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CONTENTS

ON THE EXPECTATION OF THE PRODUCT OF FOUR MATRIX-VALUED GAUSSIAN RANDOM VARIABLES

Peter H.M. Janssen (*)
Petre Stoica (**)

Abstract: The formula for the expectation of the product of four scalar real Gaussian random variables is generalized to matrix-valued (real or complex) Gaussian random variables. As an application of the extended formula, we present a simple derivation of the covariance matrix of instrumental variable (IV) estimates of parameters in multivariate linear regression models.

(*) Faculty of Electrical Engineering, Eindhoven University of Technology (EUT), P.O. Box 513, NL-5600 MB Eindhoven, the Netherlands.

(**) Facultatea de Automatica,Institutul Politehnic Bucuresti, Splaiul Independentei 313, R-77206 Bucharest, Romania.

Mailing address:
P.H.M. Janssen, Eindhoven University of Technology,
Faculty of Electrical Engineering,
Measurement and Control Group,
P.O. Box 513, NL-5600 MB Eindhoven,
The Netherlands

1 Introduction

The expectation of the product of four real scalar random variables $\{x_i\}$, $i=1,\ldots,4$, which are jointly Gaussian distributed can be simply expressed in terms of first- and second-order moments (E denotes expectation) (see e.g. [1],[2])

$$E(x_1x_2x_3x_4) = E(x_1x_2)\ E(x_3x_4) + E(x_1x_3)\ E(x_2x_4) + \\ + E(x_1x_4)\ E(x_2x_3) - 2\ E(x_1)E(x_2)E(x_3)E(x_4) \qquad (1.1)$$

This relationship plays an important role in determining the (asymptotic) variances and covariances of the estimates of correlation and spectral density functions of stationary stochastic processes ([3]-[6]), as well as of several parameter estimates (see e.g. [3], [7], [13]).
In this note formula (1.1) is generalized to matrix-valued real or complex random variables which are Gaussian distributed. Using the Kronecker product notation, the generalized formula is expressed in a compact form (section 2).
As an application of the extended formula, we derive (in section 3) the asymptotic covariance matrix of IV estimates of parameters in multivariate linear regression models. The extended formula may find other applications in multivariate analysis and system identification.

2 The main results

In order to state our main result we first need to introduce some definitions (see e.g. [8], [13]):

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $(m*n)$ and $(p*r)$ matrices, respectively. The Kronecker product of A and B is defined by

$$A\otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots\cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots\cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots\cdots & a_{mn}B \end{bmatrix} \qquad (2.1)$$

Denote the vector having "1" at the s-th position and zero elsewhere by e_s. The dimension of e_s will be clear from the context.

Finally, we introduce the "vec" operation on a matrix, consisting of stacking the columns of a matrix on top of each other. If A is a m*n matrix with columns denoted by $A_{*1}, A_{*2}, \dots, A_{*n}$, then

$$\mathrm{Vec}(A) := \begin{bmatrix} A_{*1} \\ A_{*2} \\ \vdots \\ A_{*n} \end{bmatrix} \tag{2.2}$$

We can now state a generalization of (1.1) for matrix-valued real Gaussian random variables.

<u>Theorem 1</u>:

Let A,B,C,D be matrices of dimension (p*q), (q*r), (r*s) and (s*t). Assume that the entries of these matrices are real random variables which (jointly) have a multivariate Gaussian distribution. Then the following result holds:

$$E\{ABCD\} = E\{AB\}.E\{CD\} + \sum_{k=1}^{r} [E\{e_k^T C \otimes A\}].[E\{D \otimes Be_k\}]$$

$$+ E\{A[E\{BC\}]D\} - 2E\{A\}E\{B\}E\{C\}E\{D\} \tag{2.3a}$$

An alternative expression for the second term in the r.h.s. of (2.3a) is:

$$\sum_{\ell=1}^{q} \sum_{m=1}^{s} [E\{Ae_\ell e_m^T C^T\}][E\{B^T e_\ell e_m^T D\}] \tag{2.3b}$$

For r=1 the expression (2.3a) can be simplified to:

$$E\{ABCD\} = E\{AB\}.E\{CD\} + E\{C \otimes A\} \; E\{D \otimes B\} + E\{A[E\{BC\}]D\}$$

$$- 2E\{A\}E\{B\}E\{C\}E\{D\} \tag{2.4}$$

Proof:

