[Journal of Stochastic Analysis](https://digitalcommons.lsu.edu/josa)

Volume 2 | Number 3 Article 11

September 2021

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Journal of Stochastic Analysis
Vol. 2, No. 3 (2021) Article 11 (21 pages)
DOI: 10.31390/josa.2.3.11

ON THE EXPONENTIAL MOMENTS OF ADDITIVE PROCESSES

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Dedicated to the memory of Professor Hiroshi Kunita

Abstract. A theorem on the exponential moments of general $\mathbb{R}$-valued additive processes will be established. A condition that implies the integrability of the exponential of additive processes will be proposed and furthermore the representation of their exponential moments by their characteristics will be shown.

In the previous paper [1], the same problem as above has been investigated in the case when the underlying additive processes have the structure of semimartingales. In this paper, another proof for this case will be presented. It will be more inherent and simpler than the previous one. Moreover, the result will be generalized to the case when the underlying additive processes do not necessarily have the structure of semimartingales.

1. Introduction

In this paper, we will establish a theorem on the exponential moments of general $\mathbb{R}$-valued additive processes.

Let $(X_t)_{t\in[0,T]}$, $T \in (0,\infty)$, be an $\mathbb{R}$-valued additive process, that is, a real-valued stochastic process with independent increments. We will propose a condition under which the exponential of additive process (e^{X_t}) can be integrable and furthermore represent the expectation $E[\mathrm{e}^{X_t}]$ by the characteristics.

It is a simple but fundamental problem in the probability theory because the exponential moment $E[\mathrm{e}^{X_t}]$ can be regarded as the Laplace transform at 1 of the law of X_t. In the case when (X_t) is a Lévy process, that is, a stochastically continuous stochastic process with stationary independent increments, a complete answer to this problem is stated as Theorem 25.17 in [8]. Furthermore, in the previous paper [1], we have discussed the case when (X_t) has the structure of semimartingale. See Theorem 1 in [1]. This result plays a fundamental rôle in determining the minimal entropy martingale measure for $(S_0\,\mathrm{e}^{X_t})$ with positive constant S_0. See [2] for the details. On the other hand, in [6], it is pointed out that the result of [1] can be used to extend their main result on moderate deviations

Received 2021-1-6; Accepted 2021-7-29; Communicated by S. Aida, D. Applebaum, Y. Ishikawa, A. Kohatsu-Higa, and N. Privault.

2010 *Mathematics Subject Classification.* Primary 60G51; Secondary 60H05.

Key words and phrases. Exponential moment, additive process, process with independent increments, semimartingale, Lévy process.

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for additive processes without fixed jump discontinuities to the one for additive processes with fixed jump discontinuities.

The purpose of this paper is to generalize these results above to the case when the underlying additive processes do not necessarily have the structure of semimartingales. A main result, Theorem 2.1, will be stated in Section 2.

A proof of Theorem 2.1 will be given in Section 3. The first part (Section 3.1) will deal with the case when the additive process (X_t) is also a semimartingale. The content is regarded as another proof of Theorem 1 in [1]. The proof given there heavily depends on a result in the theory of semimartingale, Theorem 3.2 in [5], whereas the proof given here is more inherent and simpler than the previous one. The second part (Section 3.2) will deal with the extention to the case when (X_t) is not a semimartingale.

2. Exponential Moments of Additive Processes

Let $(X_t)_{t\in[0,T]}$, $T>0$, be an $\mathbb{R}$-valued additive process defined on a probability space $(\Omega,\mathscr{F},P)$ equipped with a filtration $(\mathscr{F}_t)$, that is an increasing and right-continuous family of sub-σ-fields of $\mathscr{F}$. To be precise, (X_t) is an $\mathbb{R}$-valued adapted càdlàg process with $X_0=0$ that has independent increments: for all $s\le t$, the increment X_t-X_s is independent of $\mathscr{F}_s$. In [4], such a process as (X_t) is called a PII (a process with independent increments) ([4] Definition II.4.1 (p.101)).

Let $(C_t, n(dtdx), B_t)$ be the characteristics, in the sense of Theorem II.5.2 in [4] (pp.114-115), of (X_t) associated with the truncation function $h(x):=xI_{\{|x|\le 1\}}(x)$ on $\mathbb{R}$. This means that the law of (X_t) is characterized by the following formula, which is an extension of the Lévy-Khinchin formula: For any $\xi\in\mathbb{R}$ and $s\le t$,

$$\begin{aligned}E[\mathrm{e}^{i\xi(X_t-X_s)}] = \exp\Big[&-\frac{1}{2}\xi^2(C_t-C_s)+i\xi(B_t-B_s)\\&+\int_{(s,t]}\int_{\mathbb{R}\setminus\{0\}}\big(\mathrm{e}^{i\xi x}-1-i\xi h(x)\big)I_{J^c}(u)\,n(dudx)\Big]\\&\times\prod_{u\in(s,t]}\Big\{\mathrm{e}^{-i\xi\Delta B_u}\big[1+\int_{\mathbb{R}\setminus\{0\}}\big(\mathrm{e}^{i\xi x}-1\big)\,n(\{u\},dx)\big]\Big\},\end{aligned}\tag{2.1}$$

where $i=\sqrt{-1}$ and $J:=\{t>0;\ n(\{t\},\mathbb{R}\setminus\{0\})>0\}$ denotes the set of all fixed times of discontinuity of (X_t). Also, A^c denotes the complement of the set A. As fundamental properties of characteristics, the following facts are known:

- $(C_t, n(dtdx), B_t)$ are deterministic, since (X_t) has independent increments.
- $$\int_{(0,T]}\int_{\mathbb{R}\setminus\{0\}}(|x|^2\wedge 1)I_{J^c}(u)\,n(dudx)<\infty,\tag{2.2}$$
  where $\alpha\wedge\beta:=\min\{\alpha,\beta\}$ for $\alpha,\beta\in\mathbb{R}$, and $n(\{u\},\mathbb{R}\setminus\{0\})\le 1$ ([4] II.5.5-(i),(iii),(v) (p.114)).
- (B_t) is a càdlàg function ([4] II.5.3 (p.114)). Note that (B_t) is not necessarily a function with finite variation on $[0,T]$.
  $$\Delta B_u := B_u - B_{u-} = \int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u\},dx),\tag{2.3}$$

where $B_{u-} := \lim_{v\uparrow u} B_v$ ([4] II.5.5-(v) (p.114)).

The following property also comes from the formula (2.1):

- The law of the random variable ΔX_u is
$$n(\{u\}, dx) + (1 - n(\{u\}, \mathbb{R}))\delta_0(dx), \tag{2.4}$$
where $\delta_0(dx)$ denotes the Dirac measure at the origin 0 ([4] Theorem II.5.2 - a) (p.115)).

The purpose of this paper is to establish the following theorem:

Theorem 2.1. *Let $(X_t)_{t\in[0,T]}$, $T > 0$, be an $\mathbb{R}$-valued additive process defined on a probability space $(\Omega, \mathscr{F}, P)$ equipped with a filtration $(\mathscr{F}_t)$, and let $(C_t, n(dtdx), B_t)$ be the characteristics of (X_t) associated with the truncation function $h(x) := xI_{\{|x|\le 1\}}(x)$. Suppose that*
$$\int_{(0,T]}\int_{\{x>1\}} \mathrm{e}^x\, n(dudx) < \infty. \tag{2.5}$$
Then, for all $t \in [0, T]$,
$$\begin{aligned} E[\mathrm{e}^{X_t}] = \exp\Big[\frac{1}{2}C_t + B_t + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^x - 1 - h(x)\big) I_{J^c}(u)\, n(dudx)\Big] \\ \times \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u}\Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx)\Big]. \end{aligned} \tag{2.6}$$

The meaning of this theorem is clear: under the integrability condition (2.5) the exponential moment of $E[\mathrm{e}^{X_t}]$ is represented by the characteristics $(C_t, n(dtdx), B_t)$ as (2.6). It is regarded as a representation of the Laplace transform at 1 of the law of X_t. In the case when (X_t) is a Lévy process, that is, a stochastically continuous stochastic process with stationary independent increments, a corresponding result is stated as a part of Theorem 25.17 in [8]. Furthermore, in the previous paper [1], we have discussed the case when (X_t) has the structure of semimartingale. See Theorem 1 in [1]. This result plays a fundamental rôle in determining the minimal entropy martingale measure for $(S_0\,\mathrm{e}^{X_t})$ with positive constant S_0. See [2] for the details. On the other hand, in [6], it is pointed out that the result of [1] can be used to extend their main result on moderate deviations for additive processes without fixed jump discontinuities to the one for additive processes with fixed jump discontinuities. See Concluding remarks (i) in [6] (p.651).

