \frac{0.09}{t^{\frac{1}{4}}(t-\frac{1}{3})^{\frac{1}{8}}}\left[1 - \left(\frac{3}{5}\right)^{\frac{2}{5}(u(t)+u'(t)+D_q^{\frac{1}{4}}u(t)+I_q^{\frac{1}{5}}u(t))}\right] = 0 \tag{30}
$$

Table 6 Some numerical results of $\Lambda > 1$ in equation (28) in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$

n	$q = \frac{1}{7}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	m_0	$M(\alpha, b)$	Λ	m_0	$M(\alpha, b)$	Λ	m_0	$M(\alpha, b)$	Λ
1	0.9656	4.8544	5.6396	0.9772	2.8302	9.7893	1.389	0.5474	71.942
2	0.9642	4.8685	5.6151	0.9493	3.0915	8.706	1.2733	0.7566	47.7144
3	0.964	4.8705	5.6116	0.9364	3.2248	8.2327	1.1967	0.9602	35.3353
4	0.964	4.8708	5.6113	0.9294	3.292	8.0044	1.1419	1.1541	28.0524
5	0.964	4.8709	5.6112	0.9251	3.3258	7.8864	1.1007	1.336	23.3586
6	0.964	4.8709	5.6112	0.923	3.3427	7.8287	1.0685	1.505	20.129
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	0.964	4.8709	5.6112	0.9209	3.3596	7.7716	0.9452	2.4896	10.7641
16	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.9396	2.5526	10.4363
17	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.9346	2.6089	10.1567
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
95	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.8927	3.0744	8.2325
96	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.8927	3.0744	8.2325
97	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.8926	3.0744	8.2315
98	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.8926	3.0744	8.2315
99	0.964	4.8709	5.6112	0.9209	3.3597	7.7714	0.8926	3.0744	8.2315

Table 7 Some numerical results of $\tau(\alpha, b) < 1$ in equation (29) in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$. Note that $(a) = \Gamma_q(\beta_2 + 1)$, $(b) = \Gamma_q(2 - \beta_1)$, and $(c) = M(\alpha, b)$

n	$q = \frac{1}{7}$				$q = \frac{1}{2}$				$q = \frac{8}{9}$			
	(a)	(b)	(c)	τ	(a)	(b)	(c)	τ	(a)	(b)	(c)	τ
1	0.9661	0.9656	4.8544	0.1742	0.9965	0.9772	2.8302	0.1005	1.6936	1.389	0.5474	0.016
2	0.9644	0.9642	4.8685	0.1749	0.9565	0.9493	3.0915	0.1117	1.4923	1.2733	0.7566	0.0231
3	0.9641	0.964	4.8705	0.175	0.9382	0.9364	3.2248	0.1175	1.3628	1.1967	0.9602	0.0302
4	0.9641	0.964	4.8708	0.175	0.9294	0.9302	3.292	0.1205	1.2721	1.1419	1.1541	0.0373
5	0.9641	0.964	4.8709	0.175	0.9251	0.9272	3.3258	0.122	1.205	1.1007	1.336	0.044
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.9641	0.964	4.8709	0.175	0.921	0.9243	3.3586	0.1234	1.031	0.9905	2.0511	0.072
11	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3591	0.1235	1.0124	0.9783	2.158	0.0763
12	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3594	0.1235	0.9967	0.9681	2.2544	0.0802
13	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3595	0.1235	0.9833	0.9593	2.3413	0.0838
14	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3596	0.1235	0.9719	0.9517	2.4194	0.087
15	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3596	0.1235	0.962	0.9452	2.4896	0.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
62	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3597	0.1235	0.8929	0.8989	3.072	0.1146
63	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3597	0.1235	0.8929	0.8989	3.0723	0.1147
64	0.9641	0.964	4.8709	0.175	0.9209	0.9242	3.3597	0.1235	0.8928	0.8989	3.0725	0.1147

for $t \in \overline{J}$, with boundary condition $u'(0) = u(\frac{1}{2})$ and $u(1) = \int_0^{\frac{1}{7}} u(s)\,ds$. Put $\alpha = \frac{5}{2}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{1}{5}$, $a = \frac{1}{2}$, $b = \frac{1}{7}$, and $k_1 = 1$. Note that

$$\begin{aligned} M(\alpha, b) &= \max\left\{ \frac{3}{(1-b)\Gamma_q(\alpha)}, \frac{2}{(1-b)\Gamma_q(\alpha-1)} \right\} \\ &= \max\left\{ \frac{3}{(1-\frac{1}{7})\Gamma_q(\frac{5}{2})}, \frac{2}{(1-\frac{1}{7})\Gamma_q(\frac{3}{2})} \right\}. \end{aligned} \tag{31}$$