For ease of reference we first state the following results which can be readily verified (see e.g. [8],[13]):

Let X,Z,Y be matrices of dimensions (m*n),(n*p) and (p*r) respectively; then

$$(X \otimes Y)^T = (X^T \otimes Y^T) \tag{2.5}$$

$$\mathrm{Vec}(XZY) = (Y^T \otimes X)\ \mathrm{Vec}\ Z \tag{2.6}$$

Next we note that for $1 \leq i \leq p$, $1 \leq j \leq t$:

$$(E\{ABCD\})_{ij} = E\ e_i^T ABCDe_j = E \sum_{k=1}^{r} e_i^T ABe_k e_k^T CDe_j$$

$$= E\Big\{ \sum_{k=1}^{r} \Big(\sum_{\ell=1}^{q} e_i^T Ae_\ell e_\ell^T Be_k \Big)\Big(\sum_{m=1}^{s} e_k^T Ce_m e_m^T De_j \Big)\Big\}$$

$$= \sum_{k=1}^{r} \sum_{\ell=1}^{q} \sum_{m=1}^{s} E\{ e_i^T Ae_\ell e_\ell^T Be_k e_k^T Ce_m e_m^T De_j \} \tag{2.7}$$

Using the formula (1.1) we thus obtain

$$(E\{ABCD\})_{ij} = \sum_{k=1}^{r} \sum_{\ell=1}^{q} \sum_{m=1}^{s} \{ E(e_i^T Ae_\ell e_\ell^T Be_k)\ E(e_k^T Ce_m e_m^T De_j) +$$

$$+ E(e_i^T Ae_\ell e_k^T Ce_m)\ E(e_\ell^T Be_k e_m^T De_j) + E(e_i^T Ae_\ell e_m^T De_j)\ E(e_\ell^T Be_k e_k^T Ce_m)$$

$$- 2E(e_i^T Ae_\ell)\ E(e_\ell^T Be_k)\ E(e_k^T Ce_m)\ E(e_m^T De_j) \}$$

$$= e_i^T \Big\{ \sum_k \Big[E\Big(\sum_\ell Ae_\ell e_\ell^T B \Big) \Big] e_k e_k^T \Big[E\Big(\sum_m Ce_m e_m^T D \Big) \Big] \Big\} e_j$$

$$+ \sum_k \sum_\ell \sum_m E\{ (e_i^T Ae_\ell)(e_k^T Ce_m) \}\ E\{ (e_k^T B^T e_\ell)(e_j^T D^T e_m) \}$$

$$+ \sum_\ell \sum_m E\Big\{ (e_i^T Ae_\ell)\Big(\sum_k E[e_\ell^T Be_k e_k^T Ce_m] \Big)(e_m^T De_j) \Big\}$$

$$-2e_i^T \Big\{ \sum_k \Big[\sum_\ell (E\{A\} e_\ell e_\ell^T E\{B\}) \Big]\ e_k e_k^T \Big[\sum_m (E\{C\} e_m e_m^T E\{D\}) \Big] \Big\} e_j$$

$$=: A_1 + A_2 + A_3 + A_4 \tag{2.8}$$

It can easily be verified that

$$A_1 = e_i^T [E\{AB\} \cdot E\{CD\}] e_j \tag{2.9}$$

$$A_2 = e_i^T \sum_{\ell=1}^{q} \sum_{m=1}^{s} [E\{Ae_\ell e_m^T C^T\}][E\{B^T e_\ell e_m^T D\}] e_j \tag{2.10}$$

$$A_3 = e_i^T E\{A[E\{BC\}]D\} e_j \tag{2.11}$$

$$A_4 = -2e_i^T [E\{A\} \; E\{B\} \; E\{C\} \; E\{D\}] e_j \tag{2.12}$$

An alternative expression for A_2 can be obtained as follows.
Using the fact that for (q*s) matrices R and T:

$$\sum_{\ell=1}^{q} \sum_{m=1}^{s} R_{\ell m} T_{\ell m} = [\mathrm{Vec}(R)]^T [\mathrm{Vec}(T)] \tag{2.13}$$

we obtain

$$A_2 = \sum_k [\mathrm{Vec}(E\{(e_i^T A)^T \; e_k^T C\})]^T \; [\mathrm{Vec}(E\{(e_k^T B^T)^T \; e_j^T D^T\})] \tag{2.14}$$

