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On the Extended Nature of Edge States of Quantum Hall Hamiltonians

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Abstract. Properties of eigenstates of one-particle Quantum Hall Hamiltonians localized near the boundary of a two-dimensional electron gas - so-called edge states - are studied. For finite samples it is shown that edge states with energy in an appropriate range between Landau levels remain extended along the boundary in the presence of a small amount of disorder, in the sense that they carry a non-zero chiral edge current. For a two-dimensional electron gas confined to a half-plane, or to a domain in the plane satisfying a certain geometric condition, the Mourre theory of positive commutators is applied to prove absolute continuity of the energy spectrum well in between Landau levels, corresponding to edge states.

1 Introduction and summary of results

In this paper, we study two-dimensional electron gases in a uniform magnetic field perpendicular to the plane, in the presence of a small amount of disorder. The integer quantum Hall effect, discovered by von Klitzing [1], is the phenomenon that when the Fermi energy of the electron gas is well in between two Landau levels, the Hall conductance is equal to an integer multiple of e^2/h.

Under the assumption of negligibly small electron-electron interactions, the integer quantum Hall effect can be derived from a simple one-electron picture. For an appropriate choice of sample geometry, described by a potential confining the electrons to the sample, and for a small amount of disorder, one can analyze, qualitatively, the energy spectrum of the corresponding one-particle Hamiltonian. In particular, as we show in this paper, eigenenergies well in between Landau levels correspond to eigenstates localized near, but extended along, the boundary of the sample, so called edge states. Those edge states carry a non-zero chiral edge current. Given a small voltage drop between two parallel components of the boundary, the edge states corresponding to the two boundary components will be filled somewhat asymmetrically with electrons. The result is a net Hall current parallel to the boundary and proportional to the voltage drop. The proportionality factor is the Hall conductivity. If the Fermi energy of the electron gas is well in between two Landau levels, and if the voltage drop is small compared to the energy gap between two adjacent Landau levels and to the Zeeman energy of the magnetic moment of an electron, the spectral properties of the Hamiltonian yield a Hall conductivity equal to e^2/h times the number of Landau levels below the Fermi energy, which is an integer. An argument of this sort, based on a clever use of gauge invariance, was first given by Laughlin [2] and subsequently refined by many other people (see e.g. [3], [4]). The idea that the Hall current is supported by edge states first appeared in a paper of Halperin [3]. The fundamental role of edge currents in the integer *and* the fractional quantum Hall effect was later understood in terms of a gauge anomaly cancellation mechanism in [5] and [6]. In this paper, we provide a rigorous analysis of one important detail underlying Halperin's argument, namely of the question whether, and in what sense, the edge states are indeed extended states.

Because we neglect electron-electron interactions, the magnetic moment of the electron turns out to be essentially irrelevant in our analysis, and we thus neglect electron spin. The one-electron Hamiltonian is therefore given by

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + V. \tag{1.1}$$

In (1.1), m is the mass of an electron, e is its charge, $\vec{A}$ is an electromagnetic vector potential corresponding to a constant magnetic field $\vec{B} = \operatorname{curl}\vec{A}$, and $V = V_0 + gV_d$ is an external potential consisting of an edge potential, V_0, that confines the electron to the sample, and a disorder potential, gV_d, corresponding to the presence of random impurities. The factor g, a "coupling constant", is a measure for the strength of the disorder. The potential V_0 can be replaced by appropriate boundary conditions in the definition of the covariant Laplacian, $(\vec{p} - e\vec{A})^2$, which prevent an electron from leaving the sample; see, for example, [17], and section 6 of the present paper.

The location of the energy spectrum of the one-particle Hamiltonian (1.1) is indicated in figure 1. This spectrum consists of a part corresponding to "bulk states" and a part corresponding to "edge states". The former is located near the Landau levels, which are broadened by the disorder potential. Most of the bulk states are localized, but close to each Landau level, there are eigenvalues corresponding to extended bulk states. It is well known that in order to *observe* quantum Hall *plateaux*, one needs to have localized bulk states. The energy spectrum corresponding to edge states is located in the intervals between the broadened Landau levels. For a sample covering the entire plane, the intervals between the broadened Landau levels would be spectral gaps.

The edge states of clean samples ($g = 0$) are well understood. For a bounded sample and weak disorder, one may use analytic perturbation theory in the disorder potential, gV_d, in order to analyze the edge states. Unfortunately, as the sample size increases, the spacing between eigenvalues of H corresponding to edge states becomes smaller and smaller, and, as a consequence, the convergence radius of the perturbation series in g becomes smaller and smaller. Perturbation theory cannot be used in the limit of an infinitely large sample.

The relation between quantization of the Hall conductivity and the extended nature of edge states is reviewed in section 2. Our *definition* of extended edge states for bounded samples is that they carry a non-vanishing chiral current. It is this property that plays an essential role in our analysis of the quantization of

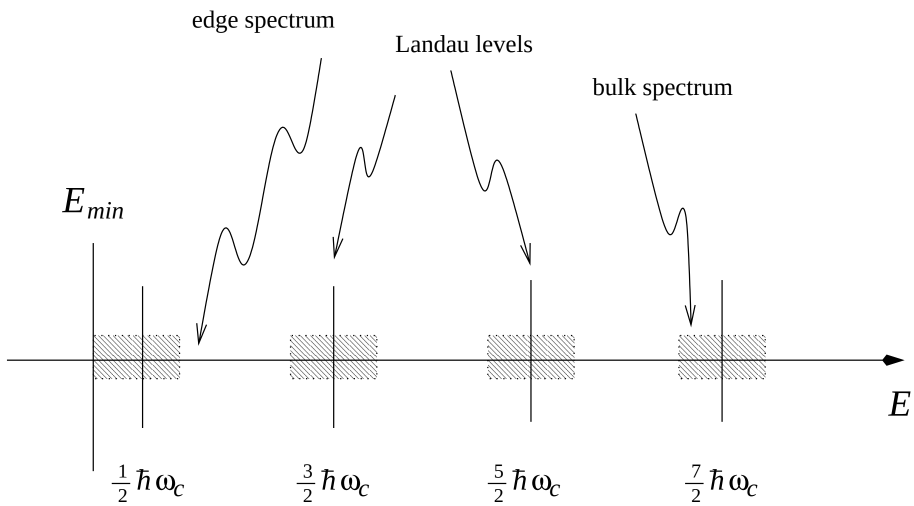

Figure 1: Bulk and edge spectrum of H. The energy scale is the cyclotron frequency $\hbar\omega_c = eB/m$.

the Hall conductivity. In sections 4, 5, and 6 the extended nature of edge states is established on the basis of arguments which are valid for sufficiently weak disorder, i.e. for $|g| < g_*$, but our bounds on g_* are *uniform* in the sample size. In sections 2 and 4, our sample has the original Laughlin cylinder geometry, but our proofs can be adapted for other sample shapes, such as the Corbino disc geometry used by Halperin [3], which we treat in an appendix.

For infinite samples, the natural definition of "extended states" is that they correspond to absolutely continuous spectrum. In section 5, we consider the case of a two-dimensional electron gas confined to a half-plane by a smooth but steep edge potential, and prove that the energy spectrum well in between Landau levels is absolutely continuous for weak disorder. It turns out that our bound on the allowed strength of the disorder becomes smaller as the edge is made steeper. In section 6, we treat an "infinitely steep" edge directly by introducing Dirichlet boundary conditions in the definition of the covariant Laplacian. We show that for weak disorder, the edge states are again extended states. Our proofs can be extended to more general domains than the half-plane, provided they satisfy a certain geometric condition. The proofs in sections 5 and 6 are based on an application of the Mourre theory of positive commutators [7], which is briefly presented in section 3.

For both finite and infinite samples, the extended nature of the edge states is analyzed with the help of the so-called "guiding center" of cyclotron motion. The commutator of the coordinate of the guiding center along the edge with the Hamiltonian is given by the derivative of the potential in the direction perpendicular to the edge*. The proofs are reduced to showing that this commutator is positive on states with energy well in between Landau levels. Instead of the guiding center, one can also use the coordinate of the particle itself along the edge as conjugate operator in the sense of Mourre theory. For the problem with Dirichlet boundary conditions, De Bièvre and Pulé [20] have shown that this allows to relax the assumptions on the disorder potential. It turns out that a Mourre estimate for one commutator is equivalent to a Mourre estimate for the other with the same lower bound on the commutator, but the techniques used to prove the estimate are different. In section 6, combining the two ideas, we use the coordinate of the particle itself as conjugate operator in the sense of Mourre theory, but prove the positivity of the commutator by considering the coordinate of the guiding center.

Recently, Macris, Martin, and Pulé (see [19]) have studied the half-plane case by a somewhat different method. They rule out the existence of eigenvalues between the broadened Landau levels by showing that the expectation value of the derivative of the potential in the direction perpendicular to the edge would be positive in an assumed eigenstate with energy between the broadened Landau levels. This would contradict the fact that the expectation value of a commutator with the Hamiltonian in an energy eigenstate must vanish by the virial theorem. To prove the positivity of the commutator in an assumed eigenstate, for weak disorder, they estimate the decay of edge state eigenfunctions into the edge with the help of Brownian motion techniques. Our use of the conjugate operator method allows us to exclude not only point spectrum, but also singular continuous spectrum. Furthermore, whereas the estimates for smooth potentials tend to fail in the limit of an infinitely steep edge, we also treat the problem with Dirichlet boundary conditions, and for more general domains than the half-plane.

2 The Laughlin argument revisited

In this section, we review the argument leading to the integer quantization of the Hall conductivity, motivating our interest for the extended nature of edge states. In order to keep our analysis as simple as possible, we consider the cylinder geometry used in the original Laughlin argument [2]. The Hall current flows along the circumference and the Hall voltage is measured between two edge circles (see figure 2). In addition to the homogeneous magnetic field perpendicular to the surface of the cylinder, there is a "magnetic flux tube", Φ, at the axis of the cylinder.

