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On the existence of solutions to a problem in multidimensional segmentation

by

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ABSTRACT. — We prove the existence of a minimizer for a multidimensional variational problem related to the Mumford-Shah approach to computer vision.

Key words : Mumford-Shah, computer vision.

RÉSUMÉ. — On démontre l'existence d'un minimum pour un problème variationnel en dimension n, voisin de celui que Mumford et Shah ont postulé à la base de la vision artificielle.

INTRODUCTION

In a recent paper [M-S], to which we refer for additional information, D. Mumford and J. Shah suggest a variational approach to the study of image segmentation in computer vision. In particular, they prove (*see* Theorem 5.1 of [M-F] and also [Mo-S]) the existence of minimizers of the following functional:

$$E_0(f, \Gamma) = \int_R (f-g)^2 + \nu_0 . \text{lengtht}(\Gamma)$$

where R denotes an open plane rectangle, g is a continuous function on the closure $\bar{R}$ of R, ν_0 is a given positive constant and where Γ, f vary in a suitable class of curves contained in R and, respectively, in the class of locally constant functions on $R \setminus \Gamma$.

In the present paper we will prove an existence result for a similar minimum problem with an open subset of $\mathbf{R}^n$, $n \geqq 2$ as base domain. According to [DeG], such a minimum problem pertains to the class of "minimal boundary problems"; a related problem of "free discontinuity type" has recently been studied in [DeG-C-L].

Specifically, we shall demonstrate the following:

MAIN THEOREM. — *Let* $n \in \mathbf{N}$, $n \geqq 2$, Ω *open* $\subset \mathbf{R}^n$, $0 < \lambda < +\infty$, $1 \leqq p < +\infty$, $g \in L^p(\Omega) \cap L^\infty(\Omega)$.

Then there exists at least one pair (K, u) *minimizing the functional*

$$F(K, u) = \lambda \int_{\Omega \setminus K} |u-g|^p \, dx + H^{n-1}(K \cap \Omega)$$

defined for every K *closed* $\subset \mathbf{R}^n$ *and for every* $u \in C^1(\Omega \setminus K)$ *such that* $\nabla u \equiv 0$ *in* $\Omega \setminus K$.

Here we denote by H^{n-1} the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ (*see* § 1 below). Notice the equivalence of conditions $u \in C^1(\Omega \setminus K)$ and $\nabla u \equiv 0$ in $\Omega \setminus K$ to the requirement that u be constant on every connected component of $\Omega \setminus K$.

Regularity properties of the minimizing pair (K, u) will be considered in a subsequent paper [M-T] (*see* also [M-S], Theorem 5.2, and [Mo-S], [Alm], [DeG-C-T], [T1]). Here we only observe that:

(i) $K \cap \Omega$ *is countably* $(H^{n-1}, n-1)$ *rectifiable*, i. e. (*see* e. g. [F], [S]) there exists a sequence $\{ S_h \}$ of class C^1 hypersurfaces such that

$$H^{n-1}(K \setminus (\bigcup_h S_h)) = 0;$$

(ii) *there exists a new pair* (K′, u') *minimizing* F *which satisfies*

$$K' \subset K, \quad H^{n-1}((K \setminus K') \cap \Omega) = 0, \quad K' = \overline{K' \cap \Omega}, \quad u' = u \quad \text{in } \Omega \setminus K$$

and for which there holds

$$\liminf_{\rho \to 0} \rho^{1-n} H^{n-1}(K' \cap B_{x,\rho}) > 0$$

for every $x \in \Omega \cap K'$, where $B_{x, \rho}$ denotes the open ball of radius $\rho > 0$ centered at x. Notice that, given (K, u), the new pair (K', u') is uniquely determined by the preceding conditions (ii).

We now give an outline of the proof of the Main Theorem.

Firstly, we derive some results concerning partitions of an open subset Ω of $\mathbf{R}^n$ in sequences of sets of finite perimeter (among them, a compactness theorem). Next, we investigate how such partitions are related to the class $SBV_{loc}(\Omega)$ of special bounded variation functions, introduced in [DeG-A]. Then we prove the existence of minimizers of a suitable functional G, defined on locally constant functions of $SBV_{loc}(\Omega)$.

By a straightforward generalization of results proved in [C-T2] in the context of "finite partitions", we can show that the jump set of a minimizer of G is essentially closed in Ω. As a consequence, we can prove that G and F have the same minimum value, and we show that from every minimizer of G we get a minimizing pair for F and *vice versa.*

The plan of the exposition is as follows. In section 1 we collect a few properties of Hausdorff measure, the perimeter of a set, and the space $SBV_{loc}(\Omega)$, and we study the relations between partitions in sets of finite perimeter and locally constant SBV functions.

In section 2 we introduce the new functional G, prove the existence and closure property of minimizers of G, and conclude the proof of the Main Theorem. A few explicit examples, showing the effect of dropping the boundedness assumption on g (the "grey-level image" in applications to computer vision), are added to the end of section 2.

We observe that the problem treated in [DeG-C-L] is also a multidimensional version of a variational problem suggested by Mumford and Shah in the context of image segmentation. Moreover, the general philosophy underlying the papers [DeG-C-L] and the present one is the same: the two problems have a similar formulation, and both are solved by recourse to a weak reformulation in the SBV framework, followed by a closure theorem.

This last point is delicate in both papers, and the methods used in establishing this result are quite different (though ultimately based on appropriate *decay estimates*): they seem not to be conveyable from one setting to another. Results on harmonic functions, of basic importance in [DeG-C-L], are often meaningless in our context (we are working with piecewise constant functions); while the recourse to partitions is crucial here but not appropriate for the analysis of [DeG-C-L].

In conclusion, quoting from [DeG], it can be said that the two works "show a sort of parallelism", each one exhibiting its own distinct features, methods and results.

Finally, we would like to thank prof. E. De Giorgi for helpful discussions during the preparation of this work.

1. PARTITIONS IN SETS OF FINITE PERIMETER AND FUNCTIONS OF CLASS SBV

In the following, we denote by Ω an open set of $\mathbf{R}^n$, $n \geqq 2$ and by $B_{x,\rho}$ the open ball centered at $x \in \mathbf{R}^n$ and of radius $\rho > 0$:

$$B_{x,\rho} = \{ y \in \mathbf{R}^n : |x-y| < \rho \}$$

When $x=0$, we write B_ρ instead of $B_{0,\rho}$. $\mathbf{B}(\Omega)$ is the family of Borel subsets of Ω.

For $E \subset \mathbf{R}^n$, $\bar{E}$ and ∂E are respectively the closure and topological boundary of E, diam E is the diameter of E and χ_E its characteristic function, which is 1 on E and 0 on the complementary set E^c. The notation $E \subset\subset \Omega$ means that $\bar{E}$ is a compact subset of Ω.

