ON THE EXISTENCE OF WEAK SOLUTIONS OF DIFFERENTIAL EQUATIONS IN NONREFLEXIVE BANACH SPACES

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Introduction

The study of the Cauchy problem for differential equations in a Banach space relative to the strong topology has attracted much attention in recent years [2,4,5,7]. This study has taken two different directions. One direction is to impose compactness type conditions that guarantee only existence and the corresponding results are extensions of the classical Peano's Theorem. The other approach is to utilize dissipative type conditions that assure existence and uniqueness of solutions, and the corresponding results are extensions of the classical Picard's Theorem.

However, a similar study of the Cauchy problem in a Banach space relative to the weak topology has lagged behind. Recently Szep [7] proved Peano's Theorem in the weak topology for differential equations in a reflexive Banach space and his main tool is the Eberlein-Šmulian Theorem which assures the weak compactness of a closed set in the weak topology. In this paper we wish to prove this theorem in arbitrary Banach spaces, imposing weak compactness type conditions. For this purpose we introduce the notion of measure of weak noncompactness which is parallel to the Kuratowski measure of noncompactness and develop its properties. We also impose weak dissipative type conditions and prove an existence and uniqueness theorem.

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1. *Preliminaries*

Throughout this paper, $(E,||\cdot||)$ will denote a real Banach space, E^* the dual space, that is the set of all continuous linear functionals on E, and $E\omega$, the space E endowed with the weak topology. A subbase for $E\omega$ is $B = \{S[x,\phi,r] : x \in E,\ \phi \in E^*,\ r > 0\}$ where $S[x,\phi,r] = \{y \in E : |\phi(x - y)| < r\}$.

It should be noted that $\rho_\phi(x,y) = |\phi(x - y)|$ is a pseudometric, so that $\{\rho_\phi : \phi \in E^*\}$ generates a uniformity on E. The weak topology is the same as the uniform topology generated by $\{\rho_\phi : \phi \in E^*\}$. We state the following results in terms of the weak topology, however in general these can be stated for uniform topologies [6].

<u>*Definition 1.1.*</u> A subset A of E_ω is totally bounded if and only if for all $\phi \in E^*$ and $\varepsilon > 0$, A can be covered by a finite number of ϕ balls of radius ε.

<u>*THEOREM 1.1.*</u> A set A is compact in the weak topology if and only if it is weakly complete and totally bounded.

Since our purpose is to work in $E\omega$, we list some necessary definitions below for completeness.

<u>*Definition 1.2.*</u> Let $\{y_n\}$ be a sequence in E, $\{x_n(t)\}$ be a sequence of functions mapping the interval $I \subset R$, the real line, into E, and f a function from $I \times A \subset R \times E$ into E. We say that

1) $\{y_n\}$ converges weakly to $y \in E$ if $\{\phi(y_n)\}$ converges to $\phi(y)$ for all $\phi \in E^*$;

2) $\{x_n(t)\}$ converges weakly uniformly to $x(t)$ where $x : I \to E$ if for $\varepsilon > 0$, $\phi \in E^*$ there exists $N = N(\phi,\varepsilon)$ such that $n > N$ implies $|\phi[x_n(t) - x(t)]| < \varepsilon$ for all $t \in I$;

3) $\{y_n\}$ is weakly Cauchy if given $\varepsilon > 0$, $\phi \in E^*$ there exists $N = N(\phi,\varepsilon)$ such that n, $m > N$ implies $|\phi(y_n - y_m)| < \varepsilon$;

4) E is weakly complete if every weakly Cauchy sequence converges weakly to a point in E;

5) $x(t)$ is weakly continuous at t_0 if $\phi(x(t))$ is continuous at t_0 for each $\phi \in E^*$;

6) $x(t)$ is weakly (Reimann) integrable on $[a,b]$ if there is $x_0 \in E$ such that $\phi(x_0) = \int_a^b \phi(x(s))ds$ for each $\phi \in E^*$, and we write $x_0 = \int_{a^w}^b x(s)ds$;

7) $x(t)$ is weakly differentiable at t_0 if there exists a point in E, denoted $x'(t_0)$, such that $\phi(x'(t_0)) = (\phi x)'(t_0)$ for each $\phi \in E^*$;

