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ON THE EXISTENCE OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Let B be an N-dimensional Brownian motion. We treat a stochastic differential equation based on B,

$$(\mathrm{I}) \qquad dX(s) = \alpha(X(s))\,dB(s)+\gamma(X(s))\,ds$$

where $\alpha(x)=(\alpha_{ij}(x))$ is an $N\times N$-matrix and $\gamma(x)=(\gamma_i(x))$ an N-dimensional vector. In the case where coefficients are bounded and Borel measurable, Krylov [6] proved the existence of solutions under the condition that α is uniformly elliptic. But in the stochastic control, we consider a stochastic differential equation of the following form,

$$dX(t) = \alpha(B(t), X(t))\,dB(t)+\gamma(X(t))\,dt\,.$$

This equation can be regarded as follows,

$$(*) \qquad \begin{cases} dX(t) = \alpha(X(t), Y(t))\,dB(t)+\gamma(X(t))\,dt \\ Y(t) = B(t) \end{cases}$$

In this case, the uniform ellipticity of the coefficient of Brownian part is not valid.

The purpose of this article is to seek some weaker conditions of solvability which can be applied to (*). Our result on the equation (I) is the following. Let f be a function on R^N and K a compact subset of R^{N-l}. Let us define ${}^K f(\xi)$ and $|||f|||_{p,\Gamma,K}$ by

$${}^K f(\xi) = \sup_{\eta\in K} |f(\xi,\eta)|\,, \quad \xi\in R^l$$

and

$$|||f|||_{p,\Gamma,K} = ||{}^K f||_{L_p(\Gamma)}$$

where Γ is a Borel subset of R^l.

Theorem 1. *Suppose that there exist a non-negative integer l, a positive constant $p(>2l)$ and a non-negative bounded Borel function μ, such that*

(A1) *the submatrix* $(\alpha)_l=(\alpha_{ij})_{ij=1\cdots l}$ *is symmetric and*

$$\sum_{ij=1}^{l}\alpha_{ij}(x)t_i t_j \geq \mu(x)|t|^2, \quad t\in R^l,\ x\in R^N\ ,$$

(A2) *for any compact set* K *of* R^{N-l}, ${}^{K}(1/\mu)$ *belongs to* L_p^{loc} [1] *and*

(A3) *for any* $\eta\in R^{n-l}$, $\alpha_{ij}(\cdot,\eta)$ *and* $\gamma_i(\cdot,\eta)$ *are in* L_Q^{loc}, *where* $Q>6pl/p-2l$ $(\equiv 6q)$.

Furthermore we assume that, for any compact sets $\Gamma(\subset R^l)$ *and* $K(\subset R^{N-l})$, *we can take a continuous increasing function* $W_{\Gamma,K}$ *on* $[0\ \infty)$, *so that* $W(0)=0$ *and*

(A4) $|\zeta(\xi,\eta_1)-\zeta(\xi,\eta_2)|\leq W_{\Gamma,K}(|\eta_1-\eta_2|)$, $\xi\in\Gamma$, η_1, $\eta_2\in K$, *where* $\zeta=\alpha_{ij}$ *and* γ_i.

Then we have a solution X *which is starting at* $x\in R^N$ *and defined up to an explosion time* $e(>0)$. *This means that we have a positive random variable* $e(0<e\leq\infty)$ *and two processes* X *and* B, *on a probability space* $(\Omega, \boldsymbol{B}, P)$ *with an increasing family of Borel fields* $F_t(\subset \boldsymbol{B})$, *such that*

(i) B *is an* F_t*-Brownian motion starting at* 0

(ii) $X(t)\chi_{[0\ e)}(t)$ [2] *is* F_t*-measurable and continuous in* $t(<e)$ *with probability* 1,

(iii) $X(t)=x+\int_0^t\alpha(X(s))\,dB(s)+\int_0^t\gamma(X(s))\,ds, \quad t<e$

and

(iv) $\overline{\lim_{t\uparrow e}}\ |X(t)|=\infty$ *for* $e<\infty$, *and if* α *and* γ *are locally bounded, then*

$$\lim_{t\uparrow e}|X(t)|=\infty \qquad \text{for}\quad e<\infty\ .$$

We shall prove it in §2 and some complementary examples, which have 1-dimensional aspect, will be treated in §3. In §4 we consider the stochastic differential equation on $\bar{D}=\{x\in R^N,\ x_1\geq 0\}$, with reflecting barrier,

$$\text{(II)}\quad \begin{aligned} X_1(t) &= x_1+\int_0^t\alpha_1(X(s))\,dB(s)+\int_0^t\gamma_1(X(s))\,ds+\phi(t)\\ X_i(t) &= x_i+\int_0^t\alpha_i(X(s))\,dB(s)+\int_0^t\gamma_i(X(s))\,ds, \quad i=2\cdots N\ . \end{aligned}$$

Under analogous conditions as Theorem 1, we can solve (II), namely

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1) L_p^{loc} is the set of all functions which are p-th integrable on any compact set.

2) χ_A is the indicator of A.

Theorem 2. *Suppose that α and γ, defined on $\bar{D}$, satisfy the conditions* (A1)~(A4)[3] *with $l\geq 1$. Then we have a solution (X, ϕ), defined up to an explosion time $e(>0)$. Namely we have a positive random variable $e(0<e\leq\infty)$ and processes X, ϕ and B, on a probability space $(\Omega, \boldsymbol{B}, P)$ with an increasing family of Borel fields $F_t(\subset \boldsymbol{B})$, such that*

(i) *B is an F_t-Brownian motion starting at* 0

(ii) *$X(t)\chi_{[0e)}(t)$ is F_t-measurable and continuous in $t(<e)$, with probability* 1,

(iii) *$X_1(t)\geq 0$ and $\phi(t)$ is non-decreasing, continuous up to e, flat off $\{t<e; X_1(t)=0\}$ and $\phi(0)=0$*

(iv) *(X, ϕ, B) satisfies* (II) *up to e,*

and

(v) *$\overline{\lim_{t\uparrow e}} |X(t)| = \infty$ for $e<\infty$, and if α and γ are locally bounded, then*

$$\lim_{t\uparrow e} |X(t)| = \infty \qquad \text{for} \quad e<\infty\,.$$

In §5, we shall discuss on a two dimensional stochastic differential equation with boundary conditions. On a stochastic differential equation of this type, we can get nice informations in [4], [12], [13] and [14]. Consider a stochastic differential equation on $\bar{D}=\{x\in R^2,\ x_1\geq 0\}$

$$\text{(III)}\quad \begin{aligned} X_1(t) &= x_1+\int_0^t \alpha_1(X(s))\,dB(s)+\int_0^t \gamma_1(X(s))\,ds+\phi(t) \\ X_2(t) &= x_2+\int_0^t \alpha_2(X(s))\,dB(s)+\int_0^t \gamma_2(X(s))\,ds+\int_0^t \hat{\alpha}(X_2(s))\,dM(s) \\ &\qquad +\int_0^t \hat{\gamma}(X_2(s))\,d\phi(s)\,. \end{aligned}$$

Krylov [7] proved the existence of a solution if α is uniformly elliptic and $\hat{\alpha}=0$ (oblique derivatives).

Theorem 3. *Let the coefficients be locally bounded. Suppose that*

(A1) *α is symmetric and*

$$\sum_{i,j=1}^{2} \alpha_{ij}(x)t_i t_j \geq \mu(x)|t|^2\geq 0$$

where $\dfrac{1}{\mu}\in L_{16}^{loc}$, and

(A2) *$\hat{\alpha}$ is non-negative and $\dfrac{1}{\hat{\alpha}}\in L_p^{loc}(R^1)$ with $p>2$.*

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3) We shall understand that Γ turns to $\Gamma\cap\{x\in R^l;\ x_1\geq 0\}$.

Then we have a solution (X, ϕ, M), defined up to an explosion time e. Namely we have a positive random variable $e(0<e\leq\infty)$ and processes X, ϕ, B and M, on a probability space $(\Omega, \boldsymbol{B}, P)$ with an increasing family of Borel fields $F_t(\subset \boldsymbol{B})$ such that

(i) $(B(t), M(t))$ *is an* F_t*-martingale starting at* 0, *such that*

$$\langle B_i, B_j\rangle=\delta_{i,j}t, \quad \langle B_i, M\rangle=0 \quad i=1, 2, \quad \langle M, M\rangle=\phi(t),$$

where $\langle\ \rangle$ *is the variation process*, [9]

(ii) $X(t)\chi_{[0e)}(t)$ *is* F_t*-measurable and continuous in* $t(<e)$, *with probability* 1,

(iii) $X_1(t)\geq 0$ *and* $\phi(t)$ *is non-decreasing, continuous up to* e, *flat off* $\{t<e; X_1(t)=0\}$ *and* $\phi(0)=0$

(iv) (X, ϕ, B, M) *satisfies* (III) *up to* e *and*

(v) $\lim_{t\uparrow e}|X(t)|=\infty \qquad$ *for* $\quad e<\infty$.

In order to prove Theorems 1~3, we use similar methods as Krylov and our results may be regarded as complementary to the work of Krylov.