The Lebesgue measure of $E \subset \mathbf{R}^n$ is denoted by $|E|$; we now recall the definition of Hausdorff m-dimensional measure in $\mathbf{R}^n$ $(m \geqq 0)$:

$$\text{(1.1)} \qquad H^m(E) = \lim_{\varepsilon \to 0} H^m_\varepsilon(E)$$

where

$$\text{(1.2)} \qquad H^m_\varepsilon(E) = 2^{-m}\omega_m \times \inf\left\{ \sum_{h=1}^{\infty} (\operatorname{diam} E_h)^m : E \subset \bigcup_{h=1}^{\infty} E_h,\ \operatorname{diam} E_h < \varepsilon \right\}$$

$$\omega_m = \Gamma^m(1/2)/\Gamma(1+m/2)$$

and where Γ is Euler's "gamma function"; when m is a positive integer, ω_m coincides with the m-dimensional volume of the unit ball of $\mathbf{R}^m$ (*see* e. g. [F], 2.10.2, [S], § 2); in particular: $\omega_n = |B_{0,1}|$. The following useful result is proved e. g. in [S], Theorem 3.5:

LEMMA 1.1. — *Let* $E \in \mathbf{B}(\Omega)$ *and assume that*

$$(*) \qquad H^m(E \cap K) < +\infty$$

for every compact $K \subset \Omega$. *Then*

$$H^m\{ x \in \Omega \setminus E : \limsup_{\rho \to 0} \rho^{-m} H^m(E \cap B_{x,\rho}) > 0 \} = 0.$$

We now introduce some more notation. The set of *points of density* $\alpha \in [0, 1]$ of $E \in \mathbf{B}(\mathbf{R}^n)$ is denoted by $E(\alpha)$:

$$\text{(1.3)} \qquad E(\alpha) = \{ x \in \mathbf{R}^n : \lim_{\rho \to 0} |E \cap B_{x,\rho}| / |B_{x,\rho}| = \alpha \}$$

The *perimeter* of E in Ω is defined by:

$$P(E, \Omega) = \sup \left\{ \int_E \operatorname{div} \varphi(x)\, dx : \varphi \in C^1_0(\Omega; \mathbf{R}^n),\ |\varphi| \leqq 1 \right\}$$

and coincides with $H^{n-1}(\partial E \cap \Omega)$ in case of sets with regular boundary in Ω. When $P(E, \Omega) < +\infty$ we say that E has finite perimeter in Ω; in this case $D\chi_E$ (the distributional gradient of the characteristic function of E) is a vector-valued measure with finite total variation $|D\chi_E|$ on Ω:

$$|D\chi_E|(\Omega) = P(E, \Omega).$$

The derivative of $D\chi_E$ with respect to $|D\chi_E|$ allows one to introduce the notion of *interior unit normal* ν_E to E at any point of the *reduced boundary* $\partial * E$ of E:

$$\partial * E \cap \Omega = \{ x \in \Omega : \text{there exists } \nu_E(x) = \lim_{\rho \to 0} D\chi_E(B_{x,\rho}) / |D\chi_E|(B_{x,\rho}) \text{ and } |\nu_E(x)| = 1 \}$$

We refer to [DeG-C-P], [G], [M-M] for a complete exposition of the theory of sets of finite perimeter, we only recall a few basic properties, which will be useful in the sequel:

Assuming $P(E, \Omega) < +\infty$ *we have:*

(1.4) $$P(E, \Omega) = H^{n-1}(\partial * E \cap \Omega)$$

(1.5) $$\partial * E \cap \Omega \subset E(1/2) \cap \Omega, \qquad H^{n-1}[E(1/2) \cap \Omega \setminus \partial * E] = 0$$

(1.6) $$H^{n-1}(\Omega \setminus [E(0) \cup E(1) \cup E(1/2)]) = 0$$

In the proof of Lemma 1.4 below we will use the following result, which is proved in [DeG-C-P], Cap. IV, Def. 2.1 and Theorem 4.5:

LEMMA 1.2. – *There exist two constants* $K_1(n)$, $K_2(n)$ *depending only on the dimension n, such that if* $E \in \mathbf{B}(\mathbf{R}^n)$ *verifies*

$$E(0) \cap E = \varnothing, \qquad E(1) \subset E, \qquad |E| \leqq K_1(n)\,\varepsilon^n \qquad (\varepsilon > 0)$$

then

$$H_\varepsilon^{n-1}(E) \leqq K_2(n)\, P(E, \mathbf{R}^n).$$

We now define the notion of *Borel partition* of a given set:

DEFINITION 1.3. – Let $B \in \mathbf{B}(\mathbf{R}^n)$; we say that the sequence $\{ E_i \}$ is a Borel partition of B if and only if

$$E_i \in \mathbf{B}(\mathbf{R}^n), \quad \forall\, i \in \mathbf{N}, \qquad E_i \cap E_j = \varnothing \text{ when } i \neq j, \qquad \bigcup_{i=1}^{\infty} E_i = B.$$

More generally, we could require that $|E_i \cap E_j| = 0$ when $i \neq j$ and $|B \setminus \bigcup_i E_i| = 0$, assuming of course $E_i \subset B$, $\forall\, i$.

The following Lemma (and subsequent remark) is of basic importance: roughly speaking, it says that "most" of Ω is constituted by "interior" (density 1) and "boundary" (density 1/2) points of the sets in the partition, and that "most" of such boundary points form the interface between pairs of the partitioning sets. The partition corresponding to the usual construction of the Cantor set [*i. e.*, the sequence of "middle thirds" of the unit interval $(0,1)\subset\mathbf{R}$ together with the Cantor set itself] shows the need of assumption (*) below.

LEMMA 1.4. – *Let* $\{E_i\}$ *be a Borel partition of the open set* Ω *of* $\mathbf{R}^n$, *satisfying*

$$(*)\qquad \sum_{i=1}^{\infty} P(E_i,\Omega)<+\infty$$

Then:

$$\text{(i)}\qquad H^{n-1}\left(\Omega\setminus\bigcup_{i=1}^{\infty}[E_i(1)\cup E_i(1/2)]\right)=0$$

$$\text{(ii)}\quad H^{n-1}(E_i(1/2)\cap\Omega)=\sum_{j\neq i} H^{n-1}[E_i(1/2)\cap E_j(1/2)\cap\Omega],\qquad \forall i\in\mathbf{N}.$$

Proof. – Defining

$$Z=\Omega\cap\bigcap_i E_i(0)$$
$$M_i=\Omega\cap E_i(1/2)\cap\bigcap_{j\neq i} E_j(0)$$
$$M=\bigcup_i M_i$$
$$G_h=\bigcup_{i\geqq h} E_i$$

we have indeed

$$Z\subset G_h(1)\cap\Omega,\qquad \forall h$$
$$M_i\subset G_h(1/2)\cap\Omega,\qquad \forall h>i.$$

Moreover, for $h\to+\infty$ we have

$$P(G_h,\Omega)\leqq\sum_{i\geqq h} P(E_i,\Omega)\to 0$$

[thanks to hypothesis (*)] and

$$|G_h\cap B|\to 0,\qquad \forall \text{ ball } B\subset\subset\Omega.$$

From (1.1), (1.2), (1.4), (1.5) and from Lemma 1.2 we thus obtain

$$H^{n-1}(Z\cup M)=0,$$

which gives (i), (ii), on the account of (1.6). ■

Remark 1.5. – Recalling (1.4), (1.5), conclusion (ii) of Lemma 1.4 also yields:

$$\text{(iii)}\qquad \sum_{i=1}^{\infty} P(E_i,\Omega)=2H^{n-1}\left[\bigcup_{i=1}^{\infty}(E_i(1/2)\cap\Omega)\right]$$

Next we state a theorem which embodies a compactness and a semicontinuity result for Borel partitions.