8) F is weakly weakly continuous at (t_0,x_0) if given $\varepsilon > 0$, $\phi \in E^*$ there exists $\delta = \delta(\phi,\varepsilon)$ and $\mu = \mu(\phi,\varepsilon)$ a weakly open set continuing x_0 such that $|\phi(f(t,x) - f(t_0,x_0))| < \varepsilon$ whenever $|t - t_0| < \delta$ and $x \in \mu$;

9) f is weakly weakly uniformly continuous if given $\varepsilon > 0$ and $\phi \in E^*$ there exists $\delta = \delta(\phi,\varepsilon)$ and $\{\phi_i : \phi_i \in E^* \ \ i = 1,2,\ldots,n_{(\phi,\varepsilon)}\}$ such that $|\phi(f(t,x) - f(s,y))| < \varepsilon$ whenever $|t - s| < \delta$ and $|\phi_i(x - y)| < \delta$ $i = 1,2,\ldots,n(\phi,\varepsilon)$;

10) the family $\{x_n(t)\}$ is said to be weakly equicontinuous if given $\varepsilon > 0$, $\phi \in E^*$ there exists $\delta = \delta(\phi,\varepsilon)$ such that $|\phi(x_n(t) - x_n(s))| < \varepsilon$ whenever $|t - s| < \delta$.

The following facts result from the definitions and are stated below for convenience

1) If $x(t)$ is weakly continuous and Ew is weakly complete then $x(t)$ is weakly integrable.

2) If $x(t)$ is weakly differentiable then $x(t)$ is weakly continuous.

3) If $x(t)$ is weakly continuous and $F(t) = \int_{a^w}^{t} x(s)ds$ then $F'(t) = x(t)$ where $F'(t)$ is the weak derivative of F.

In its usual form the Ascoli-Arzela Theorem deals with a family of functions into a metric space. The following Ascoli-Arzela type theorem can be proved in a manner precisely parallel to that of the usual theorem and hence we merely state it.

THEOREM 1.2. Let F be a weakly equicontinuous family of functions from $I = [t_0, t_0 + a] \subset R$ to E. Let $\{x_n\}$ be a sequence in F such that for each $t \in I$, $\{x_n(t)\}$ is weakly pre-compact. Then there exists a subsequence $\{x_{n_k}\}$ which converges weakly uniformly on I to a weakly continuous function $x(t)$.

2. *Measure of Weak Noncompactness*

Here we define the notion of a measure of weak noncompactness of a bounded set in E which is suitable for our purpose and develop several of the necessary properties of such a measure. Our definition, as will be seen, is parallel to the Kuratowski measure of noncompactness. Other forms of measure of weak noncompactness are known [1,6] but are not convenient for our discussion.

<u>*Definition 2.1.*</u> For a bounded subset A of E, the measure of weak noncompactness $\beta(A)$ is a real valued function defined by

$$\beta(A) = inf\{d > 0 : \text{ for each } \phi \in E^*, \ ||\phi|| = 1 \text{ there exist}$$

$$x_1, x_2, \ldots, x_{n_\phi} \text{ such that } A \quad \bigcup_{i=1}^{n} \{x : |\phi(x - x_i)| \leq d\}$$

The following lemma is concerned with the properties of β.

<u>*Lemma 2.1.*</u> Let A and B be bounded subsets of E and $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then

(1) if $A \quad B$ then $\beta(A) \leq \beta(B)$;

(2) $\beta(A) = \beta(\overline{A}^w)$ where $\overline{A}^w$ denotes the weak closure of A;

(3) if E_w is weakly complete then $\beta(A) = 0$ if and only if $\overline{A}^w$ is weakly compact;

(4) $\beta(A \cup B) = max\{\beta(A), \beta(B)\}$;

(5) $\beta(A) = \beta(Co\ A)$;

(6) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(7) $\beta(\{x_n\}) - \beta(\{y_n\}) \leq \beta(\{x_n - y_n\})$;

(8) $\beta(x + A) = \beta(A)$ where $x \in E$;

(9) $\beta(tA) = t\beta(A) \quad t \geq 0$;

(10) given $\varepsilon > 0$ if for each $\phi \in E^*$ $||\phi|| = 1$ there exists $N = N(\phi,\varepsilon)$ such that $n > N$ implies $|\phi(x_n)| < \varepsilon$ then $\beta(\{x_n\}) \leq \varepsilon$;

(11) if $\{X_n\}$ is a sequence of nonvoid weakly closed subsets of E, X_1 is bounded, $X_1 \supset X_2 \supset \ldots \supset X_n \supset \ldots$ and $\lim_{n\to\infty} \beta(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

Proof. Except for (3) and (10), the proofs of the rest of the properties of β are similar to the proofs of the corresponding properties of the Kuratowski measure of noncompactness and hence we merely indicate the proofs of (3) and (10).