The cylinder is characterized by two length scales, the radius, R, and the distance, L, between the two edges. Both lengths play a role in the mathematical analysis. Increasing R reduces the spacing between edge state eigenvalues of the Hamiltonian, and thus limits the applicability of perturbation theory to analyze the edge states. On the other hand, L influences the tunneling probability between two edges. Physically, we expect that for weak disorder, the tunneling probability

*This is in the case of an edge potential, for Dirichlet boundary conditions, see section 6.

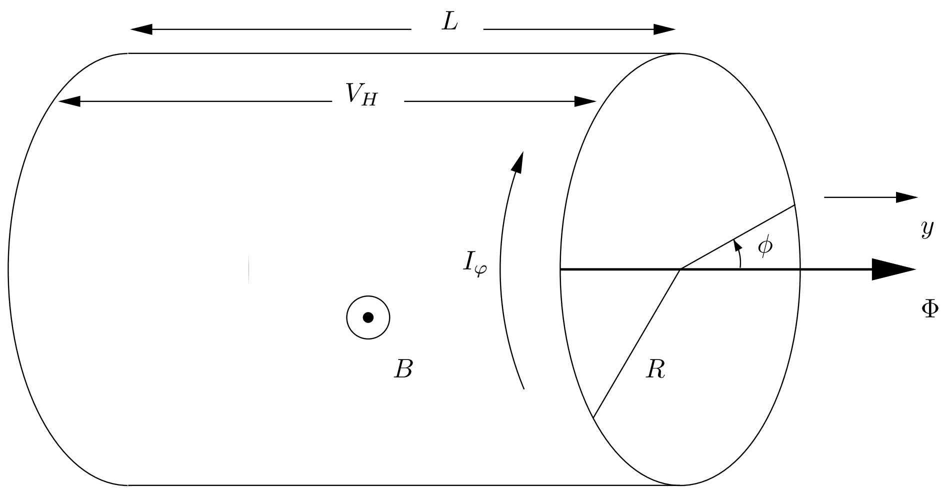

Figure 2: Cylinder geometry

per edge length for states with energy well in between Landau levels is suppressed exponentially in L/l_c, where l_c is the cyclotron length. In this paper, we shall not provide rigorous bounds on those tunneling rates, but only deal with the problem connected to the spacing of eigenvalues. Possible tunneling between edges will be avoided by considering only one edge, or, equivalently, taking one edge to infinity.

In the Corbino disc or annulus geometry introduced by Halperin [3], one cannot completely eliminate the tunneling problem, because, even if one considers only the outer edge of the annulus, the flux tube at the center is comparable to having a second edge, in the sense that for generic Φ, there are eigenvalues between Landau levels. Without precise estimates on tunneling probabilities between inner and outer edge, we can only show that edge states are extended for small $|\Phi|$. The argument for the Corbino disc is carried out in appendix A.

The coordinate along the axis of the cylinder will be denoted by y, and the coordinate perpendicular to it will be $x = R\varphi$, where $0 < \varphi \leq 2\pi$. The magnetic field is pointing radially outward, and the vector potential is chosen in φ-direction $A_\varphi = -By + \Phi/2\pi R$. Of course, the magnetic field can only be homogeneous on the two-dimensional surface of the cylinder, since otherwise the Maxwell equations would be violated. In these coordinates, the Hamiltonian is

$$H = \frac{1}{2m}\left(-\partial_y^2 + \left(\frac{1}{\mathrm{i}R}\partial_\varphi - e\left(-By + \frac{\Phi}{2\pi R}\right)\right)^2\right) + V(y,\varphi), \tag{2.1}$$

in units where $\hbar = 1$. We start with $V = 0$, that is, with a cylinder infinite in y-direction and without disorder. The states can be labeled by the angular momentum quantum number l and the Landau band index n. The energy depends only on n through $E_{n,l} = (n + 1/2)\omega_c$. In the y-direction, the eigenfunctions are harmonic oscillator wave functions, localized near $y_0(l - e\Phi/2\pi) = (-l + e\Phi/2\pi)/eBR$. Changing Φ does not affect the energy of the states, but only their position along the cylinder. In particular, a change of Φ by $2\pi/e$ maps states with angular momentum l on those with angular momentum $l-1$. This "spectral flow" produced by the change in Φ plays an important role in the following arguments.

For a symmetric confining potential $V_0 = V_0(y)$, it is possible to continue labeling the states by l and n and to qualitatively discuss the dependence of the energy $E_{n,l}(\Phi)$ on l and Φ. The one-dimensional Hamiltonian for the motion in y-direction that results after separating the angular momentum, is analytic in the parameter $l - e\Phi/2\pi$. Therefore, $E_{n,l}(\Phi) = E_n(l - e\Phi/2\pi)$ are analytic functions, and the spectral flow of eigenstates with changing Φ is preserved by a symmetric V_0,

$$E_{n,l}(\Phi + 2\pi/e) = E_{n,l-1}(\Phi). \tag{2.2}$$

Furthermore, all eigenstates are well localized in the y-direction and the localization position, $y_0(l - e\Phi/2\pi)$, is also an analytic function, which is monotonically decreasing as can be seen by inspection of the Hamiltonian (2.1). $E_{n,l}(\Phi)$ can lie between Landau levels only if $y_0(l-e\Phi/2\pi)$ comes close to an edge of the cylinder, that is, only for edge states. For each n, it is possible to identify those l which correspond to states at the left and right edge. Large positive l correspond to the left edge, and large negative l to the right edge.

Consider the current carried by an (n, l)-state in φ-direction,

$$I_{\varphi,n,l} = -\frac{\mathrm{d}E_{n,l}(\Phi)}{\mathrm{d}\Phi}. \tag{2.3}$$

Under the assumption that V_0 is monotonically increasing as one leaves the sample on either edge of the cylinder, so that it correctly describes the confining of the electron gas to the sample, it is easy to see that $I_{\varphi,n,l}$ has a definite sign for states localized at either edge. Edge states carry a chiral edge current.

The main goal of our present work is to show that the edge states remain extended in the sense that they carry a chiral edge current if the sample contains a small amount of disorder. To motivate our interest for edge states, we now show that the chirality of our edge states implies the integer quantization of the Hall conductivity. Our argument is of a very general character and can be applied independently of a labeling of states by angular momentum and Landau band index. It is only to identify the integer $\nu = \sigma_H/(e^2/h)$ as the number of Landau levels below the Fermi energy that one must consider a situation where the Landau band index is a good quantum number. In our general argument, edge states are labeled by an index α, with corresponding energies $E_\alpha(\Phi)$.

We are interested in calculating the Hall conductance, σ_H, when the Fermi energy, E_F, of the electron gas on the surface of the cylinder lies well in between two Landau levels. We assume that the disorder is sufficiently small so that Landau bands are still well defined in the bulk, and that there is no bulk spectrum in the vicinity of the Fermi energy. Experimentally, the Quantum Hall effect is observed in macroscopic samples as plateaux in the Hall conductance as a function of the magnetic field or carrier density. Therefore, the Fermi energy has to remain in between Landau levels, such that σ_H is quantized, for a sufficiently wide range of magnetic field or carrier density. This can only be achieved if there exist localized bulk states at E_F. This fact is very well known, but we shall not make the attempt to solve the analytical problems connected with localized bulk states. For the rest of the argument, assume that the only occupied energy levels near the Fermi energy correspond to edge states. This edge spectrum will then be discrete. A small voltage drop, $0 < eV_H = \mu_r - \mu_l \ll \omega_c$, between the left and right edge of the sample is taken into account by assuming that states localized at the right edge are occupied up to μ_r, and states localized the left edge up to μ_l. The Hall current is the current induced by this asymmetrical filling of edge states, in excess of the current carried by the electron gas in the ground state. If, for simplicity, we assume that $\mu_l = E_F$, the Hall current is carried by electrons on the right edge only. We denote by $\mathcal{I}$ the set of labels α of occupied edge states with $\mu_l \leq E_\alpha(0) \leq \mu_r$. The Hall current can then be written as

$$I_\varphi = \sum_{\alpha\in\mathcal{I}} -\frac{\mathrm{d}E_\alpha(\Phi)}{\mathrm{d}\Phi}|_{\Phi=0}. \tag{2.4}$$

As mentioned above, we shall neglect effects due to tunneling between the edges. On physical grounds, we expect that the tunneling rates are suppressed exponentially in L/l_c, and tunneling will play no role when describing measurements performed on laboratory scales. We therefore take the limit $L \to \infty$ in (2.4), and consider only the right edge. The rigorous justification of this restriction to a sample with only one edge is a rather difficult problem in localization theory which is not considered here. The next step in our argument is then to replace the expression (2.4), where $\mathrm{d}E/\mathrm{d}\Phi$ is evaluated at $\Phi = 0$ by the average over a range of $2\pi/e$,

$$I_\varphi = \sum_{\mathcal{I}} -\frac{e}{2\pi}\int_0^{2\pi/e} \mathrm{d}\Phi \frac{\mathrm{d}E_\alpha(\Phi)}{\mathrm{d}\Phi}. \tag{2.5}$$

It is this step which would fail in a sample with two edges at a finite distance L from each other, because resonances would necessarily occur at intermediate values of Φ. On physical grounds, expression (2.5) is a good approximation to (2.4) (in the limit $L \to \infty$) provided the number of contributing states, $|\mathcal{I}|$, is large, which is equivalent to a small spacing between successive eigenvalues, or to a large radius R.