We will prove Theorem 2.1 in Section 3. In the first part (Section 3.1), we will discuss the case when the additive process (X_t) is also a semimartingale. The content is regarded as another proof of Theorem 1 in [1]. The proof given there heavily depends on a result in the theory of semimartingale, Theorem 3.2 in [5], whereas the proof given here might be more inherent and simpler than the previous one. In the second part (Section 3.2), we will investigate the case when (X_t) is not a semimartingale to complete our proof of Theorem 2.1.

3. Proof of Theorem 2.1

3.1. The case of PII-semimartingales. In this subsection, we will give a proof of Theorem 2.1 in the case when the additive process (X_t) has the structure of semimartingale. In [4], such a process is simply called a PII-semimartingale ([4] (p.106)).

We first introduce the canonical representation of (X_t) associated with the truncation function h:

$$X_t = X_t^c + B_t + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} h(x)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} \check{h}(x)\,N(dudx). \tag{3.1}$$

Here, (X_t^c) is a continuous (local) martingale with $X_0^c = 0$ and $\langle X^c\rangle_t = C_t$. $N(dudx)$ denotes the counting measure of the jumps of (X_t):

$$N((0,t],A) := \sharp\{u \in (0,t];\ \Delta X_u := X_u - X_{u-} \in A\}$$

for $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, where $X_{u-} := \lim_{v\uparrow u} X_v$ and $\mathcal{B}(\mathbb{R}\backslash\{0\})$ is the Borel σ-field on $\mathbb{R}\backslash\{0\}$. We denote by $\widetilde{N}(dudx) := N(dudx) - n(dudx)$ the compensated measure of $N(dudx)$. Also, we set $\check{h}(x) := x - h(x) = xI_{\{|x|>1\}}(x)$. See [4] Theorem II.2.34 (p.84) for the canonical representation of semimartingales. Since (X_t) is assumed to be a semimartingale in this subsection, the integrability (2.2) of $n(dudx)$ is strengthened as follows:

$$\int_{(0,T]}\int_{\mathbb{R}\backslash\{0\}} (|x|^2 \wedge 1)\,n(dudx) < \infty, \tag{3.2}$$

([4] II.2.13 (p.77)).

Moreover, we decompose (X_t) more finely as follows:

$$X_t = X_t^c + B_t + X_t^{d,c} + X_t^{d,d}, \tag{3.3}$$

where

$$X_t^{d,c} := \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} h(x)I_{J^c}(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} \check{h}(x)I_{J^c}(u)\,N(dudx) \tag{3.4}$$

$$X_t^{d,d} := \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} h(x)I_J(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} \check{h}(x)I_J(u)\,N(dudx). \tag{3.5}$$

The first stage of our proof is to show the following property:

Proposition 3.1. *The processes (X_t^c), $(X_t^{d,c})$ and $(X_t^{d,d})$ are independent.*

In order to prove Proposition 3.1, we prepare the following lemma:

Lemma 3.2. *Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}\backslash\{0\}$ be arbitrarily fixed and set*

$$Z_t := \xi_1 X_t^c + \xi_2 X_t^{d,c} + \xi_3(X_t^{d,d} + B_t). \tag{3.6}$$

Then the characteristics $(C_t^Z, n^Z(dudz), B_t^Z)$ of (Z_t) associated with h are given as follows:

$$C_t^Z = \xi_1^2 C_t; \tag{3.7}$$

$$n^Z((0,t],A) = \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} I_A(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u))\,n(dudx); \tag{3.8}$$

$$\begin{aligned} B_t^Z = \xi_3 B_t + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} \big((h(\xi_2 x) - \xi_2 h(x))I_{J^c}(u) \\ + (h(\xi_3 x) - \xi_3 h(x))I_J(u)\big)\,n(dudx). \end{aligned} \tag{3.9}$$

Proof. Note that

$$\begin{aligned}
&\xi_2 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{J^c}(u)\, \widetilde{N}(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(\xi_2 x) I_{J^c}(u)\, \widetilde{N}(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 h(x) - h(\xi_2 x)\big) I_{J^c}(u)\, N(dudx) \\
&\quad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 h(x) - h(\xi_2 x)\big) I_{J^c}(u)\, n(dudx)
\end{aligned}$$

and that

$$\begin{aligned}
&\xi_2 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_{J^c}(u)\, N(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(\xi_2 x) I_{J^c}(u)\, N(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 \check{h}(x) - \check{h}(\xi_2 x)\big) I_{J^c}(u)\, N(dudx).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\xi_2 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{J^c}(u)\, \widetilde{N}(dudx) + \xi_2 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_{J^c}(u)\, N(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(\xi_2 x) I_{J^c}(u)\, \widetilde{N}(dudx) + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(\xi_2 x) I_{J^c}(u)\, N(dudx) \\
&\quad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 h(x) - h(\xi_2 x)\big) I_{J^c}(u)\, n(dudx),
\end{aligned}$$

because

$$\begin{aligned}
&\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 h(x) - h(\xi_2 x)\big) I_{J^c}(u)\, N(dudx) \\
&\qquad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_2 \check{h}(x) - \check{h}(\xi_2 x)\big) I_{J^c}(u)\, N(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} (\xi_2 x - \xi_2 x) I_{J^c}(u)\, N(dudx) \\
&= 0.
\end{aligned}$$

By the same way, we have

$$\begin{aligned}
&\xi_3 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_J(u)\, \widetilde{N}(dudx) + \xi_3 \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_J(u)\, N(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(\xi_3 x) I_J(u)\, \widetilde{N}(dudx) + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(\xi_3 x) I_J(u)\, N(dudx) \\
&\quad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\xi_3 h(x) - h(\xi_3 x)\big) I_J(u)\, n(dudx).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\xi_2 X_t^{d,c} + \xi_3 X_t^{d,d} \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(h(\xi_2 x) I_{J^c}(u) + h(\xi_3 x) I_J(u)\big)\, \widetilde{N}(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\check{h}(\xi_2 x) I_{J^c}(u) + \check{h}(\xi_3 x) I_J(u)\big)\, N(dudx) \\
&\quad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big((\xi_2 h(x) - h(\xi_2 x)) I_{J^c}(u) + (\xi_3 h(x) - h(\xi_3 x)) I_J(u)\big)\, n(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u))\, \widetilde{N}(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u)\big)\, N(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big((h(\xi_2 x) - \xi_2 h(x)) I_{J^c}(u) + (h(\xi_3 x) - \xi_3 h(x)) I_J(u)\big)\, n(dudx).
\end{aligned} \tag{3.10}$$