If $\overline{A}^w$ is weakly compact it is totally bounded which implies $\beta(\overline{A}^w) = 0$ and thus by (2) $\beta(A) = 0$. If $\beta(A) = 0$ then $\beta(\overline{A}^w) = 0$ which implies $\overline{A}^w$ is totally bounded. Since $\overline{A}^w$ is closed and the space is weakly complete $\overline{A}^w$ is also complete. Thus by Theorem 1.1 $\overline{A}^w$ is weakly compact. This proves (3).

To prove (10), consider $\phi \in E$ with $||\phi|| = 1$. The spheres of radius ε about the points $0, x_1, x_2, \ldots, x_{n(\phi,\varepsilon)}$ cover the entire sequence $\{x_n\}$.

We need a mean value theorem in this set up similar to the one in the strong topology. For this purpose we first prove the following Theorem.

THEOREM 2.2. Let E be a real Banach space, $A \subset E$ and $x \in E$. If for all $\phi \in E^*$, $\phi(x) \in \phi(A)$ then $x \in \overline{co}\, A$.

Proof. Let x_0 be any fixed element of A. Let $M = \overline{co}(A - \{x_0\})$. Clearly M is closed, convex and $0 \in M$. Suppose $x - x_0 \notin M$. Then by a theorem due to S. Mazur [8] there exists $\phi \in E^*$ such that $\phi(x - x_0) > 1$ and $\phi(z) \leq 1$ on M. But by hypothesis there exists $y \in A$ such that $\phi(x) = \phi(y)$. Then $\phi(x - x_0) = \phi(y - x_0)$. Note that $y - x_0 \in M$. We reach a contradiction by observing that $1 < \phi(x - x_0) = \phi(y - x_0) \leq 1$ and therefore $x - x_0 \in M$. Now given $\varepsilon > 0$ there exist $\{y_i\}$ A, $\{\alpha_i\}$ with $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$ such that $|x - x_0 - \sum_{i=1}^{n} \alpha_i(y_i - x_i)| < \varepsilon$. But $|x - x_0 - \sum_{i=1}^{n} \alpha_i(y_i - x_0)| = |x - \sum_{i=1}^{n} \alpha_i y_i| < \varepsilon$ and therefore $x \in \overline{co}\, A$.

THEOREM 2.3. Suppose $x \in C_w[[t_0,t_0 + a],E]$ and $x(t)$ is weakly differentiable on $[t_0,t_0 + a]$. If $t \in [t_0,t_0 + a)$ and $h > 0$ such that $t + h < t_0 + a$ then $\frac{x(t + h) - x(t)}{h} \in \overline{co}\{x'(s) : s \in [t,t + h]\}$.

Proof. The weak differentiability of $x(t)$ implies that $\phi(t)$ is differentiable. For each $\phi \in E^*$, by the mean value theorem there exists $t_\phi \in [t,t + h]$ such that

$$\frac{\phi x(t + h) - \phi x(t)}{h} = (\phi x)'(t_\phi) = \phi x'(t_\phi).$$

Letting $A = \{x'(s) : s \in [t,t + h]$, we have

$\phi\left(\frac{x(t + h) - x(t)}{h} \right) \in \phi(A)$ for all $\phi \in E^*$.

Consequently by applying Theorem 2.2 we can conclude

$$\frac{x(t+h)-x(t)}{h} \in \overline{co}\{x'(s) : s \in [t,t+h]\}.$$

3. *Existence of Solutions*

This section of the paper contains our main results. Throughout this section we will assume that $E\omega$ is weakly complete.

Consider the Cauchy problem

(3.1) $$x' = f(t,x), \quad x(t_0) = x_0,$$

where we will assume that

(H_1) f is weakly weakly continuous on R_0 and $||f(t,x)|| \leq M$ on R_0 where $R_0 = [(t,x) : t_0 \leq t \leq t_0 + a$ and $||x - x_0|| \leq b]$.