Because the spectrum of H is invariant under a change of Φ by $2\pi/e$, there is a bijective map $\alpha \mapsto \beta(\alpha)$ with $E_\alpha(2\pi/e) = E_{\beta(\alpha)}(0)$. Then

$$I_\varphi = -\frac{e}{2\pi}\Big(\sum_{\beta(\mathcal{I})\setminus\mathcal{I}} E_\alpha(0) - \sum_{\mathcal{I}\setminus\beta(\mathcal{I})} E_\alpha(0)\Big). \tag{2.6}$$

Taking (2.6) as our expression for the Hall current, we now exploit the chirality of the edge states. For samples with only one edge, our analysis in subsequent sections implies that states corresponding to an energy in between Landau levels are localized near the edge of the sample and that the current they carry satisfies an estimate of the form

$$\frac{C}{R} \geq \frac{\mathrm{d}E_\alpha(\Phi)}{\mathrm{d}\Phi} \geq \frac{C'}{R}, \tag{2.7}$$

where C and C' are non-zero constants whose *common sign* depends upon whether we consider a right or a left edge, the other being at infinity, and R is the size of the sample. It is easy to see that (2.7) is satisfied in the situation of a clean sample ($g = 0$) described above. These inequalities imply that $C/R \geq \big(E_{\beta(\alpha)}(0) - E_\alpha(0)\big)e/2\pi \geq C'/R > 0$, if α corresponds to an edge state at the right edge. Therefore,

$$E_\alpha(0) = \begin{cases} \mu_r + O(1/R) & \text{if } \alpha \in \beta(\mathcal{I}) \setminus \mathcal{I} \\ \mu_l + O(1/R) & \text{if } \alpha \in \mathcal{I} \setminus \beta(\mathcal{I})\,. \end{cases} \tag{2.8}$$

Because β is bijective, $|\mathcal{I} \setminus \beta(\mathcal{I})| = |\beta(\mathcal{I}) \setminus \mathcal{I}|$, and recalling that $\mu_r - \mu_l = eV_H$, we obtain that

$$\frac{\sigma_H}{(e^2/h)} = |\beta(\mathcal{I}) \setminus \mathcal{I}|\,(1 + O(1/R)) \tag{2.9}$$

is an integer, where $\sigma_H = I_\varphi/V_H$ is the Hall conductance.

This argument goes back essentially to ideas of Laughlin [2] and Halperin [3]. In the form presented above, it completely clarifies the universal character of the integer quantum Hall effect. The effect is not related to any particular geometry or symmetry of the sample. Our argument rests on the identification of the Hall current, as given by (2.5) or (2.6), and on the estimate (2.7), which we prove under various hypotheses in subsequent sections.

Let us summarize the approximations made in the derivation: Taking the limit $L \to \infty$ is justified by the exponential suppression of tunneling rates, and making this rigorous requires localization theory techniques. Taking the "thermodynamic limit" plays an important role also in the argument where the Hall conductance is identified with a charge index (see [18]). Indeed, the charge index vanishes in the generic situation with two edges. In our argument, the expression for the Hall current simplifies when we average over Φ, and this averaging is still possible, if more subtle, in a situation with two edges ($\mathcal{I}$ will in general depend on Φ). Precise estimates of the tunneling rates would make the transition from (2.4) to (2.5) rigorous. After having taken the limit $L \to \infty$, the averaging over Φ and the replacement of $E_\alpha(0)$ with μ_r or μ_l for $\alpha \in \beta(\mathcal{I}) \setminus \mathcal{I}$ or $\alpha \in \mathcal{I} \setminus \beta(\mathcal{I})$, respectively, become increasingly good approximations as the spacing between edge state eigenvalues decreases with increasing sample size, R. Here, finite size corrections are in principle easier to estimate than the tunneling rates.

The definition of extended states as states whose energy increases (or decreases) monotonously when Φ is varied is consistent with the fact that an eigenfunction of H whose support does not surround the flux will, up to a phase factor, not be affected by a change of Φ, since, in a simply connected region, the influence of Φ can always be gauged away.

Before closing this section, we explain how the integer $\nu = \sigma_H/(e^2/h)$ can be identified as the number of Landau bands below the Fermi energy, if n and l are good quantum numbers. For clean samples, this follows immediately from (2.2) and (2.6). We now show that this identification is still possible after inclusion of a small amount of disorder, gV_d.

In the generic situation with only *one* edge, the eigenvalues of the clean sample, $E_{n,l}(\Phi)$, will be non-degenerate for all Φ. One may then appeal to analytic perturbation theory to continue labeling the states by n and l in the presence of disorder, with energies $E_{n,l}(\Phi, g)$ which are analytic functions in g, for $|g|$ small enough. By assumption, $E_{n,l}(\Phi, 0) = E_{n,l}(\Phi)$. Because the eigenfunctions are also analytic in g, labels (n, l) that correspond to edge states for $g = 0$ will also correspond to states localized at the edge for $|g|$ non-zero, but small. Thus, we propose to show the analog of equation (2.2),

$$E_{n,l}(\Phi + 2\pi/e, g) = E_{n,l-1}(\Phi, g), \tag{2.10}$$

which will imply the extended nature of the edge states and thereby the integer quantum Hall effect, with ν equal to the number of Landau bands below the Fermi energy (see figure 3).

The shift in energy due to the disorder, $E_{n,l}(\Phi, g) - E_{n,l}(\Phi, 0)$, is of the order of $|g|$, by the Feynman-Hellmann theorem, and (2.2) then immediately implies that $E_{n,l}(\Phi+2\pi/e, g)-E_{n,l-1}(\Phi, g)$ is also of the order of $|g|$. A change of Φ by $2\pi/e$ can be compensated by a gauge transformation and does not change the spectrum at all. Therefore, there must be an n' and an l' with $E_{n,l}(\Phi + 2\pi/e, g) = E_{n',l'}(\Phi, g)$. But if, without disorder, the energies are sufficiently far apart and non-degenerate, this is only possible for $n' = n$ and $l' = l - 1$, for small $|g|$, i.e., for weak disorder. This implies (2.10).

The perturbative argument for the spectral flow may break down when the radius of the cylinder becomes very large, because the spacing between eigenvalues at $g = 0$ decreases with increasing R. But, as we have already noted, finite size corrections become small at least inversely proportional to R. It is therefore desirable to have an argument that establishes the spectral flow with a bound on the allowed strength of the disorder that is *uniform* in R. Such an argument will be provided in section 4. The rigorous proofs establishing the bounds in (2.7) are contained in section 5.

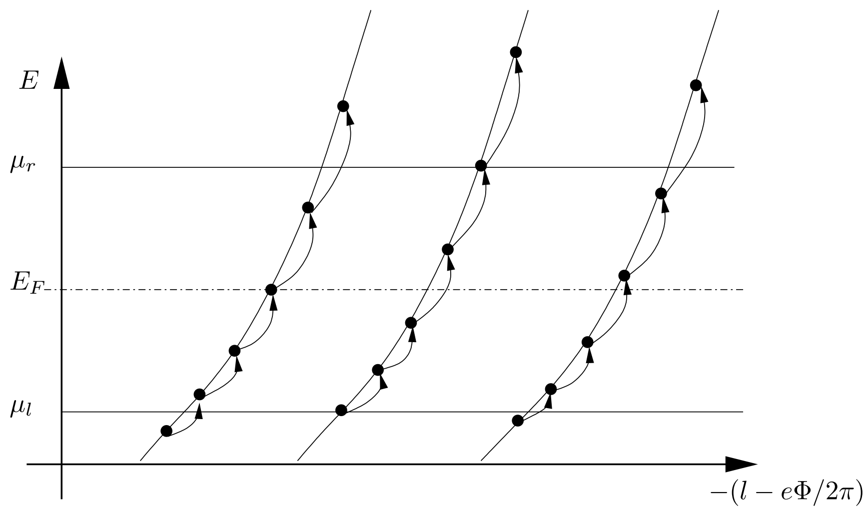

Figure 3: The spectral flow of edge states under variation of Φ. Each line represents a Landau band with eigenvalues symbolized by a dot. μ_l and μ_r are the chemical potentials on the left and right edge of the cylinder, respectively.

3 Methods and tools

3.1 The guiding center

It is well known that the classical cyclotron orbit of a charged particle in a homogeneous magnetic field drifts under the influence of an electrostatic potential. This can be seen most simply by considering the "guiding center" of the motion, which is the center of a circle with cyclotron radius $r = v/\omega_c = v/(eB/m)$ that passes through the position of the particle. The velocity, v, of the particle is given by

$$\vec{v} = \frac{1}{m}(\vec{p} - e\vec{A}). \tag{3.1}$$

This yields for the guiding center

$$\vec{Z} = \vec{r} + (\vec{p} - e\vec{A}) \times \frac{\hat{B}}{B}, \tag{3.2}$$

where $\hat{B}$ is a unit vector in B-direction perpendicular to the plane.

It is easily established that the equation of motion of $\vec{Z}$ in a potential V is

$$\dot{\vec{Z}} = \frac{\hat{B}}{B} \times \operatorname{grad} V. \tag{3.3}$$

The separation of the motion of the guiding center from the cyclotronic motion will be used in section 4 and appendix A to get an estimable expression for the azimuthal current carried by any eigenstate of the Hamiltonian. In sections 5 and 6, the coordinate of the guiding center along the edge is used to establish estimates needed in Mourre theory, to which we turn now.

3.2 Positive commutators and absolutely continuous spectrum

If in a classical Hamiltonian system, one can show that for some orbit the Poisson bracket of some coordinate K with the Hamiltonian H remains bounded away from zero, $\dot{K} = \{K, H\} \geq \alpha > 0$, for all times, one can conclude that the motion is extended along this coordinate. The quantum mechanical counterpart of this simple statement is the following: Assume there exists a "conjugate operator", A, such that the commutator of A with the Hamiltonian is positive on some energy interval Δ,

$$E_\Delta(H)\,[H, \mathrm{i}A]\,E_\Delta(H) \geq \alpha E_\Delta(H), \tag{3.4}$$

with $\alpha > 0$, and where $E_\Delta(H)$ denotes the spectral projector of H on Δ. Noting that if ψ is an eigenstate of H, we have $(\psi, [H, \mathrm{i}A]\,\psi) = 0$ by the virial theorem, we can conclude from (3.4) that H can not have an eigenvalue in the interval Δ. It was first proved by Mourre [7] that under additional regularity assumptions on H and its commutators with A, the spectrum of H is actually purely absolutely continuous on Δ. Equation (3.4) is termed a Mourre estimate.