Here, since $X_u - B_u = X_u^c + X_u^{d,c} + X_u^{d,d}$,

$$\begin{aligned}
\Delta Z_u &= \xi_2 \Delta X_u^{d,c} + \xi_3 (\Delta X_u^{d,d} + \Delta B_u) \\
&= \begin{cases} \xi_2 \Delta (X-B)_u, & u \in J^c \\ \xi_3 (\Delta (X-B)_u + \Delta B_u), & u \in J \end{cases} \\
&= \begin{cases} \xi_2 \Delta X_u, & u \in J^c \\ \xi_3 \Delta X_u, & u \in J. \end{cases}
\end{aligned}$$

Hence, if we denote by $N^Z(dudz)$ the counting measure of the jumps of (Z_t), we have

$$N^Z((0,t], A) = \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} I_A(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u))\, N(dudx),$$

which implies that

$$\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u)\big)\, N(dudx) = \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(z)\, N^Z(dudz)$$

and that the compensator $n^Z(dudz)$ is given by (3.8). Also, we denote by $\widetilde{N}^Z(dudz)$ the compensated measure: $\widetilde{N}^Z(dudz) := N^Z(dudz) - n^Z(dudz)$. Then, we see that

$$\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(\xi_2 x I_{J^c}(u) + \xi_3 x I_J(u))\, \widetilde{N}(dudx) = \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(z)\, \widetilde{N}^Z(dudz).$$

Thus, we see from (3.8) and (3.10) that

$$\begin{aligned}
Z_t = \xi_1 X_t^c + \Big\{ &\xi_3 B_t \\
&+ \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big((h(\xi_2 x) - \xi_2 h(x)) I_{J^c}(u) + (h(\xi_3 x) - \xi_3 h(x)) I_J(u)\big)\, n(dudx) \Big\}
\end{aligned}$$

$$+\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} h(z)\,\widetilde{N}^Z(dudz)+\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} \check{h}(z)\,N^Z(dudz), \tag{3.11}$$

which gives the canonical representation of (Z_t) associated with h. Therefore, it is easy to see that C_t^Z and B_t^Z are given by (3.7) and (3.9), respectively. □

Note that $J^Z = J$ under the assumption $\xi_3 \neq 0$, since

$$\begin{aligned} J^Z &:= \{u>0; n^Z(\{u\},\mathbb{R}\backslash\{0\})>0\} \\ &= \{u>0; \int_{\mathbb{R}\backslash\{0\}} I_{\mathbb{R}\backslash\{0\}}(\xi_3 x I_J(u))\, n(\{u\},dx)>0\} \\ &= \{u>0; I_J(u) n(\{u\},\mathbb{R}\backslash\{0\})>0\} \\ &= J. \end{aligned}$$

Proof of Proposition 3.1. In order to prove Proposition 3.1, it is sufficient to show that for all $\xi_k \in \mathbb{R}$ $(k=1,2,3)$

$$\begin{aligned} &E[\mathrm{e}^{i\{\xi_1(X_t^c-X_s^c)+\xi_2(X_t^{d,c}-X_s^{d,c})+\xi_3(X_t^{d,d}-X_s^{d,d})\}}] \\ &= E[\mathrm{e}^{i\xi_1(X_t^c-X_s^c)}]\times E[\mathrm{e}^{i\xi_2(X_t^{d,c}-X_s^{d,c})}]\times E[\mathrm{e}^{i\xi_3(X_t^{d,d}-X_s^{d,d})}]. \end{aligned} \tag{3.12}$$

Without loss of generality, we may assume that $\xi_k \neq 0$ for any $k=1,2,3$.

Combining the formula (2.1) for (Z_t) of (3.6) and Lemma 3.2, we have

$$\begin{aligned} &E[\mathrm{e}^{i\{\xi_1(X_t^c-X_s^c)+\xi_2(X_t^{d,c}-X_s^{d,c})+\xi_3(X_t^{d,d}-X_s^{d,d})+\xi_3(B_t-B_s)\}}] \\ &= E[\mathrm{e}^{i(Z_t-Z_s)}] \\ &= \exp\Big[-\frac{1}{2}(C_t^Z-C_s^Z)+i(B_t^Z-B_s^Z) \\ &\qquad +\int_{(s,t]}\int_{\mathbb{R}\backslash\{0\}}\big(\mathrm{e}^{iz}-1-ih(z)\big)I_{(J^Z)^c}\,n^Z(dudz)\Big] \\ &\quad\times\prod_{u\in(s,t]}\mathrm{e}^{i\Delta B_u^Z}\big[1+\int_{\mathbb{R}\backslash\{0\}}\big(\mathrm{e}^{iz}-1\big)\,n^Z(\{u\},dz)\big] \\ &= \exp\Big[-\frac{1}{2}\xi_1^2(C_t-C_s)+i\xi_3(B_t-B_s) \\ &\qquad +i\int_{(s,t]}\int_{\mathbb{R}\backslash\{0\}}\big((h(\xi_2 x)-\xi_2 h(x))I_{J^c}(u)+(h(\xi_3 x)-\xi_3 h(x))I_J(u)\big)\,n(dudx) \\ &\qquad +\int_{(s,t]}\int_{\mathbb{R}\backslash\{0\}}\big(\mathrm{e}^{i\xi_2 x}-1-ih(\xi_2 x)\big)I_{J^c}(u)\,n(dudx)\Big] \\ &\quad\times\prod_{u\in(s,t]}\mathrm{e}^{-i(\xi_3\Delta B_u+\int_{\mathbb{R}\backslash\{0\}}(h(\xi_3 x)-\xi_3 h(x))\,n(\{u\},dx))} \\ &\qquad\qquad\times\big[1+\int_{\mathbb{R}\backslash\{0\}}\big(\mathrm{e}^{i\xi_3 x}-1\big)\,n(\{u\},dx)\big] \\ &= \exp\Big[-\frac{1}{2}\xi_1^2(C_t-C_s)+i\xi_3(B_t-B_s) \\ &\qquad +\int_{(s,t]}\int_{\mathbb{R}\backslash\{0\}}\big(\mathrm{e}^{i\xi_2 x}-1-i\xi_2\,h(x)\big)I_{J^c}(u)\,n(dudx)\Big] \end{aligned}$$

$$\times \prod_{u\in(s,t]} \mathrm{e}^{-i\xi_3 \Delta B_u}\big[1+\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{i\xi_3 x}-1\big)\, n(\{u\},dx)\big]. \tag{3.13}$$

On the other hand, it follows from the formula (2.1) that

$$\begin{aligned} &E[\mathrm{e}^{i\xi(X_t-X_s)-i\xi(B_t-B_s)}] \\ &= \exp\Big[-\frac{1}{2}\xi^2(C_t-C_s)+\int_{(s,t]}\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{i\xi x}-1-i\xi h(x)\big) I_{(J)^c}\, n(dudx)\Big] \\ &\quad \times \prod_{u\in(s,t]} \mathrm{e}^{-i\xi \Delta B_u}\big[1+\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{i\xi x}-1\big)\, n(\{u\},dx)\big]. \end{aligned} \tag{3.14}$$

(1) Take $n\equiv 0$ and $\xi=\xi_1$ in (3.14). Then, since $X^{d,c}=X^{d,d}\equiv 0$, we have

$$E[\mathrm{e}^{i\xi_1(X^c_t-X^c_s)}] = \exp\Big[-\frac{1}{2}\xi_1^2(C_t-C_s)\Big]. \tag{3.15}$$

(2) Take $C\equiv 0$, $J=\emptyset$ and $\xi=\xi_2$ in (3.14). Then, since $X^c=X^{d,d}\equiv 0$, we have

$$E[\mathrm{e}^{i\xi_2(X^{d,c}_t-X^{d,c}_s)}] = \exp\Big[\int_{(s,t]}\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{i\xi_2 x}-1-i\xi_2\, h(x)\big) I_{J^c}(u)\, n(dudx)\Big]. \tag{3.16}$$