Looking at the hypothesis (H_1) it may seem more natural to impose boundedness on f in terms of each $\phi \in E^*$. However, by the Uniform Boundedness Theorem it is known that a subset A of a normed space is bounded if and only if the set $\{\phi(x) : x \in A\}$ is bounded for each $\phi \in E^*$.

The technique for proving existence results generally follows a three step procedure. First, a sequence of approximate solutions is constructed. Second, it is shown that the sequence converges. Third, it is proved that the limit function is a solution. Two of the steps, namely constructing a sequence of approximate solutions for (3.1) and proving that the limit function is a solution of (3.1) are straight forward. Assertions concerning these two steps are given in the following lemmas.

Lemma 3.1. Let the assumption (H_1) be satisfied and let $\{\varepsilon_n\} > 0$ such that $\varepsilon_n \to 0$ as $n \to \infty$ be given. Then there exists a sequence of approximate solutions $\{x_n(t)\}$ satisfying

(i) $x_n(t_0) = x_0$;

(ii) $\{x_n(t)\}$ is weakly equicontinuous and uniformly bounded on $[t_0, t_0 + \alpha]$;

(iii) $x_n'(t) = f(t, x_n(t - \varepsilon_n))$, $t \in [t_0, t_0 + \alpha]$, where $\alpha = min(a, b/M)$.

Lemma 3.2. Let the assumption (H_1) be satisfied. Suppose that $\{x_n(t)\}$ converge weakly to $x(t)$ on $[t_0, t_0 + \alpha]$. Then $x(t)$ is a solution of the Cauchy problem (3.1).

For a proof of Lemmas 3.1 and 3.2 see [7].

We shall now concentrate our attention to show the convergence of the sequence of approximate solutions. As in the case of existence of strong solutions, we utilize weak compactness type conditions and weak dissipative type conditions to achieve this goal. Note that if we assume that E is reflexive, we do not need any additional assumptions for proving an existence result because of the Eberlein Šmulian Theorem. This is precisely the result in [7].

(a) Weak Compactness Type Conditions

Here we shall employ the measure of weak noncompactness β discussed in section 2, to impose conditions on f. Specifically, let us first prove the following result.

THEOREM 3.1. Let the hypothesis (H_1) hold. Suppose further that $\beta(f(I \times A)) \le g(\beta(A))$ where $I = [t_0, t_0 + \alpha]$, $A \subset R_0$ and $g \in C[R^+, R^+]$.

Assume that $u(t) \equiv 0$ is the unique solution of $u' = g(u)$, $u(t_0) = 0$ on $[t_0, t_0 + \alpha]$. Then there exists a solution $x(t)$ for the problem (3.1) on $[t_0, t_0 + \alpha]$ where $\alpha = min(a, b/M)$.

Proof. Let $\{x_n(t)\}$ be the sequence of approximate solutions of (3.1) constructed in Lemma 3.1. In view of Theorem 1.2 and Lemma 3.2, it is sufficient to show $\beta(\{x_n(t)\}_{n=1}^{\infty}) = 0$. We define $m(t) = \beta(\{x_n(t)\}_{n=1}^{\infty})$ and note that $m(t_0) = 0$. The continuity of $m(t)$ is clear from the equicontinuity of the sequence $\{x_n(t)\}$ and property (7) of β in Lemma 2.1. Now

$$D^+m(t) = \lim_{\tau \to 0^+} \sup_{h \in [0,\tau]} \frac{m(t+h) - m(t)}{h}$$

$$\leq \lim_{\tau \to 0^+} \sup_{h \in [0,\tau]} \frac{\beta(\{x_n(t+h) - x_n(t)\})}{h}, \text{ by property (7) of } \beta.$$

Using the mean value Theorem 2.3, we get

$$D^+m(t) \leq \lim_{\tau \to 0^+} \sup_{h \in [0,\tau]} \beta\left(\bigcup_{n=1}^{\infty} \overline{co} \{x_n'(s) : s \in [t, t+b]\} \right)$$

$$\leq \lim_{\tau \to 0^+} \beta\left(\overline{co} \bigcup_{n=1}^{\infty} \{f(s, x_n(s - \varepsilon_n) : s \in [t, t+\tau]\} \right)$$

$$= \lim_{\tau \to 0^+} \beta\left(\bigcup_{n=1}^{\infty} \{f(s, v_n(s - \varepsilon_n) : s \in [t, t+\tau]\} \right)$$

$$\leq \lim_{\tau \to 0^+} \beta\left(\bigcup_{n=1}^{\infty} \{f(Ix\{x_n(s - \varepsilon_n) : s \in [t, t+\tau]\} \right)$$

$$\leq \lim_{\tau \to 0^+} g\left(\beta\left(\bigcup_{n=1}^{\infty} \{x_n(s - \varepsilon_n) : s \in [t, t+\tau]\} \right)\right)$$