The assumptions on H have been subsequently relaxed considerably ([8, 9, 10], see also [11]). For the treatment of the problem with a smooth, steep edge potential in section 5, the original assumptions of Mourre can be verified. Those are (see [7]):

(i) H and A are self-adjoint operators with domains $\mathcal{D}(H)$ and $\mathcal{D}(A)$. $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H.

(ii) The unitary group $e^{\mathrm{i}Aa}$ generated by A leaves $\mathcal{D}(H)$ invariant and for all $\psi \in \mathcal{D}(H)$

$$\sup_{|a|<1} \left\| H e^{\mathrm{i}Aa} \psi \right\| < \infty.$$

(iii) The quadratic form $[H, \mathrm{i}A]$ which is defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$, is bounded below and closable; the associated self-adjoint operator admits a domain containing $\mathcal{D}(H)$.

(iv) The quadratic form $\big[[H, \mathrm{i}A], \mathrm{i}A\big]$ is form-bounded by $|H|^2$.

Following Mourre's work, the main efforts have gone in the direction of reducing the regularity assumptions described in (iii) and (iv). A very complete treatment of necessary and sufficient conditions for applicability of Mourre theory can be found in [10].

It is also possible to apply Mourre theory with a non self-adjoint conjugate operator. Such situations are, for instance, treated in references [12], [13], and, implicitly, also in [14]. However, a general treatment of Mourre theory with non self-adjoint conjugate operator, in particular as regards the regularity assumptions, is to our knowledge still missing. In the case of Dirichlet boundary conditions, the coordinate of the guiding center is non self-adjoint. Still, it is possible to use it as a conjugate operator to prove absolute continuity of the spectrum [15, 16]. In section 6, we shall circumvent those technical difficulties, using ideas suggested in [20].

3.3 Decay of edge state eigenfunctions

In the case of bounded samples, the decay of edge state eigenfunctions into the bulk can be proved with the help of the equation

$$\psi = (E - H_0 - V_d)^{-1} V_0 \psi, \tag{3.5}$$

where ψ is an eigenfunction of H with energy E in the gaps of the bulk spectrum, and $H_0 = (\vec{p} - e\vec{A})^2/2m$, $H = H_0 + V_0 + V_d$.

The free resolvent $(E - H_0)^{-1}$ can be calculated explicitly in coordinate space representation, and it has Gaussian decay. Using this decay, equation (3.5), and the fact that $V_0\psi$ is supported at the edge, one can show that the eigenfunction decays exponentially into the bulk of the sample.

More technical details can be found in appendix A.

4 Extended edge states in large cylindrical samples

In section 2, the notion of an "extended edge state" for arbitrarily large, bounded samples was defined as one that carries a chiral edge current. The central estimate needed for the derivation of the integer quantum Hall effect is (2.7). In the present section, we show that if ψ is an eigenstate of the Hamiltonian H, with an energy between Landau levels, and if the disorder is weak, the current carried by ψ is non-vanishing and has a definite sign.

We assume the cylinder geometry described in section 2, with an edge potential confining the electron to the region $y < 0$. The current in φ-direction can be written as

$$I_\varphi = -\frac{dE}{d\Phi} = \left(\psi, \frac{2}{2\pi R}\Big(\frac{\partial_\varphi}{\mathrm{i}R} - \frac{\Phi}{2\pi R} + By\Big)\psi\right). \tag{4.1}$$

Here and from now on, units are chosen in which $m = 1/2$ and $e = 1$. Because ψ is an eigenstate of H, the expectation value of

$$[H, \mathrm{i}p_y] = -B\Big(\frac{\partial_\varphi}{\mathrm{i}R} - \frac{\Phi}{2\pi R} + By\Big) - \partial_y V \tag{4.2}$$

in ψ vanishes by the virial theorem. This yields the equation

$$I_\varphi = -\frac{1}{2\pi BR}\big(\psi, \partial_y V\psi\big). \tag{4.3}$$

Equation (4.3) can be interpreted as saying that the current arises solely from the motion of the guiding center, whereas the cyclotronic motion does not contribute to the current when averaged over the whole cylinder.

It is now intuitively clear why the current is chiral and non-vanishing. $\big(\psi, \partial_y V\psi\big)$ gets a positive contribution from the edge potential, and a contribution of indefinite sign from the disorder. If the disorder is weak, and because an edge state is localized near the edge, the contribution from the edge potential dominates the one from the disorder potential, so that the claim follows. The calculations needed to rigorously establish the lower bound on $\big(\psi, \partial_y V\psi\big)$ can be found in section 5 (proof of Theorem 1).

On the other hand $(\partial_\varphi/\mathrm{i}R - \Phi/2\pi R + By)\psi$ is bounded in norm by the Hamiltonian. Equations (4.1) and (4.3) therefore imply the estimate (2.7).

5 Steep edge potentials

In the one particle picture that is the basis for our arguments in section 2, the edge of the sample can be naturally modelled by a smooth, but steep edge potential. In this section, we shall consider a half-plane geometry. The magnetic field B points in the z-direction, the sample is infinite in the x-direction, and the electron gas is confined by a wall to the region $y < 0$.

The half-plane can be viewed as the limiting case $R \to \infty$ of the cylinder geometry with one edge. The spectrum of the Hamiltonian is not purely discrete anymore and it is not possible to induce a spectral flow by changing a flux Φ. The definition of extended states is that the corresponding spectrum is absolutely continuous.

The edge potential, V_0, is assumed to vanish for $y < 0$ and to rapidly increase for $y > 0$. The total potential is $V = V_0 + V_d$, and the Hamiltonian is

$$H = (\vec{p} - \vec{A})^2 + V = H_0 + V = H_0 + V_0 + V_d, \tag{5.1}$$

where we have again chosen units with $e = 1$ and $m = 1/2$. The vector potential is taken in the Landau gauge, $A_x = -By$, $A_y = A_z = 0$. We show absolute continuity for parts of the spectrum of H located between Landau levels, using Mourre theory with the x-coordinate of the guiding center as conjugate operator.

The case $V = V_0$, that is without disorder potential, is standard. In the Landau gauge, the y-coordinate of the guiding center $Z_y = -p_x/B$ is a cyclic coordinate. After a Fourier transformation in the x-direction, the problem splits into one-dimensional Hamiltonians H_k indexed by the constant of motion $k = -BZ_y$. Those have spectrum $E_n(k)$, where n is the Landau band index. For $k \to \infty$, one can easily establish $E_n(k) \to (2n+1)B$, while for $k \to -\infty$, we have $E_n(k) \to \infty$.

Also $E_n(k)$ is analytic as a function of k, so this implies that the spectrum of the full Hamiltonian is absolutely continuous (see [23], Theorem XIII.16).

After introducing a disorder potential, it is worthwhile to first estimate the changes in the location of the spectrum. If the disorder is a random potential satisfying certain reasonable assumptions, it is known that the almost sure spectrum of the full Hamiltonian contains the spectrum of the clean Hamiltonian as a subset [24]. More details about this will be found in appendix B. The situation is of course more complicated for an arbitrary deterministic potential, but the results about continuity of the spectrum do not depend upon existence of spectrum.

In the chosen gauge, our conjugate operator is $\Pi = BZ_x = p_y + Bx$. The commutator with H is $[H, i\Pi] = -\partial_y V$, so that one rather has a "negative commutator" than a positive one, but this does obviously not hinder the application of Mourre theory.

In addition to establishing a Mourre estimate, we need to assume that the edge potential allows the verification of conditions (i) to (iv) from section 3.2. We shall not make the attempt to present the optimal conditions on V_0, but simply note that (iii) and (iv) are valid, for example, if the potential is an upper bound for its own derivatives. Assumption (i) is trivially valid since the C^∞ function with compact support, C_c^∞, form a core for both H and Π. As for (ii), note that up to a phase factor, the group generated by Π are the translations in y-direction. If the edge potential $V_0 = V_0(y)$ does not increase too fast, for example subexponentially, so that an estimate of the form $V_0(y+\alpha) \le CV_0(y)$ holds uniformly in y, the domain of V_0 is invariant under those translations, and with it also the domain of H, since the domain of the kinetic energy is trivially invariant. As noted above, extensions of the Mourre theory allow the treatment of much more singular potentials.

For the following theorem, we assume an unbounded edge potential, vanishing for $y < 0$, with $V_0'(y) \ge 0$ for all y and $\inf\{V_0'(y); y \ge b\} > 0$ for all $b > 0$. We discuss afterwards how the assumption that V_0 is unbounded can be avoided.

Theorem 1 (Mourre estimate). *Assume $E \notin \sigma(H_0) = \{(2n+1)B, n \in \mathbb{N}_0\}$. Then there is a constant δ, such that if the disorder potential satisfies $|V_d| \le \delta$, there is an open interval $\Delta \ni E$ and a positive constant α with*

$$- E_\Delta(H)\,[H, i\Pi]\,E_\Delta(H) \ge \alpha E_\Delta(H). \tag{5.2}$$

The strategy for the proof is clear: Since $-[H, i\Pi] = \partial_y V_0 + \partial_y V_d$, one first establishes the estimate considering only $\partial_y V_0$, but including V_d in the Hamiltonian. This is the content of proposition 1, which yields a bound $E_\Delta(H)\partial_y V_0 E_\Delta(H) \ge \tilde{\alpha} E_\Delta(H)$. Then one estimates $|\partial_y V_d|$ on Δ by E and δ, and Theorem 1 follows if $|E_\Delta\,\partial_y V_d\,E_\Delta| < \tilde{\alpha}$. It is also possible to introduce another constant δ' to control the derivative $|\partial_y V_d| \le \delta'$ separately, so that (5.2) follows for $\delta' < \tilde{\alpha}$. This allows a somewhat more generous choice of δ.

We point out again that the proof of Theorem 1 can be transferred, without any changes, to the cylinder geometry to prove that $\big(\psi, \partial_y V \psi\big)$ is positive if ψ is an energy eigenstate with energy well in between Landau levels.

