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Abstract We provide simple necessary and sufficient conditions for the existence of a standard Young tableau of a given shape and major index r mod n, for all r. Our result generalizes the $r = 1$ case due essentially to Klyachko [11] and proves a recent conjecture due to Sundaram [32] for the $r = 0$ case. A byproduct of the proof is an asymptotic equidistribution result for "almost all" shapes. The proof uses a representation-theoretic formula involving Ramanujan sums and normalized symmetric group character estimates. Further estimates involving "opposite" hook lengths are given which are well-adapted to classifying which partitions $\lambda \vdash n$ have $f^\lambda \leqslant n^d$ for fixed d. We also give a new proof of a generalization of the hook length formula due to Fomin-Lulov [4] for symmetric group characters at rectangles. We conclude with some remarks on unimodality of symmetric group characters.

1. Introduction

We assume basic familiarity with the combinatorics of Young tableaux and the representation theory of the symmetric group. For further information and definitions, see [6], [28], or [22].

Let $\lambda \vdash n$ be an integer partition of size n, and let $\mathrm{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ. We write λ' for the transpose (or conjugate) of λ. Let $\operatorname{maj} T$ denote the major index of $T \in \mathrm{SYT}(\lambda)$. We are chiefly interested in the counts

$$a_{\lambda,r} \coloneqq \#\{T \in \mathrm{SYT}(\lambda) : \operatorname{maj} T \equiv_n r\}$$

where r is taken mod n. To avoid giving undue weight to trivial cases, we take $n \geqslant 1$ throughout. Work due to Klyachko and, later, Kraśkiewicz–Weyman, gives the following.

Theorem 1.1 ([11, Proposition 2], [14]). *Let $\lambda \vdash n$ and $n \geqslant 1$. The constant $a_{\lambda,1}$ is positive except in the following cases, when it is zero:*

- *$\lambda = (2,2)$ or $\lambda = (2,2,2)$;*
- *$\lambda = (n)$ when $n > 1$; or $\lambda = (1^n)$ when $n > 2$.*

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Indeed, the counts $a_{\lambda,r}$ can be interpreted as irreducible multiplicities as follows, a result originally due to Kraśkiewicz–Weyman. Let C_n be the cyclic group of order n generated by the long cycle $\sigma_n \coloneqq (12\cdots n) \in S_n$, let S^λ be the Specht module of shape $\lambda \vdash n$, and let $\chi^r \colon C_n \to \mathbb{C}^\times$ be the irreducible representation given by $\chi^r(\sigma_n^i) \coloneqq \omega_n^{ri}$ where ω_n is a fixed primitive nth root of unity and $r \in \mathbb{Z}/n$. Let $\langle -, - \rangle$ denote the standard scalar product for complex representations.

Theorem 1.2 (see [14, Theorem 1]). *With the above notation, we have*

$$\langle S^\lambda, \chi^r\uparrow_{C_n}^{S_n}\rangle = a_{\lambda,r} = \langle \chi^r, S^\lambda\downarrow_{C_n}^{S_n}\rangle.$$

Moreover, $a_{\lambda,r}$ depends only on λ and $\gcd(n, r)$.

Remark 1.3. Kraśkiewicz-Weyman gave the first equality in Theorem 1.2, and the second follows by Frobenius reciprocity. Klyachko [11, Proposition 2] actually determined which S^λ contain faithful representations of C_n in agreement with Theorem 1.1. One may see through a variety of methods that $\chi^r\uparrow_{C_n}^{S_n}$ depends up to isomorphism only on $\gcd(r, n)$.

The manuscript [14] was long-unpublished, the delay being largely due to Klyachko having already given a significantly more direct proof of their main application, relating $\chi^1\uparrow_{C_n}^{S_n}$ to free Lie algebras, though we have no need of this connection. For a more modern and unified account of these results, see [20, Theorems 8.8-8.12].

The following recent conjecture due to Sundaram was originally stated in terms of the multiplicity of S^λ in $1\uparrow_{C_n}^{S_n}$.

Conjecture 1.4 ([32]). . *Let $\lambda \vdash n$ and $n \geqslant 1$. Then $a_{\lambda,0}$ is positive except in the following cases, when it is zero: $n > 1$ and*

- $\lambda = (n-1, 1)$
- $\lambda = (2, 1^{n-2})$ *when n is odd*
- $\lambda = (1^n)$ *when n is even.*

Conjecture 1.4 is the $r = 0$ case of the following theorem, which is our main result.

Theorem 1.5. *Let $\lambda \vdash n$ and $n \geqslant 1$. Then $a_{\lambda,r}$ is positive except in the following cases, when it is zero: $n > 1$ and*

- $\lambda = (2, 2)$, $r = 1, 3$; *or* $\lambda = (2, 2, 2)$, $r = 1, 5$; *or* $\lambda = (3, 3)$, $r = 2, 4$;
- $\lambda = (n-1, 1)$ *and* $r = 0$;
- $\lambda = (2, 1^{n-2})$, $r = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even;} \end{cases}$
- $\lambda = (n)$, $r \in \{1, \ldots, n-1\}$;
- $\lambda = (1^n)$, $r \in \begin{cases} \{1, \ldots, n-1\} & \text{if } n \text{ is odd} \\ \{0, \ldots, n-1\} - \{\frac{n}{2}\} & \text{if } n \text{ is even.} \end{cases}$

Equivalently, using Theorem 1.2, every irreducible representation appears in each $\chi^r\uparrow_{C_n}^{S_n}$ or $S^\lambda\downarrow_{C_n}^{S_n}$ except in the noted exceptional cases.

M. Johnson [9] gave an alternative proof of Klyachko's result, Theorem 1.1, involving explicit constructions with standard tableaux. Kovács–Stöhr [13] gave a different proof using the Littlewood–Richardson rule which also showed that $a_{\lambda,1} > 1$ implies $a_{\lambda,1} \geqslant \frac{n}{6} - 1$. Our approach is instead based on normalized symmetric group character estimates. It has the benefit of yielding both more general and vastly more precise estimates for $a_{\lambda,r}$.

Our starting point is the following character formula. See Section 3 for further discussion of its origins and a generalization. Let $\chi^\lambda(\mu)$ denote the character of S^λ at a permutation of cycle type μ. We write $\ell^{n/\ell}$ for the rectangular partition $(\ell, \ldots, \ell)$ with ℓ columns and n/ℓ rows. Write $f^\lambda \coloneqq \chi^\lambda(1^n) = \dim S^\lambda = \#\,\mathrm{SYT}(\lambda)$.

Theorem 1.6. *Let $\lambda \vdash n$ and $n \geqslant 1$. For all $r \in \mathbb{Z}/n$,*

$$\frac{a_{\lambda,r}}{f^\lambda} = \frac{1}{n} + \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell \neq 1}} \frac{\chi^\lambda(\ell^{n/\ell})}{f^\lambda} c_\ell(r)$$

where

$$c_\ell(r) \coloneqq \mu\left(\frac{\ell}{\gcd(\ell, r)}\right) \frac{\phi(\ell)}{\phi(\ell/\gcd(\ell, r))}$$

is a Ramanujan sum, μ is the classical Möbius function, and ϕ is Euler's totient function.

We estimate the quotients in the preceding formula using the following result due to Fomin and Lulov.

Theorem 1.7 ([4, Theorem 1.1]). *Let $\lambda \vdash n$ where $n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| \leqslant \frac{s!\ell^s}{(n!)^{1/\ell}} (f^\lambda)^{1/\ell}.$$

The character formula in Theorem 1.6 and the Fomin-Lulov bound are combined below to give the following asymptotic uniform distribution result.