(ii) $\beta(\{x + hf(hx) : x \in A) - \beta(A) \leq hg(t,\beta(A))$ for $h > 0$ sufficiently small, $t \in [t_0, t_0 + \alpha]$ and $A \subset R_0$, where $g \in C[R^+ \times R^+, R]$;

(iii) $u(t) = 0$ is the only solution on $u' = g(t,u)$, $u(t_0) = 0$ on $[t_0, t + \alpha]$ then the conclusion of Theorem 3.] is true.

Proof. We proceed as in Theorem 3.1 and set $m(t) = \beta(\{x_n(t)\})$. In this case we don't use mean value Theorem 2.3 and hence part of the proof is different. Now,

$$D^+m(t) = \overline{lim}_{h\to 0^+} \frac{1}{h} \beta\ \{x_n(t + h)\}) - \beta(\{x_n(t)\})$$

$$\leq \overline{lim}_{h\to 0^+} \frac{1}{h} \beta(\{x_n(t) + hf(t,x_n(t))\}) - \beta(\{x_n(t)\})$$

$$+ \overline{lim}_{h\to 0^+} \frac{1}{h} \beta(\{x_n(t + h)\}) - \beta(\{x_n(t) + hf(t,x_n(t))\}),$$

Hence by using the assumption we get

$$D^+m(t) \leq g(t,m(t)) + \overline{lim}_{h\to 0^+} \beta\left(\frac{x_n(t + h) - x_n(t) - hf(t,x_n(t)}{h} \right).$$

We shall next show that

$$\overline{lim}_{h\to 0^+} \beta\left(\frac{x_n(t + h) - x_n(t) - hf(t,x_n(f))}{h} \right) = 0.$$

For this purpose note that

$$\frac{1}{h}(x_n(t+h) - x_n(t) - hf(t,x_n(t)) = \frac{1}{h}\int_t^{t+h} [f(s,x_n(s-\varepsilon_n)) - f(t,x_n(t)]ds$$

given $\hat{\varepsilon} > 0$ $\phi \in E^*$ with $||\phi|| = 1$, by the weak weak uniform continuity of f, there exists $\delta = \delta(\phi,\hat{\varepsilon}) > 0$ and $\{\phi_i \mid i = 1,2,\ldots p$ such that $|\phi(f(t,x) - f(t,y))| < \hat{\varepsilon}$ whenever $|s - t| < \delta$ and $|\phi_i(x - y)| < \delta$ $i = 1,2,\ldots p$. By the equicontinuity of the family $\{x_n(t)\}$ there exist γ_i such that $|\phi_i(x_n(t) - x_n(s))| < \delta$ whenever $|t - s| < \gamma_i$. Let $\gamma = \min \frac{\gamma i}{2}$. Choose $h = \min\{\delta,\gamma\}$ and choose N sufficiently larger so that if $n > N$ then $\varepsilon_n < \gamma$. Thus on $[t,t+h]$ we have $|t - s| < \delta$ and $|\phi_i(x_n(s - \varepsilon_n) - x_n(t)| < \delta$. Hence we have

$$\left|\frac{1}{h}\phi\, x_n(t+h) - x_n(t) - hf(t,x_n(t))\right| \leq \frac{1}{h}\int_t^{t+h} \Big|\phi\, f(s,x_n(s-\varepsilon_n))$$

$$- f(t,x_n(t))\Big| ds \leq \frac{1}{h}\cdot h \cdot \hat{\varepsilon}$$

By property (10) of Lemma 2.1 this yields

$$\beta\left(\frac{x_n(t+h) - x_n(t) - hf(t,x_n(t))}{h}\right) < \hat{\varepsilon}$$

But this is true for arbitrary $\hat{\varepsilon}$ and h sufficiently small. Therefore

$$\overline{\lim_{h\to 0^+}}\, \beta\left(\frac{x_n(t+h) - x_n(t) - hf(t,x_n(t))}{h}\right) = 0.$$