Table 8 Some numerical results of $M(\alpha, b)$ in equation (31) in Example 2 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$

n	$q = \frac{1}{7}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	$A_1(\alpha, b)$	$A_2(\alpha, b)$	$M(\alpha, b)$	$A_1(\alpha, b)$	$A_2(\alpha, b)$	$M(\alpha, b)$	$A_1(\alpha, b)$	$A_2(\alpha, b)$	$M(\alpha, b)$
1	3.2787	2.4151	3.2787	2.4764	2.3414	2.4764	0.479	1.3777	1.3777
2	3.2878	2.4195	3.2878	2.7051	2.4394	2.7051	0.662	1.5636	1.5636
3	3.2891	2.4201	3.2891	2.8217	2.487	2.8217	0.8402	1.7122	1.7122
4	<u>3.2893</u>	2.4202	3.2893	2.8805	2.5105	2.8805	1.0098	1.8343	1.8343
5	3.2893	2.4202	3.2893	2.9101	2.5222	2.9101	1.169	1.9364	1.9364
6	3.2893	2.4202	3.2893	2.9249	2.528	2.9249	1.3169	2.0229	2.0229
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	3.2893	2.4202	3.2893	2.9396	2.5338	2.9396	2.0486	2.3729	2.3729
14	3.2893	2.4202	3.2893	2.9397	2.5338	<u>2.9397</u>	2.117	2.4009	2.4009
15	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.1784	2.4255	2.4255
16	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.2335	2.4472	2.4472
17	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.2828	2.4662	2.4662
18	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.327	2.4831	2.4831
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
88	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.69	2.6139	2.69
89	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.69	2.6139	2.69
90	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.69	2.6139	2.69
91	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.6901	2.6139	<u>2.6901</u>
92	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.6901	2.6139	2.6901
93	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.6901	2.6139	2.6901
94	3.2893	2.4202	3.2893	2.9397	2.5338	2.9397	2.6901	2.6139	2.6901

Table 8 shows the values of $M(\alpha, b)$ in equation (31). Define

$$\omega(t, u_1, \ldots, u_4) = \frac{0.09}{t^{\frac{1}{4}}(t - \frac{1}{3})^{\frac{1}{8}}}\left[1 - \left(\frac{3}{5}\right)^{\frac{2}{5}(u_1 + \cdots + u_4)}\right],$$

$\mu(t) = \frac{0.09}{t^{\frac{1}{4}}(t-\frac{1}{3})^{\frac{1}{8}}}$, and

$$\Theta(u_1, \ldots, u_4) = 1 - \left(\frac{3}{5}\right)^{\frac{2}{5}(u_1 + \cdots + u_4)}.$$

One can easily see that Θ is nondecreasing in all its components and $\Theta(w, w, w, w) \geq 0$ for all $w \geq 0$. Assume that $(u_1, \ldots, u_4)$ and $(v_1, \ldots, v_4)$ belong to $\overline{\mathcal{B}}^4$ and $u_i \geq v_i \geq 0$ for $i = 1, \ldots, 4$. Since $(\frac{3}{5})^{v_i} \geq (\frac{3}{5})^{u_i}$,

$$\left(\frac{3}{5}\right)^{v_i}\left[\left(\frac{3}{5}\right)^{v_i} - \left(\frac{3}{5}\right)^{u_i}\right] \leq \left(\frac{3}{5}\right)^{v_i} - \left(\frac{3}{5}\right)^{u_i},$$

and so

$$\left(\frac{3}{5}\right)^{v_i}\left[\left(\frac{3}{5}\right)^{v_i} - \left(\frac{3}{5}\right)^{u_i}\right] \leq \left(\frac{3}{5}\right)^{v_i}\left[1 - \left(\frac{3}{5}\right)^{u_i - v_i}\right].$$

Thus,

$$\left[1 - \left(\frac{3}{5}\right)^{u_i}\right] - \left[1 - \left(\frac{3}{5}\right)^{v_i}\right] \leq 1 - \left(\frac{3}{5}\right)^{u_i - v_i}.$$

By replacing u_i, v_i with $\frac{2}{5}\sum_{i=1}^{4} u_i$, $\frac{2}{5}\sum_{i=1}^{4} v_i$, respectively, we get

$$\left[1-\left(\frac{3}{5}\right)^{\frac{2}{5}\sum_{i=1}^{4} u_i}\right]-\left[1-\left(\frac{3}{5}\right)^{\frac{2}{5}\sum_{i=1}^{4} v_i}\right]\leq 1-\left(\frac{3}{5}\right)^{\frac{2}{5}(\sum_{i=1}^{4} u_i - v_i)}.$$