2. Proof of Theorem 1

For simplicity we may assume that the starting point $x=0$. We shall apply the following regularization of function f which satisfies (A3) and (A4), [2]. Let us define ω_h^1 and ω_h^2 by

$$\omega_h^1(\xi)=\begin{cases} e^{-|\xi|^2/h^2-|\xi|^2}\Big/h^l\displaystyle\int_{|u|<1} e^{-|u|^2/1-|u|^2}du, & \text{for} \quad |\xi|<h \\ 0 & \text{for} \quad |\xi|\geq h\end{cases}$$

and

$$\omega_h^2(\eta)=\begin{cases} e^{-|\eta|^2/h^2-|\eta|^2}\Big/h^{N-l}\displaystyle\int_{|v|<1} e^{-|v|^2/1-|v|^2}dv & \text{for} \quad |\eta|<h \\ 0 & \text{for} \quad |\eta|\geq h\end{cases}$$

Put $\omega_h(\xi, \eta)=\omega_h^1(\xi)\omega_h^2(\eta)$ and

$$F_h(\xi, \eta)=\int_{|x|<h,\ |y|<h} f(\xi-x, \eta-y)\,\omega_h(x, y)\,dx\,dy$$

for $\xi\in R^l$, $\eta\in R^{N-l}$ and $0<h<1$. Then we have

Lemma 1.

(i) $F_h\in C^\infty(R^N)$

(ii) *For any compact sets* $\Gamma(\subset R^l)$ *and* $K(\subset R^{N-l})$,

$$\sup_{\xi\in\Gamma} |F_h(\xi, \eta_1)-F_h(\xi, \eta_2)| \leqq W_{V(\Gamma), V(K)}(|\eta_1-\eta_2|)$$

where $V(A)$ *means the closed* $1-$*neighbourhood of* A, *and*

(iii) $|||f(\cdot, \eta)-F_h(\cdot, \eta)|||_{q, \Gamma, K}\to 0$ *as* $h\to 0$,

where $q=\dfrac{pl}{p-2l}$.

Proof. Let $\eta_1\cdots\eta_j$ be an ε-net of K w.r.t. to $W_{V(\Gamma), V(K)}$. Put $g_h(\xi, \eta)=$ f$(\cdot, \eta)*\omega_h^1(\xi)$. Then

(1) $||g_h(\cdot, \eta)-\mathrm{f}(\cdot, \eta)||_{L_q(\Gamma)}\to 0 \qquad \eta\in K$.

By (A4) we have

(2) $\sup_{\xi\in\Gamma} |g_h(\xi, \eta)-g_h(\xi, \eta')| \leq W_{V(\Gamma), K}(|\eta-\eta'|), \qquad \eta, \eta'\in K$

Since $\{\eta_i\}$ is an ε-net, by (2),

(3) $|g_h(\xi, \eta)-f(\xi, \eta)| \leq 2\varepsilon+\sum |g_h(\xi, \eta_i)-f(\xi, \eta_i)|, \qquad \xi\in\Gamma$

Hence, by (1) and (3), we have, for small h,

(4) $|||g_h-f|||_{p, \Gamma, K}<3\varepsilon$.

On the other hand, by (2),

(5) $\sup_{\xi\in\Gamma} |g_h(\xi, \eta)-F_h(\xi, \eta)| \leq W_{V(\Gamma), V(K)}(h), \qquad \eta\in K$.

Therefore, by (4) and (5),

$$|||F_h-f|||_{q, \Gamma, K}\leq|||F_h-g_h|||_{q, \Gamma, K}+|||g_h-f|||_{q, \Gamma, K}\to 0 \qquad \text{as} \quad h\to 0 .$$

Take a following approximate stochastic differential equation; Let $\Gamma_k(\subset R^l)$ and $K_k(\subset R^{N-l})$ be compact subsets such that $S_k\subset\Gamma_k\times K_k\subset S_{k+1}$ where $S_k=\{|x|<2^k\}$. Using Lemma 1, we can choose a small h_k so that

(6) $|||f-F_{h_k}|||_{q, \Gamma_k, K_k}<\dfrac{1}{k}$

where $f=\alpha_{ij}$ and γ_i. Let $\alpha_{ij}^{(k)}$ and $\gamma_i^{(k)}$ be smooth and bounded functions such that

(7) $\alpha_{ij}^{(k)}=(\alpha_{ij}*\omega_{h_k})+C_k\delta_{ij} \qquad \text{on} \quad V(S_{k+1})$

where $C_k(>0)$ tends to 0 and

(8) $\gamma_i^{(k)}=\gamma_i*\omega_{h_k} \qquad \text{on} \quad V(S_{k+1})$

Then, by (A1).

$$\sum_{ij=1}^{l} \alpha_{ij}^{(k)}(x) t_i t_j \geq (\mu * \omega_{h_k} + C_k) |t|^2 \qquad \text{on} \quad V(S_{k+1}).$$

Putting $a^{(k)} = \alpha^{(k)} * {}^t\alpha^{(k)}$ [4], we see that

$$\sum_{ij=1}^{l} a_{ij}^{(k)}(x) t_i t_j \geq (\mu * \omega_{h_k} + C_k)^2 |t|^2 \qquad \text{on} \quad V(S_{k+1}).$$

Hence

$$det(a^{(k)})_l \geq (\mu * \omega_{h_k} + C_k)^{2l}.$$

So, by virtue of the convexity of $\frac{1}{x}$, $(x>0)$,

$$\frac{1}{det(a^{(k)})_l} \leq \left(\frac{1}{\mu * \omega_{h_k}}\right)^{2l} \leq \left(\frac{1}{\mu} * \omega_{h_k}\right)^{2l} \leq \left(\frac{1}{\mu}\right)^{2l} * \omega_{h_k}.$$

Therefore, we have

$$(9) \qquad \left|\left|\left| \frac{1}{det(a^k)_l} \right|\right|\right|_{p/2l, \Gamma, K} \leq \left|\left|\left| \frac{1}{\mu} \right|\right|\right|^{2l}_{p, V(\Gamma), V(K)}, \qquad \Gamma \times K \supset \Gamma_k \times K_k.$$

It is also clear

$$(10) \qquad |||\gamma_i^{(k)}|||_{q, \Gamma, K} \leq |||\gamma_i|||_{q, V(\Gamma), V(K)}, \qquad \Gamma \times K \supset \Gamma_k \times K_k$$

We define $\mathcal{A}_k$ by

$\mathcal{A}_k = \{(A, C)$; A is an $N \times N$ matrix valued smooth and bounded function on R^N such that $(A)_l$ is symmetric, uniformly elliptic and

$$\left|\left|\left| \frac{1}{det(A)_l} \right|\right|\right|_{p/2l, \Gamma, K} \leq \left|\left|\left| \frac{1}{\mu} \right|\right|\right|^{2l}_{p, V(\Gamma), V(K)}, \qquad \Gamma \times K \subset \Gamma_k \times K_k.$$

C is an N-vector valued smooth and bounded function on R^N such that

$$|||C_i|||_{q, \Gamma, K} \leqq |||\gamma_i|||_{q, V(\Gamma), V(K)} \qquad \Gamma \times K \subset \Gamma_k \times K_k.$$

A stochastic differential equation with coefficient (A, C) $(\in \mathcal{A}_k)$ has the unique diffusion solution X whose generator is given by

$$= \frac{1}{2} \sum_{ij} (A(x) {}^t A(x))_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i C_i(x) \frac{\partial}{\partial x_i}.$$

Let $\tau_\rho(x)$ be the hitting time of X for $\partial S\rho$, $(\rho < k)$. Now we shall repeat the following inequality, [1], [6].

4) ${}^t\alpha$ means the transpose matrix of α.

Lemma 2. *Let $f(\geq 0)$ be smooth. Then*

$$E\int_0^{\tau_\rho} f(X(s))\,ds \leq M_1 |||f|||_{q,\Gamma,K}, \qquad \Gamma\times K \supset S_\rho . \tag{11}$$

where $M_1 = M_1\Big(N,\, l,\, \rho,\, \Big|\Big|\Big|\frac{1}{\mu}\Big|\Big|\Big|_{p,V(\Gamma_\rho),V(K_\rho)},\, |||\gamma_i|||_{q,V(\Gamma_\rho),V(K_\rho)}\Big)$.

Proof. Put $\mathfrak{G}_\varepsilon = \frac{1}{2}\Sigma(A^tA+\varepsilon I)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \Sigma C_i \frac{\partial}{\partial x_i}$.

By the method of stochastic differential equation, we can construct a diffusion Y_ε with generator $\mathfrak{G}_\varepsilon$. On the other hand, we have the smooth function u of the boundary value problem;

$$\mathfrak{G}_\varepsilon u = f \quad \text{on} \quad S_\rho, \qquad u = 0 \quad \text{on} \quad \partial S_\rho .$$

Moreover Aleksandrov's inequality [1] tells us that

$$|u(x)| \leq M_1 |||f|||_{q,\Gamma,K} \qquad \text{on} \quad \bar{S}_\rho . \tag{12}$$

By a formula on stochastic differentials [5], we have

$$u(Y_\varepsilon(t\wedge\tau_\rho)) - u(0) = \text{martingale} + \int_0^{t\wedge\tau_\rho} f(Y_\varepsilon(s))\,ds .$$

where τ_ρ is the hitting time of Y_ε to ∂S_ρ. Appealing to (12), we see that (11) holds for Y_ε. Since $\{Y_\varepsilon, \varepsilon>0\}$ is totally bounded in Prohorov topology and $\mathfrak{G}_\varepsilon$ tends to $\mathfrak{G}$, Y_ε tends to Y in Prohorov topology, when ε tends to 0. Hence, by Skorohod's theorem, we have a sequence $\{Z_n\}$ and Z such that Z_n (resp. Z) has the same law as Y_{ε_n} (resp. X) and with probability 1, $Z_n(t)$ tends to $Z(t)$ uniformly on any bounded set of t. Let $\sigma^{(n)}$ and σ be the hitting times of Z_n and Z for ∂S_ρ. Then $\underline{\lim}_n \sigma_n \geq \sigma$. Therefore, by the continuity of f, we have Lemma 2.