Theorem 1.8. *For all $\lambda \vdash n \geqslant 1$ and all r,*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| \leqslant \frac{2n^{3/2}}{\sqrt{f^\lambda}}. \tag{1}$$

In Section 4 we use "opposite hook lengths" to give a lower bound for f^λ, Corollary 4.13. These bounds, together with a somewhat more careful analysis involving the character formula, Stirling's approximation, and the Fomin-Lulov bound, are used to deduce both our main result, Theorem 1.5, and the following more explicit uniform distribution result.

Theorem 1.9. *Let $\lambda \vdash n$ be a partition where $f^\lambda \geqslant n^5 \geqslant 1$. Then for all r,*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^2}.$$

In particular, if $n \geqslant 81$, $\lambda_1 < n - 7$, and $\lambda'_1 < n - 7$, then $f^\lambda \geqslant n^5$ and the inequality holds.

Indeed, the upper bound in Theorem 1.9 is quite weak and is intended only to convey the flavor of the distribution of $(a_{\lambda,r})_{r=0}^{n-1}$ for fixed λ. One may use Roichman's asymptotic estimate [21] of $|\chi^\lambda(\ell^s)|/f^\lambda$ to prove exponential decay in many cases. Moreover, one typically expects f^λ to grow super-exponentially, i.e. like $(n!)^\epsilon$ for some $\epsilon > 0$ (see [16] for some discussion and a more recent generalization of Roichman's result), which in turn would give a super-exponential decay rate in Theorem 1.9. We have no need for such explicit, refined statements and so have not pursued them further.

Theorem 1.7 is based on the following generalization of the hook length formula (the $\ell = 1$ case), which seems less well-known than it deserves. We give an alternate proof of Theorem 1.10 in Section 5 along with further discussion. A *ribbon* is a connected skew shape with no 2×2 rectangles. For $\lambda \vdash n$, write $c \in \lambda$ to mean that c is a cell in λ. Further write h_c for the *hook length* of c and write $[n] \coloneqq \{1, 2, \ldots, n\}$.

Theorem 1.10 ([8, 2.7.32]; see also [4, Corollary 2.2]). *Let $\lambda \vdash n$ where $n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| = \frac{\prod_{\substack{i \in [n] \\ i \equiv_\ell 0}} i}{\prod_{\substack{c \in \lambda \\ h_c \equiv_\ell 0}} h_c} \tag{2}$$

whenever λ can be written as s successive ribbons of length ℓ (i.e. whenever the ℓ-core of λ is empty), and 0 *otherwise.*

Other work on q-analogues of the hook length formula has focused on algebraic generalizations and variations on the hook walk algorithm rather than evaluations of symmetric group characters. For instance, an application of Kerov's q-analogue of the hook walk algorithm [10] was to prove a recursive characterization of the right-hand side of (5) below. See [2, §6] for a relatively recent overview of literature in this direction.

The rest of the paper is organized as follows. In Section 2, we recall earlier work. In Section 3 we discuss and generalize Theorem 1.6. In Section 4, we use symmetric group character estimates and a new estimate involving "opposite hook products," Proposition 4.5, to deduce our main results, Theorem 1.5 and Theorem 1.9. We give an alternative proof of Theorem 1.10 in Section 5. In Section 6, we briefly discuss unimodality of symmetric group characters in light of Proposition 4.5.

2. Background

Here we review objects famously studied by Springer [27, (4.5)] and Stembridge [31] and give further background for use in later sections. All representations will be finite-dimensional over $\mathbb{C}$.

Continuing our earlier notation, $\lambda \vdash n$ is a partition of size n, $\mathrm{SYT}(\lambda)$ is the set of standard Young tableaux of shape λ and which has cardinality f^λ, $(12\cdots n)$ is the long cycle in the symmetric group S_n, S^λ is the irreducible S_n-module (Specht module) of shape λ with character at an element of cycle type μ given by $\chi^\lambda(\mu)$, $c \in \lambda$ denotes a cell in the Ferrers diagram of λ, and h_c denotes the hook length of that cell.

Let G be a finite group, $g \in G$ a fixed element of order n, M a finite dimensional G-module, and ω_n a fixed primitive nth root of unity. Suppose $\{\omega_n^{e_1}, \omega_n^{e_2}, \ldots\}$ is the multiset of eigenvalues of g acting on M. The multiset $\{e_1, e_2, \ldots\}$ lists the *cyclic exponents* of g on M; these integers are well-defined mod n. Following [31], define the corresponding "modular" generating function as

$$P_{M,g}(q) \coloneqq q^{e_1} + q^{e_2} + \cdots \qquad (\mathrm{mod}\ (q^n - 1)).$$

Write $\chi^M(g)$ to denote the character of M at g. Note that

$$P_{M,g}(\omega_n^s) = \chi^M(g^s), \tag{3}$$

so that for instance $P_{M,g}(q)$ depends only on the conjugacy class of g. When $G = S_n$ and $g \in S_n$ has cycle type $\mu \vdash n$, we write $P_{M,\mu}(q) \coloneqq P_{M,g}(q)$.

Theorem 2.1 (see [31, Theorem 3.3] and [14]). *Let $\lambda \vdash n$. The cyclic exponents of $(12\cdots n)$ on S^λ are the major indices of* $\mathrm{SYT}(\lambda)$*, mod n, and*

$$\begin{aligned} P_{S^\lambda,(n)}(q) &\equiv \sum_{T \in \mathrm{SYT}(\lambda)} q^{\operatorname{maj} T} \\ &\equiv \sum_{r \mid n} a_{\lambda,r} \left(\sum_{\substack{1 \leqslant i \leqslant n \\ \gcd(i,n)=r}} q^i \right) \qquad (\textit{mod } (q^n-1)). \end{aligned} \tag{4}$$

Remark 2.2. Stembridge gave the first equality in Theorem 2.1. Equality of the first and third terms follows immediately from Kraśkiewicz-Weyman's work using Theorem 1.2 and the observation that the multiplicity of χ^r in $S^\lambda\!\downarrow^{S_n}_{C_n}$ is the number of times r appears as a cyclic exponent of $(12\cdots n)$ in S^λ.

We also recall Stanley's q-analogue of the hook length formula.

Theorem 2.3 ([28, 7.21.5]). *Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then*

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\operatorname{maj}(T)} = \frac{q^{d(\lambda)}[n]_q!}{\prod_{c \in \lambda} [h_c]_q} \tag{5}$$

where $d(\lambda) \coloneqq \sum (i-1)\lambda_i$, $[n]_q! \coloneqq [n]_q[n-1]_q \cdots [1]_q$, and $[a]_q \coloneqq 1 + q + \cdots + q^{a-1} = \frac{q^a-1}{q-1}$.

The representation-theoretic interpretation of the coefficients $a_{\lambda,r}$ in Theorem 1.2 is related to the following result due independently to Lusztig (unpublished) and Stanley. We record it to give our results context, though it will not be used in our present work. For $\lambda \vdash n$ and $i \in \mathbb{Z}$, define

$$b_{\lambda,i} \coloneqq \#\{T \in \mathrm{SYT}(\lambda) : \operatorname{maj} T = i\}$$

so that $\sum_{k \in \mathbb{Z}} b_{\lambda, i+kn} = a_{\lambda,i}$.

Theorem 2.4 ([29, Proposition 4.11]). *Let $\lambda \vdash n$. The multiplicity of S^λ in the ith graded piece of the type A_{n-1} coinvariant algebra is $b_{\lambda,i}$.*

Indeed, the second equality in Theorem 1.2 follows from Theorem 2.4 and [27, Prop. 4.5]. See also [1, p. 3059] for a more recent refinement of Theorem 2.4 and some further discussion.