Proposition 1. *Assume $E \notin \sigma(H_0)$. Then there is a constant δ, such that if the disorder satisfies $|V_d| \leq \delta$, there is an open interval $\Delta \ni E$ and a positive constant $\tilde{\alpha}$ with*

$$E_\Delta(H)\,\partial_y V_0\,E_\Delta(H) \geq \tilde{\alpha} E_\Delta(H). \tag{5.3}$$

This proposition is at the heart of the matter, and its proof will be presented in some detail.

We have to show that $\big(\psi, V_0'\psi\big) \geq \tilde{\alpha}\,\|\psi\|^2$ with $\tilde{\alpha} > 0$ holds for all ψ with $\psi = E_\Delta(H)\psi$†. Obviously $\big(\psi, V_0'\psi\big) \geq 0$ is non-negative. The intuition is that if $\big(\psi, V_0'\psi\big)$ goes to 0, then $\big(\psi, V_0\psi\big)$ is also small, whence ψ is supported in the bulk and cannot be an edge state, so that $\psi = E_\Delta(H)\psi$ is impossible. The problem is to estimate $\big(\psi, V_0'\psi\big)$ in terms of $\big(\psi, V_0\psi\big)$.

Proof of proposition 1. Let $\eta = \mathrm{dist}(E, \sigma(H_0))$, so that for all $\phi \in \mathcal{D}(H_0)$ in the domain of H_0, $\|(E - H_0)\phi\| \geq \eta\,\|\phi\|$ holds. The condition we put on δ is $\eta > \delta$. Then E lies in the gaps of the bulk Hamiltonian $H_0 + V_d$.

Choose an $\epsilon > 0$ with $\eta > \epsilon > \delta$ and a smooth "cutoff" function $j = j(y)$ with $1 \geq j \geq 0$, $j(y) = 1$ for $y \leq b$ for some $b > 0$ and $\sup(|j(y)V(x,y)|) \leq \epsilon$. This is possible because of the assumptions on V_0 and because of $|V_d| \leq \delta < \epsilon$. (see figure 4).

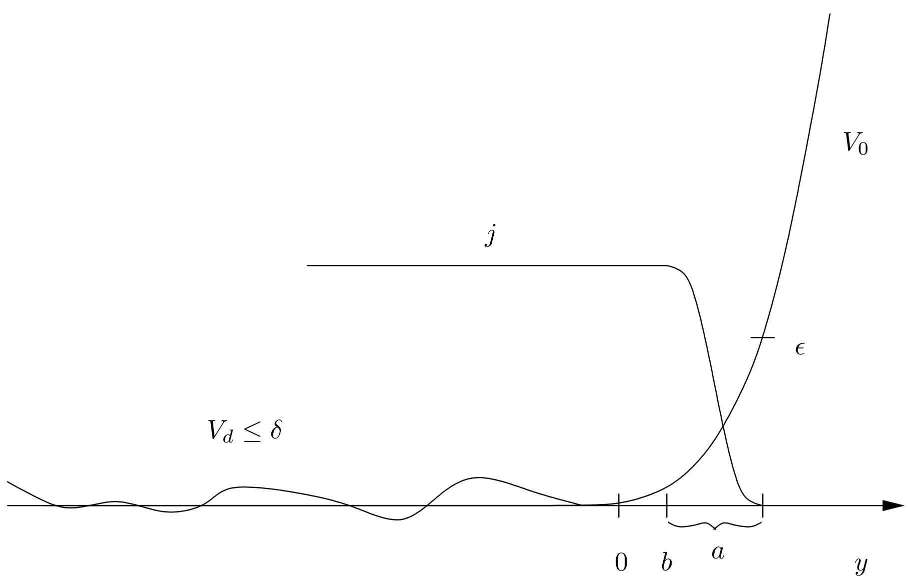

Figure 4: The cutoff function j

†From now on, a $'$ will denote a derivative with respect to y.

We control j and its derivatives by introducing the following finite constants:

$$\begin{aligned} C_1 &= \sup\{1/V_0'(y)\} : y \in \operatorname{supp}(1-j)\}\,, \\ C_2 &= \sup\{(j''(y))^2/V_0'(y) : y \in \operatorname{supp}(1-j)\}\,, \\ C_3 &= \sup\{(j'(y))^2/V_0'(y) : y \in \operatorname{supp}(1-j)\}\,, \\ C_4 &= \sup\{|j'(y)|\} \end{aligned} \tag{5.4}$$

Keeping track of these different constants will later allow to determine the dependence of the estimates on the steepness of the potential. Now let $\Delta \ni E$ be an interval around E, and $\psi = E_\Delta(H)\psi$. Then $\tilde{\psi} = j\psi \in \mathcal{D}(H_0)$ and the assumption on E yields the estimate

$$\eta\,\|j\psi\| \le \|(E-H_0)j\psi\| \le \|(E-H)j\psi\| + \underbrace{\|Vj\psi\|}_{\le \epsilon\,\|\psi\|}\,. \tag{5.5}$$

A bound of $\|j\psi\|$ in the other direction is obtained from

$$\|(1-j)\psi\|^2 = \big(\psi,(1-j)^2\psi\big) \le C_1\big(\psi, V_0'\psi\big) \tag{5.6}$$

and is

$$\|j\psi\| \ge \|\psi\| - \|(1-j)\psi\| \ge \|\psi\| - C_1^{1/2}\big(\psi, V_0'\psi\big)^{1/2}. \tag{5.7}$$

$Hj = jH - 2\mathrm{i}(p_y - A_y)j' + j''$ yields

$$\|(E-H)j\psi\| \le \|j(E-H)\psi\| + 2\,\|(p_y - A_y)j'\psi\| + \|j''\psi\|\,. \tag{5.8}$$

The terms on the right hand side can be controlled in terms of $|\Delta|$ and $\big(\psi, V_0'\psi\big)$ as follows:

$$\begin{aligned} \|j''\psi\|^2 &= \big(\psi,(j'')^2\psi\big) \le C_2\big(\psi, V_0'\psi\big), \\ \|j(E-H)\psi\| &\le |\Delta|\,\|\psi\|\,, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \|(p_y - A_y)j'\psi\|^2 &= \big(\psi, j'(p_y - A_y)^2 j'\psi\big) \\ &\le \big(\psi, j'Hj'\psi\big) + \delta\big(\psi,(j')^2\psi\big) \end{aligned} \tag{5.10}$$

because $V + \delta, (p_x - A_x)^2 \ge 0$. Using $j'Hj' = ({j'}^2 H + H{j'}^2)/2 + (j'')^2$, the first term on the right hand side of (5.10) can be further estimated as

$$\begin{aligned} \big(\psi, j'Hj'\psi\big) &= \frac{1}{2}\big(j'\psi, j'H\psi\big) + \frac{1}{2}\big(j'H\psi, j'\psi\big) + \big(\psi,(j'')^2\psi\big) \\ &\le \|j'\psi\|\,\|j'H\psi\| + \big(\psi,(j'')^2\psi\big). \end{aligned} \tag{5.11}$$

Since

$$\|j'\psi\|^2 \le C_3\big(\psi, V_0'\psi\big), \tag{5.12}$$

$$\|j'H\psi\| \le \|j'(E-H)\psi\| + E\,\|j'\psi\| \le C_4\,|\Delta|\,\|\psi\| + EC_3^{1/2}\big(\psi, V_0'\psi\big)^{1/2}, \tag{5.13}$$

equations (5.10) and (5.11) imply

$$\|(p_y - A_y)j'\psi\|^2 \le \big(\psi, V_0'\psi\big)[EC_3 + C_2 + \delta C_3] + \big(\psi, V_0'\psi\big)^{1/2} C_3^{1/2} C_4\, |\Delta|\, \|\psi\|\,,$$

and thus

$$\begin{aligned}\|(p_y - A_y)j'\psi\| \;&\le \big(\psi, V_0'\psi\big)^{1/2}[EC_3 + C_2 + \delta C_3]^{1/2} \\ &\qquad + \big(\psi, V_0'\psi\big)^{1/4} C_3^{1/4} C_4^{1/2}\, |\Delta|^{1/2}\,. \end{aligned} \tag{5.14}$$

We now combine (5.5), (5.7), (5.8), (5.9) and the last inequality to:

$$\begin{aligned}\eta\Big(\|\psi\| - C_1^{1/2}\big(\psi, V_0'\psi\big)^{1/2}\Big) &\le \eta\, \|j\psi\| \\ &\le \|(E-H)j\psi\| + \epsilon\, \|\psi\| \\ &\le |\Delta|\, \|\psi\| + \epsilon\, \|\psi\| + C_2^{1/2}\big(\psi, V_0'\psi\big)^{1/2} + \\ &\qquad + 2\big(\psi, V_0'\psi\big)^{1/2}\big[(E+\delta)C_3 + C_2\big]^{1/2} \\ &\qquad\quad + 2\big(\psi, V_0'\psi\big)^{1/4} C_3^{1/4} C_4^{1/2}\, |\Delta|^{1/2}\, \|\psi\|^{1/2} \end{aligned} \tag{5.15}$$

Abbreviating $D_1 = C_2^{1/2} + 2\big[(E+\delta)C_3 + C_2\big]^{1/2}$, $D_2 = 2C_3^{1/4}C_4^{1/2}$ and $D_3 = C_1^{1/2}$, the conclusion is that for all $\psi = E_\Delta(H)\psi$:

$$\begin{aligned}(\eta - |\Delta| - \epsilon)\, \|\psi\| &\le \big(\psi, V_0'\psi\big)^{1/2}(D_1 + \eta D_3) + D_2\big(\psi, V_0'\psi\big)^{1/4} |\Delta|^{1/2}\, \|\psi\|^{1/2} \\ &\le 2\big(\psi, V_0'\psi\big)^{1/2}(D_1 + \eta D_3) + (\lambda - 1)\, |\Delta|\, \|\psi\|\,, \end{aligned} \tag{5.16}$$

where $\lambda - 1 = D_2^2/4(D_1 + \eta D_3)$. Since $\eta - \epsilon > 0$, and by making $|\Delta|$ small enough, one finally gets:

$$\big(\psi, V_0'\psi\big) \ge \left[\frac{\eta - \lambda\, |\Delta| - \epsilon}{2(D_1 + \eta D_3)}\right]^2 \|\psi\|^2 =: \tilde{\alpha}\, \|\psi\|^2\ . \tag{5.17}$$