(3) Take $C\equiv 0$ and $J^c=\emptyset$ and $\xi=\xi_3$ in (3.14). Then, since $X^c=X^{d,c}\equiv 0$, we have

$$E[\mathrm{e}^{i\xi_3(X^{d,d}_t-X^{d,d}_s)}] = \prod_{u\in(s,t]} \mathrm{e}^{-i\xi_3\Delta B_u}\times\big[1+\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{i\xi_3 x}-1\big)\, n(\{u\},dx)\big]. \tag{3.17}$$

Therefore, combining these relations (3.15)$\sim$(3.17) with (3.13), we have

$$\begin{aligned} &E[\mathrm{e}^{i\{\xi_1(X^c_t-X^c_s)+\xi_2(X^{d,c}_t-X^{d,c}_s)+\xi_3(X^{d,d}_t-X^{d,d}_s)+\xi_3(B_t-B_s)\}}] \\ &= E[\mathrm{e}^{i\xi_1(X^c_t-X^c_s)}]\times E[\mathrm{e}^{i\xi_2(X^{d,c}_t-X^{d,c}_s)}]\times E[\mathrm{e}^{i\xi_3(X^{d,d}_t-X^{d,d}_s)}]\times \mathrm{e}^{i\xi_3(B_t-B_s)}, \end{aligned}$$

which immediately implies the equation (3.12). Thus, we have proved Proposition 3.1. □

We are now on the second stage of our proof of semimartingale case. Owing to Proposition 3.1, the proof will be completed if we establish the exponential moments of X^c_t, $X^{d,c}_t$ and $X^{d,d}_t$, respectively; they will be shown as Propositions 3.3, 3.4 and 3.12, respectively.

Proposition 3.3. *For all $t\in(0,T]$,*

$$E[\mathrm{e}^{X^c_t}] = \mathrm{e}^{\frac{1}{2}C_t}. \tag{3.18}$$

Proof. (3.15) implies that the law of X^c_t is the normal distribution with mean 0 and variance C_t. Hence, it is easy to see that (3.18) holds. □

Proposition 3.4. *For all $t\in(0,T]$,*

$$E[\mathrm{e}^{X^{d,c}_t}] = \exp\Big[\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{x}-1-h(x)\big) I_{J^c}(u)\, n(dudx)\Big]. \tag{3.19}$$

We will prove this proposition by deviding into several pieces: Lemmas 3.5 $\sim$ 3.10. The outline of the proof is similar to that of the proof of Theorem 25.17 in [8]. However, note that the stationarity of increments is assumed in the theorem but it is not assumed here.

Lemma 3.5. *Let*

$$\begin{aligned} X_t^{d,c,1} &:= \int_{(0,t]} \int_{\{|x|>1\}} x I_{J^c}(u)\, N(dudx)\,; \\ X_t^{d,c,0} &:= X_t^{d,c} - X_t^{d,c,1}. \end{aligned}$$

(1) $(X_t^{d,c,0})$ *and* $(X_t^{d,c,1})$ *are independent.*
(2) *For all* $\xi \in \mathbb{R}$,

$$E[\mathrm{e}^{i\xi X_t^{d,c,1}}] = \exp\Big[\int_{(0,t]} \int_{\{|x|>1\}} \big(\mathrm{e}^{i\xi x} - 1\big)\, I_{J^c}(u)\, n(dudx)\Big].$$

Proof. (1) It is immediate from Proposition 4′ in [3] (p.65).
(2) It is nothing but a special case of the formula (2.1). □

Lemma 3.6. *Fix* $t \in (0,T]$ *and let* μ_t^1 *be the law of* $X_t^{d,c,1}$. *Then,*

$$\mu_t^1 = \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (n_t^1)^{*k}, \tag{3.20}$$

where

$$\begin{aligned} \lambda &:= n((0,t] \cap J^c, \{|x| > 1\}); \\ n_t^1(dx) &:= I_{\{|x|>1\}}(x)\, n((0,t] \cap J^c, dx); \end{aligned}$$

$*k$ *denotes the* k*-fold convolution.*

Proof. By Lemma 3.5-(2), we see that

$$\begin{aligned} \mathscr{F}[\mu_t^1](\xi) &:= E[\mathrm{e}^{i\xi X_t^{d,c,1}}] \\ &= \exp\Big[\int_{(0,t]} \int_{\{|x|>1\}} \big(\mathrm{e}^{i\xi x} - 1\big)\, I_{J^c}(u)\, n(dudx)\Big] \\ &= \exp\Big[\int_{\mathbb{R}} \big(\mathrm{e}^{i\xi x} - 1\big)\, n_t^1(dx)\Big] \\ &= \exp\Big[\lambda \int_{\mathbb{R}} \big(\mathrm{e}^{i\xi x} - 1\big)\, \bar{n}_t^1(dx)\Big] \\ &= \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \Big(\int_{\mathbb{R}} \mathrm{e}^{i\xi x}\, \bar{n}_t^1(dx)\Big)^k \\ &= \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (\mathscr{F}[\bar{n}_t^1](\xi))^k \\ &= \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathscr{F}[(\bar{n}_t^1)^{*k}](\xi) \end{aligned}$$

$$= \mathscr{F}\Big[\mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}(n_t^1)^{*k}\Big](\xi),$$

where $\bar{n}_t^1(dx) := n_t^1(dx)/\lambda$. Therefore, from the uniqueness of the Fourier transform, we obtain the conclusion (3.20). □

Lemma 3.7. *For all $t \in (0,T]$, $\mathrm{e}^{X_t^{d,c,1}}$ is integrable and*

$$E[\mathrm{e}^{X_t^{d,c,1}}] = \exp\Big[\int_{(0,t]}\int_{\{|x|>1\}}\big(\mathrm{e}^x - 1\big)\, I_{J^c}(u)\, n(dudx)\Big]. \tag{3.21}$$

Proof. By Lemma 3.6, we see that

$$\begin{aligned}
E[\mathrm{e}^{X_t^{d,c,1}}] &= \int_{\mathbb{R}} \mathrm{e}^x\, \mu_t^1(dx) \\
&= \int_{\mathbb{R}} \mathrm{e}^x\, \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}(n_t^1)^{*k}(dx) \\
&= \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{R}} \mathrm{e}^x\, (n_t^1)^{*k}(dx) \\
&= \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{e}^{x_1+\cdots+x_k}\, n_t^1(dx_1)\cdots n_t^1(dx_k) \\
&= \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\int_{\mathbb{R}} \mathrm{e}^x\, n_t^1(dx)\Big)^k \\
&= \mathrm{e}^{-\lambda}\exp\Big[\int_{\mathbb{R}} \mathrm{e}^x\, n_t^1(dx)\Big] \\
&= \exp\Big[\int_{(0,t]}\int_{\{|x|>1\}}\big(\mathrm{e}^x - 1\big)\, I_{J^c}(u)\, n(dudx)\Big].
\end{aligned}$$

Hence, by the assumption (2.5), we obtain the conclusion:

$$E[\mathrm{e}^{X_t^{d,c,1}}] = \exp\Big[\int_{(0,t]}\int_{\{|x|>1\}}\big(\mathrm{e}^x - 1\big)\, I_{J^c}(u)\, n(dudx)\Big] < \infty. \qquad \square$$

Lemma 3.8. *Fix $t \in (0,T]$ and let μ_t^0 be the law of $X_t^{d,c,0}$. Then, for $\xi \in \mathbb{R}$,*

$$\begin{aligned}
\mathscr{F}[\mu_t^0](\xi) &:= E[\mathrm{e}^{i\xi X_t^{d,c,0}}] \\
&= \exp\Big[\int_{(0,t]}\int_{\{|x|\le 1\}}\big(\mathrm{e}^{i\xi x} - 1 - i\xi x\big) I_{J^c}(u)\, n(dudx)\Big].
\end{aligned} \tag{3.22}$$

Proof. It is a special case of the formula (2.1). □

Lemma 3.9. *For each $t \in (0,T]$, $\mathscr{F}[\mu_t^0](\xi)$ can be extended as an entire function on $\mathbb{C}$.*

Proof. We set

$$f(\xi) := \int_{(0,t]}\int_{\{|x|\le 1\}}\big(\mathrm{e}^{i\xi x} - 1 - i\xi x\big)\, I_{J^c}(u)\, n(dudx).$$

It is sufficient to show that $f(\xi)$ can be extended as an entire function on $\mathbb{C}$.