Finally, we have need of the so-called Ramanujan sums.

Definition 2.5. *Given $j \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$, the corresponding Ramanujan sum is*

$$c_j(s) \coloneqq \textit{the sum of the } s\textit{th powers of the primitive } j\textit{th roots of unity.}$$

For instance, $c_4(2) = i^2 + (-i)^2 = -2 = \mu(4/2)\phi(4)/\phi(2)$. The equivalence of this definition of $c_j(s)$ and the formula in Theorem 1.6 is classical and was first given by Hölder; see [12, Lemma 7.2.5] for a more modern account. These sums satisfy the well-known relation

$$\sum_{v \mid n} c_v(n/s) c_r(n/v) = \begin{cases} n & r = s \\ 0 & r \neq s \end{cases} \tag{6}$$

for all $s, r \mid n$ [12, Lemma 7.2.2].

3. Generalizing the Character Formula

In this section we discuss Theorem 1.6 and present a straightforward generalization. We begin with a proof of Theorem 1.6 similar to but different from that in [3]. It is included chiefly because of its simplicity given the background in Section 2 and because part of the argument will be used below in Section 5.

Proof of Theorem 1.6. Pick $s \mid n$, so $(12\cdots n)^s$ has cycle type $((n/s)^s)$. Evaluating (4) at $q = \omega_n^s$ gives

$$\chi^\lambda((n/s)^s) = P_{S^\lambda,(n)}(\omega_n^s) = \sum_{r|n} a_{\lambda,r} c_{n/r}(s) \tag{7}$$

since $(\omega_n^s)^i = (\omega_n^i)^s$ and ω_n^i is a primitive $n/\gcd(i,n)$th root of unity. Equation (7) gives a system of linear equations, one for each s such that $s \mid n$, and with variables $a_{\lambda,r}$ for each $r \mid n$. The coefficient matrix is $C \coloneqq (c_{n/r}(s))_{s|n,r|n}$. For example, the $s = n$ linear equation reads

$$f^\lambda = \chi^\lambda(1^n) = \sum_{r|n} a_{\lambda,r}\phi(n/r),$$

which follows immediately from the fact that $f^\lambda = \sum_{r=0}^{n-1} a_{\lambda,r}$ and that $a_{\lambda,r}$ depends only on $\gcd(r,n)$.

As it happens, the coefficient matrix C is nearly its own inverse. Precisely,

$$(c_{n/r}(s))^2_{s|n,r|n} = n\,I, \tag{8}$$

where I is the identity matrix with as many rows as positive divisors of n. It is easy to see that (8) is equivalent to the identity (6) above. Using (8) to invert (7) gives

$$a_{\lambda,r} n = \sum_{s|n} \chi^\lambda((n/s)^s) c_{n/s}(r).$$

For the $s = n$ term, we have $c_1(r) = 1$ and $\chi^\lambda(1^n) = f^\lambda$. Tracking this term separately, dividing by n and replacing s with $\ell \coloneqq n/s$ now gives Theorem 1.6, completing the proof. $\square$

Variations on Theorem 1.6 have appeared in the literature numerous times in several guises, sometimes implicitly (see [3, Théorème 2.2], [11, (7)], or [28, 7.88(a), p. 541]). In this section we write out a precise and relatively general version of these results which explicitly connects Theorem 1.6 to the well-known corresponding symmetric function expansion due to H. O. Foulkes. Let Ch denote the Frobenius characteristic map and let p_λ denote the power symmetric function indexed by the partition λ.

Theorem 3.1 ([5, Theorem 1]). *Suppose $\lambda \vdash n \geqslant 1$ and $r \in \mathbb{Z}/n$. Then*

$$\mathrm{Ch}\,\chi^r\!\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{\ell|n} c_\ell(r) p_{(\ell^{n/\ell})}. \tag{9}$$

The following straightforward result, essentially implicit in [28, 7.88(a), p. 541], connects and generalizes Theorem 3.1 and Theorem 1.6.

Theorem 3.2. *Let H be a subgroup of S_n and let M be a finite-dimensional H-module with character $\chi^M \colon H \to \mathbb{C}$. Then*

$$\mathrm{Ch}\,M\!\uparrow_H^{S_n} = \frac{1}{|H|}\sum_{\mu\vdash n} c_\mu p_\mu \tag{10}$$

and, for all $\lambda \vdash n$,

$$\langle M\uparrow_H^{S_n}, S^\lambda\rangle = \frac{1}{|H|}\sum_{\mu\vdash n} c_\mu \chi^\lambda(\mu), \tag{11}$$

where

$$c_\mu := \sum_{\substack{h\in H\\ \tau(h)=\mu}} \chi^M(h)$$

and $\tau(\sigma)$ denotes the cycle type of the permutation σ.

Proof. Write $N := M\uparrow_H^{S_n}$. By definition (see [28, p. 351]),

$$\operatorname{Ch} N = \sum_{\mu\vdash n} \frac{\chi^N(\mu)}{z_\mu} p_\mu \tag{12}$$

where z_μ is the order of the stabilizer of any permutation of cycle type μ under conjugation. From the induced character formula (see [24, 7.2, Prop. 20]), we have

$$\chi^N(\sigma) = \frac{1}{|H|} \sum_{\substack{a\in S_n\\ \text{s.t. } a\sigma a^{-1}\in H}} \chi^M(a\sigma a^{-1}).$$

Say $\tau(\sigma) = \mu$. Each $a\sigma a^{-1} = h \in H$ with $\tau(h) = \mu$ appears in the preceding sum z_μ times, since σ and h are conjugate and z_μ is also the number of ways to conjugate any fixed permutation with cycle type μ to any other fixed permutation with cycle type μ. Hence

$$\chi^N(\mu) = \frac{1}{|H|} \sum_{\substack{h\in H\\ \tau(h)=\mu}} z_\mu \chi^M(h). \tag{13}$$

Equation (10) now follows from (12) and (13). Equation (11) follows from (10) in the usual way using the fact (see [28, (7.76)]) that $p_\mu = \sum_\lambda \chi^\lambda(\mu) s_\lambda$. □

Note that (10) specializes to Theorem 3.1 and (11) specializes to Theorem 1.6 when $M = \chi^r$. In that case, the only possibly non-zero c_μ arise from $\mu = (\ell^{n/\ell})$ for $\ell \mid n$.

One may consider analogues of the counts $a_{\lambda,r}$ obtained by inducing other one-dimensional representations of subgroups of S_n. Motivated by the study of so-called higher Lie modules, there is a natural embedding of reflection groups $C_a \wr S_b \hookrightarrow S_{ab}$. A classification analogous to Klyachko's result, Theorem 1.1, was asserted for $b = 2$ by Schocker [23, Theorem 3.4], though the "rather lengthy proof" making "extensive use of routine applications of the Littlewood-Richardson rule and some well-known results from the theory of plethysms" was omitted. By contrast, our approach using Theorem 3.2 may be pushed through in this case using an appropriate generalization of the Fomin-Lulov bound, such as [16, Theorem 1.1], resulting in analogues of Theorem 1.5 and Theorem 1.9. Our approach begins to break down when b is large relative to $n = ab$ and (11) has many terms. However, we have no current need for such generalizations and so have not pursued them further.