Some generalization of (11) is obtained by Krylov [8]. From Lemma 2, we get, by a routine

$$E\int_0^{\tau_\rho} \chi_{A\times R^{N-l}}(X(s))\,ds = 0$$

where A is a null set of R^l. Moreover

Lemma 2′. *For a function $f(\geq 0)$ which satisfies* (A3) *and* (A4), *we have*

$$E\int_0^{\tau_\rho} f(X(s))\,ds \leq M_1 |||f|||_{q,\Gamma,K}. \qquad \Gamma\times K \supset S_\rho . \tag{13}$$

REMARK. • Let Y_n be a solution of $(A_n, C_n) \in \mathcal{A}_n$. If $Y_n(\cdot\wedge\tau_\rho(Y_n))$

tends to Y, then (13) holds for Y, where τ_ρ should be understood as $\underline{\lim}_n \tau_\rho(Y_n)$.

Let $X^{(m)}$ be a solution for coefficients $\alpha^{(m)}$ and $\gamma^{(m)}$ and $\tau_\rho^{(m)}$ the hitting time of $X^{(m)}$ for ∂S_ρ. Put $X_\rho^{(m)} \equiv X^{(m)}(t\wedge\tau_\rho^{(m)})$.

Proposition 1. *$\{X_\rho^{(m)}, m=1. 2\cdots\}$ is totally bounded in Prohorov topology.*

Proof. We shall drop the suffix ρ, for simplicity. By a formula on stochastic differentials

$$R(t) \equiv E|X_i^{(m)}(t)-X_i^{(m)}(s)|^2$$
$$\leq E\int_{s\wedge\tau^{(m)}}^{t\wedge\tau^{(m)}} a_{ii}^{(m)}(X^{(m)}(u))+2|X_i^{(m)}(u)-X_i^{(m)}(s)|\,|\gamma_i^{(m)}(X^{(m)}(u))|\,du\,.$$

Hence,

$$R(t) \leq (E\int_0^{t\wedge\tau^{(m)}} (a_{ii}^{(m)})^\delta du)^{1/\delta}|t-s|^{1-1/\delta}+\int_s^t R(u)\,du+(E\int_0^{t\wedge\tau^{(m)}} (\gamma_i^{(m)})^{2\delta})^{1/\delta}|t-s|^{1-1/\delta}.$$

Recalling (A3) and Lemma 2, we have

$$R(t) \leq \int_s^t R(u)\,du+M_2|t-s|^{1-1/\delta}\,. \tag{14}$$

where $\delta=\dfrac{Q}{2q}>3$ and M_2 is independent of m. Henceforth M_i denotes a constant which does not depend on m. From (14), we have

$$R(t) \leq M_2|t-s|^{1-1/\delta}e^{t-s}\,. \tag{15}$$

By the same method, we obtain

$$E|X_i^{(m)}(t)-X_i^{(m)}(s)|^3$$
$$\leq M_3 E\int_{s\wedge\tau^{(m)}}^{t\wedge\tau^{(m)}} |X_i^{(m)}(u)-X_i^{(m)}(s)|\,a_{ii}^{(m)}+|X_i^{(m)}(u)-X_i^{(m)}(s)|^2|\gamma_i^{(m)}|\,du\,.$$

the 1st term

$$\leq (E\int_s^t |X_i^{(m)}(u)-X_i^{(m)}(s)|^2 du)^{1/2}(E\int_0^{t\wedge\tau^{(m)}} a_{ii}^{(m)\delta})^{1/\delta}|t-s|^{1/2-1/\delta}$$

and

the 2nd term

$$\leq 2\rho E\int_{s\wedge\tau^{(m)}}^{t\wedge\tau^{(m)}} |X_i^{(m)}(u)-X_i^{(m)}(s)|\,|\gamma_i^{(m)}|\,du$$
$$\leq 2\rho(E\int_s^t |X_i^{(m)}(u)-X_i^{(m)}(s)|^2)^{1/2}(E\int_0^{t\wedge\tau^{(m)}} |\gamma_i^{(m)}|^\delta)^{1/\delta}|t-s|^{1/2-1/\delta}\,.$$

Hence, we have by (15)

$$E|X_i^{(m)}(t)-X_i^{(m)}(s)|^3 \leq M_4(T)|t-s|^{3/2(1-1/\delta)} \qquad \text{for} \quad |t-s|<T.$$

Since $\frac{3}{2}\left(1-\frac{1}{\delta}\right)>1$, we get Prop. 1.

Because the value of $\tau_\rho^{(m)}$ is in the compact set $[0\infty]$, $\{\tau_\rho^{(m)}, \text{m}=1, 2.\cdots\}$ is totally bounded in Prohorov topology. Hence putting $\mathfrak{X}^{(m)}=((X_\rho^{(m)}, \tau_\rho^{(m)}, B), \rho=1, 2\cdots)$, we have

Proposition 2. $\{\mathfrak{X}^{(m)}, m=1, 2\cdots\}$ *is totally bounded in Prohorov topology*[5]

Therefore we can take a convergent subsequence $\mathfrak{X}^{(m_j)}$ (for simplicity, we say m again), and by Skorohod's theorem, construct $\mathfrak{Z}^{(m)}=((Z_\rho^{(m)}, \eta_\rho^{(m)}, B_\rho^{(m)}), \rho=1, 2\cdots)$ and $\mathfrak{Z}=((Z_\rho, \eta_\rho, B_\rho), \rho=1, 2\cdots)$, on a suitable probability space, so that $\mathfrak{Z}^{(m)}$ has the same law as $\mathfrak{X}^{(m)}$ and $\mathfrak{Z}^{(m)}$ tends to $\mathfrak{Z}$. So, we have the following consistency,

(i) $Z_\rho^{(m)}(t) = Z_{\rho+1}^{(m)}(t) \quad t \leq \eta_\rho^{(m)}$,

(ii) $\eta_\rho^{(m)}$ is the hitting time of $Z_{\rho'}^{(m)}$ $(\rho' \geq \rho)$ for ∂S_ρ,

(iii) $Z_\rho^{(m)}(t+\eta_\rho^{(m)}) = Z_\rho^{(m)}(\eta_\rho^{(m)}), \quad \eta_{\rho+1}^{(m)} > \eta_\rho^{(m)}$, and

(iv) $B_\rho^{(m)}$ is a Brownian motion and ρ-independent, (we say $B^{(m)}$).

Moreover, with probability 1, $Z_\rho^{(m)}$ (resp. $B^{(m)}$) tends to Z_ρ (resp. B) uniformly on any compact interval and $\eta_\rho^{(m)}$ tends to η_ρ. Therefore Z_ρ is continuous, $Z_\rho(t)=Z_{\rho+1}(t)$, $t<\eta_\rho$ $Z_\rho(\eta_\rho)\in\partial S_\rho$ and $\eta_\rho \leq \eta_{\rho+1}$. Let us define F_t, e and Z by $F_t=\bigvee_\rho \boldsymbol{B}_{0\ t}(Z_\rho, B)$, $e=\lim_{\rho\uparrow\infty} \eta_\rho$ and $Z(t)=Z_\rho(t)$, $t<\eta_\rho$. Then B is an F_t-Brownian motion and Z is continuous up to e. Now we shall show that (Z, e) is our wanted one.

Lemma 3. *Let* τ_ρ *be the hitting time of* Z *for* ∂S_ρ. *Then* $e=\lim_{\rho\uparrow\infty} \tau_\rho$.

Proof. Since $\tau_\rho \leq \eta_\rho$, it is enough to show $\eta_\rho \leq \tau_{\rho+\varepsilon}$ $(\varepsilon>0)$. For $t<\eta_\rho$, we have $t<\eta_\rho^{(m)}$ for large m. Hence $Z_\rho^{(m)}(s)\in S_\rho$ for $s \leqq t$. So, $t<\tau_{\rho+\varepsilon}$.

From Lemma 3, e is an F_t-stopping time since τ_ρ is so. Hence $Z(t)\chi_{[0e)}(t)$ is F_t-measurable.