Hence, $\Theta(u_1,\dots,u_4)-\Theta(v_1,\dots,v_4)\leq\Theta(u_1-v_1,\dots,u_4-v_4)$. On the other hand, we have

$$\omega(t,u_1,\dots,u_4)-\omega(t,v_1,\dots,v_4)\leq\mu(t)\Theta(u_1-v_1,\dots,u_4-v_4)$$

and

$$\lim_{w\to 0^+}\frac{\Theta(w,w,w,w)}{w}=\lim_{w\to 0^+}\frac{1-(\frac{3}{5})^{4\times\frac{2}{5}w}}{w}=-4\left(\frac{2}{5}\right)\left[\ln\left(\frac{3}{5}\right)\right]=0.81<1.$$

Now, by using Theorem 6, the fractional q-integro-differential pointwise defined equation (30) has a solution.

Example 3 Consider the fractional q-integro-differential equation

$$D_q^{\frac{5}{2}}u(t)+\frac{0.01}{g(t)}F\bigl(u(t),u'(t),D_q^{\frac{1}{3}}u(t),I_q^{\frac{2}{3}}u(t)\bigr)+2=0\quad (t\in\overline{J}) \tag{32}$$

with boundary condition $u'(0)=u(\frac{1}{2})$ and $u(1)=\int_0^{\frac{4}{5}}u(s)\,ds$, where $g:\overline{J}\to[0,\infty)$ is defined by $g(t)=0$ whenever $t\in Q\cap\overline{J}$ and $g(t)=\sqrt{t}$ whenever $t\in Q^c\cap\overline{J}$ and the map $F:\mathbb{R}^4\to[0,\infty)$ is defined by

$$F(u_1,\dots,u_4)=\begin{cases}\frac{1}{2}\sum_{i=1}^{4}\frac{\|u_i\|}{1+\|u_i\|}, & u_1,\dots,u_4\in[0,29],\\ |\sin(u_1+\cdots+u_4)|, & u_1,\dots,u_4\in(-\infty,0),\\ \frac{-29}{48}(\frac{1}{4}\sum_{i=1}^{4}u_i-24), & u_1,\dots,u_4\in[29,30],\\ 0, & \text{otherwise.}\end{cases}$$

Put $\alpha=\frac{5}{2}$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{2}{3}$, $a=\frac{1}{2}$, and $b=\frac{4}{5}$. Then we have

$$\begin{aligned}M(\alpha,b)&=\max\left\{\frac{3}{(1-b)\Gamma_q(\alpha)},\frac{2}{(1-b)\Gamma_q(\alpha-1)}\right\}\\&=\max\left\{\frac{3}{(1-\frac{4}{5})\Gamma_q(\frac{5}{2})},\frac{2}{(1-\frac{4}{5})\Gamma_q(\frac{3}{2})}\right\}\end{aligned} \tag{33}$$

and

$$m_0=\min\bigl\{\Gamma_q(2-\beta_1),\Gamma_q(\beta_2+1)\bigr\}=\min\left\{\Gamma_q\left(\frac{5}{3}\right),\Gamma_q\left(\frac{5}{3}\right)\right\}=\Gamma_q\left(\frac{5}{3}\right). \tag{34}$$

Tables 9 and 10 show the values of $M(\alpha,b)$ and m_0 in equations (33) and (34), respectively. By simple checking, we can see that $\|F\|_\infty=1$. Let $n_1=1$. Define the maps $\psi(u_1,\dots,u_4):=$

Table 9 Some numerical results of $M(\boldsymbol{\alpha}, b)$ in equation (33) in Example 3 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$

n	$q = \frac{1}{7}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	$A_1(\boldsymbol{\alpha}, b)$	$A_2(\boldsymbol{\alpha}, b)$	$M(\boldsymbol{\alpha}, b)$	$A_1(\boldsymbol{\alpha}, b)$	$A_2(\boldsymbol{\alpha}, b)$	$M(\boldsymbol{\alpha}, b)$	$A_1(\boldsymbol{\alpha}, b)$	$A_2(\boldsymbol{\alpha}, b)$	$M(\boldsymbol{\alpha}, b)$
1	14.0518	10.3504	14.0518	10.6131	10.0347	10.6131	2.0529	5.9045	5.9045
2	14.0906	10.3693	14.0906	11.5932	10.4546	11.5932	2.8373	6.7012	6.7012
3	14.0962	10.372	14.0962	12.0928	10.6587	12.0928	3.6008	7.3381	7.3381
4	14.097	10.3723	14.097	12.345	10.7594	12.345	4.3279	7.8613	7.8613
5	14.0971	10.3724	14.0971	12.4717	10.8094	12.4717	5.0102	8.299	8.299
6	14.0971	10.3724	14.0971	12.5352	10.8344	12.5352	5.6437	8.6696	8.6696
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	14.0971	10.3724	14.0971	12.5987	10.8592	12.5987	9.9727	10.6418	10.6418
19	14.0971	10.3724	14.0971	12.5987	10.8592	12.5987	10.1419	10.7056	10.7056
20	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	10.2931	10.7619	10.7619
21	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	10.4282	10.8118	10.8118
22	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	10.5487	10.8559	10.8559
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
105	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.529	11.2026	11.529
106	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.529	11.2026	11.529
107	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.5291	11.2026	11.5291
108	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.5291	11.2026	11.5291
109	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.5291	11.2026	11.5291
110	14.0971	10.3724	14.0971	12.5988	10.8592	12.5988	11.5291	11.2026	11.5291