4. Proof of the Main Results

We now turn to the proofs of Theorem 1.5, Theorem 1.8, and Theorem 1.9. We begin by combining the Fomin–Lulov bound and Stirling's approximation, which quickly gives Theorem 1.8. We then use somewhat more careful estimates to give a sufficient condition, $f^\lambda \geqslant n^3$, for $a_{\lambda,r} \neq 0$. Afterwards we give an inequality between hook length products and "opposite" hook length products, Proposition 4.5, from which we classify λ for which $f^\lambda < n^3$. Theorem 1.5 follows in almost all cases, with the remainder being handled by brute force computer verification and case-by-case analysis. Theorem 1.9 will be similar, except the bound $f^\lambda < n^5$ will be used.

Lemma 4.1. *Suppose $\lambda \vdash n = \ell s$. Then*

$$\ln \frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \left(1 - \frac{1}{\ell}\right) \left[\frac{1}{2} \ln n - \ln f^\lambda + \ln \sqrt{2\pi}\right] + \frac{\ell}{12n} - \frac{1}{2} \ln \ell. \tag{14}$$

Proof. We apply the following version of Stirling's approximation [26, (1.53)]. For all $m \in \mathbb{Z}_{>0}$,

$$\left(m + \frac{1}{2}\right) \ln m - m + \ln \sqrt{2\pi} \leqslant \ln m! \leqslant \left(m + \frac{1}{2}\right) \ln m - m + \ln \sqrt{2\pi} + \frac{1}{12m}.$$

The Fomin–Lulov bound, Theorem 1.7, gives

$$\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \frac{\frac{n}{\ell}! \ell^{n/\ell}}{(n!)^{1/\ell} (f^\lambda)^{1-1/\ell}}.$$

Combining these gives

$$\begin{aligned}
\ln \frac{|\chi^\lambda(\ell^s)|}{f^\lambda} &\leqslant \ln \left(\frac{n}{\ell}\right)! + \frac{n}{\ell} \ln \ell - \frac{1}{\ell} \ln n! - \left(1 - \frac{1}{\ell}\right) \ln f^\lambda \\
&\leqslant \left(\frac{n}{\ell} + \frac{1}{2}\right) \ln \frac{n}{\ell} - \frac{n}{\ell} + \ln \sqrt{2\pi} + \frac{\ell}{12n} + \frac{n}{\ell} \ln \ell \\
&\quad - \frac{1}{\ell} \left(\left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} \right) - \left(1 - \frac{1}{\ell}\right) \ln f^\lambda \\
&= \frac{1}{2} \ln \frac{n}{\ell} + \ln \sqrt{2\pi} + \frac{\ell}{12n} - \frac{1}{2\ell} \ln n - \frac{\ln \sqrt{2\pi}}{\ell} - \left(1 - \frac{1}{\ell}\right) \ln f^\lambda.
\end{aligned}$$

Rearranging this final expression gives (14). □

We may now prove Theorem 1.8.

Proof of Theorem 1.8. For $2 \leqslant \ell \leqslant n$, applying simple term-by-term estimates to (14) gives

$$\ln \frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \frac{1}{2} \ln n - \frac{1}{2} \ln f^\lambda + \ln \sqrt{2\pi} + \frac{1}{12} - \frac{\ln 2}{2}.$$

Consequently,

$$\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant C \sqrt{\frac{n}{f^\lambda}}$$

where $C = \sqrt{\pi} \exp(1/12) \approx 1.93 < 2$. The Ramanujan sums $c_\ell(r)$ have the trivial bound $|c_\ell(r)| \leqslant \ell \leqslant n$. The estimate in Theorem 1.8 now follows immediately from Theorem 1.6. □

Lemma 4.2. *Pick $\lambda \vdash n$ and $d \in \mathbb{R}$. Suppose for all $1 \neq \ell \mid n$ where λ may be written as $s \coloneqq n/\ell$ successive ribbons each of length ℓ that*

$$\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \frac{1}{n^d \phi(\ell)}. \tag{15}$$

Then for all $r \in \mathbb{Z}/n$,

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^d}.$$

Proof. By Theorem 1.6, we must show
$$\frac{1}{n}\left|\sum_{\substack{\ell|n\\ \ell\neq 1}} \frac{\chi^\lambda(\ell^s)}{f^\lambda} c_\ell(r)\right| < \frac{1}{n^d}.$$
Using the explicit form for $c_\ell(r)$ in Theorem 1.6 and the fact that n has fewer than n proper divisors, it suffices to show
$$\left|\frac{\chi^\lambda(\ell^s)}{f^\lambda}\phi(\ell)\right| \leqslant \frac{1}{n^d}$$
for all $\ell \mid n$, $\ell \neq 1$, so the result follows from our assumption (15). □

Corollary 4.3. *Let $\lambda \vdash n$. If $f^\lambda \geqslant n^3 \geqslant 1$, then $a_{\lambda,r} \neq 0$.*

Proof. Equation (14) gives
$$\ln\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \left(1-\frac{1}{\ell}\right)\left[-\frac{5}{2}\ln n + \ln\sqrt{2\pi}\right] + \frac{\ell}{12n} - \frac{1}{2}\ln\ell. \tag{16}$$
At $\ell = 2$, the right-hand side of (16) is less than $\ln\frac{1}{\phi(2)n}$ for $n \geqslant 3$. At $\ell = 3, 4, 5$, the same expression is less than $\ln\frac{1}{\phi(\ell)n}$ for $n \geqslant 4, 3, 5$, respectively. At $\ell \geqslant 6$, applying simple term-by-term estimates to (16) gives
$$\ln\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant -\left(1-\frac{1}{6}\right)\frac{5}{2}\ln n + \ln\sqrt{2\pi} + \frac{1}{12} - \frac{1}{2}\ln 6 \tag{17}$$
which is less than $\ln\frac{1}{n^2}$ for $n \geqslant 4$. Thus, Lemma 4.2 applies with $d = 1$ for all $n \geqslant 5$, so that
$$\left|\frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n}\right| < \frac{1}{n},$$
and in particular $a_{\lambda,r} \neq 0$. The cases $1 \leqslant n \leqslant 4$ remain, but they may be easily checked by hand. □

We next give techniques that are well-adapted to classifying $\lambda \vdash n$ for which $f^\lambda < n^d$ for fixed d. We begin with a curious observation, Proposition 4.5, which is similar in flavor to [4, Theorem 2.3]. It was also recently discovered independently by Morales–Panova–Pak as a corollary of the Naruse hook length formula for skew shapes; see [18, Proposition 12.1]. See also [19] for further discussion and an alternate proof of a stronger result by F. Petrov.

Definition 4.4. *Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ as a set of cells (in French notation)*
$$\lambda = \{(a, b) \in \mathbb{Z}\times\mathbb{Z} : 1 \leqslant b \leqslant m, 1 \leqslant a \leqslant \lambda_b\}.$$
Given a cell $c = (a, b) \in \lambda \subset \mathbb{N}\times\mathbb{N}$, the opposite hook length h_c^{op} at c is $a + b - 1$. For instance, the unique cell in $\lambda = (1)$ has opposite hook length 1, and the opposite hook length increases by 1 for each north or east step.

It is easy to see that $\sum_{c\in\lambda} h_c^{\mathrm{op}} = \sum_{c\in\lambda} h_c$. On the other hand, we have the following inequality for their products.

Proposition 4.5. *For all partitions λ,*
$$\prod_{c\in\lambda} h_c^{\mathrm{op}} \geqslant \prod_{c\in\lambda} h_c.$$
Moreover, equality holds if and only if λ is a rectangle.

Proof. If λ is a rectangle, the multisets $\{h_c^{\mathrm{op}}\}$ and $\{h_c\}$ are equal, so the products agree. The converse will be established in the course of proving the inequality. For that, we begin with a simple lemma.