THEOREM 1.6. – *Let* $\{E_{h,i}\}_{h,i}$ *and* $\{t_{h,i}\}_{h,i}$ *be sequences of Borel sets of* $\mathbf{R}^n$ *and, respectively, of real numbers such that*

$$E_{h,i} \cap E_{h,j} = \varnothing \quad \text{when } i \neq j \qquad \forall h \in \mathbf{N}$$

$$\bigcup_{i=1}^{\infty} E_{h,i} = \Omega, \qquad \forall h \in \mathbf{N}$$

$$\sum_{i=1}^{\infty} P(E_{h,i}, \Omega) \leqq K_3 < +\infty, \qquad \forall h \in \mathbf{N}$$

$$|t_{h,i}| \leqq K_4 < +\infty, \qquad \forall i, h \in \mathbf{N}$$

Then there exists a Borel partition $\{E_{\infty,i}\}_i$ *of* Ω *and a sequence* $\{t_{\infty,i}\}_i \subset \mathbf{R}$ *such that, passing to a subsequence if necessary:*

(i) $$\sum_{i=1}^{\infty} t_{h,i} \chi_{E_{h,i}} \to \sum_{i=1}^{\infty} t_{\infty,i} \chi_{E_{\infty,i}} \text{ in } L^1_{\text{loc}}(\Omega), \quad \text{as } h \to +\infty;$$

(ii) $$\sum_{i=1}^{\infty} P(E_{\infty,i}, \Omega) \leqq \liminf_{h \to +\infty} \sum_{i=1}^{\infty} P(E_{h,i}, \Omega).$$

Proof. – Denote by $\xi = \{E_i\}$ a generic Borel partition of Ω satisfying

$$\sum_{i=1}^{\infty} P(E_i, \Omega) \leqq K_3$$

and by ψ a fixed function such that

$$\psi(x) > 0, \quad \forall x \in \mathbf{R}^n, \quad \psi \in C^0(\mathbf{R}^n), \qquad \int_{\mathbf{R}^n} \psi\, dx < +\infty.$$

Rearranging the elements of ξ if necessary, we can and shall assume that

$$\int_{E_i} \psi\, dx + P(E_i, \Omega) \geqq \int_{E_{i+1}} \psi\, dx + P(E_{i+1}, \Omega), \qquad \forall i \in \mathbf{N}$$

It follows that for every $j \in \mathbf{N}$ and every ball $B \subset\subset \Omega$

$$(*) \quad \int_{\mathbf{R}^n} \psi\, dx + K_3 \geqq \sum_{i=1}^{j} \left[\int_{E_i} \psi\, dx + P(E_i, \Omega) \right] \geqq j \varepsilon |E_j \cap B| + j P(E_j, B)$$

where $\varepsilon = \varepsilon(\psi, B) > 0$ is such that $\psi(x) \geqq \varepsilon$ on B. For every $j \geqq j_0$ (depending only on K_3, ψ, B) we have thus:

$$|E_j \cap B) \leqq |B|/2$$

whence

$$P(E_j, B) \geqq c(n) |E_j \cap B|^{(n-1)/n}$$

owing to the isoperimetric inequality relative to balls in $\mathbf{R}^n$ (*see* [F], [G]).

Going back to (*) we find $\forall j \geqq j_0$

$$|E_j \cap B| \leqq c j^{n/(1-n)}$$

i. e.

$$\sum_{i=j+1}^{\infty} |E_i \cap B| \leqq c' j^{1/(1-n)}$$

where c, c' denote constants depending on n, K_3, ψ.

If we call ξ_h the partition $\{ E_{h,i} : i \in \mathbf{N} \}$ of Ω given in the Theorem, then, arguing as above, we can assume that

$$(**) \qquad \sum_{i=j+1}^{\infty} |E_{h,i} \cap B| \leqq c' j^{1/(1-n)}$$

for every $h \in \mathbf{N}$, every ball $B \subset\subset \Omega$ and every $j \geqq j_0 = j_0(K_3, \psi, B)$, with c' independent of h.

Since by assumption $P(E_{h,i}, \Omega) \leqq K_3$, $\forall h$, i, by a standard diagonalization argument (*see* [DeG-C-P], [M-M]) we can extract from $\{ \xi_h \}$ a subsequence (not relabeled) such that $\forall i$:

$$(***) \qquad \begin{cases} E_{h,i} \to E_{\infty,i} & \text{in } L^1_{\mathrm{loc}}(\Omega) \\ t_{h,i} \to t_{\infty,i} & \text{in } \mathbf{R} \end{cases}$$

as $h \to +\infty$.

Clearly, $|E_{\infty,i} \cap E_{\infty,j}| = 0$ when $i \neq j$, while for every ball $B \subset\subset \Omega$ and every j:

$$\left| B \setminus \bigcup_{i=1}^{j} (E_{\infty,i} \cap B) \right| = \lim_{h \to +\infty} \sum_{i=j+1}^{\infty} |E_{h,i} \cap B|$$

which on the account of (**) yields

$$\left| \Omega \setminus \bigcup_{i=1}^{\infty} E_{\infty,i} \right| = 0.$$

The sequence $\{ E_{\infty,i} \}$ is therefore a Borel partition of Ω (recall the comment following Def. 1.3), and (i), (ii) follow immediately from (***) and the semicontinuity of the perimeter. ■

Following [DeG-A], we now introduce the space $SBV(\Omega)$ of special bounded variation functions Ω. To this purpose, we recall some notation.

When $u : \Omega \to \mathbf{R}$ is a Borel function, $x \in \Omega$ and $z \in \tilde{\mathbf{R}} = \mathbf{R} \cup \{ \infty \}$ we say that

$$z = \operatorname*{aplim}_{y \to x} u(y) \quad \Leftrightarrow \quad g(z) = \lim_{\rho \to 0} \fint_{B_\rho} g(u(x+y))\, dy$$

for every $g \in C^0(\tilde{\mathbf{R}})$; for $z \in \mathbf{R}$, the previous definition coincides with the one given in [F]. We denote by S_u the *jump set* of u:

$$S_u = \{ x \in \Omega : \operatorname*{aplim}_{y \to x} u(y) \text{ does not exist} \} \tag{1.7}$$

For $x \in \Omega \setminus S_u$, we set $\tilde{u}(x) = \operatorname*{aplim}_{y \to x} u(y)$.

When $x \in \Omega$, $z \in \tilde{\mathbf{R}}$, $\nu \in \mathbf{R}^n$ with $|\nu| = 1$, we say that z is the *exterior trace* of u at x in the direction ν, and write $z = \mathrm{tr}^+(x, u, \nu)$, if and only if

$$g(z) = \lim_{\rho \to 0} \fint_{B_\rho \cap \{ y : \nu . y > 0 \}} g(u(x+y))\, dy, \qquad \forall g \in C^0(\tilde{\mathbf{R}})$$

(where . is the inner product in $\mathbf{R}^n$); the *interior trace* is then defined by putting $\mathrm{tr}^-(x, u, \nu) = \mathrm{tr}+(x, u, -\nu)$.

When both traces are finite at x, we can denote by $u^+(x)$ the greatest and by $u^-(x)$ the least value of the traces. When $x \in \Omega \setminus S_u$ and $\tilde{u}(x) \in \mathbf{R}$, we say that u *is approximately differentiable at* x if and only if there exists a vector $\nabla u(x) \in \mathbf{R}^n$ such that

$$\operatorname*{aplim}_{y \to x} \frac{|u(y) - \tilde{u}(x) - \nabla u(x) . (y-x)|}{|y-x|} = 0;$$

in this case, $\nabla u(x)$ is called the *approximate gradient* of u at x.

In order to study functions taking on a countable number of values, we introduce some more notation.

When $u : \Omega \to \mathbf{R}$ is a Borel function and $t \in \mathbf{R}$ we set

$$U_t = \{ x \in \Omega : u(x) = t \} \tag{1.8}$$

and say that

$$t = \operatorname*{aplim}_{y \to x} u(y) \quad \Leftrightarrow \quad x \in U_t(1) \tag{1.9}$$

Moreover, we set

$$\begin{aligned} \tilde{S}_u &= \{ x \in \Omega : \operatorname*{aplim}_{y \to x} u(y) \text{ does not exist} \} \\ \tilde{\tilde{u}}(x) &= \operatorname*{aplim}_{y \to x} u(y), \qquad \forall x \in \Omega \setminus \tilde{S}_u. \end{aligned} \tag{1.10}$$

When $x \in \Omega$, $t \in \mathbf{R}$, $\nu \in \mathbf{R}^n$ with $|\nu| = 1$, we say that

$$t = \tilde{\mathrm{tr}}^+(x, u, \nu) \quad \Leftrightarrow \quad x \in (U_t \cap S^+_{x,\nu})(1/2) \tag{1.11}$$

where $S^+_{x,\nu}$ is the halfspace $\{ y \in \mathbf{R}^n : (y-x) . \nu > 0 \}$; we define similarly $\tilde{\mathrm{tr}}^-(x, u, \nu)$, and denote by $\tilde{u}^+$, $\tilde{u}^-$ the greatest and least value between $\tilde{\mathrm{tr}}^+$ and $\tilde{\mathrm{tr}}^-$.

