On the existence of special metrics in complex geometry

by

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Dedicated to Marcel Friedmann on the occasion of his 80th birthday

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0. Introduction

Every complex manifold X admits a smooth hermitian metric and, in fact, a huge space of them. To study the manifold it is often useful to pick from this collection a particularly nice metric. For example, if X is algebraic, one can always choose the metric to be Kählerian, and often to be Kähler-Einstein (cf. Yau [15]).

The topological and analytic consequences of the existence of a Kähler metric are strong and have been well understood for some time. Recently a characterization of which complex manifolds admit such metrics has been given [8]. However, relatively little is known about how to choose a good metric in general, and this paper represents a first step towards understanding this question. We shall here define and characterize a class of complex manifolds which admit a special type of hermitian metric. This class contains the Kähler manifolds as well as many important categories of non-Kähler manifolds, including, for example: 1-dimensional families of Kähler varieties, the "twistor spaces" constructed from self-dual riemannian 4-manifolds, and complex solvmanifolds. We shall carry out an extensive analysis of this class of manifolds. It is hoped that our results will be important in the further study which now seems clearly worthwhile.

We approach the problem via the torsion. Recall that each hermitian metric h on X has an associated canonical hermitian connection with a torsion tensor T_h. This can be thought of as a $(2,0)$-form with values in the tangent bundle TX, or alternatively, as a 1-form with values in endomorphisms of TX. It is a classical fact that h is Kählerian if and only if $T_h=0$, and in general such metrics do not exist.

Therefore, it is natural in the general case to search for metrics whose torsion satisfies some weaker condition. Invariant theory suggests the following. Associated to h is a real-valued 1-form $\tau_h=\mathrm{trace}(T_h)$ obtained by taking the trace of the endomorphism-valued 1-form T_h. This will be called the *torsion 1-form* of h. Metrics, h, for which $\tau_h=0$ will be called *balanced,* and the existence of such metrics will be the principal concern of this paper.

We remark that any hermitian metric h has an associated positive $(1,1)$-form ω_h called the *Kähler form* of h. (The forms h and ω_h are essentially equivalent.) It is well known that

$$h \quad \text{is Kählerian} \quad \text{iff} \quad d\omega_h=0.$$

In this spirit we have that

$$h \quad \text{is balanced} \quad \text{iff} \quad d^{*h}\omega_h=0.$$

Here d^{*h} is the formal adjoint of d in the metric h; hence the equation $d^{*h}\omega_h=0$ is *non-linear* in h.

In complex dimension two the conditions of being balanced and Kählerian are equivalent. However, in all dimensions $\geqslant 3$ there exist compact balanced manifolds which carry no Kähler metric. This is true, for example, of certain complex solvmanifolds. (See section 6.)

The condition of being balanced is, in a strong sense, *dual* to that of being Kähler, and this duality appears over and over again in our work. A simple example of this is the following. It is classical and easy to prove that if X is Kählerian and if there exists a holomorphic immersion $f\colon Y\to X$, then Y is Kähler. The dual statement is also true. That is, *if X is balanced and if there exists a holomorphic submersion $f\colon X\to Y$, then Y is balanced.* (This is proved in § 1.) Thus, the Kähler property is induced on sub-objects and the balanced (or "co-Kähler") property projects to quotient objects.

The beginning of this paper is devoted to discussing the basic properties of balanced manifolds. In section 2 we examine the relation of Kählerian and balanced metrics to the complex Dirac operators $\mathscr{D}$ and $\bar{\mathscr{D}}$ introduced in [11]. It is shown that

$$h \quad \text{is Kählerian} \quad \text{iff} \quad \mathscr{D}^2 = 0$$
$$h \quad \text{is balanced} \quad \text{iff} \quad \bar{\mathscr{D}} = \mathscr{D}^*$$

where $\mathscr{D}^*$ denotes the formal adjoint of $\mathscr{D}$.

In section 3 we analyze the behavior of the torsion form under conformal changes of metric. This is particularly relevant to the study of metrics whose torsion form is closed. We show (under mild hypotheses) that in the conformal class of each such metric there is a *uniquely determined* metric whose torsion $(1,0)$-form (the $(1,0)$-component of T_h) is ∂-closed and holomorphic.

In section 4 we formulate and prove one of the main results of the paper: *a complete intrinsic characterization of those compact complex manifolds which admit balanced metrics*. Not every complex manifold X carries a balanced metric. This can be seen as follows. If ω is the Kähler form of such a metric, and if $n = \dim(X)$, then $d(\omega^{n-1}) = 0$. It follows that every compact complex hypersurface in X must represent a non-trivial class in $H_{2n-2}(X;\mathbf{R})$. Thus, for example, Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$, $p+q>0$, are not balanced. This necessary condition is easily strengthened as follows. Every d-closed positive $(n-1,n-1)$-current (which is not zero) represents a non-zero class in $H_{2n-2}(X;\mathbf{R})$. In fact, every closed deRham current of dimension $2n-2$ whose $(n-1,n-1)$-component is positive and non-zero represents a non-zero class in $H_{2n-2}(X;\mathbf{R})$. (See § 4 for definitions.) A complex n-manifold with this last property is called *homologically balanced*. The principal result is the following.

THEOREM A. *A compact complex manifold X admits a balanced metric if and only if it is homologically balanced.*

This result will be used, in a forthcoming paper, to prove that the set of balanced manifolds is *open* i.e., *in a family* or "moduli space", *the subset of those manifolds which admit balanced metrics is open.*

The second main result concerns the problem of inductively constructing balanced metrics (cf. § 5). The statement is as follows.

THEOREM B. *If a compact complex manifold X admits an essential holomorphic map with balanceable fibers onto a complex curve, then X can be balanced.*

A map $f: X \to C$ (with irreducible fibers) is *essential* if the generic fibre is not an $(n-1,n-1)$-boundary, or dually, if the f^*-image of a fundamental cocycle on C is not a $(1,1)$-coboundary. Very often (and always in dimension 2) this simply means that the f^*-image of the fundamental cohomology class of C is not zero.

The hypothesis that f be essential is necessary as one can easily see from the Hopf surface $S^1 \times S^3$ which admits a holomorphic map onto S^2 but is not balanceable.

Since in complex dimension 2 balanceable is equivalent to Kähler, we have the following, which is also proved in [8].

COROLLARY. *A compact complex surface which admits an essential holomorphic map onto a complex curve, is Kählerian.*

Here "essential" means basically that the generic fibre is not homologous to zero, i.e., that the f^*-image of the fundamental cohomology class of the curve is not zero.

We point out that in Theorem B and its corollary, the maps need *not* be submersions. The fibers can degenerate to singular fibers.

Theorem B indicates an area where balanced manifolds are natural and important, namely in the study of families of varieties. Given a 1-parameter family of Kähler manifolds, it is not always true that the total space of the family is Kählerian (even when the family is topologically trivial). From a construction of E. Calabi (cf. § 6), one obtains a family of complex tori over a curve, which is topologically a product, and which admits no Kähler metric. Nevertheless, it does carry a balanced metric by Theorem B. Thus, balanced metrics seem naturally adapted to the study of curves in moduli spaces.

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1. Hermitian geometry: torsion and its trace

Suppose X is a compact complex manifold and let $J: TX \to TX$ denote its almost complex structure. We consider on X a riemannian metric g with the property that at each point $x \in X$,

$$g(V, W) = g(JV, JW) \tag{1.1}$$

for all $V, W \in T_x X$. Such a metric will be called *hermitian*. Associated to it is an exterior 2-form

$$\omega(V, W) = g(V, JW) \tag{1.2}$$

called the Kähler form of g. Combining these gives the standard *complex hermitian metric*

$$h = g + i\omega \tag{1.3}$$

which has the property that $h(JV, W)=ih(V, W)=-h(V, JW)$.

We view a *connection* on X in the customary way as a differential operator $D\colon \Gamma(TX)\to\Gamma(T^*X\otimes TX)$ with the property that

$$D_V(fW) = (Vf)\, W+fD_V W \tag{1.4}$$

for all vector fields $V, W\in\Gamma(TX)$ and all functions $f\in C^\infty(X)$. Associated to each such D is its *torsion tensor* $T^D\in\Gamma(\Lambda^2T^*X\otimes TX)$,

$$T^D_{V,W} = D_V W-D_W V-[V, W] \tag{1.5}$$

for $V, W\in\Gamma(TX)$.

The tensors above will be extended complex multilinearly to the complexification of TX. We recall the natural decomposition

$$TX\otimes \mathbf{C} = T^{1,0}\oplus T^{0,1} \tag{1.6}$$

into the $+i$ and $-i$ eigenbundles of J, and note that for $V\in TX$,

$$V^{1,0} = \tfrac{1}{2}(V-iJV) \quad\text{and}\quad V^{0,1} = \tfrac{1}{2}(V+iJV). \tag{1.7}$$

In a local complex coordinate system $(z^1, \dots, z^n)$ on X a $(1,0)$-vector field $\varphi\in\Gamma(T^{1,0})$ can be expressed as

$$\varphi = \sum_{j=1}^{n} a_j\frac{\partial}{\partial z^j} \tag{1.8}$$

where $a_1, \dots, a_n$ are local complex-valued functions. Such a $(1,0)$-field is said to be *holomorphic* if these functions are holomorphic.

A given connection D on X is said to *preserve the riemannian metric* g if $D(g)=0$, i.e., if

$$Ug(V, W) = g(D_U V, W)+g(V, D_U W) \tag{1.9}$$

for all vector fields U, V, W. The connection is said to *preserve the complex structure* if $D(J)=0$, i.e., if

$$D_V(JW) = J(D_V W) \tag{1.10}$$

for all vector fields V, W.

There are two connections canonically associated to a given hermitian metric g on X. The first is simply the *riemannian connection* ∇^0, which is characterized by requiring that it preserve g and that its torsion vanish. The second is the *hermitian connection* ∇, which is characterized by requiring that it preserve g *and* J and that the $(1,1)$-part of the torsion vanish, i.e.,

$$T^{\nabla}_{JV,JW} = T^{\nabla}_{V,W} \tag{1.11}$$

for all V, W. It is a standard result (cf. [14]) that (1.11) is equivalent to requiring that

$$\nabla_{V^{0,1}}\varphi = 0 \tag{1.12}$$

for all local holomorphic vector fields φ (and arbitrary $(0,1)$-directions $V^{0,1}$). This can be rewritten as

$$\nabla_{JV}\varphi = i\nabla_V\varphi \qquad (1.12)$$

for all φ holomorphic and V real.

When the canonical riemannian and hermitian connections coincide, the metric is called *Kähler*. This property is equivalent to the condition that $T^{\nabla}\equiv 0$. As we shall see in a moment, it is also equivalent to the condition

$$d\omega = 0. \tag{1.13}$$

It is not difficult to see that for some X, there is no Kähler metric.

This leads one to ask whether there is a weaker condition which is always satisfied by some hermitian metric on each compact complex manifold. It is natural to try to formulate such a condition in terms of the torsion tensor T^{∇}.