Lemma 4.6. *Let $x_1 \geqslant \cdots \geqslant x_m \geqslant 0$ and $y_1 \geqslant \cdots \geqslant y_m \geqslant 0$ be real numbers. Then*

$$\prod_{i=1}^{m}(x_i + y_i) \leqslant \prod_{i=1}^{m}(x_i + y_{m-i+1}).$$

Moreover, equality holds if and only if for all i either $x_i = x_{m-i+1}$ or $y_i = y_{m-i+1}$.

Proof. If $m = 1$, the result is trivial. If $m = 2$, we compute

$$(x_1 + y_2)(x_2 + y_1) - (x_1 + y_1)(x_2 + y_2) = (x_1 - x_2)(y_1 - y_2) \geqslant 0.$$

The result follows in general by pairing terms i and $m - i + 1$ and using these base cases. $\square$

Returning to the proof of the proposition, the strategy will be to break up h_c and h_c^{op} in terms of (co-)arm and (co-)leg lengths, and apply the lemma to each column of λ when computing $\prod h_c$, or equivalently to each row of λ when computing $\prod h_c^{\mathrm{op}}$. More precisely, let $c = (a, b) \in \lambda$. Take $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. Define the *co-arm length* of c as a, the *co-leg length* of c as b, the *arm length* of c as $\alpha \coloneqq \alpha(a, b) \coloneqq \lambda_b - a + 1$, and the *leg length* of c as $\beta \coloneqq \beta(a, b) \coloneqq \lambda'_a - b + 1$; see Figure 1. With these definitions, we have $h_c^{\mathrm{op}} = a + b - 1$ and $h_c = \alpha + \beta - 1$. We now compute

$$\begin{aligned}
\prod_{c\in\lambda} h_c^{\mathrm{op}} = \prod_{(a,b)\in\lambda}(a + b - 1) &= \prod_{b}\prod_{a=1}^{\lambda_b}(a + b - 1) \\
&= \prod_{b}\prod_{a=1}^{\lambda_b}((\lambda_b + 1 - a) + b - 1) = \prod_{a}\prod_{b=1}^{\lambda'_a}(\alpha + b - 1) \\
&\geqslant \prod_{a}\prod_{b=1}^{\lambda'_a}(\alpha + (\lambda'_a + 1 - b) - 1) \\
&= \prod_{(a,b)\in\lambda}(\alpha + \beta - 1) = \prod_{c\in\lambda} h_c,
\end{aligned}$$

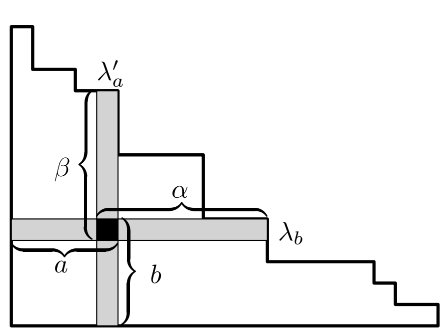

Figure 1. Arm length α, co-arm length a, leg length β, co-leg length b for $c = (a, b) \in \lambda$. The hook length is $h_c = \alpha+\beta-1$ and the opposite hook length is $h_c^{\mathrm{op}} = a + b - 1$

where Lemma 4.6 is used for the inequality with $i \coloneqq b$, $m \coloneqq \lambda'_a$, $x_i \coloneqq \alpha - 1 = \lambda_b - a$, $y_i \coloneqq \lambda'_a + 1 - b$. Moreover, if equality occurs, then since the y_i strictly decrease, we must have $\lambda_1 = \lambda_m$ for all a, forcing λ to be a rectangle. ☐

It would be interesting to find a bijective explanation for Proposition 4.5. The appearance of rectangles is particularly striking. Note, however, that $n!/\prod_{c\in\lambda} h_c^{\mathrm{op}}$ need not be an integer. In any case, we continue towards Theorem 1.5.

Definition 4.7. *Define the diagonal preorder on partitions as follows. Declare $\lambda \lesssim^{\mathrm{diag}} \mu$ if and only if for all $i \in \mathbb{P}$,*

$$\#\{c \in \lambda : h_c^{\mathrm{op}} \geqslant i\} \leqslant \#\{d \in \mu : h_d^{\mathrm{op}} \geqslant i\}.$$

Note that $\lesssim^{\mathrm{diag}}$ is reflexive and transitive, though not anti-symmetric, so the diagonal preorder is not a partial order. For example, the partitions $(3,1)$, $(2,2)$, and $(2,1,1)$ all have the same number of cells with each opposite hook length. A straightforward consequence of the definition is that

$$\lambda \lesssim^{\mathrm{diag}} \mu \qquad \Rightarrow \qquad \prod_{c\in\lambda} h_c^{\mathrm{op}} \leqslant \prod_{d\in\mu} h_d^{\mathrm{op}}. \tag{18}$$

Hooks are maximal elements of the diagonal preorder in a sense we next make precise.

Definition 4.8. *Let $\lambda \vdash n$ for $n \geqslant 1$. The diagonal excess of λ is*

$$N(\lambda) \coloneqq |\lambda| - \max_{c\in\lambda} h_c^{\mathrm{op}}.$$

For instance, $\lambda = (3,3)$ has opposite hook lengths ranging from 1 to 4, so $N((3,3)) = 6 - 4 = 2$.

The following simple observation will be used shortly.

Proposition 4.9. *Let $\lambda \vdash n$ for $n \geqslant 1$. Take $\pi\colon \lambda \to \mathbb{P}$ via $\pi(c) \coloneqq h_c^{\mathrm{op}}$. Then the fiber sizes $|\pi^{-1}(i)|$ are unimodal, and are indeed of the form*

$$1 = |\pi^{-1}(1)| < \cdots < m = |\pi^{-1}(m)| \geqslant |\pi^{-1}(m+1)| \geqslant \cdots$$

for some unique $m \geqslant 1$.

Proof. This follows quickly by considering the largest staircase shape contained in λ. Indeed, m is the number of rows or columns in such a staircase. ☐

Example 4.10. If $\lambda \vdash n$ is a hook, the sequence of fiber sizes in Proposition 4.9 is

$$1 < 2 \geqslant 2 \geqslant 2 \cdots \geqslant 2 \geqslant 1 \geqslant \cdots \geqslant 1 \geqslant 0 \geqslant \cdots$$

where there are $N(\lambda)$ two's and $n - N(\lambda)$ non-zero entries. In particular, $N(\lambda) + 1 \leqslant n - N(\lambda)$, i.e. $2N(\lambda) + 1 \leqslant n$.

Proposition 4.11. *Let $\lambda \vdash n$ for $n \geqslant 1$. Set*

$$N \coloneqq \begin{cases} N(\lambda) & \text{if } 2N(\lambda)+1 \leqslant n \\ \lfloor \frac{n-1}{2} \rfloor & \text{if } 2N(\lambda)+1 > n. \end{cases} \tag{19}$$

Then

$$\lambda \lesssim^{\mathrm{diag}} (n-N, 1^N). \tag{20}$$

In particular, if $2N(\lambda)+1 \leqslant n$, then the hook $(n-N(\lambda), 1^{N(\lambda)})$ is maximal for the diagonal preorder on partitions of size n with diagonal excess $N(\lambda)$.