□

Proof of Theorem 1. The missing piece is the estimate of $\partial_y V_d = -[V_d, \mathrm{i}p_y]$ on the energy interval Δ.

$$\begin{aligned}\big|\big(\psi, [V_d, \mathrm{i}p_y]\psi\big)\big| &= \big|\big(\psi, [V_d, i(p_y - A_y)]\psi\big)\big| \\ &= \big|\big(V_d\psi, (p_y - A_y)\psi\big) - \big((p_y - A_y)\psi, V_d\psi\big)\big| \\ &\le 2\delta\, \|\psi\|\, \|(p_y - A_y)\psi\|\ . \end{aligned} \tag{5.18}$$

For $\psi = E_\Delta(H)\psi$ with $\Delta \ni E$,

$$\begin{aligned}\|(p_y - A_y)\psi\|^2 &= \big(\psi, (p_y - A_y)^2\psi\big) \leq \big(\psi, (H+\delta)\psi\big) \\ &\leq (E + |\Delta| + \delta)\, \|\psi\|^2\end{aligned} \tag{5.19}$$

so that

$$\big|\big(\psi, [V_d, \mathrm{i}p_y]\psi\big)\big| \leq 2\delta(E + |\Delta| + \delta)^{1/2}\, \|\psi\|^2\,. \tag{5.20}$$

With the additional condition on δ,

$$2\delta(E + |\Delta| + \delta)^{1/2} < \tilde{\alpha}, \tag{5.21}$$

the proof is complete. □

We now discuss the dependence of the estimate on the assumptions about the disorder and edge potentials.

As mentioned above, one can relax the constraints on δ by introducing another constant δ' and imposing $|V_d'| \leq \delta' < \tilde{\alpha}$ with $\tilde{\alpha}$ as in (5.17). It is actually enough if V_d' is small near the edge, as one can easily show in a manner similar to the above by introducing a partition of unity separating the regions where V_d is smooth from those where it is rougher. The length scale is of course set by the cyclotron length $l_c = 1/\sqrt{B}$: If V_d varies strongly on a scale of l_c, it is better to use $\delta\sqrt{E}$ as in (5.21) to control $|V_d'|$. If V_d is smooth on this scale, the use of a δ' is more appropriate.

In an alternative approach, using the x-coordinate of the particle itself as conjugate operator, similarly to section 6, it is possible to avoid assumptions on the derivative of the potential altogether. We will not make this explicit here.

We now turn to the dependence of the estimates on the edge potential. $\tilde{\alpha}$ as defined in (5.17) depends not only on the disorder potential via δ, but also on V_0 via the constants in the denominator, which are constrained by $|jV| < \epsilon$: Assume V_0 increases from 0 to ϵ on a length scale a. Then j must go from 1 to 0 on this scale (see figure 4), and for small a, the constants vary as:

$$C_1 \sim \frac{a}{\epsilon}\,, \qquad C_2 \sim \frac{1}{\epsilon a^3}\,, \qquad C_3 \sim \frac{1}{\epsilon a}\,, \qquad D_1^2 \sim \frac{1}{\epsilon a^3}\,, \qquad D_3^2 \sim \frac{a}{\epsilon}\,. \tag{5.22}$$

Together with (5.17), small a or a steep potential implies $\tilde{\alpha} \sim (\eta - \epsilon)^2 \epsilon a^3$. A steeper edge thus seems to allow less disorder. This problem is unexpected, since in the classical case also a hard wall leads to extended edge states. Using dimensional analysis, one can argue that any direct estimate of $\big(\psi, V_0'\psi\big)$ in terms of η, which is roughly the same as $\big(\psi, V_0\psi\big)$, will have a dependence on a that makes it fail when a tends to zero.

Before we return to this problem in the next section by analyzing the problem with Dirichlet boundary conditions, we indicate what must be changed in the case of a bounded edge potential.

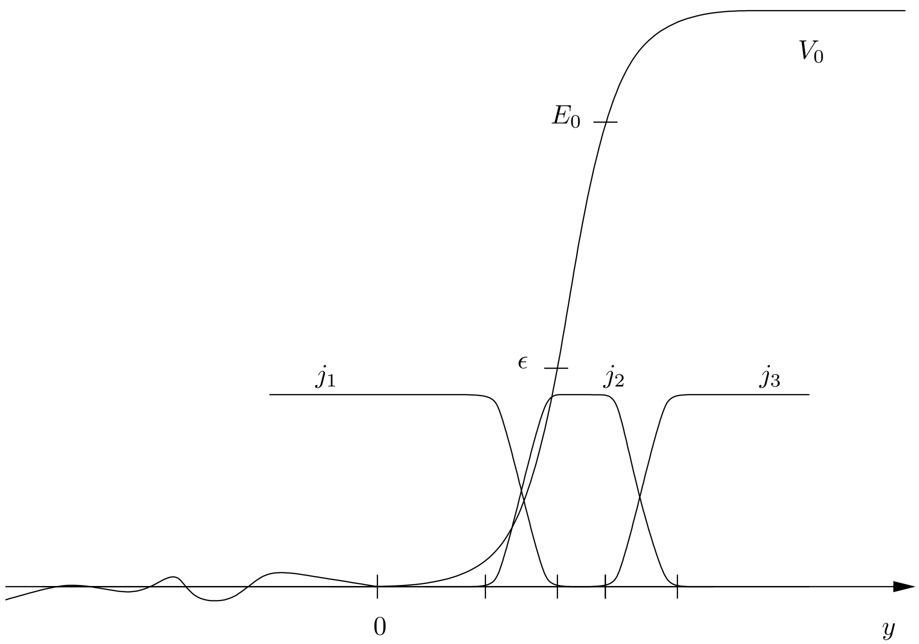

Figure 5: Partition of unity for bounded V_0

We assume that the edge potential levels off above some height E_0, but still with $V_0' \geq 0$. Let $\eta = \min(\operatorname{dist}(E, \sigma(H_0)), E_0 - E)$, and choose $\delta < \epsilon < \eta$ as above. Introduce a partition of unity according to figure 5, satisfying $\sup(|j_1 V|) \leq \epsilon$ and $V_0 \geq E_0 > E$ on $\operatorname{supp} j_3$. The condition $\inf\{V_0'(y) : y \geq b\} > 0$ for all $b > 0$ is replaced by the condition $\inf\{V_0'(y) : y \in \operatorname{supp}(j_2)\} > 0$.

From the proof of proposition 1, we know that for small $|\Delta|$,

$$\begin{aligned} \eta \, \|j_1 \psi\| &\leq \|(E - H_0) j_1 \psi\| \\ &\leq \lambda \, |\Delta| \, \|\psi\| + \epsilon \, \|j_1 \psi\| + C\big(\psi, V_0' \psi\big)^{1/2}. \end{aligned} \tag{5.23}$$

The constants C and λ might change subsequently, but are independent of ψ, δ, and ϵ. Since $\eta - \epsilon \leq V_0 - E + V_d$ on $\operatorname{supp}(j_3)$,

$$\begin{aligned} (\eta - \epsilon) \, \|j_3 \psi\|^2 &= (\eta - \epsilon)\big(\psi, j_3^2 \psi\big) \\ &\leq \big(\psi, j_3 (H - E) j_3 \psi\big) \\ &\leq \|j_3 \psi\| \, \|j_3 (H - E) \psi\| + \big(\psi, (j_3')^2 \psi\big) \\ &\leq \|j_3 \psi\| \, |\Delta| \, \|\psi\| + \|j_3 \psi\| \, C\big(\psi, V_0' \psi\big)^{1/2}. \end{aligned} \tag{5.24}$$

Therefore,

$$\eta \|j_3\psi\| \le |\Delta| \, \|\psi\| + \epsilon \|j_3\psi\| + C\big(\psi, V_0'\psi\big)^{1/2}. \tag{5.25}$$

Together with

$$\|\psi\| \le \|j_1\psi\| + \|j_2\psi\| + \|j_3\psi\| , \tag{5.26}$$

$$\|j_2\psi\| \le C\big(\psi, V_0'\psi\big)^{1/2}, \tag{5.27}$$

equations (5.23) and (5.25) yield

$$\eta \|\psi\| \le \epsilon \|\psi\| + \lambda |\Delta| \, \|\psi\| + C\big(\psi, V_0'\psi\big)^{1/2} \tag{5.28}$$

so that one gets an estimate of $\big(\psi, V_0'\psi\big)$ as above.

6 Dirichlet boundary conditions

In this section, we analyze the problem of extended edge states with the smooth edge potential replaced with Dirichlet boundary conditions. The fact that the conjugate operator that was used in section 5 is not self-adjoint in this situation makes special care necessary on the technical side. It is very likely that Mourre theory can be extended in some generality to the case when the conjugate operator is not self-adjoint. Elements of this theory can be found in [12, 13, 14]. That Mourre theory can be used to show absolute continuity of the edge spectrum with the x-coordinate of the guiding center, Z_x, as conjugate operator, has been demonstrated in the preprint version of this paper [16], see also [15], which contains the first proof of absolute continuity of the spectrum as well as the first analysis of Dirichlet boundary conditions. Here, we shall resolve the problem as follows, using ideas suggested in reference [20]. We apply Mourre theory with the x-coordinate of the particle itself along the edge as conjugate operator, which is manifestly self-adjoint. The positivity of the commutator with the Hamiltonian, $[H, \mathrm{i}x]$ is reduced to the positivity of the commutator $[H, \mathrm{i}Z_x]$. Classically, only Z_x is monotonically increasing in time, but quantum mechanically, the positivity of either of the two commutators is equivalent to the positivity of the other.