We begin by computing the torsion in a local complex coordinate system $(z^1, \ldots, z^n)$. Suppose the complex hermitian metric is given in these coordinates by

$$h = \sum h_{i,\bar j}\, dz^i \otimes d\bar z^j; \quad h_{i,\bar j} = h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) \tag{1.14}$$

and write the hermitian connection as

$$\nabla\left(\frac{\partial}{\partial z^j}\right) = \sum_{k=1}^{n} \omega_{jk} \otimes \frac{\partial}{\partial z^k}. \tag{1.15}$$

By condition (1.12) the complex-valued 1-forms ω_{jk} are all of type $(1,0)$; and from condition (1.9), $dh_{j\bar k} = \Sigma_\alpha(\omega_{j\alpha} h_{\alpha\bar k} + h_{j\bar\alpha}\bar\omega_{\alpha k})$ for all j, k. Hence, writing $dh_{j\bar k} = \partial h_{j\bar k} + \bar\partial h_{j\bar k}$ and identifying types, we retrieve the standard result that

$$\omega = \partial H \cdot H^{-1} \tag{1.16}$$

where $\omega = ((\omega_{jk}))$ and $H = ((h_{j\bar{k}}))$.

The torsion T^{∇}, considered as a TX-valued 2-form, has no $(1,1)$-component, and (since it is real) its $(2,0)$ and $(0,2)$ components are complex conjugates of one another.

PROPOSITION 1.1. *The* $(2,0)$-*component of the torsion associated to the canonical hermitian connection of the metric* $h = \Sigma\, h_{j\bar{k}}\, dz^j \otimes d\bar{z}^k$ *can be written as*

$$T = \sum_{j,k,l} T^l_{jk}\, dz^j \wedge dz^k \otimes \frac{\partial}{\partial z^l}$$

where

$$T^l_{jk} = \sum_{\alpha} \left\{ \frac{\partial h_{k\bar{\alpha}}}{\partial z^j} h^{\bar{\alpha} l} - \frac{\partial h_{j\bar{\alpha}}}{\partial z^k} h^{\bar{\alpha} l} \right\} \tag{1.17}$$

and where $((h^{\bar{j}k}))$ *denotes the inverse of the matrix* $((h_{j\bar{k}}))$.

Proof. This is a direct consequence of (1.15) and (1.16).

COROLLARY 1.2. *Let* h *and* T *be as in Proposition* 1.1, *and consider the exterior* $(2,1)$-*form*

$$\mathbf{T} = \sum T_{jk\bar{l}}\, dz^j \wedge dz^k \wedge d\bar{z}^l$$

where $T_{jk\bar{l}} = i\, \Sigma_{\alpha}\, T^{\alpha}_{jk}\, h_{\alpha\bar{l}}$. *Then*

$$d\omega = \mathbf{T} + \bar{\mathbf{T}} \tag{1.18}$$

where $\omega = i\, \Sigma\, h_{j\bar{k}}\, dz^j \wedge d\bar{z}^k$ *is the Kähler form associated to* h.

Note that (1.18) is equivalent to the two equations:

$$\partial\omega = \mathbf{T} \quad \text{and} \quad \bar{\partial}\omega = \bar{\mathbf{T}}. \tag{1.19}$$

Clearly $\mathbf{T} = 0$ (i.e., the metric is Kähler) if and only if $d\omega = 0$.

We are now prepared to enunciate one of the central concepts of the paper.

Definition 1.3. The *torsion* $(1,0)$-*form* of the complex hermitian metric h is defined as $\tau = \Sigma\, \tau_k\, dz^k$ where

$$\tau_k \equiv \sum_j T^j_{kj} \tag{1.20}$$

and where the T^l_{jk} are given by (1.17).

It is sometimes useful to consider the real form $\tau_{\mathbf{R}}=\tau+\bar{\tau}$ which we shall call the *torsion 1-form* of h. This is clearly obtained by contraction of the torsion tensor.

Definition 1.4. A hermitian metric on a complex manifold is said to be *balanced* if its torsion $(1,0)$-form vanishes identically.

PROPOSITION 1.5. *Let h be a complex hermitian metric with torsion $(1,0)$-form τ and Kähler form ω. Then*

$$i\tau=\bar{\partial}^*\omega \tag{1.21}$$

where $\bar{\partial}^*$ denotes the formal hermitian adjoint of the operator $\bar{\partial}$ with respect to the metric h.

Proof. A straightforward computation using, say, [10, p. 97] gives the result.

Note that since ω is real, equation (1.21) is equivalent to: $-i\bar{\tau}=\partial^*\omega$, and also to the real equation:

$$\tau_{\mathbf{R}}=(d^C)^*\omega \tag{1.22}$$

where $d^C\equiv i(\bar{\partial}-\partial)$.

Since $\bar{\partial}^*=-\ast\partial\ast$ and since $\ast\omega=\omega^{n-1}/(n-1)!$, we conclude the following.

THEOREM 1.6. *Let h be a complex hermitian metric with Kähler form ω on an n-dimensional complex manifold. Then the following are equivalent.*

(1) *The metric h is balanced.*
(2) *$P^*\omega=0$ where P^* is any of the operators $\partial^*, \bar{\partial}^*, (d^C)^*$, or d^*.*
(3) *$P(\omega^{n-1})=0$ where P is any of the operators $\partial, \bar{\partial}, d^C$, or d.*

Note that *when $n=2$*, $\ast\omega=\omega$ and so *the conditions of being balanced and Kähler coincide*. Moreover, condition (3) of Theorem 1.6 gives the following general restriction on the existence of balanced metrics.

COROLLARY 1.7. *Suppose X is an n-dimensional complex manifold which admits a balanced metric. Then every compact complex subvariety of dimension $n-1$ in X represents a non-zero class in $H_{2n-2}(X;\mathbf{R})$.*

Proof. Let V be such a subvariety and observe that

$$\frac{1}{(n-1)!}\int_V \omega^{n-1}=\text{volume}(V)\neq 0.$$

Example 1.8. Consider the complex structures introduced by Calabi and Eckmann [3] on $S^{2p+1}\times S^{2q+1}$. These have the property that the product of the Hopf mappings

$$\pi\colon S^{2p+1}\times S^{2q+1}\to \mathbf{P}^p(\mathbf{C})\times\mathbf{P}^q(\mathbf{C})$$

is holomorphic. Hence, there are plenty of compact complex submanifolds of codimension-one. (Take π^{-1} of such an object in $\mathbf{P}^p(\mathbf{C})\times\mathbf{P}^q(\mathbf{C})$.) Since the homology is zero in dimension $2p+2q$, we see that these manifolds support no balanced metrics.

Corollary 1.7 can be strengthened to include all positive, d-closed currents $T\in\mathscr{E}'_{n-1,n-1}(X)$. (See Section 4.)

This seems an appropriate place to record some of the "functorial" properties of being balanced. (For convenience, we shall say a complex manifold is "balanced" if it admits a balanced metric.)

PROPOSITION 1.9. *Let X and Y be complex manifolds.*

(i) *If X and Y are balanced, then the product $X\times Y$ is balanced.*

(ii) *If X is balanced and if there exists a proper holomorphic submersion of X onto Y, then Y is balanced.*

Proof. Suppose X and Y each admit balanced hermitian metrics, and let ω_X and ω_Y denote the respective Kähler forms of these metrics. Then $\omega=\omega_X+\omega_Y$ is the Kähler form of the product metric on $X\times Y$; and if $n=\dim(X)$ and $m=\dim(Y)$, then

$$\omega^{n+m-1}=\binom{n-1}{n+m-1}\omega_X^{n-1}\,\omega_Y^{m}+\binom{n}{n+m-1}\omega_X^{n}\,\omega_Y^{m-1}.$$

By Theorem 1.6 we have that $d(\omega^{n+m-1})=0$ and so $X\times Y$ is balanced. This proves (i).

Suppose now that X is balanced and that $f\colon X\to Y$ is a proper holomorphic submersion onto Y. Let ω_X be the Kähler form of a balanced metric on X and consider the closed form $\Omega_X=\omega_X^{n-1}$. Since f is proper, we can consider the 'push-forward" $\Omega_Y\equiv f_*(\Omega_X)$ of Ω_X considered as a current of dimension 2. Then Ω_Y is simply the $(2m-2)$-form obtained by integration over the fibers of f. Since f_* commutes with d and since $d\Omega_X=0$, we have that $d\Omega_Y=0$.

Now Ω_X is a strictly positive $(n-1,n-1)$-form; that is, with respect to any $\mathbf{C}$-basis $\varepsilon_1,\dots,\varepsilon_n$ of any complex cotangent space $T_x^{1,0}X$, we have that

$$\Omega_X=i^{n-1}\sum_{j,k=1}^{n}a_{jk}\,\varepsilon_1\wedge\bar\varepsilon_1\wedge\dots\wedge\widehat{\varepsilon_j}\wedge\bar\varepsilon_j\wedge\dots\wedge\varepsilon_k\wedge\widehat{\bar\varepsilon_k}\wedge\dots\wedge\varepsilon_n\wedge\bar\varepsilon_n$$

where $((a_{jk}))$ is a positive definite matrix. Furthermore, since f is a holomorphic submersion, the push forward Ω_Y is a strictly positive $(m-1, m-1)$-form on Y. This can be seen as follows. Fix $y \in Y$ and fix a C-basis $\varepsilon_1, \ldots, \varepsilon_m$ for $T_y^{1,0} Y$. At each point x in the fibre $F=f^{-1}(y)$ we can lift the forms $\varepsilon_1, \ldots, \varepsilon_m$ and complete to a basis $\varepsilon_1, \ldots, \varepsilon_n$ of $T_x^{1,0} X$. We can do this so that $\nu = i^{n-m} \varepsilon_{m+1} \wedge \bar{\varepsilon}_{m+1} \wedge \ldots \wedge \varepsilon_n \wedge \bar{\varepsilon}_n$, when restricted to F, is any given smooth volume element. Then Ω_Y at y is written as

$$\Omega_Y = i^{m-1} \sum_{j,k=1}^{m} \tilde{a}_{jk} \varepsilon_1 \wedge \bar{\varepsilon}_1 \wedge \ldots \wedge \widehat{\varepsilon_j} \wedge \bar{\varepsilon}_j \wedge \ldots \wedge \varepsilon_k \wedge \widehat{\bar{\varepsilon}_k} \wedge \ldots \wedge \varepsilon_m \wedge \bar{\varepsilon}_m$$

where

$$\tilde{a}_{jk} = \int_F a_{jk} \nu$$

for each j, k. Note that $((\tilde{a}_{jk}))_{j,k=1}^m$ is again positive definite.

It is shown in § 4 that Ω_Y can therefore be written uniquely as $\Omega_Y = \omega_Y^{m-1}$, where ω_Y is a strictly positive $(1, 1)$-form. (See 4.8 forward.) Of course, ω_Y uniquely determines the hermitian metric $h_Y = g + i\omega_Y$ on Y by the formula $g(V, W) = \omega_Y(JV, W)$. This completes the proof.