Using the relation (2.6) we have:

$$\mathrm{Vec}[(e_i^T A)^T e_k^T C] = \mathrm{Vec}(A^T e_i e_k^T C) = (C^T e_k \otimes A^T) \; \mathrm{Vec}(e_i) \tag{2.15}$$

and

$$\mathrm{Vec}[(e_k^T B^T)^T e_j^T D^T] = \mathrm{Vec}(Be_k e_j^T D^T) = (D \otimes Be_k) \; \mathrm{Vec}(e_j^T) \tag{2.16}$$

Inserting (2.15) and (2.16) into (2.14) we obtain, by using (2.5),

$$A_2 = e_i^T \{ \sum_{k=1}^{r} [E\{e_k^T C \otimes A\}] \cdot [E\{D \otimes Be_k\}] \} e_j \tag{2.17}$$

and the proof of (2.3a,b) is concluded.
If r = 1 then $e_k^T C = C$ and $Be_k = B$, and the expression (2.4) follows from (2.3a). Thus the proof is finished. ■

Theorem 1 provides a compact expression for the expectation of the product of four real matrix-valued random variables having a Gaussian distribution. An application of this result is presented in the next section. Another application is presented in the following.

In what follows, let us relax the assumption that the matrix random variables A,B,C and D are real-valued. In other words, A,B,C,D may consist of elements which are complex-valued variables. The assumption of Gaussianity is maintained. This means that the real and imaginary parts of the entries of A,B,C,D are assumed to be jointly Gaussian distributed. Under these conditions we claim that <u>the formulas of theorem 1 continue to hold for complex A,B,C,D matrices</u>. To prove this claim it would clearly be necessary and sufficient to show that the scalar formula (1.1) holds for (scalar) complex Gaussian random variables as well. This is shown in the next lemma:

<u>Lemma 1</u>:
Let the complex scalar-valued random variables x_1, x_2, x_3 and x_4 be jointly Gaussian distributed (that is to say, their real and imaginary parts are joint Gaussian random variables). Then the formula (1.1) applies.

<u>Proof</u>: It should, in principle, be possible to prove the assertion of the lemma by making use of formula (1.1) for real variables. However, the calculations involved appear to be <u>very</u> tedious.
A much simpler proof can be obtained by using theorem 1.
Let the real and imaginary parts of a variable x be denoted by $\bar{x}$ and $\tilde{x}$, respectively. To each variable x we associate the real-valued matrix

$$X = \begin{bmatrix} \bar{x} & -\tilde{x} \\ \tilde{x} & \bar{x} \end{bmatrix} \tag{2.18}$$

We denote this association by the symbol "$\sim$":

$$X \sim x \tag{2.19}$$

It can easily be verified that for two complex variables x and y,

$$X\,Y = \begin{bmatrix} \overline{xy} & -\widetilde{xy} \\ \widetilde{xy} & \overline{xy} \end{bmatrix} = Y\,X \tag{2.20}$$

In other words, the matrix XY (or YX) is associated with the variable xy.

Using formula 2.3 (with expression (2.3b) for the second term), we can write (let X_i denote the matrix (2.18) associated with x_i):

$$E\{X_1X_2X_3X_4\} = T_1 + T_2 + T_3 + T_4 \tag{2.21}$$

where

$$T_1 = (EX_1X_2)(EX_3X_4) \sim (Ex_1x_2)(Ex_3x_4) \tag{2.22}$$

$$T_2 = \left(E\begin{bmatrix}\bar{x}_1\\ \tilde{x}_1\end{bmatrix}\begin{bmatrix}\bar{x}_3 & \tilde{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\bar{x}_2\\ -\tilde{x}_2\end{bmatrix}\begin{bmatrix}\bar{x}_4 & -\tilde{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}\bar{x}_1\\ \tilde{x}_1\end{bmatrix}\begin{bmatrix}-\tilde{x}_3 & \bar{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\bar{x}_2\\ -\tilde{x}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_4 & \bar{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}-\tilde{x}_1\\ \bar{x}_1\end{bmatrix}\begin{bmatrix}\bar{x}_3 & \tilde{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\tilde{x}_2\\ \bar{x}_2\end{bmatrix}\begin{bmatrix}\bar{x}_4 & -\tilde{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}-\tilde{x}_1\\ \bar{x}_1\end{bmatrix}\begin{bmatrix}-\tilde{x}_3 & \bar{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\tilde{x}_2\\ \bar{x}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_4 & \bar{x}_4\end{bmatrix}\right)$$