Proposition 3. $Z(t) = \int_0^t \alpha(Z(s))\, dB(s) + \int_0^t \gamma(Z(s))\, ds, \quad t<e$

5) $\mathfrak{X}^{(m)}$ is a $(C^N[0\infty)\times[0\infty]\times C^N[0\infty))^{\psi_0}$-valued random variable with distance D; $D((\text{f}_\rho, \text{x}_\rho, \text{g}_\rho)\ \rho=1, 2\cdots), ((\text{f}_\rho{}', \text{x}_\rho{}', \text{g}_\rho{}')\ \rho\cdots 1.\ 2\cdots) = \sum_{\rho=1}^{\infty} \frac{1}{2^\rho} \frac{d(\text{f}_\rho, \text{x}_\rho, \text{g}_\rho), (\text{f}_\rho{}', \text{x}_\rho{}', \text{g}_\rho{}'))}{1+d(f_\rho, x_\rho, g_\rho), (f_\rho{}', x_\rho, g')}$ where

$$d(f, x, g)(f', x', g') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{t\leq n} |f(t)-f'((|t+|g(t)-g'(t)|}{1+\sup_{t\leq n} |f(t)-f'(t)|+g(t)-g'(t)|} + tg^{-1}|x-x'|$$

Proof. Using Lemma 2, we shall show the convergence of each term of the following equation,

$$Z^{(m)}(\mathrm{t}) = \int_0^t \alpha^{(m)}(Z^{(m)}(s))\,dB^{(m)}(s) + \int_0^t \gamma^{(m)}(Z^{(m)}(s))\,ds\,.$$

$$\begin{aligned} R_m &\equiv E\left|\int_0^{t\wedge\eta_\rho(m)} \gamma_i^{(m)}(Z^{(m)}(\mathrm{s}))\,ds - \int_0^{t\wedge\eta_\rho} \gamma_i(Z(s))\,ds\right| \\ &\le E\left|\int_0^{t\wedge\eta_\rho(m)} (\gamma_i^{(m)}-\gamma_i^{(k)})(Z^{(m)}(s))\,ds\right| + E\left|\int_0^{t\wedge\eta_\rho} (\gamma_i^{(k)}-\gamma_i)(Z(s))\,ds\right| \\ &\quad + E\left|\int_0^{t\wedge\eta_\rho(m)} \gamma_i^{(k)}(Z^{(m)}(s))\,ds - \int_0^{t\wedge\eta_\rho} \gamma_i^{(k)}(Z(s))\,ds\right| \\ &\le M_1 |||\gamma_i^{(m)}-\gamma_i^{(k)}|||_{q,\Gamma,K} + M_1|||\gamma_i^{(k)}-\gamma_i|||_{q,\Gamma,K} + \text{3rd term} \end{aligned}$$

Since $\gamma_i^{(k)}$ is continuous, the 3rd term tends to 0. Hence $R^m \to 0$ as $m\to\infty$.

In the same way, we can show that

$$E\left|\int_0^{t\wedge\eta_\rho(m)} \alpha_{ij}^{(m)}(Z^{(m)}(s))\,dB_j^{(m)}(s) - \int_0^{t\wedge\eta_\rho} \alpha_{ij}(Z(s))\,dB(s)\right|^2 \to 0.$$

Therefore we have Prop. 3.

Proposition 4. *If α and γ are locally bounded, then*

$$\lim_{t\uparrow e} |Z(t)| = \infty \qquad \textit{for} \quad e<\infty\,.$$

Proof. We shall apply a similar method as [11]. Suppose

(16) $\qquad P(A)>0$

where $A=\{\underline{\lim}_{t\uparrow e}|Z(t)|<\infty,\ e<\infty\}$. Since $\overline{\lim}_{t\uparrow e}|Z(t)|=\infty$ by Lemma 3, we have two large constants $M>m$ such that

(17) $\qquad P(|Z(t)|>M,\ |Z(s)|<m$ for infinitely often t, s up to e, $e<\infty)>0\,.$

Difine t_i and t_i' by

$$\begin{aligned} t_1 &= \inf\{t;\,|Z(t)|=m\} \\ t_1' &= \inf\{t>t_1;\,|Z(t)|=M\} \\ t_2 &= \inf\{t>t_1';\,|Z(t)|=m\}\,, \qquad \text{etc.} \end{aligned}$$

Since t_k is an F_t-stopping time, $B(\cdot+t_k)-B(t_k)$ is a Brownian motion and

(18) $$Z(t+t_k) = Z(t_k) + \int_0^t \alpha(Z(s+t_k))\,dB(t+t_k) + \int_0^t \gamma(Z(s+t_k))\,ds,\quad t<e-t_k\,.$$

Appealing to "$\mathrm{E}\left|\int_0^{t\wedge(t_k'-t_k)} f(Z(n+t_k))\,dB_j(u+t_k)\right|^4 \le \text{const.}\ t^2$ for a locally bounded function f" we have

$$(19) \qquad P(\max_{s\leq t} |\int_0^{s\wedge(t_k'-t_k)} f(Z(u+t_k)dB_j(u+t_k)| > C) \leq \text{const. } \frac{t^2}{C^4}$$

If $t_{k+1}-t_k<d$, then $\max_{s<d\wedge(t_k'-t_k)} |Z(s+t_k)-Z(t_k)| \geq M-m$. Hence, by (18) and (19),

$$\begin{aligned}
&P(t_{k+1}-t_k<d/t_k<\infty)\\
&\leq P(\max_{s\leq d\wedge(t_k'-t_k)} |Z(s+t_k)-Z(t_k)| \geq M-m/t_k<\infty)\\
&\leq \sum_{i=1}^N P\Big(\max_{s\leq_p\wedge(t_k'-t_k)} |Z_i(s+t_k)-Z_i(t_k)| \geq \frac{M-m}{N}/t_k<\infty\Big)\\
&\leq \sum_{ij=1}^N P\Big(\max_{s\leq d\wedge(t_k'-t_k)} |\int_0^s \alpha_{ij}(Z(u+t_k))dB_j(u+t_k)| \geq \frac{M-m}{2N^2}/t_k<\infty\Big)\\
&+ \sum_{i=1}^N P\Big(\max_{s\leq d\wedge(t_k'-t_k)} |\int_0^s \gamma_i(Z(u+t_k))ds| \geqq \frac{M-m}{2N}/t_k<\infty)\\
&\leq \text{const. } d^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_k P\Big(t_{k+1}-t_k<\frac{1}{k}, e<\infty\Big) \leq \sum_k P\Big((t_{k+1}-t_k<\frac{1}{k}, t_k<\infty\Big)\\
&\leq \sum_k \frac{P(t_{k+1}-t_k<1/k, t_k<\infty)}{P(t_k<\infty)} < \infty .
\end{aligned}$$

Namely, if $e<\infty$, then $t_{k+1}-t_k \geq \frac{1}{k}$ for large k.
This contradicts with (17).

3. Examples

1. $dX(t)=\alpha(X(t))dB(t), \quad N=1$.

If α is a positive and bounded function and $\frac{1}{\alpha}$ is not square integrable near 0, i.e. $\frac{1}{\alpha} \notin L_2(-\varepsilon, \varepsilon)$ for any $\varepsilon>0$, then there is no solution starting at 0.

Proof. Let X be a solution. Let us define $t(t)$ by $\int_0^t \alpha^2(X(s))ds$. Then $t(t)$ is strictly increasing and $\beta(t)=X(t^{-1}(t))$ is a Brownian motion up to $t(\infty)$ (>0). Let $f(t, x)$ be a Brownian local time, i.e.

$$f(t, x) = \lim_{\varepsilon\downarrow 0} \frac{\int_0^t X_{(x-\varepsilon, x+\varepsilon)}(\beta(s))ds}{2\varepsilon} .$$

Then f is continuous in (t, x) and $f(t, 0)>0$ for any $t>0$. For $t<t(\infty)$, we have

$$t^{-1}(t)=\int_0^t \frac{1}{\alpha^2(X(t^{-1}(u)))}du=\int_0^t \frac{1}{\alpha^2(\beta(u))}du$$
$$=\int_{-\infty}^{\infty}\frac{f(t, x)}{\alpha^2(x)}dx \geq \inf_{|x|<\varepsilon} f(t, x)\int_{-\varepsilon}^{\varepsilon}\frac{1}{\alpha^2(x)}dx .$$

Since we have a small $\varepsilon>0$ such that $\inf_{|x|<\varepsilon} f(t, x)>0$, "$\frac{1}{\alpha}\notin L_2(-\varepsilon, \varepsilon)$" means "$t^{-1}(t)=\infty$" namely, "$t(t)\equiv 0$, for $t>0$". This implies "$\alpha(X(s))\equiv 0$". But this contradicts with the positivity of α.

2. $(e=0)$.

$$\begin{cases} X_1(t)=B_1(t) \\ X_2(t)=B_2(t)+\int_0^t \gamma_2(X_1(s))\,ds \end{cases}$$

If γ_2 is non-negative and not integrable near 0, then an explosion occurs immediately.

Proof. Let $f(t, x)$ be the local time of B_1.
Then

$$\int_0^t \gamma_2(B_1(s))\,ds=\int_{-\infty}^{\infty}\gamma_2(x)f(t, x)\,dx \geq \inf_{|x|<\varepsilon} f(t, x)\int_{-\varepsilon}^{\varepsilon}\gamma_2(x)\,dx .$$

Therefore $X_2(t)=\infty$ for $t>0$.

3. random acceleration [3].

$$\begin{cases} X_1(t)=x_1+\int_0^t \alpha(X_2(s))\,dB(s) \\ X_2(t)=x_2+\int_0^t X_1(s)\,ds \end{cases}$$

This motion means that αdB is a random acceleration whose coefficient α depends only on the position X_2. Assume that $C_1\leq\alpha\leq C_2$ with positive constants C_1 and C_2. Then we can obtain a solution by the method of time substitution, without continuity of α. Moreover the law of solution is unique.