Hence we have $D^+m(t) \leq g(t,m(t))$, $t \in [t_0,t_0+\alpha]$. By the comparison theorem [3] this implies $m(t) \leq r(t)$ $t \in [t_0,t_0+\alpha]$ where $r(t)$ is the maximal solution of $u' = g(t,u)$ $u(t_0) = 0$. By hypothesis $r(t) \equiv 0$ which implies $m(t) \equiv 0$ and the proof is complete.

$$\leq g\left(\lim_{t\to 0+} \beta\left(\bigcup_{n=1}^{\infty} \{x_n(s - \varepsilon_n) : s \in [t, t + \tau]\}\right)\right)$$

$$= g\,(\beta\{x_n(t - \varepsilon_n)\}_{n=1}^{\infty})$$

given $\varepsilon > 0$. By the weak equicontinuity of the family $\{x_n(t)\}$, given $\phi \in E^*$ for ε_n sufficiently small or equivalently n sufficiently larger $|\phi(x_n(t - \varepsilon_n) - x_n(y))| < \varepsilon$. By property (10) of Lemma 2.1 we conclude that $\beta(\{x_n(t - \varepsilon_n) - x_n(t)\}) < \varepsilon$. But ε was arbitrary so that $\beta(\{x_n(t - \varepsilon_n) - x_n(t)\}) = 0$.

Since $\{x_n(t - \varepsilon_n)\} = \{x_n(t - \varepsilon_n) - x_n(t) + x_n(t)\}$ $\{x_n(t - \varepsilon_n) + y_n(t)\} + \{x_n(t)\}$ it follows $\beta(\{x_n(t - \varepsilon_n)\} \leq \beta(\{x_n(t - \varepsilon_n) - x_n(t)\}) + \beta(\{x_n(t)\}) = 0 + \beta(\{x_n(t)\})$. Similarly $\beta(\{x_n(t)\}) \leq \beta(\{x_n(t - \varepsilon_n)\})$ and therefore $\beta(\{x_n(t - \varepsilon_n)\}) = \beta(\{x_n(t)\}$. Consequently we obtain $D^+m(t) \leq g(m(t))$, $t \in [t_0, t_0 + \alpha]$. This implies by the comparison theorem [3] that $m(t) \leq r(t)$ where $r(t)$ is the maximal solution of $u' = g(u)$ $u(t_0) = 0$. Since by assumption $r(t) \equiv 0$ we have $m(t) \equiv 0$. The proof of the theorem is thus complete.

If we wish to weaken the compactness type assumption in Theorem 3.1, namely $\beta(f(I \times A)) \leq g(\beta(A))$, we need to strengthen continuity assumption on f. The next result is concerned with this situation.

THEOREM 3.2. Assume that

(i) f is weakly weakly uniformly continuous on R_0 and that $||f(t,x)|| \leq M$ on R_0;

(B) Weak Dissipative Type Conditions

Here we shall utilize a weak dissipative type condition on f to prove existence and uniqueness of solutions of the problem (3.1).

Theorem 3.3. Suppose that

(i) f is weakly weakly uniformly continuous on R_0 and

$||f(t,x)|| \leq M$ on R_0.

(ii) for each $\phi \in E^*$ with $||\phi|| = 1$,

$$\overline{\lim_{h\to 0^+}} \frac{|\phi[x - y + h(f(t,x) - f(t,y))]| - |\phi(x - y)|}{h} \leq g(t, |\phi(x - y)|)$$

where $g \in C[[t_0, t_0 + a] \times [0, 2b], R_+]$;

(iii) $u(t_0) \equiv 0$ is the only solution $u' = g(t,u)$ $u(t_0) = 0$ on $[t_0, t_0 + a]$

Then there exists a unique solution $x(t)$ of (3.1) on $[t_0, t_0 + \alpha]$ where $\alpha = min\{a, b/M\}$.

Proof. As before, let $\{x_n(t)\}$ be the family of approximate solutions for (3.1) constructed as in Lemma 3.1. We want to show that $\{x_n(t)\}$ are weakly cauchy u given $\phi, \hat{\varepsilon}$ we need to find N such that $n, m > N$ implies $|\phi(x_n(t) - x_m(t))| < \hat{\varepsilon}$.