$$=\left(E\begin{bmatrix}\bar{x}_1\\ \tilde{x}_1\end{bmatrix}\begin{bmatrix}\bar{x}_3 & -\tilde{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\bar{x}_2\\ \tilde{x}_2\end{bmatrix}\begin{bmatrix}\bar{x}_4 & -\tilde{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}\bar{x}_1\\ \tilde{x}_1\end{bmatrix}\begin{bmatrix}\bar{x}_3 & -\tilde{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}-\tilde{x}_2\\ \bar{x}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_4 & \bar{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}-\tilde{x}_1\\ \bar{x}_1\end{bmatrix}\begin{bmatrix}\tilde{x}_3 & \bar{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}\bar{x}_2\\ \tilde{x}_2\end{bmatrix}\begin{bmatrix}\bar{x}_4 & -\tilde{x}_4\end{bmatrix}\right) +$$

$$+\left(E\begin{bmatrix}-\tilde{x}_1\\ \bar{x}_1\end{bmatrix}\begin{bmatrix}\tilde{x}_3 & \bar{x}_3\end{bmatrix}\right)\left(E\begin{bmatrix}-\tilde{x}_2\\ \bar{x}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_4 & \bar{x}_4\end{bmatrix}\right)$$

$$= E\begin{bmatrix}\bar{x}_1 & -\tilde{x}_1\\ \tilde{x}_1 & \bar{x}_1\end{bmatrix}\begin{bmatrix}\bar{x}_3 & -\tilde{x}_3\\ \tilde{x}_3 & \bar{x}_3\end{bmatrix} \cdot E\begin{bmatrix}\bar{x}_2 & -\tilde{x}_2\\ \tilde{x}_2 & \bar{x}_2\end{bmatrix}\begin{bmatrix}\bar{x}_4 & -\tilde{x}_4\\ \tilde{x}_4 & \bar{x}_4\end{bmatrix} =$$

$$= (E\,X_1X_3)\,(E\,X_2X_4) \sim (E\,x_1x_3)(E\,x_2x_4) \tag{2.23}$$

$$T_3 = E\{X_1[E\{X_2X_3\}]X_4\} = (EX_1X_4)(EX_2X_3) \sim (Ex_1x_4)(Ex_2x_3) \quad (2.24)$$

$$T_4 = -2(EX_1)(EX_2)(EX_3)(EX_4) \sim -2(Ex_1)(Ex_2)(Ex_3)(Ex_4) \quad (2.25)$$

Since $E(X_1X_2X_3X_4) \sim E(x_1x_2x_3x_4)$, the proof is completed.

■

We were unable to locate a reference containing the result of lemma 1. Only a special case of this result, which holds under a certain restriction on the Gaussian distributions of $\{x_i\}$ (see [9]-[11]), appears to be known (see e.g. [12]).

3. An application

Consider the following multivariate linear regression equation

$$y(t) = \Phi^T(t)\ \theta^* + v(t) \quad (3.1)$$

where y(t) is the n_yx1 output vector; $\theta^* \in R^{n_\theta}$ denotes the unknown parameter vector; $\Phi(t)$ is the $(n_\theta x n_y)$ regressor matrix which may contain delayed values of y(t), and v(t) is an n_y-dimensional disturbance term. We assume that the entries of $\Phi(t)$ and v(t) are real stationary stochastic processes and that Ev(t) = 0. A fairly large class of systems (for example, noisy weighting function and difference equation systems) can be represented in the form (3.1) (see e.g. [7], [13]).
Let the unknown parameter vector θ^* be estimated by the Instrumental Variable method (IV), (see [7])

$$\hat{\theta}_N := [\sum_{t=1}^{N} Z(t)\Phi^T(t)]^{-1} \sum_{t=1}^{N} Z(t)y(t) \quad (3.2)$$

where Z(t) is an IV-matrix of dimension $n_\theta x n_y$, whose entries are real stationary stochastic processes and which satisfies

$$E\ Z(T)\ \Phi^T(t) \quad \text{is non-singular} \quad (3.3)$$

$$E\, Z_{ij}(t)\, v_k(s) = 0 \quad 1 \leq i \leq n_\theta;\ 1 \leq j,k \leq n_y \quad \text{for all } s \geq t \quad (\text{or } s \leq t) \tag{3.4}$$