Put $A(x)=\int_0^x \alpha^2(x_2+y)\,dy$, (for $x<0$, $\int_0^x=-\int_x^0$), and $\xi(s)=\alpha(x_2+A^{-1}(\int_0^t x_1+\beta(u)\,du))$ where β is an F_t-Brownian motion. Let us define $t(t)$ by $\int_0^t \frac{1}{\xi^2(s)}ds$. Then $P(\forall t,\, t(t)<\infty)=1$. Hence $B(t)\equiv\int_0^{t^{-1}(t)}\frac{1}{\xi(s)}d\beta(s)$ is an $F_{t^{-1}(t)}$-Brownian motion, $\beta(t^{-1}(t))$ is $F_{t^{-1}(t)}$-measurable and

$$(1) \qquad \beta(t^{-1}(t)) = \int_0^t \xi(t^{-1}(s))dB(s), \text{ [11]}.$$

We shall calculate the right side of (1).

$$Z(s) \equiv \int_0^{t^{-1}(s)} x_1+\beta(u)du = \int_0^s (x_1+\beta(t^{-1}(v)))\xi^2(t^{-1}(v))dv$$
$$= \int_0^s (x_1+\beta(t^{-1}(v)))\alpha^2(x_2+A^{-1}(Z(v)))dv .$$

Hence Z is absolutely continuous and

$$(2) \qquad Z'(s) = (x_1+\beta(t^{-1}(s)))\alpha^2(x_2+A^{-1}(Z(s))) .$$

Therefore, recalling the definition of A, we have

$$(3) \qquad \int_0^t x_1+\beta(t^{-1}(s))ds = \int_0^t \frac{Z'(s)}{\alpha^2(x_2+A^{-1}(Z(s)))}ds = A^{-1}(Z(t)) .$$

Appealing to (1) and (3),

$$X_1(t) \equiv x_1+\beta(t^{-1}(t)) = x_1+\int_0^t \alpha(x_2+A^{-1}(Z(s)))dB(s)$$
$$= \int_0^t \alpha(x_2+\int_0^s X_1(u)du)dB(s)$$

Hence we have a solution, putting $X_2(t)=x_2+\int_0^t X_1(s)ds$.

Let (X_1, X_2) be a solution. Put $X=X_1$ and $t(t)=\int_0^t \alpha^2(x_2+\int_0^s X(u)du)ds$. Then

$$X(t) = x_1+\int_0^t \alpha(x_2+\int_0^s X(u)du)dB(s)$$

and

$$X(t) = x_1+\beta(t(t))$$

where β is a Brownian motion. Since

$$g(t) \equiv (t^{-1}(t))' = \frac{1}{\alpha^2(x_2+\int_0^{t^{-1}(s)} X(u)du)} \qquad \text{a.a.t} ,$$

we have

$$\int_0^{t^{-1}(t)} X(u)du = \int_0^{t^{-1}(t)} x_1+\beta(t(u))du = \int_0^t (x_1+\beta(v))g(v)dv .$$

Therefore the following equality holds

$$g(t) = \frac{1}{\alpha^2(x_2+\int_0^t (x_1+\beta(v))g(v)dv)} \qquad \text{a.a.t} ,$$

Setting $F(t)=\int_0^t (x_1+\beta(v))g(v)\,dv$, we have

$$F'(t)\,\alpha^2(x_2+F(t))=x_1+\beta(t) \qquad \text{a.a.t}\,,$$

Hence, with the same $A(x)$,

$$A(F(t))=\int_0^t x_1+\beta(s)\,ds\,.$$

Since A is strictly increasing, $F(t)$ is determined uniquely. Therefore g is so, for a.a.t. This implies that $t(t)$ is determined uniquely.
Hence the law of $X(t)$ $(=x+\beta(t(t)))$ is unique.

4. (pathwise unique)

$$dX(t)=\sum_{i=1}^{N}\alpha_i(X(t),\,B(t))\,dB_i(t)+\gamma(X(t))\,dt\,.$$

Suppose that α_i and γ are bounded and $\alpha_1(\cdot)\geq C>0$. If $\alpha_i(x,\,\xi_1,\,\cdots\xi_N)$ is smooth, then the solution is pathwise unique.

In order to prove, we shall extend a formula on stochastic differentials, in our convenient form [6].

Proposition. *Let e and h be bounded and non-anticipative. Put* $X(t)=x+\int_0^t e(s)\,dB+\int_0^t h(s)\,ds$. *Suppose that, for any smooth function f on* R^N,

$$(4) \qquad E\int_0^{\tau_\rho\wedge T} f(X(s))\,ds\leq M(\rho,\,T)|||f|||_{q,\Gamma,K}.$$

where τ_ρ *is the hitting time of* X *to* ∂S_ρ *and* $\Gamma\times K\supset S_\rho$, $(\Gamma\subset R^l$ *and* $K\subset R^{N-l})$. *Then for any continuous* ϕ *whose generalized derivatives* ϕ_i *and* ϕ_{ij} *satisfy* (A3) *and* (A4), *we have*

$$(5) \qquad \phi(X(t))-\phi(x)=\sum_{ij}\int_0^t \phi_i(X(s))\,e_{ij}(s)\,dB_j(s)$$
$$+\int_0^t \sum_i \phi_i(X(s))\,h_i(s)+\frac{1}{2}\sum_{ijk}\phi_{ij}(X(s))\,e_{ik}(s)\,e_{jk}(s)\,ds\,.$$

Proof. Put $\phi_h=\phi*\omega_h$. Then $\phi_{h,i}=\phi_i*\omega_h$ and $\phi_{h,ij}=\phi_{ij}*\omega_h$. Therefore $\phi_{h,i}$(resp. $\phi_{h,ij}$) tends to ϕ_i(resp. ϕ_{ij}) in $|||\ |||_{q,\Gamma,K}$ by Lemma 1 in §1. Moreover, we can easily see that $\phi_h(\xi,\,\eta)$ tends to $\phi(\xi,\,\eta)$ uniformly on any compact set of $(\xi,\,\eta)$. Since τ_ρ tends to ∞, as $\rho\to\infty$, we have, for any t,

$$(6) \qquad \phi_h(X(t))\to\phi(X(t)) \qquad \text{as} \quad h\to 0\,.$$

$$(7)\qquad \phi_h(X(t))-\phi_h(x) = \sum_{ij}\int_0^t \phi_{h,i}(X(s))e_{ij}(s)dB_j(s)$$
$$+\int_0^t \sum_i \phi_{h,i}(X(s))h_i(s)+\frac{1}{2}\sum_{ijk}\phi_{ij}(X(s))e_{ik}(s)e_{jk}(s)ds\,.$$

By the same method as in §2, we see that the right side of (7) tends to the right side of (5). Recalling (6), we have (5).

We shall show the uniqueness of Example 4. We apply a routine of the transformation of drift. Let us define $A(x, \xi)$ and $\phi(x, \xi)$ by

$$A(x, \xi) = 2\int_0^x \frac{\gamma(y)}{\alpha^2(y, \xi)}dy \qquad (\alpha^2 = \sum_{i=1}^N \alpha_i^2)\,,$$

and

$$\phi(x, \xi) = \int_0^x e^{-A(y,\xi)}dy\,.$$

Then $\phi(x, \xi)$ is strictly increasing in x and $\psi(x, \xi)\equiv\phi^{-1}(x, \xi)$ (inverse w.r. to x) satisfies the following Lipschitz condition

$$(8)\qquad |\psi(x, \xi)-\psi(y, \xi)|\leqq C(T)|x-y|,\quad \xi\in R^N,\ x, y\in[-T, T]$$

By virtue of positivity of α_1, we can show that any solution (X, β) satisfies the inequality (4). We have

$$X(t) = \int_0^t \alpha(s)d\beta(s)+\int_0^t \gamma(X(s)ds\,, \qquad (\alpha(s)\equiv(\Sigma\alpha_i^2(X(s), B(s))^{1/2})$$

with some Brownian motion β. Let $g(\geqq 0)$ be somoth function on R^1 and u the solution of boundary value problem; $u''(x)=g(x)$, $|x|\leq\rho$, $u(-\rho)=u(\rho)=0$. Then $|u_\rho(x)|$ and $|u(x)|$ are less than $M(\rho)||f||_{L_1(-\rho,\rho)}$,

$$u(X(t\wedge\tau_\rho))-u(x) = \text{martingale}+\int_0^{t\wedge\tau_\rho}\frac{1}{2}g(X(s)\alpha^2(s)+u'(X(s))\gamma(X(s))ds$$

where τ_ρ is the hitting time to $\{-\rho, \rho\}$. Hence

$$E\int_0^{t\wedge\tau_\rho} g(X(s))ds\leq M(\rho, C, T)\ ||g||_{L_1(-\rho,\rho)}\,, \qquad (t\leqq T)$$

Therefore, for any bounded Borel function g,

$$E\int_0^{t\wedge\tau_\rho} g(X(s))ds\leq M(\rho, C, T)||g||_{L_1(-\rho,\rho)}$$

Let $f(\geq 0)$ be a smooth function on R^{N+1}. Put $S=\{x\in R^N;\ |x|\leq\rho\}$ and $g(x)=\sup_{\xi\in S} f(x, \xi)$.