Remark 1.10. Note that in (ii) above, the (proper) map $f: X \to Y$ need not be a submersion. It is only necessary that the Jacobian f_* be surjective at some point along each fiber of the map.

Note also that if $f: X \to Y$ is a finite covering map, then X is balanced if and only Y is balanced.

2. The relationship to Dirac operators

When a manifold is Kählerian, we know (cf. [11]) that there are complex Dirac operators $\mathscr{D}$ and $\bar{\mathscr{D}}$ defined, with the useful properties that $\mathscr{D}^2 = 0 = \bar{\mathscr{D}}^2$ and that $\bar{\mathscr{D}}$ is the formal adjoint of $\mathscr{D}$. These operators may be defined for any manifold with a hermitian metric by using the canonical hermitian connection ∇. We will see that the first property, namely that $\mathscr{D}^2 = \bar{\mathscr{D}}^2 = 0$, is equivalent to the property that the metric be Kählerian. As we shall see, $\bar{\mathscr{D}}$ is the adjoint of $\mathscr{D}$ if and only if the metric is balanced.

Let X be a hermitian manifold. In order to define complex Dirac operators for X we introduce the complex Clifford bundle associated to X (cf. [11]). This is the bundle $\mathrm{Cl}(X)$ associated to TX, which at a point x is

$$\mathrm{Cl}(X)_x = \mathrm{Cl}(T_x X) \otimes \mathbf{C}$$

where $\mathrm{Cl}(T_xX)$ is the Clifford algebra associated to the real vector space T_xX with the quadratic form given by the underlying real inner product. If $e_1, Je_1, \ldots, e_n, Je_n$ is a (real) orthonormal basis for T_xX, then a multiplicative basis for $\mathrm{Cl}(x)_x$ is 1 and

$$\varepsilon_k = \tfrac{1}{2}(e_k - iJe_k),$$

$$\bar{\varepsilon}_k = \tfrac{1}{2}(e_k + iJe_k).$$

These basis elements all anti-commute with one-another except that

$$\varepsilon_k \bar{\varepsilon}_k + \bar{\varepsilon}_k \varepsilon_k = -1.$$

These are the only relations. The connection ∇ extends canonically to sections of $\mathrm{Cl}(X)$ as a derivation. The Dirac operators $\mathscr{D}$ and $\bar{\mathscr{D}}$ are defined on sections of $\mathrm{Cl}(X)$ by the formulas

$$\mathscr{D} = \sum \varepsilon_k \cdot \nabla_{\bar{\varepsilon}_k}$$

$$\bar{\mathscr{D}} = \sum \bar{\varepsilon}_k \cdot \nabla_{\varepsilon_k}$$

where $\cdot$ denotes Clifford multiplication.

PROPOSITION 2.1. *The following statements are equivalent.*

(i) *X is Kählerian.*

(ii) $\mathscr{D}^2 = 0$.

(iii) $\bar{\mathscr{D}}^2 = 0$.

Proof. It was proved in [11] that (i) implies (ii) and (iii). We shall show that (ii) implies (i). (The case (iii)$\Rightarrow$(i) is similar.) Now at a point x we have that

$$\mathscr{D}^2 = \sum \varepsilon_j \varepsilon_k \nabla_{\bar{\varepsilon}_j, \bar{\varepsilon}_k} \tag{2.1}$$

where for tangent vectors U and V, $\nabla_{U,V}$ is the *invariant second covariant derivative* which is defined by

$$\nabla_{U,V} = \nabla_{\tilde{U}} \nabla_{\tilde{V}} - \nabla_{\nabla_{\tilde{U}} \tilde{V}}$$

for arbitrary local vector fields $\tilde{U}$ and $\tilde{V}$ which extend U and V. This is independent of the choice of local extensions. It follows from (2.1) that

$$\mathscr{D}^2 = \sum_{j<k} \varepsilon_j \varepsilon_k \left(\nabla_{\bar{\varepsilon}_j, \bar{\varepsilon}_k} - \nabla_{\bar{\varepsilon}_k, \bar{\varepsilon}_j}\right).$$

But

$$\begin{aligned}\sum \left(\nabla_{\bar{\varepsilon}_j, \bar{\varepsilon}_k} - \nabla_{\bar{\varepsilon}_k, \bar{\varepsilon}_j}\right) &= \sum \left(\nabla_{\bar{\varepsilon}_j} \nabla_{\bar{\varepsilon}_k} - \nabla_{\bar{\varepsilon}_k} \nabla_{\bar{\varepsilon}_j} - \nabla_{[\bar{\varepsilon}_j, \bar{\varepsilon}_k]}\right) - \nabla_{\left(\nabla_{\bar{\varepsilon}_j} \bar{\varepsilon}_k - \nabla_{\bar{\varepsilon}_k} \bar{\varepsilon}_j - [\bar{\varepsilon}_j, \bar{\varepsilon}_k]\right)} \\ &= \sum R_{\bar{\varepsilon}_j, \bar{\varepsilon}_k} - \nabla_{T_{\bar{\varepsilon}_j, \bar{\varepsilon}_k}} \\ &= -\sum \nabla_{T_{\bar{\varepsilon}_j, \bar{\varepsilon}_k}}\end{aligned}$$

where R is the curvature tensor defined by

$$R_{\tilde{U}, \tilde{V}} = \nabla_{\tilde{U}} \nabla_{\tilde{V}} - \nabla_{\tilde{V}} \nabla_{\tilde{U}} - \nabla_{[\tilde{U}, \tilde{V}]}$$

for $\tilde{U}, \tilde{V}$ as before. The last equality follows because the curvature is of type $(1, 1)$, i.e., it satisfies

$$R_{JU, JV} = R_{U, V}$$

for all U, V. So we have

LEMMA 2.2.

$$\mathscr{D}^2 = -\sum_{j<k} \varepsilon_j \varepsilon_k \nabla_{T_{\bar{\varepsilon}_j, \bar{\varepsilon}_k}} = -\frac{1}{2} \sum_{j,k,l} \varepsilon_j \varepsilon_k T^l_{\bar{j}\bar{k}} \nabla_{\bar{\varepsilon}_l}.$$

Therefore, to prove Proposition 2.1, for a given l we let f be a function such that at a given point x

$$\begin{cases} \nabla_{\bar{\varepsilon}_l} f = 1 \\ \nabla_{\bar{\varepsilon}_j} f = 0 \quad \text{if } j \neq l. \end{cases}$$

Then if $\mathscr{D}^2 = 0$, we have

$$\sum_{j<k} \varepsilon_j \varepsilon_k T^l_{\bar{j}\bar{k}} = \mathscr{D}^2(f \cdot 1) = 0$$

holds for any l. But $\{\varepsilon_j \varepsilon_k : j<k\}$ are linearly independent. So

$$T^l_{\bar{j}\bar{k}} = 0$$

for all j, k, l, and therefore, X is Kähler.

PROPOSITION 2.3. *A hermitian metric on a complex manifold is balanced if and only if $\mathscr{D}$ and $\bar{\mathscr{D}}$ are formal adjoints of one another.*

Proof. Fix a point $x \in X$ and choose ε_j's as above with the property that $\nabla\varepsilon_j = 0$ at x for each j. Let s and s' be sections of $\mathrm{Cl}(X)$, and let $(\cdot, \cdot)$ denote the complex hermitian inner product naturally induced in this bundle. A straightforward computation as in [11] shows that at the point x, we have

$$(\mathscr{D}s, s') - (s, \bar{\mathscr{D}}s') = \sum \bar{\varepsilon}_j(\varepsilon_j s, s').$$

Let ∇^0 denote the canonical riemannian connection for the metric on X, and set $A = \nabla^0 - \nabla$. Let W be the complex tangent vector field (of type $(1,0)$) of X defined by the requirement that

$$(V, W) = (Vs, s')$$

for all $(1,0)$ vectors V on X. Straightforward computation now shows that at the point x, we have

$$\operatorname{div}(\bar{W}) = \sum (A_{e_j} e_j + A_{Je_j} Je_j, W) + 2\sum \bar{\varepsilon}_j(\varepsilon_j s, s'),$$

and

$$\sum_j (A_{e_j} e_j + A_{Je_j} Je_j, W) = -2\sum_j (T_{\bar{W}} \bar{\varepsilon}_j, \varepsilon_j) = -2\bar{\tau}(\bar{W}).$$

Consequently, we have that

$$(\mathscr{D}s, s') - (s, \bar{\mathscr{D}}s') \equiv \bar{\tau}(\bar{W}) \quad \text{(mod divergences)}.$$

This completes the proof.

3. Conformal changes of metric

It is certainly natural to ask how the torsion form behaves within the conformal class of a given hermitian metric. The basic result is the following.

PROPOSITION 3.1. *Suppose the hermitian metrics $\tilde{h}$ and h are conformally related by $\tilde{h} = e^u h$, where u is a smooth real-valued function. Then the corresponding torsion*

$(1,0)$-*forms are related by the equation*

$$\tilde{\tau}=\tau+(n-1)\,\partial u, \tag{3.1}$$

where n is the complex dimension of the manifold. In particular, we have that

$$\tilde{\tau}_{\mathbf{R}}=\tau_{\mathbf{R}}+(n-1)\,du. \tag{3.2}$$

COROLLARY 3.2. *The class of the torsion form* $[\tau_{\mathbf{R}}]$ *in the space* $\mathscr{E}^1(X)_{\mathbf{R}}/d\mathscr{E}^0(X)_{\mathbf{R}}$ *is a conformal invariant of the metric.*

Proof of Proposition 3.1. We express $\tilde{h}$ and h in local complex coordinates $(z^1,\dots,z^n)$ as in Proposition 1.1 Then $\tilde{h}_{j\bar{k}}=e^u h_{j\bar{k}}$ and $\tilde{h}^{jk}=e^u h^{jk}$ for all j,k. Applying equation (1.17), we find that

$$\tilde{T}^l_{jk}=\frac{\partial u}{\partial z_j}\delta_{kl}-\frac{\partial u}{\partial z_k}\delta_{jl}+T^l_{jk} \tag{3.3}$$

for all j,k,l. It follows immediately that for $\tau_j\equiv\Sigma_k T^k_{jk}$ we have

$$\tilde{\tau}_j=(n-1)\frac{\partial u}{\partial z_j}+\tau_j. \tag{3.4}$$

This completes the proof.

Note that if a metric can be chosen so that

$$d\tau_{\mathbf{R}}=0, \tag{3.5}$$

then the cohomology class $[\tau_{\mathbf{R}}]\in H^1(X;\mathbf{R})$ is a conformal invariant. It is an interesting question whether such metrics exist.

It would be stronger to require that $d\tau=0$, since $2d\tau_{\mathbf{R}}=d(\tau+\bar{\tau})$. This is equivalent to

$$\partial\tau=0 \quad\text{and}\quad \bar{\partial}\tau=0, \tag{3.6}$$

i.e., to the requirement that τ be a ∂-closed holomorphic 1-form.