Remark 1.7. – If $u:\Omega\to\mathbf{R}$ is a Borel function, then:

(i) $S_u\subset\tilde{S}_u$, $\tilde{u}(x)=\tilde{\tilde{u}}(x)$, $\forall x\in\Omega\setminus\tilde{S}_u$;

(ii) $\tilde{\tilde{u}}$ takes on a finite or countable number of values and coincides with u H^n-almost everywhere in $\Omega\setminus\tilde{\tilde{S}}_u$; reciprocally, if u takes on a finite or countable number of values then $\tilde{\tilde{u}}$ exists H^n-a. e. in Ω;

(iii) When u takes on a finite number of values, we have $S_u=\tilde{S}_u$.

As usual, we denote by $BV(\Omega)$ the space of functions having bounded total variation in Ω:

$$u\in BV(\Omega) \quad\Leftrightarrow\quad u\in L^1(\Omega)$$

and

$$\int_\Omega |Du|\equiv\sup\left\{\int u\operatorname{div}\varphi\,dx:\varphi\in C_0^1(\Omega);\mathbf{R}^n),\ |\varphi(x)|\leqq 1,\ \forall x\right\}<+\infty$$

$BV(\Omega)$ is a Banach space with norm $\|u\|_{BV(\Omega)}=\int_\Omega |u(x)|\,dx+\int_\Omega |Du|$.

Notice that $E\in\mathbf{B}(\mathbf{R}^n)$ has finite measure and finite perimeter in Ω if and only if $\chi_E\in BV(\Omega)$.

Referring to a [G], [F], [M-M] for general properties of BV functions, we recal that *if* $u\in BV(\Omega)$ *then*

(i) S_u is countably $(H^{n-1}, n-1)$ rectifiable;

(ii) $u^+(x)$, $u^-(x)$ exist for H^{n-1}-almost all $x\in S_u$;

(iii) the *coarea formula* holds:

$$\int_\Omega |Du|=\int_{-\infty}^{+\infty} P(\{x\in\Omega:u(x)<t\},\Omega)\,dt; \tag{1.12}$$

(iv) $\displaystyle\int_\Omega |Du|\geqq\int_\Omega |\nabla u|\,dx+\int_{S_u\cap\Omega}(u^+-u^-)\,dH^{n-1}$.

According to [DeG-A], we denote by $SBV(\Omega)$ the space of $BV(\Omega)$ functions for which (iv) above holds with $\geqq$ replaced by the equality sign. The following useful characterization is a simple consequence of [A1], Prop. 3.1:

$$u\in SBV(\Omega) \quad\Leftrightarrow\quad u\in BV(\Omega)$$

and

$$\int_\Omega |\nabla u|\,dx=\inf\left\{\int_{\Omega\setminus K}|Du|:K\text{ compact }\subset\Omega,\ H^{n-1}(K)<+\infty\right\}$$

(*see* also [DeG-A], [DeG]). From this it follows immediately that $SBV(\Omega)$ is closed with respect to norm convergence in BV, *i. e.*

(1.13)

$$\left.\begin{array}{l} u_h\in SBV(\Omega),\ \forall h \\ \lim\limits_{h\to+\infty}\left(\int_\Omega |u_h-u_\infty|\,dx+\int_\Omega |D(u_h-u_\infty)|\right)=0\end{array}\right\}\quad\Rightarrow\quad u_\infty\in SBV(\Omega).$$

Finally, we say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ [resp., $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$] if and only if $u \in \mathrm{BV}(\Omega')$ [resp., $u \in \mathrm{SBV}(\Omega')$] for every open $\Omega' \subset\subset \Omega$.

The following Lemma, analogous to Theorem 3.6 of [DeG-C-L], relies on the Poincaré-Wirtinger type inequality proved in [DeG-C-L], Theor. 3.1 and Remark 3.2.

LEMMA 1.8. — *If* $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ *is such that* $\nabla u = 0$ *a. e. in* Ω *and* $x \in \Omega$ *verifies*

$$(*) \qquad \lim_{\rho \to 0} \rho^{1-n} \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{B}_{x, \rho}) = 0$$

then $x \notin \tilde{\mathrm{S}}_u$.

Proof. — The inequality of Poincaré-Wirtinger type proved in [DeG-C-L] states that whenever

$$u \in \mathrm{SBV}(\mathrm{B}), \qquad \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{B}) < \mathrm{K}_5(n) |\mathrm{B}|^{(n-1)/n}$$

then one has

$$\left(\int_{\mathrm{B}} |\bar{u}(y) - \mathrm{med}(u, \mathrm{B})|^{n/(n-1)} dy \right)^{(n-1)/n} \leqq \mathrm{K}_6(n) \int_{\mathrm{B}} |\nabla u| \, dy$$

and

$$\mathrm{K}_5(n) |\mathrm{B} \cap \{ y : u(y) \neq \bar{u}(y) \}|^{(n-1)/n} < \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{B})$$

where B is a ball in $\mathbf{R}_n$, $n \geqq 2$, $\mathrm{K}_5(n)$, $\mathrm{K}_6(n)$ are positive constants and where $\bar{u}$ is a suitable truncation of u and $\mathrm{med}(u, \mathrm{B})$ is the least median of u in B: we refer to [DeG-C-L], § 3, for the precise definitions of these concepts.

In our case, hypothesis (*) implies that $\forall \varepsilon > 0$ a radius $\rho_\varepsilon > 0$ can be found such that

$$\mathrm{B}_{x, \rho_\varepsilon} \subset \Omega, \qquad \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{B}_{x, \rho}) < \varepsilon \rho^{n-1}, \qquad \forall \rho \leqq \rho_\varepsilon.$$

Since $u \in \mathrm{SBV}(\mathrm{B}_{x, \rho})$ and $\nabla u = 0$ a. e. in $\mathrm{B}_{x, \rho}$ we then get

$$(**) \quad \rho^{-n} |\mathrm{B}_{x, \rho} \cap \{ y : u(y) \neq \mathrm{med}(u, \mathrm{B}_{x, \rho}) \}| \leqq \mathrm{K}_7(n) \varepsilon^{n/(n-1)}, \qquad \forall \rho \leqq \rho_\varepsilon$$

provided ε is sufficiently small. Reducing ε if necessary, it follows that the median is constant in ρ, for ρ small enough:

$$\mathrm{med}(u, \mathrm{B}_{x, \rho}) \equiv t \in \mathbf{R}, \qquad \forall \rho \leqq \rho_\varepsilon$$

From (**) we obtain $\tilde{\tilde{u}}(x) = t$ and the proof is concluded. ■

As a straightforward consequence of Lemma 1.8 we have the following

COROLLARY 1.9. — *If* $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ *is s. t.* $\nabla u = 0$ *a. e. in* Ω, *then* $\tilde{\mathrm{S}}_u \subset \bar{\mathrm{S}}_u \cap \Omega$. *If in addition* $\mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{K}) < +\infty$ *for all compact* $\mathrm{K} \subset \Omega$, *then Lemma* 1.1 *gives immediately* $\mathrm{H}^{n-1}(\tilde{\mathrm{S}}_u \setminus \mathrm{S}_u) = 0$.