For ε sufficiently small the maximal solutions $r(t,\varepsilon)$ of $u' = g(t,u) + \varepsilon$, $u(t_0) = \varepsilon$ converge to zero as $\varepsilon \to 0$ [3]. Thus for ε sufficiently small, $\varepsilon < \varepsilon_1$, $r(t,\varepsilon) < \hat{\varepsilon}$. By the weak weak uniform continuity of f $|\phi(f(t,x) - f(s,y))| < \varepsilon_1/2$ whenever $|t - s| < \delta$ and $|\phi_i(x - y)| < \delta$ $i = 1,2,\ldots p$. By the equicontinuity of the family $|\phi_i(x_n(t) - x_n(s))| < \delta$ whenever $|t - s| < \delta_i$. Let $\gamma = \min_{i=1,2,\ldots p} \{\gamma_i\}$. Choose N such that $\varepsilon_n < \gamma$ for $n > N$.

Let $n, m > N$ and define $m(t) = |\phi(x_n(t) - x_m(t))|$. Then

$$D^+m(t) \leq \overline{\lim_{h\to 0^+}} \frac{|\phi(x_n(t) - x_m(t) + h(f(t,x_n(t) - f(t,x_m(t))| - |\phi(x_n(t) - x_m(t))|}{h}$$

$$+ \overline{\lim_{h\to 0^+}} \frac{|\phi(x_n(t+h) - x_m(t+h)) - \phi(x_n(t) - x_m(t) + h(f(t,x_n(t) - f(t,x_m(t}{h}$$

This implies that

$$D^+m(t) \leq g(m(t)) + \overline{\lim_{h\to 0^+}} \left|\frac{\phi(x_n(t+h) - x_n(t) - hf(t,x_n(t))}{h}\right|$$

$$+ \overline{\lim_{h\to 0^+}} \left|\frac{\phi(x_m(t+h) - x_m(t) - hf(t,x_m(t)))}{h}\right| .$$

But $\overline{\lim_{h\to 0^+}} \left|\frac{\phi(x_n(t+h) - x_n(t) - hf(t,x_n(t)))}{h}\right| \leq$

$$\overline{\lim_{h\to 0^+}} \left|\frac{\phi(x_n(t+h) - x_n(A - hf(t,x_n(t-\varepsilon_n)))}{h}\right|$$

$$+ \overline{\lim_{h\to 0^+}} |\phi(f(t,x_n(t-\varepsilon_n)) - f(t,x_n(t)))|$$

$$\leq 0 + \varepsilon_1/2 .$$

Similarly $\overline{\lim_{h\to 0^+}} \frac{|\phi(x_m(t+h) - x_m(t) - hf(t,x_m(t)))|}{h} \leq \varepsilon_1/2.$

Hence we have $D^+m(t) \leq g(t,m(t)) + \varepsilon_1 \quad m(t_0) = 0$. Thus $m(t) \leq r(t,\varepsilon_1) < \hat{\varepsilon}$. We have now shown that $\{\phi x_n(t)\}$ is Cauchy for each ϕ. Since the space is weakly complete $\{x_n(t)\}$ converges weakly to a function $x(t)$.

Lemma 3.2 assures that $x(t)$ is a solution of (3.1).

To see that the solution is unique, suppose that $x(t)$ and $y(t)$ are both solutions of (3.1). Let $p(t) = |\phi(x(t) - y(t))|$.

$$\overline{\lim_{h\to 0^+}} \frac{p(t+h) - p(t)}{h} \leq$$

$$\overline{\lim_{h\to 0^+}} \frac{|\phi(x(t) - y(t) + h(f(t,x(t)) - f(t,y(t)))| - |\phi(x(t) - y(t))|}{h}$$

$$+ \overline{\lim_{h\to 0^+}} \frac{|\phi(x(t+h) - x(t) - hf(t,x))|}{h} + \overline{\lim_{h\to 0^+}} \frac{|\phi(y(t+h) - y(t) - hf(t,y))|}{h}$$

Thus $D^+p(t) \leq g(t,p(t))$, $p(t_0) = 0$, $t \in [t_0, t_0 + \alpha]$. and $0 \leq m(t) \leq r(t) \equiv 0$. We conclude $m(t) = 0$ and therefore $\phi x = \phi y$ for each $\phi \in E^*$ which implies $x(t) \equiv y(t)$ completing the proof.

Remark. In the dissipative condition assumed in Theorem 3.3, we could replace g by g_ϕ for each ϕ provided $|g_p(t,u)| \leq M$ for $t \in [t_0, t_0 + \alpha]$ and $0 \leq u \leq 2b$.

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