Proof. Using Proposition 4.9, the sequence

$$D(\lambda) \coloneqq \left(|\pi^{-1}(i)|\right)_{i \in \mathbb{P}}.$$

is of the form

$$D(\lambda) = (1, 2, \ldots, m, \ldots, 0, \ldots)$$

where the terms weakly decrease starting at m. Given a sequence $D = (D_1, D_2, \ldots) \in \mathbb{N}^{\mathbb{P}}$, define $N(D) \coloneqq \sum_{i : D_i \neq 0} (D_i - 1)$. We have $N(D(\lambda)) = N(\lambda)$. Iteratively perform the following procedure starting with $D \coloneqq D(\lambda)$ as many times as possible; see Example 4.12.

(i) If $2N(D) + 1 > n$ and some $D_i > 2$, choose i maximal with this property. Decrease the ith entry of D by 1 and replace the first 0 term in D with 1.

(ii) If $2N(D) + 1 \leqslant n$ and some $D_i > 2$, choose i maximal with this property. We will shortly show that there is some $j > i$ for which $D_j = 1$. Choose j minimal with this property, decrease the ith term in D by 1, and increment the jth term by 1.

Example 4.12. Suppose $\lambda = (4, 4, 4, 4)$, so $n = 16$ and

$$D(\lambda) = (1, 2, 3, 4, 3, 2, 1, 0, \ldots),$$

which we abbreviate as $D(\lambda) = 1234321$. Applying the procedure gives the following sequences, where modified entries are underlined:

D	$N(D)$	$2N(D)+1$
1234321	9	19
$1234\underline{2}21\underline{1}$	8	17
$123\underline{3}2211\underline{1}$	7	15
$123\underline{2}22\underline{2}11$	7	15
$12\underline{2}22222\underline{2}1$	7	15

Returning to the proof, for the claim in (ii), first note that both procedures preserve unimodality and the initial 1 in $D(\lambda)$. Hence at any intermediate step, D is of the form

$$(1, D_2, D_3, \ldots, D_k, 1, \ldots, 1, 0, \ldots)$$

where $D_2, \ldots, D_k \geqslant 2$ and there are $\ell \geqslant 0$ terminal 1's. Since $2N(D) + 1 \leqslant n$, we have

$$\begin{aligned} 2N(D) + 1 = 2(D_2 - 1 + \cdots + D_k - 1) + 1 \leqslant n = 1 + D_2 + \cdots + D_k + \ell \\ \Leftrightarrow (D_2 - 2) + \cdots + (D_k - 2) \leqslant \ell, \end{aligned}$$

forcing $\ell > 0$ since by assumption some $D_i > 2$, giving the claim. The procedure evidently terminates.

In applying (i), $N(D)$ decreases by 1, whereas $N(D)$ is constant in applying (ii). For the final sequence D_{fin}, it follows that $N(D_{\text{fin}}) = N$ from (19). Both (i) and (ii) strictly increase in the natural diagonal partial order on sequences. The final sequence will be

$$D_{\text{fin}} = (1, 2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, \ldots)$$

where there are N two's and $n - N$ non-zero entries. This is precisely $D((n - N, 1^N))$ by Example 4.10, and the result follows. □

We may now give a polynomial lower bound on f^λ.

Corollary 4.13. *Let $\lambda \vdash n$ for $n \geqslant 1$ and take N as in* (19)*. For any $0 \leqslant M \leqslant N$, we have*

$$\prod_{c\in\lambda} h_c^{\mathrm{op}} \leqslant (n-M)!(M+1)!. \tag{21}$$

Moreover,

$$f^\lambda \geqslant \frac{1}{M+1}\binom{n}{M}. \tag{22}$$

Proof. Equation (21) in the case $M = N$ follows by combining (18) and (20). The general case follows similarly upon noting $(n-N, 1^N) \lesssim^{\mathrm{diag}} (n-M, 1^M)$ since $N \leqslant \lfloor \frac{n-1}{2} \rfloor$.

For (22), use Proposition 4.5 and (21) to compute

$$f^\lambda = \frac{n!}{\prod_{c\in\lambda} h_c} \geqslant \frac{n!}{\prod_{c\in\lambda} h_c^{\mathrm{op}}} \geqslant \frac{n!}{(n-M)!(M+1)!} = \frac{1}{M+1}\binom{n}{M}.$$

□

We now prove Theorem 1.5 and Theorem 1.9.

Proof of Theorem 1.5. We begin by summarizing the verification of Theorem 1.5 for $n \leqslant 33$. For $1 \leqslant n \leqslant 33$, a computer check shows that one may use Corollary 4.3 for all but 688 particular λ. However, the number of standard tableaux for these exceptional λ is small enough that the conclusion of the theorem may be quickly verified by computer. We now take $n \geqslant 34$.

Let N be as in (19). If $N \geqslant 5$, by Corollary 4.13,

$$f^\lambda \geqslant \frac{1}{6}\binom{n}{5} \geqslant n^3$$

for $n \geqslant 32$, so we may take $N \leqslant 4$. Since $\lfloor \frac{n-1}{2} \rfloor \geqslant 16 > 4 \geqslant N$, we must have $N = N(\lambda)$.

Write $\nu \oplus \mu$ to denote the concatenation of partitions ν and μ, where we assume the largest part of μ is no larger than the smallest part of ν. Using Proposition 4.9, since $n \geqslant 32$ and $N = N(\lambda) \leqslant 4$, we find that either $\lambda = (n-N) \oplus \mu$ or $\lambda' = (n-N) \oplus \mu$ for $|\mu| = N$.

To cut down on duplicate work, note that transposing $T \in \mathrm{SYT}(\lambda)$ complements the descent set of T. It follows that $b_{\lambda,i} = b_{\lambda',\binom{n}{2}-i}$, so that $a_{\lambda,r} = a_{\lambda',\binom{n}{2}-r}$. Since the statement of Theorem 1.5 also exhibits this symmetry, we may thus consider only the case when $\lambda = (n-N) \oplus \mu$.

There are twelve μ with $|\mu| \leqslant 4$. One may check that the five possible μ for $N = 4$ all result in $f^\lambda \geqslant n^3$ for $n \geqslant 34$, leaving seven remaining μ, namely

$$\mu = \varnothing, (1), (2), (1,1), (3), (2,1), (1,1,1).$$

It is straightforward (though tedious) to verify the conclusion of Theorem 1.5 in each of these cases. For instance, for $\mu = (1)$ and $\lambda = (n-1, 1)$, there are $n-1$ standard tableaux with major indexes $1, \ldots, n-1$ (alternatively, (5) results in $q[n-1]_q$). The remaining cases are omitted. □

Proof of Theorem 1.9. If $f^\lambda \geqslant n^5$, then (14) gives

$$\ln \frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant \left(1 - \frac{1}{\ell}\right)\left[-\frac{9}{2}\ln n + \ln\sqrt{2\pi}\right] + \frac{\ell}{12n} - \frac{1}{2}\ln \ell \tag{23}$$

As before one can check that the right-hand side of (23) is less than $\ln \frac{1}{\phi(\ell)n^2}$ for $\ell = 2, 3$ and $n \geqslant 3$. When $\ell \geqslant 4$, term-by-term estimates give

$$\ln \frac{|\chi^\lambda(\ell^s)|}{f^\lambda} \leqslant -\frac{9}{2}\left(1 - \frac{1}{4}\right) \ln n + \ln \sqrt{2\pi} + \frac{1}{12} - \frac{1}{2} \ln 4$$

which is less than $\ln \frac{1}{n^3}$ for $n \geqslant 3$. The first part of Theorem 1.9 now follows from Lemma 4.2 with $d = 2$ for $n \geqslant 3$. It remains true for $n = 1, 2$.