The next two lemmas investigate the relation between partitions in sets of finite perimeter and SBV functions whose approximate gradient vanishes almost everywhere.

LEMMA 1.10. – *Let* $\{U_i\}$ *be a Borel partition of the open set* $\Omega \subset \mathbf{R}^n$, *and* $\{t_i\}$ *be a bounded sequence* $\subset \mathbf{R}$ *with* $t_i \neq t_j$ *for* $i \neq j$.

If $\sum_{i=1}^{\infty} P(U_i, \Omega) < +\infty$, *then:*

$$u \equiv \sum_{i=1}^{\infty} t_i \chi_{U_i} \in SBV_{loc}(\Omega) \cap L^{\infty}(\Omega)$$

and there holds:

(i) $$\nabla u = 0 \quad a.e. \ in\ \Omega,$$

(ii) $$2 H^{n-1}(S_u) = \sum_{i=1}^{\infty} P(U_i, \Omega).$$

Proof. – For $h \in \mathbf{N}$ define

$$u_h = \sum_{i=1}^{h} t_i \chi_{U_i}, \qquad K_8 = \sup |t_i| < +\infty,$$

and observe that

$$u_h \to u \quad \text{in } L^1_{loc}(\Omega)$$

$$\int_{\Omega} |D(u-u_h)| \leq K_8 \sum_{i=h+1}^{\infty} P(U_i, \Omega)$$

(*i. e.* $\|u-u_h\|_{BV(B)} \to 0$, $\forall B \subset\subset \Omega$), so that [*see* (1.13)] $u \in SBV_{loc}(\Omega)$.

Conclusion (i) is now clear, while (ii) follows from Lemma 1.4 and Remark 1.5, since

$$x \in \bigcup_i [U_i(1/2) \cap \Omega] \qquad \text{for} \quad H^{n-1}\text{-almost all } x \in S_u;$$

$$U_i(1/2) \cap U_j(1/2) \cap \Omega \subset S_u, \qquad \forall i \neq j \ (\text{being } t_i \neq t_j) \quad \blacksquare$$

LEMMA 1.11. – *Let* $u \in SBV_{loc}(\Omega)$ *be such that* $\nabla u = 0$ a. e. *in* Ω *and* $H^{n-1}(S_u) < +\infty$. *Then there exist a Borel partition* $\{U_i\}$ *of* Ω *and a sequence* $\{t_i\} \subset \mathbf{R}$ *with* $t_i \neq t_j$ *for* $i \neq j$, *such that*

(i) $$\sum_{i=1}^{\infty} P(U_i, \Omega) < +\infty$$

(ii) $$u = \sum_{i=1}^{\infty} t_i \chi_{U_i} \quad \text{a. e. in } \Omega.$$

Proof. – Combining Remark 1.7 (i) and Corollary 1.9 we get, thanks to the hypothesis $H^{n-1}(S_u) < +\infty$:

(1.14) $$S_u \subset \tilde{S}_u, \qquad H^{n-1}(\tilde{S}_u \setminus S_u) = 0$$

so that $\tilde{\tilde{u}}$ exists a. e. in Ω [*see* (1.10)].

By virtue of Remark 1.7 (ii) we can and shall assume that

$$u=\sum_i t_i \chi_{U_i} \quad \text{a. e. in } \Omega$$

where $\{ t_i \}$ is a sequence of distinct real numbers ($t_i \neq t_j$ if $i \neq j$) and $\{ U_i \}$ is a Borel partition of Ω.

Putting for short

$$\{ u<t \} \equiv \{ x \in \Omega : u(x)<t \}$$

we notice that $\Omega \cap \{ u<t \}(1/2) \subset \tilde{S}_u$, $\forall t \in \mathbf{R}$ [*see* (1.9), (1.10)], so that from (1.14), using coarea formula (1.12), (iii) and recalling (1.4), (1.5), we get $\forall i$:

$$\text{(1.15)} \qquad \left\{ \begin{array}{l} U_i(1/2) \cap \Omega \subset \tilde{S}_u, \\ P(U_i, \Omega) \leqq H^{n-1}(S_u). \end{array} \right.$$

Now Lemma 1.4, applied to the finite partition

$$\left\{ U_1, \ldots, U_h, \Omega \setminus \bigcup_{i=1}^{h} U_i \right\}$$

of Ω, yields $\forall h$ (*see* Remark 1.5):

$$\sum_{i=1}^{h} P(U_i, \Omega) \leqq 2 H^{n-1} \left[\bigcup_{i=1}^{h} (U_i(1/2) \cap \Omega) \right] \leqq 2 H^{n-1}(S_u) < +\infty$$

by virtue of (1.14), (1.15), and (i) follows at once. ■

Remark 1.12. – In the hypotheses of Lemma 1.11 we have therefore, thanks to (1.14) and Lemma 1.4:

(i) $\quad H^{n-1}(\tilde{S}_u \setminus S_u)=0$, *i.e.* $\tilde{\tilde{u}}$ exists H^{n-1} – a.e. in $\Omega \setminus S_u$;

(ii) $\quad \tilde{u}^+, \tilde{u}^-$ exist H^{n-1} – a.e. on S_u.

2. WEAK FORMULATION OF THE MINIMUM PROBLEM AND PROOF OF THE MAIN THEOREM

With the notation of section 1, we denote by G the functional

$$G(u)=\lambda \int_\Omega |u-g|^p \, dx + H^{n-1}(S_u)$$

defined for $u \in SBV_{loc}(\Omega)$ with $\nabla u=0$ a. e. in Ω, where Ω is an open subset of $\mathbf{R}^n$, $\lambda>0$, $1 \leqq p < +\infty$, $g \in L^p(\Omega)$.

First we prove (Theorem 2.2) the existence of minimizers w of G when g is bounded, then (Theorem 2.6) that S_w is essentially closed in Ω, and from this we ultimately derive the existence of minimizers of the functional F of the introduction (Remark 2.7).

Remark 2.1. – We have at once $G(0)=\lambda \int_{\Omega} |g|^p\,dx < +\infty$; if moreover $g \in L^{\infty}(\Omega)$, then

$$
\begin{aligned}
0 &\leqq \inf\{ G(u) : u \in SBV_{loc}(\Omega),\ \nabla u = 0 \text{ a.e. in } \Omega \} \\
&= \inf\{ G(u) : u \in SBV_{loc}(\Omega),\ \nabla u = 0,\ |u(x)| \leqq \|g\|_{\infty} \text{ a.e. in } \Omega \} < +\infty
\end{aligned}
$$

(*see* [DeG-C-L], Remark 2.2), and any function w minimizing G satisfies $|w(x)| \leqq \|g\|_{\infty}$ a. e. in Ω.

As an immediate consequence of the results stated in the preceding section (Lemma 1.10 and 1.11 and Theorem 1.6; we could also apply the general compactness and semicontinuity results obtained by L. Ambrosio in [A1], [A2]) we obtain the existence of minimizers of G.

THEOREM 2.2. – *If* Ω *is open* $\subset \mathbf{R}^n$, $n \geqq 2$, $\lambda > 0$, $1 \leqq p < +\infty$, $g \in L^p(\Omega) \cap L^{\infty}(\Omega)$, *then the functional*

$$
G(u) = \lambda \int_{\Omega} |u-g|^p\,dx + H^{n-1}(S_u)
$$

achieves a finite minimum in the class of functions $u \in SBV_{loc}(\Omega)$ *such that* $\nabla u = 0$ *a. e. in* Ω. *Every* w *minimizing* G *satisfies:*

(i) $w \in L^p(\Omega) \cap L^{\infty}(\Omega)$, $|w(x)| \leqq \|g\|_{\infty}$ *a. e. in* Ω;

(ii) $w = \sum_{i=1}^{\infty} t_i \chi_{W_i}$ a. e. in Ω, $\{ W_i \}$ *being a Borel partition of* Ω;

(iii) $2\,H^{n-1}(S_w) = \sum_{i=1}^{\infty} P(W_i, \Omega) < +\infty$.