In comparison with section 5, the assumptions on the disorder potential can be relaxed in that the boundedness of the derivative is not necessary. Furthermore, the proof of the positivity of the commutator $\mathrm{i}[H, Z_y]$ does not depend on having the geometry of a half-plane, but can be extended to more general domains in the plane satisfying a certain geometric condition which we state below.

This section is organized as follows: We first give precise definitions of the occuring spaces and operators, and state the main theorems. We then prove the theorems for the half-plane, since the ideas can be made more transparent in this situation. We finally give the proofs for more general domains.

6.1 Definitions and Theorems

Let Ω be a domain in $\mathbb{R}^2$. Ω may be the half-plane $\mathbb{R}^2_- = \mathbb{R}\times\mathbb{R}_-$ or a more general domain satisying a certain *geometric assumption* (GA) stated in section 6.3. To convey an idea of the domains under consideration, we present three examples in figure 6. (GA) allows for the domains (a, b), but not for (c).

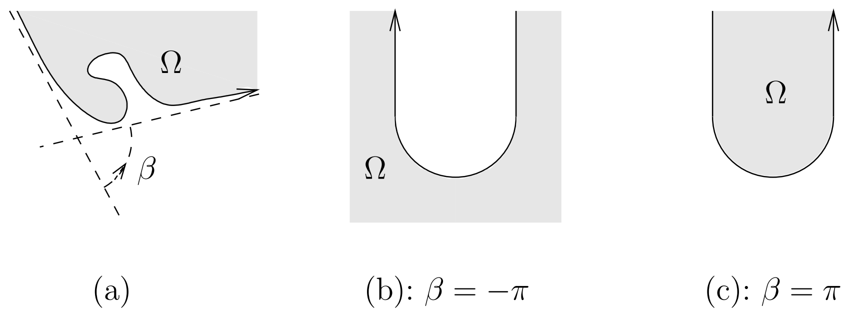

Figure 6: Examples for domains. (a,b) satisfy (GA), (c) does not.

The Hamiltonian in this section will be again H_0+V_d, where the unperturbed Hamiltonian is

$$H_0 = (-\mathrm{i}\nabla - A)^2 \tag{6.1}$$

on $\mathrm{L}^2(\Omega)$ with Dirichlet boundary conditions, i.e., with domain $\mathcal{D}(H_0) = \mathrm{W}^2(\Omega, A)$ $\cap \mathrm{W}^1_0(\Omega, A)$, where the magnetic Sobolev spaces are defined using covariant derivatives, e.g.,

$$\mathrm{W}^1(\Omega, A) = \{\psi \in \mathrm{L}^2(\Omega) \mid (-\mathrm{i}\partial_i - A_i)\psi \in \mathrm{L}^2(\Omega),\ i = x, y\}\ . \tag{6.2}$$

Note that we do not specify the gauge at this point. We shall be working in Landau gauge for the half-plane, but for more general domains this is not appropriate for obvious geometrical reasons.

Theorem 2. *Let $E/B \notin 2\mathbb{N}_0 + 1$. For sufficiently small $\|V_d\|_\infty/B$, the spectrum of H on the half–plane $\mathbb{R}^2_-$ is purely absolutely continuous near E.*

Here, and below, 'sufficiently small' is meant depending only on quantities explicitly mentioned. In particular, for Theorem 2, the bound is uniform in B.

In a classical picture, absolute continuity of the spectrum corresponds to the guiding center of the electron jumping in a definite direction along the boundary, $\partial\Omega$, of $\Omega = \mathbb{R}^2_-$, each time the electron hits the wall. If the boundary is not a straight line, then at each collision the guiding center moves forward in the direction of the tangent vector to $\partial\Omega$ at the collision point. Yet, this may be a backward motion with respect to the tangent vector at the next collision point, with the result that a classical edge trajectory can get trapped. No trapping is possible, however, if the cyclotron radius of the electron is small compared to the radius of curvature of $\partial\Omega$ (more precisely: compared to the radius of injectivity of the tubular map associated with $\partial\Omega$). Then the bouncing motion will result in an effective progression along $\partial\Omega$.

The generalization of Theorem 2 to more general domains reads as follows.

Theorem 3. *Assume Ω satisfies (GA). Fix $E/B \notin 2\mathbb{N}_0 + 1$ and let $\|V_d\|_\infty/B$ be small enough. Then, the spectrum of H is purely absolutely continuous near E, provided B is large enough.*

Finally, no trapping is possible if Ω lies on one side of the graph of a function f,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid y < f(x)\} , \tag{6.3}$$

with $f' \in \mathrm{C}^2_{\mathrm{unif}}(\mathbb{R})$.

Theorem 4. *Let $E/B \notin 2\mathbb{N}_0 + 1$, and $B_0 > 0$. For sufficiently small $\|V_d\|_\infty/B$, the spectrum of H on the domain (6.3) is purely absolutely continuous near E, provided $B \geq B_0$.*

Under the scaling $x \to \lambda x$, $B \to \lambda^{-2}B$, $V \to \lambda^{-2}V$, we have $H \to \lambda^{-2}H$. Without loss of generality we may thus set

$$B = 1$$

in Theorem 2, and the same is true for Theorem 3 if its last sentence is replaced by: *Then, the spectrum of H on $\lambda\Omega$ is purely absolutely continuous near E, provided $\lambda > 0$ is large enough.* Similarly the conclusion of Theorem 4 is *… the spectrum of H on the domain defined by $f_\lambda(x) = \lambda f(x/\lambda)$ is purely absolutely continuous near E, provided $\lambda \geq \lambda_0 = B_0^{1/2}$*. The Theorems will be proved in this form.

6.2 The half–plane

We begin the proof of Theorem 2 with some preliminaries (1.–3.):

1. We shall work in Landau gauge

$$A = (-y, 0) . \tag{6.4}$$

2. Let $p = -\mathrm{i}\nabla$. As mentioned above, the component $p_y = -\mathrm{i}\partial_y$ cannot be defined self-adjointly on $\mathrm{L}^2(\Omega)$, a fact related to the integration by parts identity

$$(p_y\omega, \rho) - (\omega, p_y\rho) = \mathrm{i}\langle\omega, \rho\rangle , \qquad (\omega, \rho \in \mathrm{W}^1(\Omega, A)) , \tag{6.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathrm{L}^2(\partial\Omega)$. Strictly speaking one should write $\langle L\omega, L\rho\rangle$ instead of $\langle\omega, \rho\rangle$, where L is the boundary trace operator, which is bounded as a map $L : \mathrm{W}^1(\Omega, A) \to \mathrm{L}^2(\partial\Omega)$. Below we shall need another, similar statement (see e.g. [25], Theorem 8.3 or [26], Section 7.50),

Theorem A. *The map*

$$\begin{aligned} L:\ \mathrm{W}^2(\Omega,A) &\to \mathrm{W}^{3/2}(\partial\Omega)\times \mathrm{W}^{1/2}(\partial\Omega)\\ \psi &\mapsto (\psi\restriction\partial\Omega, \partial_y\psi\restriction\partial\Omega) \end{aligned} \tag{6.6}$$

is continuous, has $\ker L = \mathrm{W}^2_0(\Omega,A)$*, and admits a continuous right inverse* R*:* $LR = 1$*.*

Proof. The statement in [25, 26] refers to $A = 0$; to extend it to the present situation, pick a function $j \in \mathrm{C}_0^\infty(-\infty,0]$ with $j(y) = 1$ for y near 0. Then L and R can be replaced by Lj, and jR respectively. The claim now follows, since the map $\psi \mapsto j\psi$ is bounded as a map $\mathrm{W}^2(\Omega,A) \to \mathrm{W}^2(\Omega,0)$, resp. $\mathrm{W}^2(\Omega,0) \to \mathrm{W}^2(\Omega,A)$, for the special gauge (6.4). □

3. The proof of Theorem 2 is based again on Mourre theory [7, 8, 9, 10]. We will here refer to the formulation given in [9], using the following pieces of notation: $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}$, $\rho(H) \ni z$ is the resolvent set of H, on which $R(z) := (H-z)^{-1}$, and $E_\Delta(H)$ is the spectral projection for H onto $\Delta \subset \mathbb{R}$.

Theorem B. *Let* H, A *be self–adjoint operators on* $\mathcal{H}$*, and* $E \in \mathbb{R}$*. Assume:*

(i) There is $z \in \rho(H)$ *such that the map*

$$g:\ \mathbb{R} \to \mathcal{B}(\mathcal{H})\ , \qquad s \mapsto \mathrm{e}^{\mathrm{i}sA}R(z)\mathrm{e}^{-\mathrm{i}sA} \tag{6.7}$$

is of class $\mathrm{C}^{1+\varepsilon}$ *with* $0+ \le \varepsilon \le 1$ *in the norm topology (with* $\varepsilon = 0+$ *meaning that* $g'(s)$ *is Dini continuous);*

(ii) There is an open interval $\Delta \ni E$ *and an* $\alpha > 0$ *such that*

$$E_\Delta(H)\mathrm{i}[H,A]E_\Delta(H) \ge \alpha E_\Delta(H)\ . \tag{6.8}$$

Then the spectrum of H *is purely absolutely continuous near* E*.*

According to Theorem 6.2.10 in [27], $g \in \mathrm{C}^1$ implies that $[H,A]$, defined as a quadratic form on $\mathcal{D}(H)\cap\mathcal{D}(A)$, extends to a bounded operator $[H,A] : \mathcal{D}(H) \to \mathcal{D}(H)^*$. In particular, the left hand side of (6.8) is a bounded operator on $\mathcal{H}$.

We now compute the commutator of the non self-adjoint operator $\Pi = -p_y - x$ with H_0, in the sense of quadratic forms.