Then $||g||_{L_1(-\rho,\rho)}=|||f|||_{1,(-\rho,\rho),S}$ and

$$E\int_0^{t\wedge\sigma_\rho} f(X(s), B(s))\,ds \leqslant E\int_0^{t\wedge\tau_\rho} g(X(s))\,ds \leqslant M(\rho, C, T)||g||_{L_1(-\rho,\rho)}.$$

Therefore we have (4). Moreover

$$(9)\qquad \gamma(x)\phi_x(x, \xi)+\frac{1}{2}\alpha^2(x, \xi)\phi_{xx}(x, \xi) = 0 \qquad \text{a.a.x.}$$

Therefore $Y(t)\equiv\phi(X(t), B(t))$ has the following stochastic differential

$$(10)\qquad dY(t) = \sum_{i=1}^{N} F_i(X(t), B(t))\,dB_i(t)+G(X(t), B(t))\,dt.$$

where $F_i(x, \xi)=\phi_x(x, \xi)\alpha_i(x, \xi)+\phi_{\xi_i}(x, \xi)$ and

$$G(x, \xi) = \frac{1}{2}\sum_{i=1}^{N}\phi_{\xi_i\xi_i}(x, \xi)+\phi_{x\xi_i}(x, \xi)\alpha_i(x, \xi).$$

Put $\tilde{F}_i(y, \xi)=F_i(\psi(y, \xi), \xi)$ and $\tilde{G}(y, \xi)=G(\psi(y, \xi), \xi)$. Then, by virtue of (8), $\tilde{F}$ and $\tilde{G}$ are Lipschitz in y, and the stochastic differential (10) turns out

$$(11)\qquad dY(t) = \sum_{i=1}^{N}\tilde{F}_i(Y(t), B(t))dB_i+\tilde{G}(Y(t), B(t))dt.$$

Since (11) has the unique solution, $X(t)=\phi(Y(t), B(t))$ is unique.

4. Proof of Theorem 2

In order to prove Theorem 2, we extend α and γ as follows

$$(1)\qquad \begin{aligned} &\alpha_{1j}(\tilde{x}) = -\alpha_{1j}(x) && j=2, \cdots, N\\ &\alpha_{j1}(\tilde{x}) = -\alpha_{j1}(x) && j=2, \cdots, N\\ &\alpha_{ij}(\tilde{x}) = \alpha_{ij}(x) && \text{other } (i, j)\\ &\gamma_1(\tilde{x}) = -\gamma_1(x), \quad \gamma_i(\tilde{x}) = \gamma_i(x), && i=2, \cdots, N. \end{aligned}$$

where $\tilde{x}=(-x_1, x_2\cdots x_N)$ for $x=(x_1, x_2\cdots x_N)$. Then it is clear that α and γ on R^N satisfy the conditions (A1)~(A4) of Theorem 1 with $\mu(\tilde{x})=\mu(x)$.

We shall take an approximate sequence $\alpha^{(k)}$ and $\gamma^{(k)}$ as like in § 2. Then these coefficients satisfy (1) on S_ρ, so we may modify them, so that (1) holds on R^N. Let $Y^{(k)}$ be the unique solution of (2)

$$(2)\qquad Y^{(k)}(t, x) = x+\int_0^t \alpha^{(k)}(Y^{(k)}(s))dB(s)+\int_0^t \gamma^{(k)}(Y^{(k)}(s))ds.$$

Lemma 1. *$Y^{(k)}(\cdot, \tilde{x})$ has the same law as $\tilde{Y}^{(k)}(\cdot, x)=(-Y_1^{(k)}(\cdot, x), Y_2^{(k)}(\cdot, x)\cdots Y_N^{(k)}(\cdot, x))$.*

Proof. For simplicity we shall drop the suffix k. Since coefficients α and γ are smooth, $Y(\cdot, x)$ is measurable w.r. to B and x. Using (1), we have

$$(3)\quad \begin{cases} -Y_1(t, x) = -x_1 + \int_0^t -\alpha_{11}(\tilde{Y}(s, x))dB_1(s) + \sum_{j=2}^N \int_0^t \alpha_{1j}(\tilde{Y}(s, x))dB_j(s) \\ \qquad\qquad + \int_0^t \gamma_1(\tilde{Y}(s, x))ds \\ Y_i(t, x) = x_i + \int_0^t -\alpha_{i1}(\tilde{Y}(s, x))dB_1(s) + \sum_{j=2}^N \int_0^t \alpha_{ij}(\tilde{Y}(s, x))dB_j(s) \\ \qquad\qquad + \int_0^t \gamma_i(\tilde{Y}(s, x))ds . \end{cases}$$

Putting $\tilde{B}_1(t) = -B_1(t)$ and $\tilde{B}_i(t) = B_i(t)$, we can see that $\tilde{B}$ is a Brownian motion and

$$(4)\qquad \tilde{Y}(t, x) = \tilde{x} + \int_0^t \alpha(\tilde{Y}(s, x))d\tilde{B}(s) + \int_0^t \gamma(\tilde{Y}(s, x))ds .$$

Since a solution of (4) is unique, $\tilde{Y}(\cdot, x)$ has the same law as $Y(\cdot, \tilde{x})$.

We denote, by $P_x^{(k)}$, the probability law of $Y^{(k)}(\cdot, x)$ on the path space $C^N[0\ \infty)$.

Lemma 2. $Y_+^{(k)}(t, x) \equiv (|Y_1^{(k)}(t, x)|, Y_2^{(k)}(t, x), \cdots Y_N^{(k)}(t, x))$ *is a reflecting deffusion on* $\bar{D}$ *and its generator* $\mathfrak{G}_k$ *is given by*

$$(5)\qquad \mathfrak{G}_k = \frac{1}{2}\sum a_{ij}^{(k)}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum \gamma_i^{(k)}(x)\frac{\partial}{\partial x_i}$$

Proof. Let $\boldsymbol{B}_s$ be the Borel field on $C^N[0\ \infty)$ generated by the coordinate up to s, $\{y(\tau), \tau \leq s\}$. Put $y_+(s) = (|y_1(s)|, y_2(s), \cdots y_N(s))$. For $E \subset D = \{x_1 > 0\}$ we have, by Lemma 1,

$$(6)\qquad \begin{aligned} P_x^{(k)}(y_+(t) \in E/\boldsymbol{B}_s) &= P_x^{(k)}(y(t) \in E/\boldsymbol{B}_s) + P_x^{(k)}(y(t) \in \tilde{E}/\boldsymbol{B}_s) \\ &= P_{y(s)}^{(k)}(y(t-s) \in E) + P_{y(s)}^{(k)}(y(t-s)) \in \tilde{E}) \\ &= P_{y(s)}^{(k)}(y(t-s) \in E) + P_{\tilde{y}(s)}^{(k)}(y(t-s) \ni E) \end{aligned}$$

Since $\boldsymbol{B}_s$ contains the Borel field generated by $\{y_+(\tau), \tau \leq s\}$ (6) means $\{P_x^{(k)}(y(t) \in E) + P_{\tilde{x}}^{(k)}(y(t) \in E), x \in D\}$ is the system of transition probability $Y_+^{(k)}$, by virtue of

$$P_x^{(k)}(y; y_1(t) = 0) = 1 \qquad \forall t > 0$$

Therefore we have Lemma 2.

On the other hand, the following stochastic differential equation (7) determines a reflecting deiffusion Z on $\bar{D}$ with generator (5).

$$(7)\qquad \begin{cases} Z_1^{(k)}(t, x) = x_1 + \int_0^t \alpha_1^{(k)}(Z^{(k)}(s))dB(s) + \int_0^t \gamma_1^{(k)}(Z^{(k)}(s))ds + \phi^{(k)}(t) \\ Z_i^{(k)}(t, x) = x_i + \int_0^t \alpha_i^{(k)}(Z^{(k)}(s))dB(s) + \int_0^t \gamma_i^{(k)}(Z^{(k)}(s))ds, \qquad i=2, \cdots N. \end{cases}$$

Hence $Z^{(k)}$ has the same law as $Y_+^{(k)}$.

Lemma 3. *Let f be a non-negative smooth function such that $f(\tilde{x})=f(x)$, and τ the hitting time of $Z^{(k)}(t, x)$ to ∂S_ρ. Then, for $\Gamma\times K\supset S_\rho$, we have*

$$(8)\qquad E\int_0^\tau f(Z^{(k)}(t, x))dt \leq M_1 |||f|||_{q,\Gamma,K} \qquad x\in S_\rho\cap D,$$

where M_1 is the same as it in Lemma 2 of §2.

Proof. Let σ be the hitting time of $Y^{(k)}(t, x)$ to ∂S_ρ. Then σ is the hitting time of $Y_+^{(k)}(t, x)$ to ∂S_ρ. By the symmetry of f, we have

$$E\int_0^\tau f(Z^{(k)}(t, x))dt = E\int_0^\sigma f(Y_+^{(k)}(t, x))dt = E\int_0^\sigma f(Y^{(k)}(t, x))dt.$$

Hence Lemma 2 in §2 tells us (8).