Now we want to prove that the jump set of any minimizer w of G is essentially closed in Ω, *i. e.* $H^{n-1}((\bar{S}_w \cap \Omega) \setminus S_w) = 0$: as we shall see later on, this will enable us to obtain a minimizing pair for F (as will be shown in [M-T], the sharper result $\bar{S}_w \cap \Omega = S_w$ is indeed true, as a consequence of the local finiteness of the partition $\{ W_i \}$; this seems however to require the more complex machinery of Geometric Measure Theory: monotonicity, blow-up of solutions, etc.).

We recall that $S_w \subset \tilde{S}_w \subset \bar{S}_w \cap \Omega$ and that $H^{n-1}(\tilde{S}_w \setminus S_w) = 0$ [*see* Remark 1.7 (i) and Corollary 1.9; see also Remark 1.12 (i)]. Therefore, it is clearly enough to prove that $H^{n-1}((\bar{S}_w \cap \Omega) \setminus \tilde{S}_w) = 0$. Indeed, we will prove the sharper result

$$
\tilde{S}_w = \bar{S}_w \cap \Omega. \tag{2.1}
$$

We think it possible to find a short, direct proof of this fact in this particular setting (*i. e.*, when w is a minimizer of G), perhaps by simplifying our subsequent argument. However, formula (2.1) has a much more general validity, and we think it worthy to present a general result of this kind, which might be useful in other situations.

Before stating this result, we prove that any minimizer w of G satisfies a certain local estimate, an extended form of which will constitute the basic assumption enabling us to derive (2.1).

PROPOSITION 2.3. – *Let (Theorem 2.2)* $w=\sum_{i=1}^{\infty} t_i \chi_{W_i}$ *be a minimizer of* G, *call* $T=\{t_i : i \in \mathbf{N}\}$ *so that* $d=\operatorname{diam} T \leqq 2\|g\|_\infty$ *and fix* Ω' *open with* $\Omega' \subset\subset \Omega$. *Then* $\forall A$ *open* $\subset\subset \Omega'$ *and* $\forall u \in SBV_{loc}(\Omega; T)$ *with support* $(u-w) \subset A$ *we have:*

$$(2.2) \qquad \begin{cases} H^{n-1}(S_w \cap A) < +\infty \\ H^{n-1}(S_w \cap A) \leqq H^{n-1}(S_u \cap A) + c_3 \|u-w\|_{L^{n/(n-1)}(A)} \end{cases}$$

where $c_3 = \lambda p (4\|g\|_\infty)^{p-1} |\Omega'|^{1/n}$.

Here, $SBV_{loc}(\Omega; T)$ is the class of functions $u \in SBV_{loc}(\Omega)$ such that $u(x) \in T$, $\forall x \in \Omega$.

Notice that c_3 can be made arbitrarily small by reducing $|\Omega'|$.

Proof. – The first assertion is clear, in view of Theorem 2.2 (iii). Let A be open $\subset\subset \Omega'$ and $u \in SBV_{loc}(\Omega; T)$ with support $(u-w) \subset A$; since $\nabla u = 0$ a. e. in Ω, from $G(w) \leqq G(u)$ we obtain

$$H^{n-1}(S_w \cap A) \leqq H^{n-1}(S_u \cap A) + \lambda p (4\|g\|_\infty)^{p-1} |A|^{1/n} \times \left(\int_A |u-w|^{n/(n-1)} dx \right)^{(n-1)/n}$$

thanks to the inequality $||a|^p - |b|^p| \leqq p|a-b|(|a|+|b|)^{p-1}$ which holds $\forall a, b \in \mathbf{R}$, and (2.2) follows at once. ■

We are now in a position to state our general closure result, from which the essential closure of S_w follows immediately as we have seen. In the next Theorem, we consider a function w (which *could be* in particular a minimizer of G) satisfying a generalization of the local condition (2.2) [*see* (2.3) below], where terms like $H^{n-1}(S_u \cap A)$ are replaced by $\mathscr{F}(u, A)$; $\mathscr{F}$ is essentially an integral functional of the following type:

$$\mathscr{F}(u, A) = \int_{S_u \cap A} \varphi(x, u^+, u^-, \nu)\, dH^{n-1}$$

with a bounded Borel integrand φ satisfying: $0 < c_1 \leqq \varphi \leqq c_2 < +\infty$. When $c_1 = c_2 = 1$, we reobtain in particular the Hausdorff measure of the jump set S_u.

In addition to the closure of $\tilde{S}_w$ in Ω, we will obtain a basic density estimate.

THEOREM 2.4. — *Let* Ω *be open* $\subset \mathbf{R}^n$, $n \geqq 2$, *let* T *be countable* $\subset \mathbf{R}$ *with* $d = \operatorname{diam} T < +\infty$, *and let us denote by* $\mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$ *the class of functions* $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ *such that* $u(x) \in T$, $\forall x \in \Omega$.

Let

$$\mathscr{F} : \mathrm{SBV}_{\mathrm{loc}}(\Omega; T) \times \mathbf{B}(\Omega) \to [0, +\infty]$$

be a functional satisfying the following properties:

$$(\mathrm{P}_1) \qquad c_1 \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{A}) \leqq \mathscr{F}(u, \mathrm{A}) \leqq c_2 \mathrm{H}^{n-1}(\mathrm{S}_u \cap \mathrm{A})$$

$\forall u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$, $\forall$ A *open* $\subset\subset \Omega$, *where* c_1, c_2 *are constants satisfying* $0 < c_1 \leqq c_2 < +\infty$;

(P_2) $\mathscr{F}(u, .)$ *is a positive measure on* $\mathbf{B}(\Omega)$, $\forall u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$;

(P_3) $\mathscr{F}(u, \mathrm{A}) = \mathscr{F}(v, \mathrm{A})$, $\forall$ A *open* $\subset\subset \Omega$, $\forall u, v \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$ *such that* $u(x) = v(x)$, $\forall x \in \mathrm{A}$.

Finally, let $w \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$ *be such that*

$$(2.3) \qquad \begin{cases} \mathscr{F}(w, \mathrm{A}) < +\infty \\ \mathscr{F}(w, \mathrm{A}) \leqq \mathscr{F}(u, \mathrm{A}) + c_3 \| w - u \|_{\mathrm{L}^{n/(n-1)}(\mathrm{A})} \end{cases}$$

$\forall$ A *open* $\subset\subset \Omega$, $\forall u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$ *such that support* $(w-u) \subset \mathrm{A}$, *where the constant* c_3 *satisfies* $0 \leqq c_3 < +\infty$.