PROPOSITION 3.3. *Suppose X is a compact complex manifold such that* $H^1(X;\mathscr{H})=0$, *where* $\mathscr{H}$ *denotes the sheaf of germs of pluriharmonic functions. (One could assume, for example, that* $H^1(X;\mathcal{O})=0$.) *Then in the conformal class of each hermitian metric which satisfies* (3.5), *there is a unique metric (up to homothety) which satisfies* (3.6).

Proof. Let $\tau_{\mathbf{R}}=\frac{1}{2}(\tau+\bar{\tau})$ be the torsion 1-form of a hermitian metric h, and suppose

$d\tau_{\mathbf{R}}=0$. Then taking (p, q)-components, we have that

$$\partial\tau = 0 \quad \text{and} \quad \operatorname{Re}(\bar{\partial}\tau) = 0. \tag{3.7}$$

Let $\tilde{h}=e^u h$ be a conformally related metric with torsion $(1,0)$-form $\tilde{\tau}=\tau+(n-1)\partial u$ (cf. Proposition 3.1). Clearly $\partial\tilde{\tau}=0$, and furthermore we have $\bar{\partial}\tilde{\tau}=0$ if and only if

$$dd^C u = 2i\partial\bar{\partial}u = \frac{2i}{n-1}\bar{\partial}\tau \overset{\text{def}}{=} w. \tag{3.8}$$

(We assume that $n>1$ since the case $n=1$ is trivial.) From (3.7) we know that $dw=0$ and that w is real. Furthermore, there is an isomorphism

$$H^1(X; \mathscr{H}) = \{w \in \mathscr{E}^{1,1}(X)_{\mathbf{R}}: dw = 0\}/dd^C\mathscr{E}^0(X)_{\mathbf{R}}$$

which follows from the fine resolution

$$0 \to \mathscr{H} \to \mathscr{E}^{0,0}_{\mathbf{R}} \xrightarrow{dd^C} \mathscr{E}^{1,1}_{\mathbf{R}} \xrightarrow{d} (\mathscr{E}^{2,1} \oplus \mathscr{E}^{1,2})_{\mathbf{R}} \to \ldots$$

of the sheaf $\mathscr{H}$. Since $H^1(H; \mathscr{H})=0$, we see that $w=dd^C u$ for some $u \in C^\infty(X)$. Since the kernel of dd^C on $C^\infty(X)$ is the constants, this conformal change by e^u is unique up to a multiplicative constant (i.e., a homothety). This completes the proof.

Note that if $H^1(X; \mathcal{O})=0$, then the condition $\partial\tau=0$ immediately implies that $\tau=\partial u_0$ for some u_0, and so $\bar{\partial}\tau=\bar{\partial}\partial u_0$, and the proof proceeds as above.

Note also that uniqueness up to homothety is really uniqueness if, say, we normalize the total volume of X to be 1.

As we saw in Section 1, balanced metrics, i.e., metrics with the property that $(d^C)^*\omega=0$, do not always exist (and the existence of metrics satisfying (3.5) is, at the moment, an open question). However, it seems appropriate to point out here that metrics with the property that $(dd^C)^*\omega=0$ can always be constructed.

THEOREM 3.4 (Gauduchon [5]). *Let X be a compact complex manifold of dimension $\geqslant 2$. Then each hermitian metric on X is conformally equivalent to a unique metric (up to homothety) whose Kähler form ω satisfies the conditon*

$$(dd^C)^*\omega = 0.$$

4. The characterization theorem

In order to enunciate our main theorem, we must sharpen the necessary conditions presented in Section 1 (Cor. 1.7). To do this we recall some basic facts from geometric measure theory. Let X be a compact complex manifold and let $\mathscr{E}^p(X)$ denote the space of smooth, complex-valued exterior p-forms on X with the standard C^∞-topology. The topological dual $\mathscr{E}'_p(X)\equiv\mathscr{E}^p(X)'$ is called the space of *p-dimensional* (*deRham*) *currents*. Note that the Dolbeault decomposition $\mathscr{E}^p(X)=\oplus_{r+s=p}\mathscr{E}^{r,s}(X)$ of forms determines a corresponding decomposition

$$\mathscr{E}'_p(X)=\bigoplus_{r+s=p}\mathscr{E}'_{r,s}(X) \tag{4.1}$$

of currents. The elements in $\mathscr{E}'_{r,s}(X)$ are called *currents of bidimension* (r, s).

Each operator d, d^C, ∂, and $\bar\partial$ on $\mathscr{E}^*(X)$ induces an adjoint operator (again denoted d, d^C, ∂, and $\bar\partial$ respectively) on the space $\mathscr{E}'_*(X)$. These operators continue of course, to satisfy the standard identities: $d^2=\partial^2=\bar\partial^2=0$, $d=\partial+\bar\partial$, etc. It is a classical result of deRham that there exists an isomorphism

$$H_p(\mathscr{E}'_*(X), d)\cong H_p(X;\mathbf{C}) \tag{4.2}$$

for each p.

Suppose we now consider the space $C^p(X)$ of all continuous complex-valued p-forms on X with the standard topology. The topological dual space $\mathscr{M}_p(X)\subset\mathscr{E}'_p(X)$ is called the space of p-currents which are *represented by integration* (or which are of *finite mass*) on X. There is again a decompostion $\mathscr{M}_p(X)=\oplus_{r+s=p}\mathscr{M}_{r,s}(X)$. Suppose we fix a hermitian metric on X. Then associated to each $T\in\mathscr{M}_p(X)$ there is a Radon measure $\|T\|$ on X, and a field $\vec{T}$ of unit complex p-vectors (i.e., a measureable section of $\Lambda^pTX\otimes\mathbf{C}$), defined $\|T\|$-almost everywhere, so that $T=\vec{T}\cdot\|T\|$. That is,

$$T(\varphi)=\int_X \varphi(\vec{T}_x)\,d\|T\|(x) \tag{4.3}$$

for all continuous p-forms φ.

We consider now the spaces $\hat{\mathscr{M}}_p(X)=\{T\in\mathscr{M}_p(X): dT\in\mathscr{M}_{p-1}(X)\}$ for $p\geq 0$. Then it is a basic result of Federer and Fleming [4] that there exist natural isomorphisms

$$H_p(\mathscr{M}_*(X), d)\cong H_p(X;\mathbf{C}) \tag{4.4}$$

for each p.

We now recall the notion of a positive (r, r)-current. Fix a point $x\in X$ and consider a complex (i.e., J-invariant) r-plane L in the tangent space T_xX. Choose $e_1, \ldots, e_r\in L$ such that $\{e_1, Je_1, \ldots, e_r, Je_r\}$ forms a (real) orthonormal basis of L. Then the vector

$$\xi_L = e_1 \wedge Je_1 \wedge \ldots \wedge e_r \wedge Je_r$$

is independent of the choice of the e_j's, and up to positive multiples, it is independent of the choice of hermitian metric. Note that ξ_L is real and contained in $\Lambda^{r,r}T_xX$. We call ξ_L the *(r, r)-vector associated* to L. The positive cone $P_x^{r,r} \subset \Lambda^{r,r}T_xX$ generated by all such vectors, is called the *cone of positive (r, r)-vectors* at x. Thus we have $\xi \in P_x^{r,r}$ if and only if

$$\xi = \sum c_j \xi_{L_j} \tag{4.5}$$

where $c_j \geq 0$ and L_j is a complex r-dimensional subspace of T_xX for each j.

Suppose now that ω is the Kähler form of any hermitian metric on X. Then it is easy to see that $\omega^r(\xi_L) > 0$ for each complex r-dimensional subspace $L \subset T_xX$. Hence, by (4.5) we conclude that

$$\omega^r(\xi) > 0 \quad \text{for each non-zero } \xi \in P_x^{r,r}. \tag{4.6}$$

The key concept of this section is the following.

Definition 4.1. A current $T \in \mathcal{M}_{r,r}(X)$ is said to be a *positive (r, r)-current* if

$$\vec{T}_x \in P_x^{r,r}$$

for $\|T\|$-a.a. x. The set of positive (r, r)-currents will be denoted $\mathcal{P}_{r,r}(X)$.

Note that any current $T \in \mathcal{P}_{r,r}(X)$ is real, i.e., $T = \bar{T}$. Furthermore, from (4.6) we immediately conclude the following.

PROPOSITION 4.2. *Let ω be the Kähler form of any hermitian metric on X. Then*

$$T(\omega^r) > 0$$

for each non-zero $T \in \mathcal{P}_{r,r}(X)$.

Example 4.3. Let M be a compact complex manifold of dimension r in X, and define $[M] \in \mathcal{M}_{r,r}(X)$ by $[M](\varphi) = \int_M \varphi$ for each continuous r-form φ. Then $[M] \in \mathcal{P}_{r,r}(X)$. Furthermore, by Stokes' theorem we know that $d[M] = 0$.

This generalized to r-dimensional complex subvarieties of X. (See [7], for example.)

Example 4.4. Let $\psi \in \mathcal{E}^{1,1}(X)$ be a positive $(1, 1)$-form, i.e., locally ψ can be written as $\psi=\Sigma\, a_{jk}(z)\, dz^j \wedge d\bar{z}^k$ where $A(z)\equiv((a_{jk}(z)))$ is a non-negative hermitian symmetric matrix at every point z. Define a current $[\psi]\in\mathcal{M}_{n-1,n-1}(X)$ by setting $[\psi](\varphi)=\int_X \psi\wedge\varphi$ for each continuous $(2n-2)$-form φ. Then $[\psi]\in\mathcal{P}_{n-1,n-1}(X)$ and $d[\psi]=0$ if and only if $d\psi=0$.

We are now in a position to state a general condition necessary for the existence of a balanced metric. Let $\pi_{r,r}\colon \oplus\mathcal{E}'_{r',s'}(X)\to\mathcal{E}'_{r,r}(X)$ be the natural projection, and set $d_{r,r}=\pi_{r,r}\circ d$.

PROPOSITION 4.5. *Let X be an n-dimensional complex manifold which admits a balanced metric. Then*

$$d_{n-1,n-1}\,\mathcal{E}'_{2n-1}(X)\cap\mathcal{P}_{n-1,n-1}(X)=\{0\}, \tag{4.7}$$

or equivalently, every d-closed current $T\in\mathcal{E}'_{2n-2}(X)$, such that $\pi_{n-1,n-1}T$ is non-zero and positive, represents a non-zero class in $H_{2n-2}(X;\mathbf{C})$.

Proof. Let ω be the Kähler form of the balanced metric, and recall from Theorem 1.6 that

$$d(\omega^{n-1})=0.$$

Suppose now that $T\in\mathcal{E}'_{2n-2}(X)$ satisfies: $0\neq\pi_{n-1,n-1}T\in\mathcal{P}_{n-1,n-1}(X)$. Then by Proposition 4.2

$$T(\omega^{n-1})=(\pi_{n-1,n-1}T)(\omega^{n-1})>0.$$

(We can replace T by $\pi_{n-1,n-1}T$ since $\omega^{n-1}\in\mathcal{E}^{n-1,n-1}(X)$.) Hence, if $dT=0$, then T must represent a non-zero homology class.