For the construction of solution, we can apply a similar method as in §2 by Lemma 3. Hereafter we fix a starting point x and drop it for simplicity. Let $Y_\rho^{(k)}$ be the stopped process of $Y^{(k)}$ at ∂S_ρ. Then $\{Y_\rho^{(k)}, k=1, 2\cdots\}$ is totally bounded in Prohorov topology. Hence $\{Z_\rho^{(k)}, k=1, 2\cdots\}$ is also. Therefore, putting

$$\phi_\rho^{(k)}(t) \equiv Z_{\rho.1}^{(k)}(t) - x_1 - \int_0^{t\wedge\tau_\rho^{(k)}} \alpha_1^{(k)}(Z^{(k)}(s))dB - \int_0^{t\wedge\tau_\rho^{(k)}} \gamma_1^{(k)}(Z^{(k)}(s))ds,$$

and $\mathcal{Z}^{(k)}\equiv\{(Z_\rho^{(k)}, \phi_\rho^{(k)}, \tau_\rho^{(k)}, B), \rho=1, 2\cdots\}$, we have

Proposition 1. *$\{\mathcal{Z}^{(k)}, k=1, 2\cdots\}$ is totally bounded in Prohorov topology.*

Hence we can choose a convergent subsequence $\mathcal{Z}^{(k_j)}$ (for simplicity we assume $k_j=k$) in Prohorov topology and construct a system $\mathfrak{X}^{(k)}=((X_\rho^{(k)}, \psi_\rho^{(k)}, \sigma_\rho^{(k)}, B_\rho^{(k)}), \rho=1, 2\cdots)$ which has the same law as $\mathcal{Z}^{(k)}$ and converges to $\mathfrak{X}=((X_\rho, \psi_\rho, \sigma_\rho, B_\rho)\ \rho=1, 2\cdots)$ with probability 1. For the consistency in ρ, we can define the limit process $(X(t), \phi(s), B(t))$ up to $e=\lim_{\rho\uparrow\infty}\sigma_\rho$, by $X(t)=X_\rho(t)$, $\phi(t)=\psi_\rho(t)$ and $B(t)=B_\rho(t)$ for $t<\sigma_\rho$. Putting $F_t=\bigvee_\rho \boldsymbol{B}_{0t}(X_\rho, B_\rho)$ we have a solution of (II). Condition (iii) will be show in the following Prop. 2, and the rest is same as Theorem 1.

Proposition 2. *ϕ is continuous, increasing and flat off $\{t; X_1(t)=0\}$.*

Proof. Since $\psi_\rho^{(k)}$ converges to ψ_ρ uniformly on any bounded set, ϕ is continuous and increasing. Assume that $X_1(t_0)>0$. Because of the continuity of path, there exists a positive δ such that $\inf_{|t-t_0|\geq\delta} X_1(t)>0$. Hence, for large k, $\inf_{|t-t_0|\geq\delta} X_1^{(k)}(t)>0$. Therefore $d\psi^{(k)}(t)=0$ on $|t-t_0|\leq\delta$. Hence $d\phi(t)=0$ on $|t-t_0|\leq\delta$.

5. Proof of Theorem 3

We shall recall an estimate of a derivative of solution of partial differential equation in two dimension, in our convenient form.

Lemma 1. *Let u be a solution of the boundary value problem;*

$$(1)\qquad \sum_{ij=1}^{2} a_{ij}(x)u_{ij}(x)=f \qquad on \quad G_\rho=S_\rho\cap D$$

$$u=0 \qquad on \quad \partial G_\rho$$

where a and f are smooth on $\bar{G}_\rho$ and a is symmetric and uniformly elliptic. If

$$v(x)|t|^2\leq \sum_{ij=1}^{2} a_{ij}(x)t_it_j\leq\theta|t|^2$$

with a constant θ and a non-negative function v such that $\frac{1}{v}\in L_8(\bar{G}_\rho)$, then

$$(2)\qquad \sup_{x\in\bar{G}_\rho}|u_i(x)|\leq M\|f\|_{3,G_\rho}^{1/16} \qquad if \quad \|f\|_{3,G_\rho}\leq 1$$

and

$$(3)\qquad \sup_{x\in\bar{G}_\rho}|u(x)|\leq M\|f\|_{3,G_\rho}$$

where $\|\ \ \|_{p,A}=\|\ \ \|_{L_p(A)}$ and a constant M depends only on θ, ρ and $\left\|\frac{1}{v}\right\|_{8,G_\rho}$.

For the proof of (2), we apply the same argument as [10] with detail calculations and (3) is obtained by [1] like as (12) in §2.

Suppose that $|\alpha_{ij}|$, $|\gamma_i|$, $|\hat{\alpha}|$ and $|\hat{\gamma}|$ are less than R_ρ on $\bar{G}_\rho$ and let us define $\mathfrak{A}$ by

$\mathfrak{A}=\{(A, C, \hat{A}, \hat{C})$; (i) A is a 2×2 matrix valued bounded and smooth function on $\bar{D}$, (ii) $A(x)$ is symmetric and uniform elliptic, (iii) $\sum_{ij=1}^{2} A_{ij}(x)t_it_j\geq\mu(x)|t|^2$ (iv) C is a two dimensional vector valued bounded and smooth function on $\bar{D}$, (v) $\hat{A}$ and $\hat{C}$ are bounded and smooth functions on R^1, (vi) $(1/\hat{A})\in L_p^{loc}$, (vii) $|A_{ij}|$, $|C_i|$, $|\hat{A}|$, $|\hat{C}|\leq R_{\rho+1}$ on $\bar{G}_\rho$, $(0<\rho<\infty)\}$.

Consider a stochastic differential equation (III) with coefficient $(A, C, \hat{A}, \hat{C})$ in $\mathfrak{A}$. Then we have a law unique solution X, [14],

Lemma 2. *Let τ be the hitting time of X to ∂S_ρ. Then, for any smooth* *function* $f(\geq 0)$ *on* $\bar{D}$ *we have*

$$(4) \qquad E\int_0^{\tau\wedge T} f(X(s))ds \leq M_5 \|f\|_{3,G_\rho}^{1/16} \qquad \text{if} \quad \|f\|_{3,G_\rho} \leq 1,$$

where M_5 *depends on* ρ, R_ρ, $\left\|\frac{1}{\mu}\right\|_{8,G_\rho}$ *and* T.

Proof. Let u be a solution of the boundary value problem (1), where $a(x)=\frac{1}{2}A^2(x)$. Then u is smooth. Hence, by a formula on stochastic differentials, [9], and $u(0,\cdot)=0$, we have

$$u(X(t))-u(x) = \int_0^t \sum_{ij} a_{ij}u_{ij}(X(s)) + \sum_i \gamma_i u_i(X(s))ds + \int_0^t u_1(0, X_2(s))d\phi(s) + \text{martingale}.$$

Taking the expectation

$$Eu(X(t\wedge\tau))-u(x) = E\int_0^{t\wedge\tau} f(X(s)) + \sum \gamma_i u_i(X(s))ds + E\int_0^{t\wedge\tau} u_1(0, X_2(s))d\phi(s).$$

Recalling Lemma 1, we have

$$(5) \qquad E\int_0^{t\wedge\tau} f(X(s))ds \leq M_6 \|f\|_{3,G_\rho}^{1/16}(t+E\phi(t\wedge\tau)).$$

We shall evaluate $E\phi(t\wedge\tau)$).

$$X_1^2(t\wedge\tau) = x_1^2 + 2\int_0^{t\wedge\tau} a_{11}(X(s)) + \gamma_1(X(s))X_1(s)ds + 2\int_0^{t\wedge\tau} X_1(s)d\phi(s) + \text{martingale}.$$

Since $X_1(s)d\phi(s)=0$, we have

$$EX_1^2(t\wedge\tau) = x_1^2 + 2E\int_0^{t\wedge\tau} a_{11}(X(s)) + \gamma_1(X(s))X_1(s)ds.$$

Using $|x| \leq \frac{1+|x|^2}{2}$,

$$E_1^2(t\wedge\tau) \leq M_7|x_1^2+t|e^{Mt}.$$

Hence, appealing to "$\phi(t\wedge\tau)=X_1(t\wedge\tau)-x_1-\int_0^{t\wedge\tau}\alpha_1 dB - \int_0^{t\wedge\tau}\gamma_1 ds$" we have

$$(6) \qquad E\phi(t\wedge\tau) \leq M_8(T), \qquad t \leq T.$$

Therefore (4) is derived from (5) and (6).

Lemma 3. *Let $g(\geq 0)$ be a continuous function on R^1.*

$$(7)\qquad E\int_0^{t\wedge\tau} g(X_2(s))d\phi(s) \leq M_9(T)\,||g||_{q,(-\rho,\rho)} \qquad (t\leq T),$$

where a constant M_9 depends on ρ, R_ρ and $\left\|\frac{1}{\hat{A}}\right\|_{p,(-\rho,\rho)}$ and $\frac{2}{p}+\frac{1}{q}=1$.