Then, if

$$(2.4) \qquad c_3 d \leqq n \omega_n^{1/n} c_1$$

we have:

$$\text{(i)} \qquad \tilde{\mathrm{S}}_w = \bar{\mathrm{S}}_w \cap \Omega$$

$$\text{(ii)} \qquad \liminf_{\rho \to 0} \rho^{1-n} \mathrm{H}^{n-1}(\mathrm{S}_w \cap \mathrm{B}_{x, \rho}) > 0, \qquad \forall x \in \bar{\mathrm{S}}_w \cap \Omega.$$

We emphasize that a result of this type has already been proven in [C-T2], Theorem 4.7, in the case when T is a *finite* set (*i. e.*, when the partition $\{ \mathrm{W}_i \}$ associated with w is finite: in that case one has indeed $\mathrm{S}_w = \bar{\mathrm{S}}_w \cap \Omega$). *See* also [C-T1], Section 4. A crucial tool in the proof of Theorem 2.4 is the following "decay lemma" which, roughly speaking, asserts that if a certain value $t_{i_0} \in \mathrm{T}$ is "preferred" by w in a certain annulus

$$\mathrm{A}_{r, s} = \{ x \in \mathbf{R}^n : r < |x| < r + s \} \subset\subset \Omega$$

[*see* (2.5) below for the precise meaning of this statement], then the same value is "even more preferred" in a nested annulus $\mathrm{A}_{r_1, s_1} \subset\subset \mathrm{A}_{r, s}$ [*see* (2.6), (2.7)].

LEMMA 2.5. — *With the same notation as in Theorem* 2.4, *there exist positive constants* η *and* σ (*depending only on* n, c_1, c_2, c_3 *and* d) *such that if* $w \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; T)$ *satisfies* (2.3), *if* $\mathrm{A}_{r, s} \subset\subset \Omega$ *and if for a certain* $t_{i_0} \in \mathrm{T}$ *it holds*

$$(2.5) \qquad s^{-n} |\mathrm{A}_{r, s} \setminus \mathrm{W}_{i_0}| < \eta \qquad (\mathrm{W}_{i_0} = w^{-1}\{ t_{i_0} \})$$

then there exists r_1 *satisfying* (*for* $s_1 = \sigma s$):

$$r + s/3 < r_1 < r_1 + s_1 < r + 2s/3 \tag{2.6}$$

$$s_1^{-n} \,|\, A_{r_1, s_1} - W_{i_0} \,| \leqq 2^{-1} s^{-n} \,|\, A_{r, s} \setminus W_{i_0} \,| \tag{2.7}$$

Notice that no restriction is made on c_3 [compare with (2.4)].

Lemma 2.5 is a straightforward adaptation of Lemma 4.3 of [C-T2] (which deals with the case T finite; on the account of Lemma 1.10 and 1.11, its extension to a countable T is essentially a matter of replacing finite sums by infinite series). We notice that a completely analogous result (corresponding to Lemma 4.1 of [C-T2]) can be formulated in terms of the average Hausdorff measure of the jump set as "decay parameter" [*i. e.*, using $s^{1-n} H^{n-1} (S_w \cap A_{r, s})$ instead of $s^{-n} \,|\, A_{r, s} \setminus W_{i_0} \,|$].

Proof of Theorem 2.4. — From (2.3) we get, recalling (P_1):

$$H^{n-1} (S_w \cap A) < +\infty, \qquad \forall A \text{ open } \subset\subset \Omega.$$

Since evidently $\nabla w = 0$ a. e. in Ω, from Lemmas 1.10, 1.11 of section 1 we obtain

$$\begin{aligned} & w = \sum_{i=1}^{\infty} t_i \chi_{W_i} (T = \{ t_i \},\ W_i = w^{-1} \{ t_i \}) \\ & 2 H^{n-1} (S_w \cap A) = \sum_{i=1}^{\infty} P(W_i, A) < +\infty, \qquad \forall A \text{ open } \subset\subset \Omega. \end{aligned} \tag{2.8}$$

If now $x \in \Omega \setminus \tilde{S}_w$, then by definition there exists $i_0 \in \mathbf{N}$ such that $x \in W_{i_0}(1)$; therefore, (2.5) clearly holds in a suitable annulus $A_{r, s}$ around x (we can assume $x = 0$ and take $r = 0$ and s small enough). By repeated application of Lemma 2.5 we find a nested sequence of annuli (shrinking to a sphere) where the average measure of the complementary set $W_{i_0}^c$ tends to zero. Ultimately we find a value $\bar{r} > 0$ such that $B \equiv B_{x, \bar{r}} \subset\subset \Omega$ and $\partial B \subset W_{i_0}(1)$ (*see* also Lemma 4.5 of [C-T2]).

Setting

$$u(x) = \begin{cases} w(x) & \text{if } \ x \in \Omega \setminus \bar{B} \\ t_{i_0} & \text{if } \ x \in \bar{B} \end{cases}$$

we deduce [using (2.3), (2.4), (2.8), together with (P_1)-(P_3) above and the isoperimetric inequality in $\mathbf{R}^n$] that $x \notin \tilde{S}_w$ (*see* the proof of Theorem 4.7 of [C-T2] for details). Combining this and corollary 1.9, we get conclusion (i) of the Theorem. We obtain (ii) by similar arguments, this time using the second "decay lemma" [formulated in terms of $s^{1-n} H^{n-1} (S_w \cap A_{r, s})$] quoted above. ■

In view of Proposition 2.3 and the preceding discussion, the results (i) and (ii) of Theorem 2.4 hold for any minimizer w of G. In conclusion we then have:

THEOREM 2.6. — *In the same assumptions of Theorem* 2.2, *any* w *minimizing* G *satisfies, in addition to* (i)-(iii) *of that Theorem, the following conditions*

(iv) $\tilde{S}_w = \bar{S}_w \cap \Omega$;

(v) $\liminf_{\rho \to 0} \rho^{1-n} H^{n-1}(S_w \cap B_{x,\rho}) > 0, \forall x \in \bar{S}_w \cap \Omega$;

(vi) $\tilde{w}$ *is constant on every connected component of* $\Omega \setminus \bar{S}_w$;

(vii) $H^{n-1}((\bar{S}_w \cap \Omega) \setminus S_w) = 0$.

Moreover, if w *takes on a finite number of values, then:*

(viii) $S_w = \bar{S}_w \cap \Omega$ [see *remark* 1.7 (iii)].

The role played by the assumption $g \in L^\infty(\Omega)$ in relation to conditions (vii), (viii) above is discussed in the examples at the end of the paper.

Remark 2.7. — At this point we are in a position to prove that the functional

$$F(K, u) = \lambda \int_{\Omega \setminus K} |u - g|^p \, dx + H^{n-1}(K \cap \Omega)$$

where Ω is open in $\mathbf{R}^n$, $n \geqq 2$, $\lambda > 0$, $1 \leqq p < +\infty$, $g \in L^p(\Omega) \cap L^\infty(\Omega)$, achieves its minimum in the class of pairs (K, u) with K closed $\subset \mathbf{R}^n$ and $u \in C^1(\Omega \setminus K)$ such that $\nabla u \equiv 0$ in $\Omega \setminus K$ (in this case, we say briefly that u is *locally constant* in $\Omega \setminus K$).

Indeed, we notice that

(1) if K is closed in $\mathbf{R}^n$, if u is locally constant in $\Omega \setminus K$ and if v denotes the *truncated function*

$$v = (u \wedge \|g\|_\infty) \vee (-\|g\|_\infty)$$

then

$$F(K, v) \leqq F(K, u);$$

(2) if in addition $H^{n-1}(K \cap \Omega) < +\infty$ and if u is *bounded* and locally constant in $\Omega \setminus K$, then $u \in SBV_{loc}(\Omega)$ and satisfies

$$G(u) \leqq F(K, u)$$

(in these assumptions we have indeed $S_u \subset K \cap \Omega$: *see* [DeG-C-L], Lemma 2.3).

(3) if w minimizes G (Theorem 2.2), then

$$F(\bar{S}_w, \tilde{\tilde{w}}) = G(\tilde{\tilde{w}}) = G(w)$$

(*see* Theorem 2.6 and Remark 1.7).

The Main Theorem, stated in the Introduction, follows immediately from (1)-(3) above.