Definition 4.6. A complex n-manifold X which satisfies (4.7) is said to be *homologically balanced.*

Note from the examples above that on a homologically balanced manifold, every compact complex hypersurface and every closed positive $(1,1)$-form must represent a non-trivial homology class.

THEOREM 4.7. *A compact complex n-manifold can be given a smooth balanced metric if and only if it is homologically balanced.*

An analogous result is proved in [8] for Kähler metrics. These results coincide in dimension two where the notions of being Kähler and balanced are equivalent.

Proof. It remains only to show that if X is homologically balanced, then we can construct a balanced metric.

Throughout the proof we shall work with the spaces of real currents and real-valued exterior forms, which we denote by $\mathscr{E}'_p(X)_{\mathbf{R}}$ and $\mathscr{E}^p(X)_{\mathbf{R}}$ respectively. We begin with the observation that the cone $\mathscr{P}_{n-1,n-1}(X) \subset \mathscr{E}'_{n-1,n-1}(X)_{\mathbf{R}}$ has a compact base in the weak topology. Indeed, if we fix a metric on X, then the set $B \equiv \{T \in \mathscr{P}_{n-1,n-1}(X) : \|T\|(X)=1\}$ is clearly compact in the weak topology on $\mathscr{M}_{n-1,n-1}(X)_{\mathbf{R}}$ and therefore also in the weak topology on $\mathscr{E}'_{n-1,n-1}(X)_{\mathbf{R}}$. Furthermore, we have the following fact whose proof we postpone.

LEMMA 4.8. *The space* $D \equiv d_{n-1,n-1}\, \mathscr{E}'_{2n-1}(X)_{\mathbf{R}}$ *is weakly closed.*

By the hypothesis (4.7) we know that $D \cap B = \varnothing$. Hence, by the Hahn-Banach separation theorem (see Schaeffer [12, p. 65]), there is an element $\Omega \in \mathscr{E}^{n-1,n-1}(X)_{\mathbf{R}}$ such that

(i) $\langle \Omega, T \rangle = 0$ for all $T \in D$,

(ii) $\langle \Omega, T \rangle > 0$ for all non-zero $T \in \mathscr{P}_{n-1,n-1}(X)$.

Condition (i) means that $(dS)(\Omega) = S(d\Omega) = 0$ for all $S \in \mathscr{E}'_{2n-1}(X)_{\mathbf{R}}$, and therefore,

(i)′ $d\Omega = 0$.

To interpret conditon (ii) we fix $x \in X$ and let $\xi = \xi_L \in P_x^{n-1,n-1}$ be a simple vector corresponding to a complex hyperplane $L \subset T_x X$. We define a current $T \in \mathscr{P}_{n-1,n-1}(X)$ by setting $T(\varphi) = \varphi(\xi_L)$. Condition (ii) implies that $T(\Omega) = \Omega(\xi_L) > 0$ for all such L; that is, in standard terminology,

(ii)′ Ω is a strictly positive $(n-1, n-1)$-form.

We now observe that a strictly positive $(n-1, n-1)$-form Ω can be written as

$$\Omega = \omega^{n-1} \tag{4.8}$$

where ω is a strictly positive $(1,1)$-form. To see this it suffices to observe that the $(n-1)$st power map: $\varphi \to \varphi^{n-1}$, from $\Lambda^2 T_x^* X$ to $\Lambda^{2n-2} T_x^* X$, carries the cone of strictly positive $(1,1)$-forms bijectively onto the cone of strictly positive $(n-1, n-1)$-forms at each point $x \in X$.

This is simply a question of multi-linear algebra. Let (V, J) be a complex vector space of complex dimension n, and for convenience introduce on V a real, J-invariant inner product. Then for every real (1, 1)-form φ there exists a "unitary" orthonormal basis $e_1, Je_1, \ldots, e_n, Je_n$ and real numbers $\{\lambda_j\}$ so that

$$\varphi = \sum_{j=1}^{n} \lambda_j e_j \wedge Je_j. \tag{4.9}$$

The form φ is strictly positive if and only if $\lambda_j > 0$ for each j. In a similar fashion each $(n-1, n-1)$-form Φ has a diagonalization:

$$\Phi = \sum_{j=1}^{n} \Lambda_j e_1 \wedge Je_1 \wedge \ldots \wedge \widehat{e_j \wedge Je_j} \wedge \ldots \wedge e_n \wedge Je_n \tag{4.10}$$

in some unitary basis, and Φ is strictly positive if and only if $\Lambda_j > 0$ for each j. We now observe that given a strictly positive φ as in (4.9), the form $\Phi = (1/(n-1)!)\varphi^{n-1}$ is expressed as in (4.10) with the same basis and with

$$\Lambda_j = \frac{\lambda_1 \ldots \lambda_n}{\lambda_j} \tag{4.11}$$

for each j. The injectivity is now clear. Furthermore, setting $\Lambda = \lambda_1 \ldots \lambda_n$, we see that $\Lambda_1 \ldots \Lambda_n = \Lambda^{n-1}$. Consequently, given poitive Λ_j's we set

$$\lambda_j = \frac{(\Lambda_1 \ldots \Lambda_n)^{1/(n-1)}}{\Lambda_j}$$

and thereby construct a positive preimage. Hence, the map $\varphi \to \varphi^{n-1}$ is a bijection between strictly positive cones as claimed.

The form ω given by (4.8) is the Kähler form of a unique hermitian metric g on X. Since $d(\omega^{n-1}) = 0$, we know from Theorem 1.6 that g is a balanced metric, and our argument is concluded.

Note. This type of application of the Hahn-Banach theorem was first made by Sullivan [13].

Proof of Lemma 4.8. The operator $d_{n-1,n-1}$: $\mathscr{E}'_{2n-1}(X)_{\mathbf{R}} \to \mathscr{E}'_{n-1,n-1}(X)_{\mathbf{R}}$ is the adjoint of the operator d: $\mathscr{E}^{n-1,n-1}(X)_{\mathbf{R}} \to \mathscr{E}^{2n-1}(X)_{\mathbf{R}}$, and by [12, Chapter IV, § 7] one of these operators has closed range if and only if the other does. We recall that when the range of a continuous linear operator has finite codimension, it is a closed subspace. Thus, $B \equiv d\mathscr{E}^{2n-2}(X)_{\mathbf{R}}$ is closed in $Z \equiv \mathscr{E}^{2n-1}(X)_{\mathbf{R}} \cap \ker(d)$, and therefore B is closed in $\mathscr{E}^{2n-1}(X)_{\mathbf{R}}$. (Recall that since X is compact, all cohomology groups are finite dimensional.) Furthermore, if we can prove that $B_0 \equiv d(\mathcal{E}^{n-1,n-1}(X)_{\mathbf{R}})$ has finite codimension in B, then B_0 will be closed in B (hence in $\mathcal{E}^{2n-1}(X)_{\mathbf{R}}$), and we will be done.

Every element φ in $\mathcal{E}^{2n-2}(X)_{\mathbf{R}}$ can be written uniquely as

$$\varphi = x+\varphi_0+\bar{x}$$

for $x\in\mathcal{E}^{n,n-2}(X)$ (and $\bar{x}\in\mathcal{E}^{n-2,n}(X)$) and $\varphi_0\in\mathcal{E}^{n-1,n-1}(X)_{\mathbf{R}}$. Since the Dolbeault group $\mathcal{H}^{n-2,n}(X)$ is finite dimensional, there is a finite dimensional subspace $V\subset\mathcal{E}^{n-2,n}(X)$ such that every $x\in\mathcal{E}^{n-2,n}$ can be written as

$$x = x_0+\bar{\partial}a$$

for $x_0\in V$ and for some $a\in\mathcal{E}^{n-2,n-1}(X)$. Since $d=\partial+\bar{\partial}$, we see that

$$\varphi = (x_0+\bar{\partial}a)+\varphi_0+(\bar{x}_0+\partial\bar{a}) = (x_0+\hat{\varphi}_0+\bar{x}_0)+d(a+\bar{a}) \tag{4.12}$$

where $\hat{\varphi}_0=\varphi_0-(\partial a+\bar{\partial}\bar{a})\in\mathcal{E}^{n-1,n-1}(X)_{\mathbf{R}}$. It follows immediately from (4.12) that $d\varphi=d\hat{\varphi}_0+d(x_0+\bar{x}_0)$, and therefore $B=B_0+d(V_R)$. In particular, we conclude that

$$\dim_{\mathbf{R}}(B/B_0)\leqslant\dim_{\mathbf{R}}(V)<\infty,$$

and the proof is complete.

5. Families of varieties over a curve

In this section we present a method for inductively establishing the conditions of being balanced. The theorem concerns families of varieties and should be useful in the study of deformations and moduli spaces. Roughly speaking, the result says that a compact manifold which admits a holomorphic map onto a curve such that the fibres are balanced, is itself balanced. However, this statement as it stands is not completely true. For example, the Hopf surface $S^1\times S^3$ admits a holomorphic submersion onto $\mathbf{P}^1(\mathbf{C})=S^2$ (with Kähler fibres), but is not balanced. It is not balanced because, as pointed out in § 1, the fibres of the map are homologically trivial.

This restriction is general. Suppose X is a compact complex n-manifold which admits a holomorphic map $f: X\to C$ onto a curve C. Assume for simplicity that the fibres of f are irreducible. Then each of the fibres $f^{-1}(p), p\in C$, is a positive d-closed $(n-1,n-1)$-current. Consequently, if X is balanced, no fibre can be homologous to zero. In fact, no fibre can be the $(n-1,n-1)$-part of a boundary in X. When this last condition is satisfied, we say that the map f is *essential*. If f satisfies the weaker condition that no fibre is homologous to zero, we say that f is *topologically essential*.

Before proceeding to the main result we take some time to examine these conditions.

LEMMA 5.1. *Let $\omega_C \in H^2(C;\mathbf{R})$ denote the fundamental class of the complex curve C. Then the map $f: X \to C$ is topologically essential if and only if $f^*\omega_C \neq 0$ in $H^2(X;\mathbf{R})$.*

Proof. Let $p \in C$ be any regular value of f, and let $F_p \equiv f^{-1}(p)$ denote the fibre above p. Then the class $[F_p] \in H_{2n-2}(X;\mathbf{R})$ is the Poincaré dual of $f^*\omega_C$. (To see this, represent ω_C by a 2-form ω with support in a small neighborhood of p and with $\int_C \omega = 1$. Then if σ is a smooth cycle transversal to F_p, we see that $\langle \sigma, f^*\omega_C \rangle = \int_\sigma f^*\omega =$ the intersection number of σ and F_p. Thus $[F_p]$ is the Poincaré dual of $f^*\omega_C$.) It therefore follows that $f^*\omega_C \neq 0$ in $H^2(X;\mathbf{R})$ if and only if $[F_p] \neq 0$ in $H_{2n-2}(X;\mathbf{R})$ for all regular values p. Since f is holomorphic and $\dim_{\mathbf{C}}(C)=1$, and since the fibres are irreducible, the conclusion actually holds for *all* $p \in C$. This is because, under these hypotheses, any multiple fibre F_p can be written as a limit $F_p = \lim((1/m)F_{p_j})$ in $\mathcal{P}_{n-1,n-1}(X)$, where m is some positive integer (the multiplicity of the fibre) and where $\{p_j\}_{j=1}^\infty$ is a sequence of regular values converging to p. Consequently, for any multiple fibre F_p of multiplicity m, we have that $m[F_p]=$ the Poincaré dual of $f^*\omega_C$, and the lemma is proved.