Table 10 Some numerical results of $M(\boldsymbol{\alpha}, b)$ in equation (33) in Example 3 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$. Note that $(a) = m_0$, $(b) = 0.02 \times M(\boldsymbol{\alpha}, b)$, and $(c) = 2.02 \times M(\boldsymbol{\alpha}, b)$

n	$q = \frac{1}{7}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
1	0.9738	0.281	28.3846	1.0314	0.2123	21.4385	2.1001	0.1181	11.9271
2	0.9717	0.2818	28.463	0.9796	0.2319	23.4183	1.7825	0.134	13.5364
3	0.9714	0.2819	28.4743	0.9561	0.2419	24.4275	1.5838	0.1468	14.823
4	0.9714	0.2819	28.4759	0.9448	0.2469	24.9369	1.4475	0.1572	15.8798
5	0.9714	0.2819	28.4761	0.9393	0.2494	25.1928	1.3484	0.166	16.764
6	0.9714	0.2819	28.4761	0.9365	0.2507	25.3211	1.2733	0.1734	17.5126
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	0.9714	0.2819	28.4761	0.9338	0.252	25.4494	0.9646	0.2141	21.6253
20	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.958	0.2152	21.739
21	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9522	0.2162	21.8398
22	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.947	0.2171	21.9289
23	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9426	0.2179	22.0079
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
103	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2886
104	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2886
105	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2886
106	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2886
107	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2888
108	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9086	0.2306	23.2888
109	0.9714	0.2819	28.4761	0.9338	0.252	25.4496	0.9085	0.2306	23.2888

$F(u_1, \dots, u_4)$, $\mu(t) = h(t) = \gamma(t) := 0.01t$, and $\phi(t) = \frac{1}{2}t$. If $\Omega(u_1, \dots, u_4) = 2$,

$$\omega(t, u_1, \dots, u_4) = \frac{0.01}{g(t)} T(u_1, \dots, u_4) + 2,$$

and $[\delta_1, \delta_2] = [0, 29]$, then $\omega(t, u_1, u_2, u_3, u_4) < \infty$ for $u_1, \dots, u_4 \in \overline{\mathcal{B}}$ and $t \in E := Q^c \cap \overline{J}$, $\omega(t, u_1, u_2, u_3, u_4)$ is continuous with respect to the components u_1, u_2, u_3, and u_4 for all $t \in E$,

$$\omega(t, u_1, \dots, u_4) = \frac{1}{g(t)} F(u_1, \dots, u_4) + 2 \geq h(t)\psi(u_1, \dots, u_4)$$

and

$$\begin{aligned} \left|\omega(t, u_1, \dots, u_4) - \omega(t, v_1, \dots, v_4)\right| &\leq \sum_{i=1}^{4} \frac{1}{2} \times \frac{0.01}{\sqrt{t}} \|u_i - v_i\| \\ &= \sum_{i=1}^{4} \gamma(t)\phi\bigl(\|u_i - v_i\|\bigr) \end{aligned}$$

for all $(u_1, \dots, u_4), (v_1, \dots, v_4) \in \overline{\mathcal{B}}^4$ and $t \in \overline{J}$. Note that $\phi_{m_0} \in \Psi$,

$$M(\alpha, b) \sum_{i=1}^{4} \|\gamma_i\|_1 = 4 \times 0.005 \times M(\alpha, b) = 0.02 \times M(\alpha, b) < 1, \tag{35}$$

$$\frac{\|\psi\|_m \|h\|_1^L (4 - \alpha^2 - 2\alpha)}{2\Gamma_q(\alpha)(1 - \alpha)} \geq 0 = \delta_1, \tag{36}$$

and

$$M(\alpha, b)\bigl(\|F\|_\infty \|\mu\|_1 + \|\Omega\|_1^*\bigr) \leq (1 \times 0.02 + 2)M(\alpha, b) \leq 29 = \delta_2. \tag{37}$$

Table 10 shows the values of equation (37). Now, by using Theorem 7, the fractional q-integro-differential pointwise defined equation (32) has a solution.

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