Specifically, we obtain that if w minimizes G, then $(\bar{S}_w, \tilde{\tilde{w}})$ minimizes F; *vice versa*, if (K, u) minimizes F, then necessarily $H^{n-1}(K \cap \Omega) < +\infty$

$\left(\text{since } F(\varnothing, 0)=\lambda \int_\Omega |g|^p dx < +\infty\right)$ and $u \in L^\infty(\Omega)$, thus $u \in SBV_{loc}(\Omega)$ and $S_u \subset K \cap \Omega$ [by (1) and (2) above]: it follows that u minimizes G so that $(K', u') \equiv (\bar{S}_u, \tilde{\tilde{u}})$ is a new minimizing pair for F, which satisfies all properties stated in (ii) of the Introduction [recall (v) of Theorem 2.6; also recall (1.12) (i), which yields statement (i) of the Introduction].

We conclude the present work by discussing a few examples, showing that an unbounded datum g can give rise to minimizers w of functional G satisfying $|\bar{S}_w \cap \Omega \setminus S_w| > 0$ [compare with (vii) of Theorem 2.6].

Example 2.8. – (*a*) According to [C-T2], Example (E_3) of section 3, we denote by B_i ($i \geqq 0$) the open n-ball of radius $r_i = a^{-i-2}$ ($a > 2$) centered on the x_1-axis, at the point of abscissa 2^{-i}. Let $E = \bigcup_{i=0}^{\infty} B_i$, so that the origin of $\mathbf{R}^n$ is a boundary point of E and at the same time a point of zero density for E itself. Let $\Omega = B_R$ be a ball such that $E \subset\subset \Omega$, $1 \leqq p < +\infty$ and $\lambda > 0$. Setting $g = r_i^{-1/p}$ in B_i, $g \equiv 0$ in $\Omega \setminus E$, we claim that g is the only minimizer of G, at least when $\lambda > n$.

To this aim, it will be clearly sufficient to compare g with those u which coincide with g itself on certain balls B_i', and vanish on the remaining balls B_i'', as well as on the complementary set $E^c \cap \Omega$.

We find:

$$G(u) - G(g) = \lambda \sum_i \int_{B_i''} |g|^p dx - \sum_i P(B_i'', \mathbf{R}^n) = \omega_n (\lambda - n) \sum_i{}'' r_i^{n-1}$$

which proves our claim (by $\sum_i''$ we mean the sum extended to the indices corresponding to the balls B_i'').

In this way function g defined above [which belongs to $L^q(\Omega)$, $\forall q < np$ as one readily verifies] gives rise to a minimizer w of G for which

$$\bar{S}_w \cap \Omega \setminus S_w = \{ 0 \}$$

(*b*) By a refinement of the preceding construction we can also determine a minimizer w of G for which $|\bar{S}_w \cap \Omega \setminus S_w| > 0$. To this aim, we choose a sequence $\{ x_h \}$ of points in $\mathbf{R}^n$ and a strictly increasing sequence $\{ i(h) \}$ of positive integers such that for

$$B_h = B_{x_h,\ e^{-i(h)}}$$

there holds

$$\bar{B}_h \cap \bar{B}_k = \varnothing \qquad \text{if} \quad h \neq k$$

$$E = \bigcup_{h=1}^{\infty} B_h \quad \text{bounded}$$

$$(*) \qquad |\overline{\{ x_h : h \in \mathbf{N} \}}| > 0$$

We give a sketch of the construction required in the case $n=2$. As starting points of the sequence $\{ x_h \}$ we choose the vertices of the unitary square; then we choose the middle points of the sides and the centre of the square. Correspondingly, the sequence $i(h)$ is determined by requiring that $\bar{B}_h$ be disjonted from the balls $\bar{B}_k (k<h)$ constructed before.

At this point we repete the procedure in each of the 4 squares thus determined, excluding from $\{ x_h \}$ those middle points and centers which belong to the closure of balls already constructed.

Setting $\Omega = B_R$ with R such that $E \subset\subset \Omega$, $\lambda>0$, $1 \leqq p < +\infty$, $g(x) = e^{i(h)/p}$ in B_h, $g(x)=0$ in $\Omega \setminus E$, we find by similar arguments as those used in example (*a*) above, that for $\lambda > n$ the only minimizer of G is g itself, and that $g \in L^q(\Omega)$, $\forall q < np$.

We see that $S_g = \bigcup_{h=1}^{\infty} (\partial B_h)$ and thus $|(\bar{S}_g \cap \Omega) \setminus S_g| > 0$ thanks to (*).

REFERENCES

[Alm] F. J. ALMGREN, Existence and Regularity Almost Everywhere of Solutions to Elliptic Variations Problems with Constraints, *Mem. A.M.S.*, Vol. **4**, No. 165, 1976.

[A1] L. AMBROSIO, A Compactness Theorem for a Special Class of Functions of Bounded Variation, *Boll. Un. Mat. Ital.* (to appear).

[A2] L. AMBROSIO, Existence Theory for a New Class of Variational Problems, *Arch. Rat. Mech. Anal.*, (to appear).

[A-B] L. AMBROSIO and A. BRAIDES, Functionals defined on partitions... I et II, *J. Math. Pures Appl.* (to appear).

[B-M] J. BLAT and J. M. MOREL, *Elliptic Problems in Image Segmentation* (to appear).

[C-T1] G. CONGEDO and I. TAMANINI, Note sulla regolarità dei minimi di funzionali del tipo dell'area, *Rend. Accad. Naz. XL*, **106**, Vol. XII, fasc. 17, 1988, pp. 239-257.

[C-T2] G. CONGEDO and I. TAMANINI, *Density Theorems for Local Minimizers of Area-Type Functionals* (to appear).

[DeG] E. DE GIORGI, Free Discontinuity Problems in Calculus of Variations, *Proceedings of the meeting in J. L. Lions's honour*, Paris, 1988 (to appear).

[DeG-A] E. DE GIORGI and L. AMBROSIO, Un nuovo tipo di funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei* (to appear).

[DeG-C-L] E. DE GIORGI, M. CARRIERO and A. LEACI, Existence Theorem for a minimum Problem with Free Discontinuity Set, *Arch. Rat. Mech. Anal.*, Vol. **108**, 1989, pp. 195-218.

[DeG-C-P] E. DE GIORGI, F. COLOMBINI and L. C. PICCININI, *Frontiere orientate di misura minima e questioni collegate*, Editrice Tecnico Scientifica, Pisa, 1972.

[DeG-C-T] E. DE GIORGI, G. CONGEDO and I. TAMANINI, Problemi di regolarità per un nuovo tipo di funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei* (to appear).

[F] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.

[G] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[M-M] U. MASSARI and M. MIRANDA, *Minimal Surfaces of Codimension* 1, North-Holland, Amsterdam, 1984.

[M-T] U. MASSARI, I. TAMANINI, paper in preparation.

[Mo-S] J. M. MOREL and S. SOLIMINI, Segmentation of Images by Variational Methods: a Constructive Approach, *Rev. Mat. Univ. Complutense Madrid*, Vol. 1, 1988, pp. 169-182.

[M-S] D. MUMFORD and J. SHAH, Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems, *Comm. Pure Appl. Math.*, Vol. **42**, 1989, pp. 577-685.

[S] L. SIMON, *Lectures on Geometric Measure Theory*, Center for Math. Analysis, Australian National University, Vol., **3**, 1983.

[T1] J. E. TAYLOR, The Structure of Singularities in Soap-Bubble-Like and Soap-Film-Like Minimal Surfaces, *Annals Math.*, **103**, 1976, pp. 489-539.

[T2] J. E. TAYLOR, Cristalline Variational Problems, *Bull. A.M.S.*, Vol. **84**, 1978, pp. 568-588.

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