First, we show that $f(\xi)$ can be extended as a function on $\mathbb{C}$. By the mean value theorem, for any $\xi \in \mathbb{C}$,

$$\mathrm{e}^{i\xi x} - 1 - i\xi x = \int_0^1 (1-s)(i\xi x)^2 \mathrm{e}^{i\xi x s}\, ds.$$

Hence, we see that

$$|\mathrm{e}^{i\xi x} - 1 - i\xi x| \le |\xi x|^2 \mathrm{e}^{|\mathrm{Im}\xi|\cdot|x|}.$$

Therefore,

$$\begin{aligned}&\int_{(0,t]}\int_{\{|x|\le 1\}} |\mathrm{e}^{i\xi x} - 1 - i\xi x|\, I_{J^c}(u)\, n(dudx)\\ &\quad\le |\xi|^2 \mathrm{e}^{|\mathrm{Im}\xi|}\int_{(0,t]}\int_{\{|x|\le 1\}} |x|^2\, I_{J^c}(u)\, n(dudx) < \infty.\end{aligned}$$

Next, we show that the function $f(\xi)$ is differentiable at any $\xi \in \mathbb{C}$. Since

$$\frac{\partial}{\partial \xi}\Big(\mathrm{e}^{i\xi x} - 1 - i\xi x\Big) = ix \cdot i\xi x \int_0^1 \mathrm{e}^{i\xi x s}\, ds,$$

we have

$$\Big|\frac{\partial}{\partial \xi}\Big(\mathrm{e}^{i\xi x} - 1 - i\xi x\Big)\Big| \le |\xi|\cdot|x|^2 \mathrm{e}^{|\mathrm{Im}\xi|\cdot|x|}.$$

Hence, for any $R > 0$,

$$\begin{aligned}&\int_{(0,t]}\int_{\{|x|\le 1\}} \sup_{|\xi|<R}\Big|\frac{\partial}{\partial \xi}\Big(\mathrm{e}^{i\xi x} - 1 - i\xi x\Big)\Big| I_{J^c}(u)\, n(dudx)\\ \le\quad &\Big(\sup_{|\xi|<R}|\xi|\mathrm{e}^{|\mathrm{Im}\xi|}\Big) \times \int_{(0,t]}\int_{\{|x|\le 1\}} |x|^2\, I_{J^c}(u)\, n(dudx) < \infty.\end{aligned}$$

Thus, we conclude that the function $f(\xi)$ is differentiable at any $\xi \in \mathbb{C}$, and hence holomorphic on $\mathbb{C}$. $\square$

Lemma 3.10. *For all $t \in (0,T]$, $\mathrm{e}^{X_t^{d,c,0}}$ is integrable and*

$$E[\mathrm{e}^{X_t^{d,c,0}}] = \exp\Big[\int_{(0,t]}\int_{\{|x|\le 1\}} \big(\mathrm{e}^{x} - 1 - x\big)\, I_{J^c}(u)\, n(dudx)\Big]. \tag{3.23}$$

To prove this lemma, we will apply the following fact, which is stated as Lemma 25.7 in [8] (p.161).

Lemma 3.11. *Let μ be a probability measure on $\mathbb{R}$ and suppose that the Fourier transform $\mathscr{F}[\mu](\xi)$ can be extended as an entire function on $\mathbb{C}$. Then μ has finite $\mathrm{e}^{\alpha|x|}$-moment, that is, $\int_{\mathbb{R}} \mathrm{e}^{\alpha|x|}\,\mu(dx) < \infty$ for any $\alpha > 0$.*

Proof of Lemma 3.10. As we have seen in Lemma 3.9, $\mathscr{F}[\mu_t^0](\xi)$ can be extended as an entire function on $\mathbb{C}$. Hence it follows from Lemma 3.11 (take $\mu := \mu_t^0$) that μ_t^0 has finite $\mathrm{e}^{\alpha|x|}$-moment for any $\alpha > 0$. Therefore, $\mathrm{e}^{X_t^{d,c,0}}$ is integrable. Furthermore, take $\xi := -i$ in (3.22). Then we obtain (3.23). $\square$

Proof of Proposition 3.4. We have already seen in Lemma 3.5 that $X_t^{d,c,0}$ and $X_t^{d,c,1}$ are independent. Moreover, we have shown in Lemmas 3.7 and 3.10 the integrability of $\mathrm{e}^{X_t^{d,c,0}}$ and $\mathrm{e}^{X_t^{d,c,1}}$ and explicit representation of their expectations (3.23) and (3.21). By these considerations, we conclude that $\mathrm{e}^{X_t^{d,c}} = \mathrm{e}^{X_t^{d,c,0}} \times \mathrm{e}^{X_t^{d,c,1}}$ is integrable and that

$$
\begin{aligned}
E[\mathrm{e}^{X_t^{d,c}}] &= E[\mathrm{e}^{X_t^{d,c,0}}] \times E[\mathrm{e}^{X_t^{d,c,1}}] \\
&= \exp\Big[\int_{(0,t]} \int_{\{|x|\le 1\}} \big(\mathrm{e}^x - 1 - x\big) I_{J^c}(u)\, n(dudx)\Big] \\
&\quad \times \exp\Big[\int_{(0,t]} \int_{\{|x|>1\}} \big(\mathrm{e}^x - 1\big) I_{J^c}(u)\, n(dudx)\Big] \\
&= \exp\Big[\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^x - 1 - h(x)\big) I_{J^c}(u)\, n(dudx)\Big].
\end{aligned}
$$

□

Proposition 3.12. *For all $t \in (0,T]$,*

$$
\begin{aligned}
E[\mathrm{e}^{X_t^{d,d}}] &= \prod_{u\in(0,t]\cap J} \Big\{\mathrm{e}^{-\Delta B_u}\big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx)\big]\Big\} \qquad (3.24) \\
&= \exp\Big[\sum_{u\in(0,t]\cap J} \Big\{\log\big(1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx)\big) \\
&\qquad - \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\}, dx)\Big\}\Big].
\end{aligned}
$$

Proof. Since the set J is discrete and deterministic, we can set $(0,T]\cap J := \{u_k; k \in \mathbb{N}\}$. By (2.4) and (2.5), we see that $\mathrm{e}^{\Delta X_{u_k}}$ is integrable and that

$$
\begin{aligned}
E[\mathrm{e}^{\Delta X_{u_k}}] &= \int_{\mathbb{R}} \mathrm{e}^x\, n(\{u_k\}, dx) + (1 - n(\{u_k\}, \mathbb{R})) \int_{\mathbb{R}} \mathrm{e}^x\, \delta_0(dx) \\
&= 1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u_k\}, dx). \qquad (3.25)
\end{aligned}
$$

In the sequel, in order to simplify notation, we will use the one:

$$
W_u := \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx).
$$

Also, set

$$
X_k := \frac{\mathrm{e}^{\Delta X_{u_k}}}{E[\mathrm{e}^{\Delta X_{u_k}}]} - 1.
$$

Then $\{X_k; k \in \mathbb{N}\}$ is a sequence of independent random variables with mean 0.