Lemma 5.1 can be restated by saying that a map $f: X \to C$ is topologically essential if and only if for some volume form ω on C with $\int_C \omega \neq 0$, we have that $f^*\omega \notin d\mathscr{E}^1(X)_{\mathbf{R}}$, i.e., $f^*\omega \neq da$ for any real 1-form α on X. There is a similar characterization of essential maps. Let $\pi^{1,1}: \mathscr{E}^2(X) \to \mathscr{E}^{1,1}(X)$ denote the standard projection.

LEMMA 5.2. *Let ω be a volume form on C with non-zero integral. Then the map $f: X \to C$ is essential if and only if $f^*\omega \neq \pi^{1,1}d\alpha$ for any real 1-form α on X.*

Proof. Let $p \in C$ be a regular value of f and consider the fibre F_p as a current of bidegree $(1,1)$ (and bidimension $(n-1,n-1)$). The argument for Lemma 5.1 shows that if we renormalize ω to have integral one, the $f^*\omega$ is cohomologous to F_p, i.e.,

$$f^*\omega = F_p + da \tag{5.1}$$

for some real current a of degree 1. Since both $f^*\omega$ and F_p are of bidegree $(1,1)$, we can rewrite (5.1) as

$$f^*\omega = F_p + \pi^{1,1} da. \tag{5.2}$$

Now, if f is not essential, then $F_p = \pi^{1,1} d\alpha$ for some real current α of degree 1. (Recall that "degree 1" means "dimension $2n-1$", and that $\pi^{1,1} = \pi_{n-1,n-1}$.) Hence, we have $f^*\omega = F_p + \pi^{1,1} da = \pi^{1,1}(d\alpha + da) = \pi^{1,1} d(\alpha + a)$. Since the cohomology of currents agrees with that of smooth forms, we see that we can rewrite $d(\alpha+a) = dA$ where A is a smooth 1-form, and so $f^*\omega = \pi^{1,1} dA$.

Conversely, suppose $f^*\omega = \pi^{1,1} dA$ for some smooth 1-form A. Then by (5.2) $F_p = f^*\omega - \pi^{1,1} da = \pi^{1,1} d(A-a)$, and so each non-singular fibre is the degree $(1,1)$-component (or dimension $(n-1, n-1)$-component) of a boundary. However, as observed above, a singular fibre $F_{p'}$ of multiplicity m is homologous over $\mathbf{R}$ to $(1/m) F_p$. Hence, we have that $mF_{p'} = F_p + db = F_p + \pi^{1,1} db$ for some real current b of degree 1. Thus, if one fibre is a $(1,1)$-component of a boundary, then all the fibres are. This completes the proof.

It is clear that "essential" implies "topologically essential". In fact, very often, (and perhaps always) the two conditions are equivalent. For our first result, we consider the spaces $\Gamma(\Omega^p)$ of global holomorphic p-forms on X. These form a complex, $\Omega^\cdot$, under ∂:

$$0 \to \Gamma(\Omega^1) \xrightarrow{\partial} \Gamma(\Omega^2) \xrightarrow{\partial} \Gamma(\Omega^3) \xrightarrow{\partial} \dots$$

whose pth cohomology group we denote by $H^p(\Omega^\cdot)$.

PROPOSITION 5.3. *Let $f\colon X \to C$ be a holomorphic map with irreducible fibres from a compact complex manifold onto a curve. Suppose that either of the following conditions holds*:

(i) $\dim_{\mathbf{C}}(X) = 2$

or

(ii) $H^2(\Omega^\cdot) = 0$.

Then f is essential if and only if it is topologically essential.

Proof. Suppose that f is *not* essential, and fix a volume form ω on C. By Lemma 5.2, we have $f^*\omega = \pi^{1,1} d\mathscr{S}$ for some smooth real 1-form $\mathscr{S}$. If we write $\mathscr{S} = S + \bar{S}$ where $\bar{S} \in \mathscr{E}^{0,1}(X)$, then, since $d = \partial + \bar\partial$, we have that

$$f^*\omega = \bar\partial S + \partial \bar{S}$$

and so

$$f^*\omega - d\mathcal{S} = \varphi + \bar{\varphi} \tag{5.3}$$

where $\varphi \equiv \partial S \in \mathscr{E}^{2,0}(X)$. Since $d(f^*\omega)=0$, it follows from (5.3) and the independence of bidegrees that

$$\partial\varphi = \bar{\partial}\varphi = 0, \tag{5.4}$$

i.e., φ is a ∂-closed holomorphic 2-form. Note that if we change S by a holomorphic 1-form, i.e., if we replace S by $S' = S + \sigma$ where $\bar{\partial}\sigma = 0$, then setting $\mathcal{S}' = S' + \bar{S}'$ we have $f^*\omega - d\mathcal{S}' = \varphi' + \bar{\varphi}'$ where

$$\varphi' = \varphi + \partial\sigma.$$

Consequently, if $H^2(\Omega^\cdot)=0$, we can assume that $\varphi=0$ in equation (5.3).

On the other hand if $\dim_{\mathbf{C}}(X)=2$, then by (5.3) and Stokes' theorem we have that

$$0 = \int_X (f^*\omega)^2 = 2\int_X \varphi \wedge \bar{\varphi}$$

and so $\varphi=0$.

In either case we obtain the fact that $f^*\omega = d\mathcal{S}$, and so f is not topologically essential by Lemma 5.1. This completes the proof.

There is another simple criterion for a map to be essential.

PROPOSITION 5.4. *Let $f: X \to C$ be as in Proposition 5.3. If there exists a holomorphic map $g: C \to X$ such that $f \circ g$ is not constant, then f is essential.*

Proof. Suppose f is not essential. Then equation (5.3) holds, and so, pulling down by g, we have

$$(f\circ g)^*\omega - d(g^*\mathcal{S}) = g^*\varphi + \overline{(g^*\varphi)} = 0,$$

since there are no $(2,0)$-forms on a complex curve. It follows that

$$\deg(f\circ g)\int_C \omega = \int_C (f\circ g)^*\omega = 0,$$

and so the holomorphic map $f\circ g: C \to C$ must be constant, contrary, to assumption.

We now come to the main result of this section. The proof we are about to give is fairly general, so we broaden our definition to cover what is actually established. Given a holomorphic map $f: X \to C$ onto a curve and a point $p \in C$, the fibre $F_p = f^{-1}(p)$ may, of course, be reducible. Thus, in the general situation we say that f is *essential* if no positive linear combination of components of fibres lies in the image of $d_{n-1,n-1}$. That is, no finite sum $\Sigma\, c_j[F_J]$, where $c_j \geq 0$ and F_j is an irreducible component of a fibre of f, can be the $(n-1, n-1)$-component of a boundary. If no such finite sum is a boundary, the map is called *topologically essential.*

Proposition 5.3 continues to hold in this general case, that is, Proposition 5.3 remains true without the assumption that the fibres are irreducible.

Our main theorem is the following.

THEOREM 5.5. *Suppose X is a compact complex (connected) manifold which admits an essential holomorphic map $f: X \to C$ onto a complex curve. C. If the non-singular fibres of f are balanced, then X is balanced.*

Proof. Suppose $\dim_{\mathbf{C}}(X) = n$ and consider $T \in \mathscr{P}_{n-1,n-1}(X)$ with the property that

$$T = \pi_{n-1,n-1}\, dS \tag{5.5}$$

for some $S \in \mathscr{E}'_{2n-1}(X)$. We want to prove that $T=0$.

We shall first use the hypothesis on the fibres by "slicing" the current T. Fix a point $p \in C$ and let z, $|z|<1$, be a local cordinate chart on C with $z(p)=0$. Assume p is a regular value of f and let $\Delta \cong \{|z|<\varepsilon_0\}$ be a sufficiently small disk that $N \equiv f^{-1}(\Delta)$ be a tubular neighborhood of the fiber $F_p = f^{-1}(p)$. We now choose a C^∞ product structure

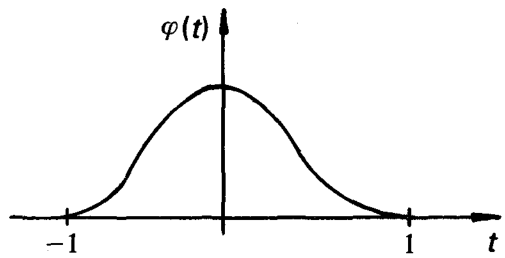
$$g: N \overset{\approx}{\to} \Delta \times F_p \tag{5.6}$$

on this tubular neighborhood with the property that the complex structure makes "infinite order contact with the Δ-factors along $\{0\} \times F_p$". By this we mean the following. Let J denote the almost complex structure on the manifold N and consider J to be carried over to $\Delta \times F$ by the diffeomorphism g. Let J_0 denote the natural product almost complex structure on $\Delta \times F_p$. Then we want the tensor $J - J_0$ to be zero to infinite order at all points of $\{0\} \times F_p$. This can be done by exponentiating the normal bundle of F_p with any hermitian (i.e. $\nabla J = 0$) connection on N.

We consider now an *approximate indentity* on C at the point p. That is, we consider the family of positive $(1,1)$-forms φ_ε on C, given by

$$\varphi_\varepsilon = \frac{i}{\varepsilon^2}\varphi\left(\frac{|z|}{\varepsilon}\right) dz \wedge d\bar{z} \tag{5.7}$$

where $\varphi \in C_0^\infty(-1,1)$ is of the form



and where

$$\int_C \varphi_\varepsilon = 1.$$

We then define the currents

$$\tilde{T}_\varepsilon \equiv f^*\varphi_\varepsilon \wedge T$$
$$\tilde{S}_\varepsilon \equiv f^*\varphi_\varepsilon \wedge S$$

and note that $\tilde{T}_\varepsilon$ is a positive $(n-2, n-2)$-current with compact support in N for all $\varepsilon < \varepsilon_0$. Furthermore, since $df^*\varphi_\varepsilon = f^*d\varphi_\varepsilon = 0$, we see that $d\tilde{S}_\varepsilon = f^*\varphi_\varepsilon \wedge dS$, and since $f^*\varphi_\varepsilon$ is of bidegree $(1,1)$, we have that $\pi_{n-2,n-2}\, d\tilde{S}_\varepsilon = f^*\varphi_\varepsilon \wedge \pi_{n-1,n-1}\, dS = f^*\varphi_\varepsilon \wedge T = \tilde{T}_\varepsilon$. That is,

$$\tilde{T}_\varepsilon = \pi_{n-2,n-2}\, d\tilde{S}_\varepsilon. \tag{5.9}$$

The masses of the currents in the family $\tilde{T}_\varepsilon$ are not necessarily uniformly bounded. Hence, we set

$$m_\varepsilon \equiv \max\{1, \mathbf{M}(\tilde{T}_\varepsilon)\}$$

where $\mathbf{M}(\ )$ denotes mass norm, and define

$$T_\varepsilon = \frac{1}{m_\varepsilon}\tilde{T}_\varepsilon \quad \text{and} \quad S_\varepsilon = \frac{1}{m_\varepsilon}\tilde{S}_\varepsilon.$$

Of course we still have that

$$T_\varepsilon = \pi_{n-2,n-2}\, dS_\varepsilon \tag{5.9$'$}$$

We now observe that the family T_ε consists of positive $(n-2, n-2)$-currents with bounded supports and bounded mass. Hence, by general compactness theorems, see

[4], we know that for any sequence $\varepsilon_m \to 0$, there is a subsequence $\{\varepsilon_{m_j}\}$ such that $T_j \equiv T_{\varepsilon_{m_j}} \to T_\infty$ (weakly) where T_∞ is a positive $(n-2, n-2)$-current with support in F_p. Furthermore, by positivity,

$$\lim_j \mathbf{M}(T_j) = \mathbf{M}(T_\infty) \tag{5.10}$$

(since $\mathbf{M}(T_\varepsilon) = T_\varepsilon(\omega^{n-2}/(n-2)!)$).