For the second part, suppose $n \geqslant 81$, $\lambda_1 < n - 7$, and $\lambda_1' < n - 7$. It follows from Proposition 4.11 that N from (19) satisfies $N \geqslant 8$. Hence by Corollary 4.13 we have

$$f^\lambda \geqslant \frac{1}{9}\binom{n}{8} \geqslant n^5.$$

□

5. Alternative Proof of the Hook Formula

The proof of Theorem 1.10 in [4] and [8] uses a certain decomposition of the r-rim hook partition lattice and the original hook length formula. We present an alternative proof following a different tradition, instead generalizing the approach to the original hook length formula in [28, Corollary 7.21.6]. A by-product of our proof is a particularly explicit description of the movement of hook lengths mod ℓ as length ℓ ribbons are added to a partition shape.

We are not at present aware of any other proofs or direct uses of Theorem 1.10, and it seems to have been neglected by the literature. Indeed, the author empirically rediscovered it and found the following proof before unearthing [4].

Proof of Theorem 1.10. Let $\lambda \vdash n$, $n = \ell s$. If λ cannot be written as s successive ribbons of length ℓ, then by the classical Murnaghan-Nakayama rule [28, Eq. (7.75)] we have $\chi^\lambda(\ell^s) = 0$, so assume λ can be so written.

Combining (4), (5), and (7) shows that we may compute $\chi^\lambda(\ell^s)$ by letting $q \to \omega_n^s$ in the right-hand side of (5). We may replace each q-number $[a]_q$ with $q^a - 1$ by canceling the $q - 1$'s, since $\lambda \vdash n$. Since ω_n^s has order ℓ, the values of $q^a - 1$ at ω_n^s depend only on a mod ℓ. Moreover, $q^a - 1$ has only simple roots, and it has a root at ω_n^s if and only if $\ell \mid a$. The order of vanishing of the numerator at $q = \omega_n^s$ is then $\#\{i \in [n] : i \equiv_\ell 0\} = s$, and the order of vanishing of the denominator is $\#\{c \in \lambda : h_c \equiv_\ell 0\}$. The following lemma ensures these counts agree. We postpone the proof to the end of this section.

Lemma 5.1. *Let $\lambda \vdash n$, $n = \ell s$, and suppose λ can be written as a sequence of s successive ribbons of length ℓ. Then for any $a \in \mathbb{Z}$,*

$$\#\{c \in \lambda : h_c \equiv_\ell \pm a\} = s \cdot \#\{a, -a \ (\textit{mod } \ell)\}.$$

Here $\#\{a, -a \ (\text{mod } \ell)\}$ is 1 if $a \equiv_\ell -a$ and 2 otherwise.

We may now compute the desired $q \to \omega_n^s$ limit by repeated applications of L'Hopital's rule. In particular, we find

$$|\chi^\lambda(\ell^s)| = \left| \lim_{q \to \omega_n^s} q^{d(\lambda)} \frac{\prod_{i \in [n]} [i]_q}{\prod_{c \in \lambda} [h_c]_q} \right| = \left| \lim_{q \to \omega_n^s} \frac{\prod\limits_{\substack{i \in [n] \\ i \not\equiv_\ell 0}} q^i - 1}{\prod\limits_{\substack{c \in \lambda \\ h_c \not\equiv_\ell 0}} q^{h_c} - 1} \right| \left| \frac{\prod\limits_{\substack{i \in [n] \\ i \equiv_\ell 0}} i\omega_n^{s(i-1)}}{\prod\limits_{\substack{c \in \lambda \\ h_c \equiv_\ell 0}} h_c \omega_n^{s(h_c - 1)}} \right| \tag{24}$$

The second factor in the right-hand side of (24) equals the right-hand side of (2), so we must show the first factor in the right-hand side of (24) is 1. For that, note that $q^a - 1$ at $q = \omega_n^s$ for $a \not\equiv_\ell 0$ is non-zero and is conjugate to $q^{-a} - 1$ at

$q = \omega_n^s$. By Lemma 5.1, it follows that the contribution to the overall magnitude due to $\{c \in \lambda : h_c \equiv_\ell a \text{ or } -a\}$ cancels with the contribution due to $\{i \in [n] : i \equiv_\ell a \text{ or } -a\}$ for each $a \not\equiv_\ell 0$. This completes the proof of the theorem. □

As for Lemma 5.1, it is an immediate consequence of the following somewhat more general result.

Lemma 5.2. *Suppose λ/μ is a ribbon of length ℓ. For any $a \in \mathbb{Z}$,*

$$\#\{c \in \mu : h_c \equiv_\ell \pm a\} + \#\{a, -a \ (\mathit{mod}\ \ell)\} = \#\{d \in \lambda : h_d \equiv_\ell \pm a\}.$$

Proof. We determine how the counts $\#\{c \in \mu : h_c \equiv_\ell \pm a\}$ change when adding a ribbon of length ℓ; see Figure 2. We define the following regions in λ, relying on French notation to determine the meaning of "leftmost," etc.

(I) Cells $c \in \mu$ where c is not in the same row or column as any element of λ/μ.
(II) Cells $c \in \mu$ which are in the same row as some element of λ/μ and are strictly left of the leftmost cell in λ/μ.
(III) Cells $c \in \mu$ which are in the same column as some element of λ/μ and are strictly below the bottommost cell of λ/μ.
(IV) Cells $c \in \lambda$ which are in both the same column and row as some element(s) of λ/μ. Region (IV) includes the ribbon λ/μ itself.

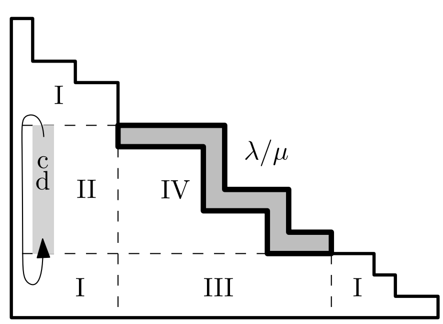

Figure 2. All regions of a partition λ where λ/μ is a ribbon

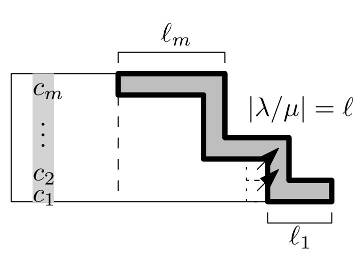

Figure 3. Regions (II) and (IV) up close

We now describe how hook lengths change in each region, mod the ribbon length ℓ, in going from μ to λ. They are unchanged in region (I). Regions (II) and (III) are similar, so we consider region (II). This region is a rectangle, which we imagine breaking up into columns. Write h_c^λ or h_c^μ to denote the hook length of a cell $c \in \mu$ as an element of λ or μ, respectively. For c in region (II), let d denote the cell in region (II) immediately below c, with wrap-around. We claim $h_c^\lambda \equiv_\ell h_d^\mu$. Given the claim, hook lengths mod ℓ in regions (II) and (III) are simply permuted in going from μ to λ, so changes to the counts $\#\{c \in \mu : h_c^\mu \equiv_\ell \pm a\}$ arise only from region (IV).