Let $\{J_N; N \in \mathbb{N}\}$ be an increasing sequence of finite subsets of $(0,T]\cap J$ that exhausts the set $(0,T]\cap J$, that is, $J_N \subset J_{N+1}$ and $\cup_N J_N = (0,T]\cap J$. Then,

$$
\lim_{N\to\infty} E[\sup_{t\in(0,T]} | \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{J_N}(u)\, \widetilde{N}(dudx)
$$

$$- \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_J(u)\, \widetilde{N}(dudx)|^2] = 0 \tag{3.26}$$

and

$$\lim_{N\to\infty} \sup_{t\in(0,T]} \Big| \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_{J_N}(u)\, N(dudx) - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_J(u)\, N(dudx)\Big| = 0 \quad \text{a.s.} \tag{3.27}$$

In fact,

$$\begin{aligned}
&E[\sup_{t\in(0,T]} | \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{J_N}(u)\, \widetilde{N}(dudx) \\
&\qquad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_J(u)\, \widetilde{N}(dudx)|^2] \\
&= E[\sup_{t\in(0,T]} | \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{(J\setminus J_N)}(u)\, \widetilde{N}(dudx)|^2] \\
&\le 4E[| \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{(J\setminus J_N)}(u)\, \widetilde{N}(dudx)|^2] \\
&= 4\Big\{ \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} |h(x)|^2 I_{(J\setminus J_N)}(u)\, n(dudx) \\
&\qquad - \sum_{u\in(0,T]\cap(J\setminus J_N)} | \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\},dx)|^2 \Big\} \\
&\le 4 \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} |h(x)|^2 I_{(J\setminus J_N)}(u)\, n(dudx),
\end{aligned}$$

where in passage from the third line to the fourth, we have used Doob's inequality. Since $|h|^2 \in L^1((0,T]\times\mathbb{R}, n(dudx))$ and $\lim_{N\to\infty} I_{(J\setminus J_N)}(u) = 0$ for each u, it follows from the dominated convergence theorem that

$$\lim_{N\to\infty} \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} |h(x)|^2 I_{(J\setminus J_N)}(u)\, n(dudx) = 0,$$

which implies (3.26).

On the other hand, it is clear that (3.27) holds, since

$$\begin{aligned}
&\sup_{t\in(0,T]} | \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_{J_N}(u)\, N(dudx) - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \check{h}(x) I_J(u)\, N(dudx)| \\
&\qquad \le \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} |\check{h}(x)| I_{(J\setminus J_N)}(u)\, N(dudx) \xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

By (3.26), there exists a subsequence $\{N'\}$ of $\mathbb{N}$ such that

$$\begin{aligned}
&\lim_{N'\to\infty} \sup_{t\in(0,T]} | \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_{J_{N'}}(u)\, \widetilde{N}(dudx) \\
&\qquad - \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} h(x) I_J(u)\, \widetilde{N}(dudx)| = 0 \quad \text{a.s.}
\end{aligned}$$

In the sequel, we denote the subsequence $\{N'\}$ by $\{N\}$ again. Now, note that

$$\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} h(x)I_{J_N}(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \check{h}(x)I_{J_N}(u)\,N(dudx)$$
$$= \sum_{k;u_k\in(0,t]\cap J_N} \big\{\Delta X_{u_k} - E[h(\Delta X_{u_k})]\big\}.$$

Hence,

$$\exp\Big[\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} h(x)I_{J_N}(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \check{h}(x)I_{J_N}(u)\,N(dudx)\Big]$$
$$= \exp\Big[\sum_{k;u_k\in(0,t]\cap J_N} \big\{\Delta X_{u_k} - E[h(\Delta X_{u_k})]\big\}\Big]$$
$$= \prod_{k;u_k\in(0,t]\cap J_N} (1+X_k) \times \prod_{k;u_k\in(0,t]\cap J_N} \frac{E[\mathrm{e}^{\Delta X_{u_k}}]}{\mathrm{e}^{E[h(\Delta X_{u_k})]}}.$$

Here, note that

$$\mathrm{e}^{E[h(\Delta X_{u_k})]} = \mathrm{e}^{\int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u_k\},dx)}.$$

By these relations and (3.25), we see that

$$\prod_{k;u_k\in(0,t]\cap J_N} (1+X_k)$$
$$= \exp\Big[\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} h(x)I_{J_N}(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \check{h}(x)I_{J_N}(u)\,N(dudx)\Big]$$
$$\times \prod_{k;u_k\in(0,t]\cap J_N} \frac{\mathrm{e}^{E[h(\Delta X_{u_k})]}}{E[\mathrm{e}^{\Delta X_{u_k}}]}$$
$$= \exp\Big[\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} h(x)I_{J_N}(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \check{h}(x)I_{J_N}(u)\,N(dudx)\Big]$$
$$\times \frac{\mathrm{e}^{\sum_{k;u_k\in(0,t]\cap J_N}\int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u_k\},dx)}}{\prod_{k;u_k\in(0,t]\cap J_N}(1+W_{u_k})}. \tag{3.28}$$

Here, recall that when N tends to infinity, the first term of the right-hand side of (3.28) converges almost surely to

$$\exp\Big[\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} h(x)I_J(u)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} \check{h}(x)I_J(u)\,N(dudx)\Big] = \mathrm{e}^{X_t^{d,d}}.$$

Also, as for the second term of the right-hand side of (3.28), we can show that

$$\frac{\mathrm{e}^{\sum_{k;u_k\in(0,t]\cap J_N}\int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u_k\},dx)}}{\prod_{k;u_k\in(0,t]\cap J_N}(1+W_{u_k})}$$
$$= \exp\Big[-\sum_{k;u_k\in(0,t]\cap J_N}\Big\{\log(1+W_{u_k}) - \int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u_k\},dx)\Big\}\Big]$$
$$\xrightarrow[N\to\infty]{} L := \exp\Big[-\sum_{u\in(0,t]\cap J}\Big\{\log(1+W_u) - \int_{\mathbb{R}\setminus\{0\}} h(x)\,n(\{u\},dx)\Big\}\Big] \tag{3.29}$$

and that $L > 0$.

To this end, it is sufficient to show that

$$\sum_{u \in (0,t] \cap J} |\log(1+W_u) - \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\}, dx)| < \infty, \tag{3.30}$$

because it implies that (3.29) holds and that

$$\sum_{u \in (0,t] \cap J} \Big\{ \log(1+W_u) - \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\}, dx) \Big\} \in \mathbb{R}$$

and hence $L > 0$.

Note that

$$\begin{aligned} &|\log(1+W_u) - \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\}, dx)| \\ &\le |\log(1+W_u) - W_u| + |\int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1 - h(x))\, n(\{u\}, dx)|. \end{aligned}$$

Furthermore,

$$\begin{aligned} |\log(1+W_u) - W_u| \le\ & |\log(1+W_u) - W_u| I_{\{|W_u| \le 1/2\}} \\ &+ |\log(1+W_u) - W_u| I_{\{|W_u| > 1/2\}}. \end{aligned}$$

Also, note that

$$\begin{aligned} &|\log(1+W_u) - W_u| I_{\{|W_u| \le 1/2\}}(u) \\ &\le C \Big\{ \int_{\{|x| \le 1\}} |x|^2\, n(\{u\}, dx) + \int_{\{x > 1\}} \mathrm{e}^x\, n(\{u\}, dx) + n(\{u\}, \{|x| > 1\}) \Big\}, \end{aligned}$$

where C is a constant that does not depend on u ((22) in [1]). Moreover,

$$\begin{aligned} &\sum_{u \in (0,t] \cap J} \Big\{ \int_{\{|x| \le 1\}} |x|^2\, n(\{u\}, dx) + \int_{\{x > 1\}} \mathrm{e}^x\, n(\{u\}, dx) + n(\{u\}, \{|x| > 1\}) \Big\} \\ &\le \int_{(0,t]} \int_{\{|x| \le 1\}} |x|^2\, n(dudx) + \int_{(0,t]} \int_{\{x > 1\}} \mathrm{e}^x\, n(dudx) + n((0,t], \{|x| > 1\}) \\ &< \infty. \end{aligned}$$