LEMMA 5.6. *For any sequence T_j as above, the limit $T_\infty = \lim_j T_j = 0$.*

COROLLARY 5.7. $\lim_{\varepsilon \to 0} T_\varepsilon = 0$.

Proof of Lemma 5.6. Consider that C^∞ retraction $\varrho: N \to F_p$ given by the composition $N \xrightarrow{g} \Delta \times F_p \xrightarrow{\text{proj}} F_p$. Since the currents T_ε and S_ε have compact support in N (for ε small) we can consider the push-forward currents $\varrho_* T_\varepsilon$ and $\varrho_* S_\varepsilon$. Note that $\varrho_* T_\infty = T_\infty$ because $\operatorname{supp} T_\infty \subset F_p$ and $\vec{T}_\infty$ is tangent to F_p at $\|T_\infty\|$-a.a. point. From equation (5.9) we have

$$\begin{aligned} \pi_{n-2,n-2}\varrho_* T_\varepsilon &= \pi_{n-2,n-2}\varrho_* d_{n-2,n-2} S_\varepsilon \\ &= \pi_{n-2,n-2}\varrho_* \Big\{ dS_\varepsilon - \sum_{r \neq s} \pi_{r,s} dS_\varepsilon \Big\} \\ &= \pi_{n-2,n-2} d(\varrho_* S_\varepsilon) + E_\varepsilon \end{aligned} \tag{5.11}$$

where E_ε is a sum of terms of the form $\pi_{n-2,n-2}\varrho_* \pi_{r,s}(dS_\varepsilon)$ for $r \neq s$. (Here $\pi_{r,s}$ denotes projection onto the subspace of (r, s)-currents.)

Suppose we can show that $E_{\varepsilon_j} \to 0$. Then since

$$\varrho_* T_j \to \varrho_* T_\infty = T_\infty$$

and since the subspace of $(n-2, n-2)$-components of boundaries in F_p is closed (Lemma 4.8), we conclude that $T_\infty = \pi_{n-2,n-2} dS$ for some real $(2n-3)$-current S on F_p. Since the fibre F_p is balanced, this implies that $T_\infty = 0$ as claimed.

It now suffices to prove that $\lim_{\varepsilon \to 0} \mathfrak{E}_\varepsilon = 0$ where $\mathfrak{E}_\varepsilon$ is any family of currents of the form

$$\begin{aligned} \mathfrak{E}_\varepsilon &= \pi_{n-2,n-2}\varrho_*(\pi_{r,s} dS_\varepsilon) \\ &= \pi_{n-2,n-2}\varrho_*(\pi_{r+1,s+1} dS \wedge \tilde{\varphi}_\varepsilon) \end{aligned}$$

where $r \neq s$, $r+s=2n-4$, and $\tilde{\varphi}_\varepsilon \equiv f^*\varphi_\varepsilon$. Let ψ be a smooth $(n-2, n-2)$-form on F_p, and let $\tilde{\psi}=\varrho^*\psi$. Then

$$\begin{aligned}\mathfrak{C}_\varepsilon(\psi) &= \pi_{r+1,s+1}(dS)\,[\tilde{\varphi}_\varepsilon \wedge \tilde{\psi}] \\ &= dS[\pi^{r+1,s+1}(\tilde{\varphi}_\varepsilon \wedge \tilde{\psi})] \\ &= dS[\tilde{\varphi}_\varepsilon \wedge (\pi^{r,s}\tilde{\psi})] \\ &= S[\tilde{\varphi}_\varepsilon \wedge d(\pi^{r,s}\tilde{\psi})]\end{aligned}$$

and so it suffices to show that

$$\tilde{\varphi}_\varepsilon \wedge d(\pi^{r,s}\tilde{\psi}) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in the C^∞ topology on N. This is an easy consequence of the following fact.

LEMMA 5.8. *The differential form $\pi^{r,s}\tilde{\psi}$ is zero to infinite order at all points of $F_p \subset N$ (i.e., all coefficients of $\pi^{r,s}\tilde{\psi}$ are zero to infinite order at points of F_p).*

Proof. Note that $\pi^{r,s}\tilde{\psi}=\pi^{r,s}\varrho^*\psi=\pi^{r,s}\varrho^*\pi^{n-2,n-2}\psi$, and so it suffices to note that the operator $\pi^{r,s}\circ\varrho^*\circ\pi^{n-2,n-2}$ (for $r\neq s$) is zero to infinite order along F_p. Let $\pi_0^{n-2,n-2}$ denote the corresponding projection for the "product" complex structure J_0 on $N\cong\Delta\times F_p$. Then we have that $\varrho^*\circ\pi^{n-2,n-2}=\pi_0^{n-2,n-2}\circ\varrho^*$ and so it suffices to show that $\pi^{r,s}\circ\pi_0^{n-2,n-2}$ dies to infinite order at points of F_p (when $r\neq s$). This is now a straightforward consequence of the fact that $J-J_0$ dies to infinite order along F_p. To see this, one should note that $\pi^{r,s}$ is pointwise projection onto the $i(r-s)$-eigenspace of J, extended as a derivation to $\Lambda^*\otimes\mathbf{C}$. In particular, $\pi^{n-2,n-2}$ is just projection onto the kernel of this extended J. Since the extended $J-J_0$ (on $\Lambda^*\otimes\mathbf{C}$) dies to infinite order at points of F_p, the claim follows easily. This completes the proof of Lemmas 5.6 and 5.8.

We have proved that for any approximate identity φ_ε at any regular value $P\in C$, the currents

$$T_\varepsilon \equiv \frac{1}{m_\varepsilon}(T\wedge f^*\varphi_\varepsilon)\to 0$$

on X. From this and from 5.10 we conclude that, in fact, the currents

$$T\wedge f^*\varphi_\varepsilon \to 0$$

in the mass-norm on X. From this we conclude the following. Set

$$X_r = f^{-1}(C-\Sigma)$$

where Σ is the set of non-regular values of f. (Note that by compactness and complex-analyticity the set $\Sigma \subset C$ is finite.) Let ω denote a volume from on C. Then the current

$$T \wedge f^*\omega \equiv 0$$

in X_r. This is equivalent to the statement that $\vec{T}_x = \vec{F}_x =$ the tangent $(n-1, n-1)$-vector to the fibre through x, for $\|T\|$-a.a. x in X_r.

That is,

$$T = \vec{F}\|T\| \quad \text{in } X_r. \tag{5.12}$$

As remarked before, this representation depends to a certain extent on a choice of metric on X. However, if we contract $\vec{T}=\vec{F}$ with $\|T\|$, considered as a $2n$-form on X, we view T as a (generalized) $(1,1)$-form on X with measure coefficients. This representation is canonical (i.e., independent of metrics) and is a standard way to view currents. In this representation the condition (5.12) can be rewritten as

$$T = g(f^*\omega) \quad \text{in } X_r, \tag{5.13}$$

where g is a positive generalized function on X_r.

Now the condition (5.5) can be reexpressed by saying that

$$T = \bar{\partial} S_0 + \partial \bar{S}_0 \tag{5.14}$$

for some $S_0 \in \mathscr{E}'_{n-1,n}(X)$. From (5.14) it is clear that T satisfies the condition

$$\partial\bar{\partial} T = 0. \tag{5.15}$$

Applying this equation to (5.13), we see that the function g is pluriharmonic, and therefore constant, in each fibre. It follows that T has the form

$$T = f^*(\mu) \quad \text{in } X_r.$$

where μ is a positive density on $C-\Sigma$ (i.e., $\mu = m\omega$ where m is a Radon measure on $C-\Sigma$.) We extend μ in the obvious way to all of C, and observe that

$$\hat{T} \equiv T - f^*(\mu)$$

is a positive $(n-1, n-1)$-current on X with support in the complex hypersurface

$\hat{\Sigma} \equiv f^{-1}(\Sigma)$. From (5.15) we see that $\hat{T}$ satisfies the equation

$$\partial\bar{\partial}\hat{T} = 0. \tag{5.16}$$

As a consequence we have the following fact whose proof we postpone.

LEMMA 5.9. *The current $\hat{T}$ is of the form*

$$\hat{T} = \sum c_j[\hat{\Sigma}_j] \tag{5.17}$$

for constants $c_j \geqslant 0$, where $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N$ are the irreducible components of the divisor $\hat{\Sigma}$.

Consequently, we have that

$$T = f^*\mu + \sum c_j[\hat{\Sigma}_j]. \tag{5.18}$$

Now any two densities on C with the same total mass are cohomologous, and so are the corresponding inverse images on X. Hence, if we modify T by an appropriate boundary dS', we have

$$T - dS' = cf^*\omega + \sum c_j[\hat{\Sigma}_j]$$

where $c = \mu(C) \geqslant 0$, and where ω is a smooth volume form on C with $\int_C \omega = 1$. As we saw in the proof of Lemma 5.2, the form $f^*\omega$ is cohomologous to a non-singular fibre F_p $(=f^*(\delta_p\,\omega))$. Consequently,

$$T - dS'' = c[F_p] + \sum c_j[\hat{\Sigma}_j]$$

for constants $c, c_j \geqslant 0$. Since $T - dS''$ is in the image of $d_{n-1,n-1}$, and since the map is essential, we conclude that $c = \mu(C) = 0$ and $c_j = 0$ for all j. It follows that $\mu = 0$, and therefore by (5.18) that $T = 0$. This completes the main argument.