The asymptotic (for large N) behaviour of $\hat{\theta}_N$ can, under the assumptions stated, be established as follows. From (3.1) and (3.2) we obtain

$$\sqrt{N}\,(\hat{\theta}_N - \theta^*) = \Big[\frac{1}{N}\sum_{t=1}^{N} Z(t)\Phi^T(t)\Big]^{-1}\Big[\frac{1}{\sqrt{N}}\sum_{t=1}^{N} Z(t)\, v(t)\Big] \tag{3.5}$$

Since

$$\frac{1}{N}\sum_{t=1}^{N} Z(t)\,\Phi^T(t) \underset{N\to\infty}{\to} R := E\, Z(t)\,\Phi^T(t) \quad (\text{wp1}) \tag{3.6}$$

we have that (see appendix 4 in [7])

$$\sqrt{N}\,(\hat{\theta}_N - \theta^*) \xrightarrow[N\to\infty]{\text{dist}} N(0, R^{-1}PR^{-T}) \tag{3.7}$$

where

$$P := \lim_{N\to\infty} E\Big(\frac{1}{N}\sum_{t=1}^{N}\sum_{s=1}^{N} Z(t)\, v(t)\, v^T(s)\, Z^T(s)\Big) \tag{3.8}$$

(i.e. $\sqrt{N}\,(\hat{\theta}_N - \theta^*)$ converges in distribution to a Gaussian random variable with zero mean and variance $R^{-1}PR^{-T}$).
An explicit expression for P (and hence for the asymptotic covariance matrix of the IV estimator (3.2)), can be found in [7], [13]. Our purpose here is to make use of theorem 1 to provide a simple derivation of that expression. In doing so we have to impose the Gaussianity assumption on the stochastic processes involved.

Using theorem 1 we get

$$\begin{aligned} E\, Z(t)\, v(t)\, v^T(s)\, Z^T(s) &= E\{Z(t)\, v(t)\}\, E\{v^T(s) Z^T(s)\} + \\ &+ E\{v^T(s) \otimes Z(t)\}\, E\{Z^T(s) \otimes v(t)\} + E\{Z(t)\, E[v(t) v^T(s)] Z^T(s)\} \\ &- 2E\{Z(t)\}E\{v(t)\}E\{v^T(s)\}\, E\{Z^T(s)\} \end{aligned} \tag{3.9}$$

Due to (3.4) and since $E\{v(t)\} = 0$, expression (3.9) simplifies to:

$$E\{Z(t)\ v(t)\ v^T(s)\ Z^T(s)\} = E\{Z(t)\ E[v(t)\ v^T(s)]Z^T(s)\} \qquad (3.10)$$

Defining

$$R_v(\tau) := E\ v(t+\tau)\ v^T(t) \qquad (3.11)$$

and inserting (3.10) into (3.8) we obtain

$$P = \lim_{N\to\infty} \frac{1}{N} \sum_{\tau=-N}^{N} (N-|\tau|)\ E\{Z(t+\tau)R_v(\tau)\ Z^T(t)\} =$$

$$= \sum_{\tau=-\infty}^{+\infty} E\{Z(t+\tau)R_v(\tau)Z^T(t)\}$$

$$- \lim_{N\to\infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau|\ E\{Z(t+\tau)R_v(\tau)Z^T(t)\} \qquad (3.12)$$

Due to the stationarity condition imposed on $Z(t)$ and $v(t)$, the limit of the second term in (3.12) can easily be shown to be zero. Thus

$$P = \sum_{\tau=-\infty}^{+\infty} E\{Z(t+\tau)\ R_v(\tau)\ Z^T(t)\} \qquad (3.13)$$

This is exactly the expression for P derived in [7], [13] via a different technique. Inserting (3.13) into $R^{-1}PR^T$ we obtain an expression for the asymptotic covariance matrix of the IV estimator $\hat{\theta}_N$.

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