Therefore, we see that

$$\sum_{u \in (0,t] \cap J} |\log(1+W_u) - W_u| I_{\{|W_u| \le 1/2\}}(u) < \infty.$$

On the other hand, if we set

$$K_t := \frac{1}{2} C_t + B_t + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^x - 1 - h(x) \big)\, n(dudx),$$

then it is a càdlàg function and $\Delta K_u = \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx) = W_u$. Hence, $\{u \in (0,T]; |W_u| > 1/2\}$ is a finite set, which implies that

$$\sum_{u \in (0,t] \cap J} |\log(1+W_u) - W_u| I_{\{|W_u| > 1/2\}}(u) < \infty.$$

Thus, we see that

$$\sum_{u\in(0,t]\cap J} |\log(1+W_u)-W_u| < \infty.$$

Similarly, we see that

$$\begin{aligned}&\sum_{u\in(0,t]\cap J} \Big|\int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1 - h(x))\, n(\{u\},dx)\Big| \\ &\le \sum_{u\in(0,t]\cap J} \Big\{ \int_{\{|x|\le 1\}} |x|^2\, n(\{u\},dx) + 2\int_{\{x>1\}} \mathrm{e}^x\, n(\{u\},dx) \\ &\qquad\qquad + 2n(\{u\},\{x<-1\})\Big\} \\ &< \infty.\end{aligned}$$

Thus, we have shown that (3.30) holds.

As a summary, we have shown that

$$\prod_{k;u_k\in(0,t]\cap J_N} (1+X_k) \xrightarrow[N\to\infty]{} \mathrm{e}^{X_t^{d,d}} \times L \quad \text{a.s.}$$

and that the limit is positive.

Therefore, applying the implication: D) $\Longrightarrow$ E) in Theorem 1 of [7], we see that

$$\prod_{k;u_k\in(0,t]\cap J_N} (1+X_k) \xrightarrow[N\to\infty]{} \mathrm{e}^{X_t^{d,d}} \times L \quad \text{in } L^1.$$

Since

$$E\Big[\prod_{k;u_k\in(0,t]\cap J_N} (1+X_k)\Big] = \prod_{k;u_k\in(0,t]\cap J_N} E[(1+X_k)] = 1,$$

we obtain

$$\begin{aligned}E[\mathrm{e}^{X_t^{d,d}}] &= 1/L \\ &= \exp\Big[\sum_{u\in(0,t]\cap J} \Big\{\log(1+W_u) - \int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\},dx)\Big\}\Big] \\ &= \prod_{u\in(0,t]\cap J} \mathrm{e}^{-\int_{\mathbb{R}\setminus\{0\}} h(x)\, n(\{u\},dx)} (1+W_u) \\ &= \prod_{u\in(0,t]\cap J} \mathrm{e}^{-\Delta B_u}\big(1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\},dx)\big).\end{aligned}$$

Thus, we have completed the proof of Proposition 3.12. □

3.2. The case of general additive processes. Throughout this subsection, $(X_t)_{t\in[0,T]}$ denotes the additive process stated in Theorem 2.1. Let us recall that we have denoted by $(C_t, n(dtdx), B_t)$ the characteristics of (X_t) associated with the truncation function $h(x) := xI_{\{|x|\le 1\}}(x)$. In order to complete our proof of Theorem 2.1, we would like to quote some facts from [4].

Proposition 3.13. *There exist a PII-semimartingale* (Y_t) *and a deterministic càdlàg function* (A_t) *with* $A_0 = 0$ *such that*

$$X_t = Y_t + A_t.$$

This result is stated as a part of Theorem II.5.1 in [4] (p.114).

As in the previous sections, we denote by $(C_t^Y, n^Y(dtdx), B_t^Y)$ the characteristics of (Y_t) associated with the truncation function h. The following result describes the relation between the characteristics of (X_t) and those of (Y_t), which is shown in the proof of Lemma II.5.14 (p.117) of [4]:

Proposition 3.14.

$$C_t = C_t^Y; \tag{3.31}$$

$$\begin{aligned} n((0,t], H) = &\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} I_H(y + \Delta A_u)\, n^Y(dudy) \\ &+ \sum_{u\in(0,t]} (1 - n^Y(\{u\}, \mathbb{R}\setminus\{0\})) I_H(\Delta A_u), \quad H \in \mathscr{B}(\mathbb{R}\setminus\{0\}); \end{aligned} \tag{3.32}$$

$$\begin{aligned} B_t = A_t + B_t^Y &+ \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \{h(y + \Delta A_u) - \Delta A_u - h(y)\}\, n^Y(dudy) \\ &+ \sum_{u\in(0,t]} \{h(\Delta A_u) - \Delta A_u\}(1 - n^Y(\{u\}, \mathbb{R}\setminus\{0\})). \end{aligned} \tag{3.33}$$

In the sequel, in order to simplify notation, we will use the one:

$$j_u(y) := h(y + \Delta A_u) - \Delta A_u - h(y).$$

The following result is also stated in the proof of Lemma II.5.14 (p.118) of [4]:

Proposition 3.15.

$$I_{J^c}(u)\, n(dudx) = I_{(J^Y)^c}(u)\, n^Y(dudx), \tag{3.34}$$

where $J^Y := \{t > 0; n^Y(\{t\}, \mathbb{R}\setminus\{0\}) > 0\}$.

We are now in a position to restart our proof of Theorem 2.1. First of all, note that $n^Y(dudy)$ satisfies the integrability condition corresponding to (2.5):

Lemma 3.16.

$$\int_{(0,T]} \int_{\{y>1\}} \mathrm{e}^y\, n^Y(dudy) < \infty.$$

Proof. Since $(A_u)_{u\in[0,T]}$ is a càdlàg function, the jumps are uniformly bounded; hence we set $M := \sup_{u\in[0,T]} |\Delta A_u|$. Then, it follows from (3.32) that

$$\begin{aligned} \int_{(0,T]} \int_{\{x>1\}} \mathrm{e}^x\, n(dudx) = &\int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{y+\Delta A_u} I_{(1,\infty)}(y + \Delta A_u)\, n^Y(dudy) \\ &+ \sum_{u\in(0,t]} (1 - n^Y(\{u\}, \mathbb{R}\setminus\{0\}))\mathrm{e}^{\Delta A_u} I_{(1,\infty)}(\Delta A_u) \\ \geq &\int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{y+\Delta A_u} I_{(1,\infty)}(y + \Delta A_u)\, n^Y(dudy) \end{aligned}$$

$$\geq \mathrm{e}^{-M} \int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{y} I_{(M+1,\infty)}(y)\, n^{Y}(dudy).$$

Hence, we have

$$\int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{y} I_{(M+1,\infty)}(y)\, n^{Y}(dudy) \leq \mathrm{e}^{M} \int_{(0,T]} \int_{\{x>1\}} \mathrm{e}^{x}\, n(dudx) < \infty.$$

On the other hand,

$$\int_{(0,T]} \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{y} I_{(1,M+1]}(y)\, n^{Y}(dudy) \leq \mathrm{e}^{M+1} n^{Y}((0,T],\{y>1\}) < \infty.$$

Thus, we see that

$$\begin{aligned}
&\int_{(0,T]} \int_{\{y>1\}} \mathrm{e}^{y}\, n^{Y}(dudy) \\
&\leq \mathrm{e}^{M+1} n^{Y}((0,T],\{y>1\}) + \mathrm{e}^{M} \int_{(0,T]} \int_{\{x>1\}} \mathrm{e}^{x}\, n(dudx) < \infty. \qquad \square
\end{aligned}$$

Owing to this lemma, we can apply Theorem 2.1 to the PII-semimartingale $(Y_t = X_t - A_t)$ and hence (2.6) holds with the characteristics $(C_t^Y, n^Y(dtdx), B_t^Y)$:

$$\begin{aligned}
E[\mathrm{e}^{Y_t}] &= \exp\Big[\frac{1}{2}C_t^Y + B_t^Y + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^{y} - 1 - h(y)\big) I_{(J^Y)^c}(u)\, n^{Y}(dudx)\Big] \\
&\quad \times \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u^Y} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{y} - 1)\, n^{Y}(\{u\},dy)\Big]. \qquad (3.35)
\end{aligned}$$

Now, in the following, we will show that the right-hand side of (2.6) is actually equal to $E[\mathrm{e}^{X_t}]$.