For the claim, let $c_1, c_2, \ldots, c_m$ be the cells of the column in region (II) containing c, listed from bottom to top; see Figure 3. Begin by comparing hook lengths at c_1 and c_2. Since $\lambda - \mu$ is a ribbon, the rightmost cell of μ in the same row as c_1 is directly left and below the rightmost cell of λ in the same row as c_2. It follows that $h_{c_1}^\mu = h_{c_2}^\lambda$. This procedure yields the claim except when $c = c_1$. In that case, $d = c_m$, and we further claim $h_{c_1}^\lambda = h_{c_m}^\mu + \ell$, which will finish the argument. Indeed, let ℓ_i denote the number of elements in $\lambda - \mu$ in the same row as c_i. Certainly $\ell = \ell_1 + \cdots + \ell_m$. Further, $h_{c_i}^\lambda = h_{c_i}^\mu + \ell_i$. Putting it all together, we have

$$\begin{aligned} h_{c_1}^\lambda &= h_{c_1}^\mu + \ell_1 = h_{c_2}^\lambda + \ell_1 \\ &= h_{c_2}^\mu + \ell_2 + \ell_1 = \cdots \\ &= h_{c_m}^\mu + \ell_m + \cdots + \ell_2 + \ell_1 = h_{c_m}^\mu + \ell. \end{aligned}$$

We now turn to region (IV). It suffices to consider the case depicted in Figure 4, where regions (I), (II), and (III) are empty. We define two more regions as follows; see Figure 4.

(A) Cells $c \in \lambda$ in the first row or column.
(B) Cells $c \in \lambda$ not in the first row or column.

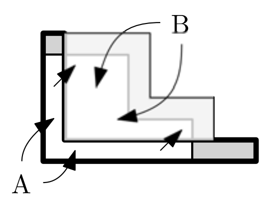

Figure 4. Regions (A) and (B) of a partition μ where λ/μ is a ribbon

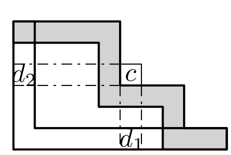

Figure 5. Adding a cell to region (B)

Region (B) is precisely μ translated up and right one square. Moreover, this operation preserves hook lengths, so changes in the counts $\#\{c \in \mu : h_c^\mu \equiv_\ell \pm a\}$ arise entirely from region (A). We have thus reduced the lemma to the statement

$$\#\{c \text{ in region (A) } : h_c^\lambda \equiv_\ell \pm a\} = \#\{a, -a \pmod{\ell}\}. \tag{25}$$

We prove (25) by induction on the size of region (B). In the base case, region (B) is empty, so λ is a hook, and the result is easy to see directly (for instance, negate the hook lengths in only the "vertical leg" to get entries of precisely $1, 2, \ldots, \ell$). For the inductive step, consider the effect of adding a cell c to region (B). Now c is in the same column as some cell d_1 in region (A) and c is in the same row as some cell d_2 in region (A); see Figure 5. Say the original hook length of d_1 is i and the original hook length of d_2 is j. It is easy to see that $i + j = \ell - 1$. Adding c to region (B) increases the hook lengths i and j each by 1, but $j + 1 \equiv_\ell -i$ and $i + 1 \equiv_\ell -j$, so the required counts remain as claimed in the inductive step. This completes the proof of the lemma and, hence, Theorem 1.10. □

We briefly contrast our approach with that of [4]. Let f_ℓ^λ be the number of ways to write λ as successive ribbons each of length ℓ. If $\lambda \vdash n = \ell s$, by the Murnaghan-Nakayama rule $\chi^\lambda(\ell^s)$ is a signed sum over terms counted by f_ℓ^λ. While there is typically cancellation in this sum, there is in fact none for rectangular cycle types [8, 2.7.26], i.e. $\chi^\lambda(\ell^s) = \pm f_\ell^\lambda$. Indeed, [4] proved Theorem 1.10 using standard rim hook tableaux instead of character evaluations, though virtually every application of their result uses the character-theoretic inequality in Theorem 1.7.

The sign of $\chi^\lambda(\ell^s)$ can be computed in terms of *abaci* as in [8, 2.7.23]. The sign may also be computed "greedily" by repeatedly removing ℓ-rim hooks from λ in any order whatsoever, which is a consequence of (among other things) the following corollary of Lemma 5.2 and Theorem 1.10. We have been unable to find part (iv) in the literature, though for the rest see [4, 2.5-2.7] and their references.

Corollary 5.3. *Let $\lambda \vdash n = \ell s$. The following are equivalent:*

(i) $\chi^\lambda(\ell^s) \neq 0$*;*
(ii) *λ can be written as successive length ℓ ribbons, i.e. the ℓ-core of λ is empty;*
(iii) *we have*

$$\#\{c \in \lambda : h_c \equiv_\ell 0\} = s;$$

(iv) *for any $a \in \mathbb{Z}$,*

$$\#\{c \in \lambda : h_c \equiv_\ell \pm a\} = s \cdot \#\{a, -a \ (\mathit{mod}\ \ell)\}.$$

Proof. (i) and (ii) are equivalent by Theorem 1.10. (ii) implies (iv) by Lemma 5.1 and (iv) implies (iii) trivially. Finally, (iii) is equivalent to (i) as follows. The expression (5) is a polynomial, so the order of vanishing at $q \to \omega_n^s$ of the numerator, namely s, is at most as large as the order of vanishing of the denominator, namely $\#\{c \in \lambda : h_c \equiv_\ell 0\}$. The limiting ratio is non-zero if and only if these counts agree, so (iii) is equivalent to (i). □

While Corollary 5.3 gives equivalent conditions for $\chi^\lambda(\ell^s) \neq 0$, [30, Corollary 7.5] gives interesting and different necessary conditions for $\chi^\lambda(\nu) \neq 0$ for general shapes ν.

6. Unimodality and $\chi^\lambda(\mu)$

We end with a brief discussion of inequalities related to symmetric group characters. In applying Proposition 4.5, we essentially replaced $\frac{n!}{\prod_{c\in\lambda} h_c}$ with $\frac{n!}{\prod_{c\in\lambda} h_c^{\mathrm{op}}}$, since the latter is order-reversing with respect to the diagonal preorder by (18). Moreover, it is relatively straightforward to mutate partitions and predictably increase or decrease them in the diagonal preorder, as in the proof of Proposition 4.11. It would be desirable to instead work directly with symmetric group characters themselves and appeal to general results about how $|\chi^\lambda(\mu)|$ increases or decreases as λ is mutated and μ is held fixed, though we have found very few concrete and no conjectural results in this direction. Any progress seems both highly non-trivial and potentially useful, so in this section we record some initial observations.

We have $\chi^{(a+1,1^b)}(1^n) = \binom{n-1}{a}$ for $a+b+1 = n$, so these values are unimodal in a. Using Theorem 1.10 shows more generally that for all $\ell \mid n$,

$$|\chi^{(a+1,1^b)}(\ell^{n/\ell})| = \binom{\frac{n}{\ell} - 1}{\lfloor \frac{a}{\ell} \rfloor}$$

which is again unimodal in a. However, $|\chi^\lambda(\ell^s)|$ does not seem to respect changes in λ under dominance order in general in any suitable sense. On the other hand, if we allow the cycle type μ to vary and consider the Kostka numbers $K_{\lambda\mu}$ as a surrogate for $|\chi^\lambda(\mu)|$ (since $K_{\lambda(1^n)} = \chi^\lambda(1^n)$), we have a series of well-known and very general inequalities. We write $K_{\lambda\mu}(t)$ for the Kostka-Foulkes polynomial and $\nu \geqslant \mu$ for dominance order. We have:

Theorem 6.1 ([25], [17], [15]; [7]). *$K_{\lambda\nu} \leqslant K_{\lambda\mu}$ for all λ if and only if $\nu \geqslant \mu$. Indeed, $\nu \geqslant \mu$ implies $K_{\lambda\nu}(t) \leqslant K_{\lambda\mu}(t)$ (coefficient-wise) for all λ.*

Question 6.2. *Are there any "nice" infinite families besides hooks and rectangles for which $|\chi^\lambda(\mu)|$ is monotonic, unimodal, or suitably order-preserving as λ varies? What about as μ varies?*

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