Proof. Put

$$K(y, z) = \begin{cases} (\rho-z)(y+\rho)/2\rho & z\geq y \\ (\rho-y)(z+\rho)/2\rho & y\geq z \end{cases}$$

and

$$v(y) = -2\int_{-\rho}^{\rho} K(y, z)g(z)\hat{A}^{-2}(z)dz\,.$$

Then v satisfies the following conditions.

$$(8)\qquad v''(y) = 2g(y)\hat{A}^{-2}(y) \qquad \text{on} \quad (-\rho, \rho)$$
$$v(-\rho) = v(\rho) = 0$$

$$(9)\qquad |v(y)| \leq M(\rho)||\hat{A}^{-1}||_{p,(-\rho,\rho)}||g||_{q,(-\rho,\rho)}$$

and

$$(10)\qquad |v'(y)| \leq M(\rho)\,||\hat{A}^{-1}||_{p,(-\rho,\rho)}||g||_{q,(-\rho,\rho)}$$

Define u by $u(x_1, x_2)=v(x_2)$ on $\bar{G}_\rho$. Then

$$(11)\qquad u_1(x) = 0\,, \qquad u_{22}(x) = 2g(x_2)\hat{A}^{-2}(x_2)$$

$$(12)\qquad |u(x)| \leq M(\rho)\,||\hat{A}^{-1}||_{p,(-\rho,\rho)}||g||_{q,(-\rho,\rho)}$$

and

$$(13)\qquad |u_2(x)| \leq M(\rho)\,||\hat{A}^{-1}|_{p,(-\rho,\rho)}||g||_{q,(-\rho,\rho)}\,.$$

Therefore we have

$$u(X(t\wedge\tau))-u(x) = \text{martingale}+\int_0^{t\wedge\tau} a_{22}(X(s))g(X_2(s))\hat{A}^{-2}(X_2(s))$$
$$+u_2(X(s))\gamma_2(X(s))ds+\int_0^{t\wedge\tau} g(X_2(s))+u_2(0, X_2(s))\hat{C}(X_2(s))d\phi(s)\,.$$

Since $a_{22}g\hat{A}^{-2}\geq 0$,

$$E\int_0^{t\wedge\tau} g(X_2(s))d\phi(s) \leq Eu(X(t\wedge\tau))-u(x)-E\int_0^{t\wedge\tau}\gamma_2(X(s))u_2(X(s))ds$$
$$-E\int_0^{t\wedge\tau} u_2(0, X_2(s))\hat{C}(X_2(s))d\phi(s)\,.$$

Hence, appealing to (6) (12) and (13), we have (7).
Therefore $E\int_0^{t\wedge\tau}\chi_N(X_2(s))d\phi(s)=0$ for a null set $N\subset(-\rho,\rho)$, by a routine, and we get

Lemma 4.

$$E\int_0^{t\wedge\tau}f(X(s))ds\leq M_5\|f\|_{3,G_\rho}^{1/16} \tag{14}$$

for any bounded Borel function f on $\bar{G}_\rho$ *with* $\|f\|_{3,G_\rho}\leq 1$.

$$E\int_0^{t\wedge\tau}g(X_2(s))d\phi(s)\leq M_9\|g\|_{q,(-\rho,\rho)} \tag{15}$$

for any bounded Borel function g on $[-\rho,\rho]$.

We shall remark that if X_n is the solution with coefficients of $\mathfrak{A}$ and, with probability 1, $\tau(X_n)$ tends to $\eta(\leq\infty)$ and $X_n(t\wedge\tau(X_n))$ tends to $X(t)$ uniformly on any bounded time interval, then (14) holds for X with same M_5, where τ means η. Moreover, if with probability 1, $\phi_n(t)$ tends to $\phi(t)$ at each t and ϕ has continuous paths, then (15) holds for X with same M_9.

We construct an approximate stochastic differential equation as like in §2, i.e. we extend α_{ij} and γ_i to R^2 as like in §4 and apply regularization, so that the coefficient is in $\mathfrak{A}$. Now we have an approximate coefficient $(\alpha^{(k)}, \gamma^{(k)}, \hat{\alpha}^{(k)}, \hat{\gamma}^{(k)})$ in $\mathfrak{A}$, such that $\alpha^{(k)}$ tends to $\alpha(x)$ almost all $x\in\bar{D}$ and $\gamma^{(k)}$, $\hat{\alpha}^{(k)}$ and $\hat{\gamma}^{(k)}$ are similar. Let $(X^{(k)}, \phi^{(k)}, B^{(k)}, M^{(k)})$ be a solution and $\tau_\rho^{(k)}$ the hitting time for ∂S_ρ. Put $\mathfrak{X}^{(k)}\equiv\{(X_\rho^{(k)}, \tau_\rho^{(k)}, B^{(k)}, \hat{B}^{(k)}, \phi_\rho^{(k)}), \rho=1,2\cdots\}$ where $\hat{B}^{(k)}(\phi^{(k)}(t))=M^{(k)}(t)$, $X_\rho^{(k)}(t)=X_\rho^{(k)}(t\wedge\tau_\rho^{(k)})$ and $\phi_\rho^{(k)}(t)=\phi^{(k)}(t\wedge\tau_\rho^{(k)})$. Then $\{\mathfrak{X}^{(k)}, k=1,2\cdots\}$ is totally bounded in Prohorov topology by the same evaluation as [12]. Therefore we have a convergent subsequence, $\mathfrak{X}^{(m_j)}$ (for simplicity, we say m again) and, by Skorohod's theorem, can construct $\mathfrak{Z}^{(k)}=\{(Z_\rho^{(k)}, \eta_\rho^{(k)}, B_\rho^{(k)}, \hat{B}_\rho^{(k)}, \psi_\rho^{(k)}), \rho=1,2\cdots\}$ and $\mathfrak{Z}=\{(Z_\rho, \eta_\rho, B_\rho, \hat{B}_\rho, \psi_\rho), \rho=1,2\cdots\}$, so that $\mathfrak{Z}^{(k)}$ has the same law as $\mathfrak{X}^{(k)}$ and $\mathfrak{Z}^{(k)}$ tends to $\mathfrak{Z}$. Moreover, the consistency holds, i.e., $B_\rho^{(k)}$ and $\hat{B}_\rho^{(k)}$ are independent of ρ and $Z_\rho^{(k)}(t)=Z_{\rho'}^{(k)}(t)$, $\psi_\rho^{(k)}(t)=\psi_{\rho'}^{(k)}(t)$ up to $\eta_\rho^{(k)}$, $(\rho'>\rho)$. Therefore we can define $Z(t)$ and $\psi(t)$ up to $e=\lim_{\rho\to\infty}\eta_\rho$ by

$$Z(t)=Z_\rho(t)\,,\quad \psi(t)=\psi_\rho(t)\,,\qquad t<\eta_\rho.$$

Since B_ρ and $\hat{B}_\rho$ are independent of ρ, denoting by B and $\hat{B}$, we can see that $(Z, e, B, \hat{B}, \Psi)$ is our wanted solution. In order to prove Theorem 3 we show only the convergence of boundary process (Lemma 5), because other parts of Theorem 3 will be proved in the same way as §2.

Lemma 5.

(16) $$\int_0^{t\wedge\eta_\rho^{(k)}} \hat{\gamma}^{(k)}(Z_2^{(k)}(s))\,d\psi^{(k)}(s) \to \int_0^{t\wedge\eta_\rho} \hat{\gamma}(Z_2(s))\,d\psi(s)$$

in probability.

(17) $$\int_0^{t\wedge\eta_\rho^{(k)}} \hat{\alpha}^{(k)}(Z_2^{(k)}(s))\,d\hat{B}^{(k)}(\psi^{(k)}(s)) \to \int_0^{t\wedge\eta_\rho} \hat{\alpha}(Z_2(s))\,d\hat{B}(\psi(s)),$$

in probability.

Proof. By virtue of Lemma 4 and its remark,

$$E\int_0^{t\wedge\eta_\rho^{(k)}} |\hat{\gamma}^{(k)}-\hat{\gamma}^{(m)}|(Z_2^{(k)})(s))d\psi^{(k)}(s) \leq M_9 ||\hat{\gamma}^{(k)}-\hat{\gamma}^{(m)}||_{q,(-\rho,\rho)}$$

$$E\int_0^{t\wedge\eta_\rho^{(k)}} |\hat{\gamma}^{(m)}-\hat{\gamma}|(Z_2(s))d\psi(s) \leq M_9 ||\hat{\gamma}^{(m)}-\hat{\gamma}||_{q,(-\rho,\rho)}.$$

Since $\hat{\gamma}^{(m)}$ is smooth, $\int_0^{t\wedge\eta_\rho} \hat{\gamma}^{(m)}(Z_\rho^{(k)}(s))d\psi^{(k)}(s)$ tends to $\int_0^{t\wedge\eta_\rho^{(k)}} \hat{\gamma}^{(m)}(Z_2(s))d\psi(s)$, with probability 1, Therefore, we have (16).

Again by Lemma 4 and its remark,

$$E(\int_0^{t\wedge\eta_\rho^{(k)}} (\hat{\alpha}^{(k)}-\hat{\alpha}^{(m)})(Z_2^{(k)}(s))d\hat{B}^{(k)}(\psi^{(k)}(s)))^2$$

$$= E\int_0^{t\wedge\eta_\rho^{(k)}} (\hat{\alpha}^{(k)}-\hat{\alpha}^{(m)})^2(Z_2^{(k)}(s))d\psi^{(k)}(s) \leq M_9 ||(\hat{\alpha}^{(k)}-\hat{\alpha}^{(m)})^2||_{q,(-\rho,\rho)}$$

$$E(\int_0^{t\wedge\eta_\rho^{(k)}} (\hat{\alpha}^{(m)}-\hat{\alpha})(Z_2(s))dB(\psi(s)))^2 \leq M_9 ||(\hat{\alpha}^{(m)}-\hat{\alpha})^2||_{q,(-\rho,\rho)}$$

Since $B^{(k)}(\psi^{(k)}(s))$ tends to $B(\psi(s))$ uniformly on any bounded time interval, $\int_0^{t\wedge\eta_\rho^{(k)}} \hat{\alpha}^{(m)}(Z_2^{(k)}(s))d\hat{B}^{(k)}(\psi^{(k)}(s))$ tends to $\int_0^{t\wedge\eta_\rho} \hat{\alpha}^{(m)}(Z_2(s))d\hat{B}(\psi(s))$ by the continuity of $\hat{\alpha}^{(m)}$. Therefore we have (17).

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