Proof of Lemma 5.9. We follow the proof given in [8] for the case of dimension two. Let x be a regular point of $\hat{\Sigma}$ and choose local coordinates $(z_1, \ldots, z_n)$ in a neighborhood U of x so that $U \cap \hat{\Sigma} \cong \{z : z_1 = 0\}$. Since $\hat{T}$ is a positive current with support in $\hat{\Sigma}$, we can write $\hat{T}$ in the form

$$\hat{T} = \sum \mu_{\alpha\beta}\, i dz_\alpha \wedge d\bar{z}_\beta$$

where the $\mu_{\alpha\beta}$'s are measures supported in $\hat{\Sigma}$, and $\mu_{\alpha\beta} = \bar{\mu}_{\beta\alpha}$. In particular, we can write

$$\hat{T} = \delta_0(z_1) \sum \mu_{\alpha\beta}(z_2, \ldots, z_n)\, idz_\alpha \wedge d\bar{z}_\beta.$$

From the fact that $\partial\bar{\partial}\hat{T}=0$ and $\hat{T}$ is real, we see that

$$\begin{aligned} 0 = &\sum_{\alpha,\beta\geqslant 2} \frac{\partial^2 \delta_0}{\partial z_1\, \partial \bar{z}_1} \mu_{\alpha\beta}\, dz_1 \wedge d\bar{z}_1 \wedge dz_\alpha \wedge d\bar{z}_\beta \\ &+2\,\mathrm{Re}\left\{ \sum_{\alpha,\beta,\gamma\geqslant 2} \frac{\partial \delta_0}{\partial z_1} \frac{\partial \mu_{\alpha\beta}}{d\bar{z}_\gamma} dz_1 \wedge d\bar{z}_\gamma \wedge dz_\alpha \wedge d\bar{z}_\beta \right\} \\ &+\delta_0 \sum_{\alpha,\beta} (\partial\bar{\partial}\mu_{\alpha\beta}) \wedge dz_\alpha \wedge d\bar{z}_\beta. \end{aligned}$$

From the independence of the derivatives of δ_0 we conclude that $\mu_{\alpha\beta}=0$ for all $\alpha,\beta\geqslant 2$ (and $\mu_{\alpha 1}$ is holomorphic for $\alpha\geqslant 2$). Positivity implies that $\mu_{\alpha 1}=\bar{\mu}_{1\alpha}=0$ for $\alpha\geqslant 2$. Hence,

$$\hat{T} = \mu_{11}\, \delta_0(z_1)\, idz_1 \wedge d\bar{z}_1$$

where $\mu_{11}=\mu_{11}(z_1, \ldots, z_n)$ is a non-negative *pluriharmonic function*. This representation holds at all regular points of the divisor $\hat{\Sigma}$. This can be reinterpreted as follows. Let $[\hat{\Sigma}]$ be the current corresponding to integration over the regular points $\mathcal{R}(\hat{\Sigma})$ of $\hat{\Sigma}$. Then away from the singular locus of $\hat{\Sigma}$, we have that

$$\hat{T} = \varphi[\hat{\Sigma}]$$

where φ is a positive pluriharmonic function $\mathcal{R}(\hat{\Sigma})$. It follows (by lifting to a desingularization of $\hat{\Sigma}$) that φ is a constant $c_j\geqslant 0$ on each component $\hat{\Sigma}_j$ of $\hat{\Sigma}$.

The current $\hat{\hat{T}}\equiv\hat{T}-\Sigma\, c_j[\hat{\Sigma}_j]$ is now a positive $(n-1, n-1)$-current supported in the singular locus of $\hat{\Sigma}$ (a variety of dimension $\leqslant n-2$). It is, furthermore, $\partial\bar{\partial}$-closed. Regular points of the singular locus are contained locally in the intersection of two hyperplanes $\{z_1=0\}$ and $\{z_2=0\}$. Applying the argument above to each plane shows that $\hat{\hat{T}}=0$ at such points. Continuing down the singular strata, we conlude that $\hat{\hat{T}}=0$. This completes the proof of the lemma and the theorem.

Our main result has the following corollary which also appears in [8].

COROLLARY 5.8. *A compact complex surface which admits a topologically essential holomorphic map onto a complex curve is Kähler.*

The main Theorem 5.5 can of course be applied inductively to any manifold which decomposes into a sequence of fibrations. A good example of such a manifold is a complex solvmanifold. We shall examine such a variety in the next section.

6. Balanced 3-folds which are not Kähler

This seems a good time to point out that in all dimensions >2, there exist non-Kähler manifolds which are balanced. The first construction here is due to Calabi [2].

His first observation was the following. Let $\mathbf{O}\cong\mathbf{R}^8$ denote the Cayley numbers and consider a smooth oriented hypersurface $M^6 \looparrowright \mathbf{R}^7=\mathrm{Im}(\mathbf{O})$, the imaginary Cayley numbers. Then there is a natural almost complex structure $J: TM\to TM$ induced by Cayley multiplication:

$$J_x(e)\equiv \nu_x\cdot e$$

where ν is the unit normal vector field to M. He shows that this structure is integrable if and only if J anticommutes with the second fundamental form of M.

He then writes $\mathrm{Im}(\mathbf{O})=\mathrm{Im}(\mathbf{H})\oplus\mathbf{H}=\mathbf{R}^3\oplus\mathbf{R}^4$ (where $\mathbf{H}=$ the quaternions), and he considers a hypersurface of the type

$$M^2\times\mathbf{R}^4 \overset{(\psi, Id)}{\looparrowright} \mathbf{R}^3=\mathbf{R}^4.$$

In this case the almost complex structure J is integrable if and only if the immersion $\psi: M^2\looparrowright\mathbf{R}^3$ is minimal. The complex structure so obtained is invariant by all translations in the $\mathbf{R}^4$-directions.

Calabi then constructs compact complex manifolds as follows. Let C be a compact Riemann surface which admits 3 holomorphic differentials $\varphi_1, \varphi_2, \varphi_3$ such that

$$\sum \varphi_j^2=0 \quad \text{and} \quad \sum |\varphi_j|^2>0.$$

Then lifting $\varphi\equiv(\varphi_1, \varphi_2, \varphi_3)$ to the universal covering $\tilde{C}\to C$, we obtain a conformal minimal immersion $\psi: \tilde{C}\to\mathbf{R}^3$ by setting

$$\psi(z)=\mathrm{Re}\left\{\int^z \varphi\right\}$$

If $g: \tilde{C}\to\tilde{C}$ denotes a covering transformation, then $\psi(gz)=\psi(z)+t_g$ for some vector $t_g\in\mathbf{R}^3$. It follows that the complex structure induced on $\tilde{C}\times\mathbf{R}^4$ by the immersion $\psi\times\mathrm{Id}$ is invariant by the covering group of C and so descends to $C\times\mathbf{R}^4$. We can further divide by a lattice Λ of translations of $\mathbf{R}^4$, and thereby produce a compact manifold X_Λ which admits a holomorphic projection $f: X_\Lambda\to C$. The fibres of this map are complex tori. Furthermore, since the natural inclusions $C\times\{x\}\hookrightarrow C\times\mathbf{R}^4$ are holomorphic (for any

$x\in\mathbf{R}^4$), we see that the bundle $f\colon X\to C$ has holomorphic cross-sections. Thus, we conclude from Theorem 5.5 that

$$\textit{The manifolds } X_\Lambda \textit{ are balanced.} \tag{6.1}$$

This will also be true of $X_\Lambda\times Y$ for any Kähler manifold Y.

However, Calabi proves in [2] that the manifolds X_Λ cannot be Kähler. His argument applies equally well to $X_\Lambda\times T$ where T is any complex torus, and thus provides examples in all dimensions.

Note. Al Gray has proved by direct computation [6] that the manifolds X_Λ carry balanced matrics.

An interesting collection of examples suggested by the referee arises from Penrose's twistor spaces. Given an oriented riemannian 4-manifold, X, there is associated a 6-manifold, τ, the socalled twistor space of X. This is an S^2 bundle over X^4 whose fibre, C_x, over a point $x\in X^4$ is the set of all almost complex structure $J\colon T_xX\to T_xX$ such that J is orthogonal and compatible with the orientation. Note that $C_x\cong SO_4/U_2\cong S^2$. The twistor space, τ, has a canonical almost complex structure which is integrable if and only if X^4 is self-dual [1]. The natural metric on τ is balanced, however, Hitchin has shown [9] that the only compact twistor spaces which are Kähler are those associated to S^4 and $P^2(C)$; namely $P^3(C)$ and the manifold of flags in C^3, $F^3(C)$, respectively. An interesting special case is the twistor space for $S^1\times S^3$. The referee points out that this gives a counterexample to the converse of Theorem 5.5. That is, the projection $\tau_{S^1\times S^3}\to S^2$ is a holomorphic map which is essential by Proposition 5.4. However, $\tau_{S^1\times S^3}$ is balanced although the fibres of the map, $S^1\times S^3$, cannot be balanced.

Another natural example, which is also known to Gray, is given by the complex Heisenberg manifold $X=G/\Gamma$ where

$$G=\left\{\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}: a,b,c\in\mathbf{C}\right\}$$

and Γ is the subgroup where a, b, and c are Gaussian integers. The map $f\colon X\to C\equiv\mathbf{C}/\mathbf{Z}[i]$, given by setting

$$f\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}=a$$

is a holomorphic surjection whose fibres are easily seen to be Kähler. The map

$$a \mapsto \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

gives a holomorphic cross-section to f, and so by Theorem 5.5 the manifold X is balanced.

Nevertheless, *this manifold is not Kähler*. This can be seen as follows. Direct calculation shows that

$$[G, G] = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbf{C} \right\}$$

and that $[\Gamma, \Gamma] \subset [G, G]$ is the subgroup with $b \in \mathbf{Z}[i]$. Thus, the exact sequences

$$0 \to [G, G] \to G \to G/[G, G] \to 0$$

$$0 \to [\Gamma, \Gamma] \to \Gamma \to \Gamma/[\Gamma, \Gamma] \to 0$$

give rise to a fibration

$$T \xrightarrow{i} X \xrightarrow{F} T \times T$$

where $T = \mathbf{C}/\mathbf{Z}[i]$ and where $F(a, b, c) = (a, c)$ and $i(b) = (0, b, 0)$. The subgroup $[G, G]$ is central in G, and so the quotient group $T_0 = [G, G]/[\Gamma, \Gamma]$ acts freely on X preserving the fibres of F. (F is a principal fibration.)

We now claim that the fibres of F are homologous to zero in X and so X cannot be Kähler. Note that $\Gamma = \pi_1(X)$, and since $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$, we conclude that the map

$$i_*: H_1(T) \to H_1(X)$$

is zero. Let $\gamma_0 = \mathbf{R}/\mathbf{Z}$ and $\gamma_1 = i\mathbf{R}/\mathbf{Z}$ be the two canonical generators of $\pi_1(T)$. Write $T = \gamma_0 \times \gamma_1$ and note that the inclusion i can be written as

$$i(\gamma_0 \times \gamma_1) = i(\gamma_0) \cdot i(\gamma_1)$$

where $\cdot$ comes from the action described above. Now since $i_* = 0$ on H_1, we see that $i(\gamma_0) = \partial(\Sigma)$ where Σ is a 2-chain in X. It follows immediately that

$$i(\gamma_0) \cdot i(\gamma_1) = \partial(\Sigma)\, i(\gamma_1) = \partial(\Sigma \cdot i(\gamma_1))$$

and so the fibre is null-homologous as claimed.

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