By the relation (3.34),

$$\begin{aligned}
&\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{x} - 1 - h(x)) I_{J^c}(u)\, n(dudx) \\
&= \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{y} - 1 - h(y)) I_{(J^Y)^c}(u)\, n^{Y}(dudy). \qquad (3.36)
\end{aligned}$$

Next, by the relation (3.32), we see that

$$\begin{aligned}
&1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{x} - 1)\, n(\{u\},dx) \\
&= 1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{y+\Delta A_u} - 1)\, n^{Y}(\{u\},dy) + (\mathrm{e}^{\Delta A_u} - 1)(1 - n^{Y}(\{u\},\mathbb{R}\setminus\{0\})) \\
&= \mathrm{e}^{\Delta A_u}\{1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{y} - 1)\, n^{Y}(\{u\},dy)\}. \qquad (3.37)
\end{aligned}$$

Hence, it follows from that (3.37) and (3.33) that

$$\begin{aligned}
&\mathrm{e}^{B_t} \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{x} - 1)\, n(\{u\},dx)\Big] \\
&= \mathrm{e}^{B_t} \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u} \mathrm{e}^{\Delta A_u} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{y} - 1)\, n^{Y}(\{u\},dy)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Big[A_t + B_t^Y + \int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(dudy) \\
&\qquad\qquad + \sum_{u\in(0,t]} \{h(\Delta A_u) - \Delta A_u\}(1 - n^Y(\{u\}, \mathbb{R}\backslash\{0\}))\Big] \\
&\quad \times \prod_{u\in(0,t]} \exp\Big[-\big(\Delta A_u + \Delta B_u^Y + \int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(\{u\}, dy) \\
&\qquad\qquad + \{h(\Delta A_u) - \Delta A_u\}(1 - n^Y(\{u\}, \mathbb{R}\backslash\{0\}))\big) + \Delta A_u\Big] \\
&\quad \times \Big[1 + \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^y - 1)\, n^Y(\{u\}, dy)\Big] \\
&= \mathrm{e}^{A_t}\mathrm{e}^{B_t^Y} \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u^Y}\Big[1 + \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^y - 1)\, n^Y(\{u\}, dy)\Big] \\
&\quad \times \exp\Big[\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(dudy) - \sum_{u\in(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(\{u\}, dy)\Big]. \qquad (3.38)
\end{aligned}
$$

Moreover, we will show that

$$
\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(dudy) = \sum_{u\in(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y)\, n^Y(\{u\}, dy). \qquad (3.39)
$$

To this end, we first show that for every $\varepsilon > 0$

$$
\begin{aligned}
&\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{\{|A_u|>\varepsilon\}}(u)\, n^Y(dudy) \\
&\quad = \sum_{u\in(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{\{|A_u|>\varepsilon\}}(u)\, n^Y(\{u\}, dy). \qquad (3.40)
\end{aligned}
$$

Since (A_u) is a càdlàg function, $J_t^{A,\varepsilon} := \{u \in (0,t]; |\Delta A_u| > \varepsilon\}$ is a finite set, and hence we set $J_t^{A,\varepsilon} = \{0 < u_1 < u_2 < \cdots < u_m < u_{m+1} = t\}$. Then, for $u \in (u_k, u_{k+1})$, $j_u(y) I_{J_t^{A,\varepsilon}}(u) = 0$. Hence, we see that

$$
\begin{aligned}
&\int_{(0,t]}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(dudy) \\
&= \sum_{k=0}^{m} \int_{(u_k,u_{k+1}]}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(dudy) \\
&= \sum_{k=0}^{m} \Big\{ \int_{(u_k,u_{k+1})}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(dudy) \\
&\qquad\qquad + \int_{\{u_{k+1}\}}\int_{\mathbb{R}\backslash\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(dudy)\Big\} \\
&= \sum_{k=0}^{m} \int_{\mathbb{R}\backslash\{0\}} j_{u_{k+1}}(y) I_{J_t^{A,\varepsilon}}(u_{k+1})\, n^Y(\{u_{k+1}\}, dy)
\end{aligned}
$$

$$= \sum_{u\in(0,t]} \int_{\mathbb{R}\setminus\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(\{u\}, dy).$$

Also, note that

$$\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} |j_u(y)|\, n^Y(dudy) < \infty$$

and that

$$\sum_{u\in(0,t]} \int_{\mathbb{R}\setminus\{0\}} |j_u(y)|\, n^Y(\{u\}, dy) < \infty,$$

(II.5.17 in [4] (p.117)). Hence, we see from the dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0} \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(dudy) = \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} j_u(y)\, n^Y(dudy)$$

and that

$$\lim_{\varepsilon\downarrow 0} \sum_{u\in(0,t]} \int_{\mathbb{R}\setminus\{0\}} j_u(y) I_{J_t^{A,\varepsilon}}(u)\, n^Y(\{u\}, dy) = \sum_{u\in(0,t]} \int_{\mathbb{R}\setminus\{0\}} j_u(y)\, n^Y(\{u\}, dy).$$

Therefore, letting $\varepsilon \downarrow 0$ in (3.40), we obtain (3.39). Thus, combining (3.39) with (3.38), we have

$$\begin{aligned} &\mathrm{e}^{B_t} \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx)\Big] \\ &= \mathrm{e}^{A_t} \mathrm{e}^{B_t^Y} \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u^Y} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^y - 1)\, n^Y(\{u\}, dy)\Big]. \end{aligned} \tag{3.41}$$

Finally, by (3.31), (3.36) and (3.41), we see that

$$\begin{aligned} &\exp\Big[\frac{1}{2} C_t + B_t + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^x - 1 - h(x)\big) I_{J^c}(u)\, n(dudx)\Big] \\ &\quad \times \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^x - 1)\, n(\{u\}, dx)\Big] \\ &= \exp\Big[\frac{1}{2} C_t^Y + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^y - 1 - h(y)\big) I_{(J^Y)^c}(u)\, n^Y(dudy)\Big] \\ &\quad \times \mathrm{e}^{A_t} \mathrm{e}^{B_t^Y} \times \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u^Y} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^y - 1)\, n^Y(\{u\}, dy)\Big] \\ &= \mathrm{e}^{A_t} \times \exp\Big[\frac{1}{2} C_t^Y + B_t^Y + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \big(\mathrm{e}^y - 1 - h(y)\big) I_{(J^Y)^c}(u)\, n^Y(dudy)\Big] \\ &\quad \times \prod_{u\in(0,t]} \mathrm{e}^{-\Delta B_u^Y} \Big[1 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^y - 1)\, n^Y(\{u\}, dy)\Big]. \end{aligned} \tag{3.42}$$

Combining (3.42) with (3.35), we see that

$$\text{the right hand side of (2.6)} = \mathrm{e}^{A_t} \times E[\mathrm{e}^{Y_t}] = E[\mathrm{e}^{X_t}].$$

Thus, we have completed our proof of Theorem 2.1.

Acknowledgment. The author is grateful to Professor Hiroshi Kunita for his warm guidance to the theory of stochastic processes.

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