Lemma 1. *For* $\varphi, \psi \in \mathcal{D}(H_0)\cap\mathcal{D}(x)$ *we have*

$$\mathrm{i}[(H_0\varphi, \Pi\psi) - (\Pi\varphi, H_0\psi)] = \langle p_y\varphi, p_y\psi\rangle\ . \tag{6.9}$$

Proof. By density we may assume that φ, ψ are of compact support. The separate contributions from the two terms in $H_0 = p_y^2 + (p_x + y)^2$ are

$$\mathrm{i}[(p_y^2\varphi, \Pi\psi) - (\Pi\varphi, p_y^2\psi)] = \langle p_y\varphi, p_y\psi\rangle \ , \tag{6.10}$$

$$\mathrm{i}[((p_x + y)^2\varphi, \Pi\psi) - (\Pi\varphi, (p_x + y)^2\psi)] = 0 \ . \tag{6.11}$$

In fact, as (6.10) is concerned, the contribution of x to the commutator is, using (6.5),

$$\mathrm{i}[(p_y\varphi, p_y x\psi) - (p_y x\varphi, p_y\psi)] = 0 \ , \tag{6.12}$$

whereas that of p_y is

$$\mathrm{i}[(p_y^2\varphi, p_y\psi) - (p_y\varphi, p_y^2\psi)] = -\langle p_y\varphi, p_y\psi\rangle \ . \tag{6.13}$$

To verify (6.11), we note that $[p_x + y, p_y + x]\psi = 0$. Hence

$$\begin{aligned}
&\big((p_x + y)^2\varphi, (p_y + x)\psi\big) = \\
&\quad = \big((p_x + y)\varphi, (p_x + y)(p_y + x)\psi\big) \\
&\quad = \big((p_x + y)\varphi, (p_y + x)(p_x + y)\psi\big) = \big((p_y + x)(p_x + y)\varphi, (p_x + y)\psi\big) \\
&\quad = \big((p_x + y)(p_y + x)\varphi, (p_x + y)\psi\big) = \big((p_y + x)\varphi, (p_x + y)^2\psi\big) \ ,
\end{aligned} \tag{6.14}$$

where, in the third equality, we used (6.5) with $(p_x + y)\varphi \restriction \partial\Omega = 0$. □

We now turn to the proof of the basic positivity estimate.

Lemma 2. *Fix $E \notin 2\mathbb{N}_0 + 1$. Then there is an open interval $\Delta \ni E$ and an $\alpha > 0$ such that*

$$|\!|\!| p_y E_\Delta(H_0)\psi |\!|\!|^2 \geq \alpha \|E_\Delta(H_0)\psi\|^2 \ , \tag{6.15}$$

where we set $|\!|\!| \cdot |\!|\!|^2 = \langle \cdot, \cdot \rangle$.

Proof. We note that $RL\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$, and write for $\psi \in \mathcal{D}(H_0)$

$$(H_0 - E)\psi = (H_0 - E)(\psi - RL\psi) + (H_0 - E)RL\psi \ . \tag{6.16}$$

Since $L(\psi - RL\psi) = 0$, the function $\psi - RL\psi \in \mathrm{W}_0^2(\Omega, A)$ can be extended to $\mathbb{R}^2$ by 0, and the Dirichlet boundary condition dropped. Hence $\|(H_0 - E)(\psi - RL\psi)\| \geq \mathrm{dist}(E, 2\mathbb{N}_0 + 1)\|\psi - RL\psi\|$. Denoting the r-th Sobolev norm by $\|\cdot\|_r$, we obtain from $\|(H_0 - E)RL\psi\| \leq \mathrm{const}\, \|RL\psi\|_2 \leq \mathrm{const}\, (|\!|\!|\psi|\!|\!|_{3/2} + |\!|\!|\partial_1\psi|\!|\!|_{1/2})$ (Theorem A) and $\psi \restriction \partial\Omega = 0$ that

$$\|(H_0 - E)\psi\| \geq \mathrm{dist}(E, 2\mathbb{N} + 1)\|\psi\| - \mathrm{const}\, |\!|\!|\partial_1\psi|\!|\!|_{1/2} \ . \tag{6.17}$$

For any $\varepsilon > 0$ we have

$$|||\partial_1\psi|||_{1/2} \le \text{const}\,(\varepsilon^{-1}|||\partial_1\psi|||_0 + \varepsilon|||\partial_1\psi|||_1)\;,$$

with $|||\partial_1\psi|||_1 \le \text{const}\,\|\psi\|_{5/2} \le \text{const}\,\|H_0^{5/4}\psi\|$. Here we used the boundedness of the map $\mathrm{W}^{5/2}(\Omega, A) \to \mathrm{W}^1(\partial\Omega)$, $\psi \mapsto \partial_1\psi \restriction \partial\Omega$ (see [25], Theorem 9.4 or [26], Theorem 7.58). We insert these estimates in equation (6.17) and use $\|H_0^{5/4}E_\Delta(H_0)\psi\| \le (|E| + |\Delta|)^{5/4}\|\psi\|$ and $\|(H_0 - E)E_\Delta(H_0)\psi\| \le |\Delta|\|\psi\|$. For sufficiently small ε and $|\Delta|$ we obtain the claim. □

Following [20], we deduce absolute continuity of the spectrum using $A = -x$ as conjugate operator in the sense of Mourre theory (Theorem B), which in contrast to Π is self–adjoint on its natural domain. The right hand side in

$$H(\tau) = \mathrm{e}^{\mathrm{i}\tau x} H \mathrm{e}^{-\mathrm{i}\tau x} = p_y^2 + (p_x - \tau + y)^2 + V_d$$

extends from $\tau \in \mathbb{R}$ to an entire analytic family of type A. Hence assumption (i) of Theorem B is fulfilled. Assumption (ii) holds because of the following lemma.

Lemma 3. *Let $E \notin 2\mathbb{N}_0 + 1$. Then there is an open interval $\Delta \ni E$ and an $\alpha > 0$ such that*

$$-E_\Delta(H)\mathrm{i}[H, x]E_\Delta(H) \ge \alpha E_\Delta(H)\;,$$

provided $\|V\|_\infty$ is small enough.

Proof. We shall prove the statement in the case $V_d = 0$ first. Because of Lemmas 1 and 2, it suffices to show that the difference $\Pi - (-x) = -p_y$ contributes arbitrarily little to the commutator. In a state $\psi = E_\Delta(H_0)\psi$, this contribution is estimated as

$$\begin{aligned}|(H_0\psi, p_y\psi) - (p_y\psi, H_0\psi)| &= |\big((H_0 - E)\psi, p_y\psi\big) - \big(p_y\psi, (H_0 - E)\psi\big)| \\ &\le 2\|(H_0 - E)\psi\|\|p_y\psi\| \le 2|\Delta|(|E| + |\Delta|)^{1/2}\|\psi\|^2\;,\end{aligned}$$

where the equality is justified by (6.5) and $\psi \in \mathcal{D}(H_0)$. The right hand side can be made arbitrarily small by letting $|\Delta| \to 0$.

We next extend the result to H and note that $H'(0) = -\mathrm{i}[H, x] = -\mathrm{i}[H_0, x]$, which is relatively bounded with respect to H_0, is unchanged. Let $\alpha > 0$ and $\Delta \ni E$ be as given by Lemma 2. We may assume Δ to be centered on E. For $\widetilde{\Delta} \ni E$, we split

$$E_{\widetilde{\Delta}}(H) = E_\Delta(H_0)E_{\widetilde{\Delta}}(H) + (1 - E_\Delta(H_0))E_{\widetilde{\Delta}}(H)\;,$$

and obtain

$$\begin{aligned}E_{\widetilde{\Delta}}(H)(H'(0) - \alpha)E_{\widetilde{\Delta}}(H) \ge{}& E_{\widetilde{\Delta}}(H)E_\Delta(H_0)(H'(0) - \alpha)E_\Delta(H_0)E_{\widetilde{\Delta}}(H) \\ &- 2\|(H'(0) - \alpha)(1 - E_\Delta(H_0))E_{\widetilde{\Delta}}(H)\|\;.\end{aligned}$$

The first term on the right hand side is non-negative, and the second line can be estimated by a constant times

$$\|H_0(1 - E_\Delta(H_0))(H_0 - E)^{-1}\| \|(H_0 - E)E_{\widetilde{\Delta}}(H)\| \le (1 + 2|E||\Delta|^{-1})(|\widetilde{\Delta}| + \|V\|_\infty) .$$

Hence,

$$E_{\widetilde{\Delta}}(H)(H'(0) - \alpha)E_{\widetilde{\Delta}}(H) \ge -\alpha/2 ,$$

for $|\widetilde{\Delta}| + \|V\|_\infty$ small enough, and therefore $E_{\widetilde{\Delta}}(H)H'(0)E_{\widetilde{\Delta}}(H) \ge (\alpha/2)E_{\widetilde{\Delta}}(H)$. □

Proof of Theorem 2. Application of Theorem B. □

6.3 General domains

We now turn to the proof of Theorem 3. We first state precisely the geometrical assumption made in Theorem 3.

Assumption. *(GA) Let $\Omega \subset \mathbb{R}^2$ be an open set.*

- *(i) Ω has the uniform C^3–property in the sense of [26];*
- *(ii) Let $\partial\Omega$ consist of finitely many connected components γ, each parametrized by its arclength s (with the induced orientation). Let there be a function $s \in \mathrm{C}^2(\overline{\Omega})$ extending arclength from $\partial\Omega$ to Ω, i.e., $s(\gamma(s')) = s'$ for $s' \in \mathbb{R}$, satisfying*

$$\|\partial_i s\|_\infty < \infty , \qquad \|\partial_i \partial_j s\|_\infty < \infty . \tag{6.18}$$

We note that, by (i), a bounded component of $\partial\Omega$ would be a closed curve, but by (ii) no such curve is allowed. Hence Ω is simply connected and unbounded. Also, (GA) is not affected by the ambiguity $s \mapsto s - s_0$ implicit in the definition of arclength.

We illustrate (GA) with an example. Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set with oriented boundary $\partial\Omega$ consisting of a single unbounded smooth curve $\gamma \in \mathrm{C}^3_{\mathrm{unif}}(\mathbb{R})$, parametrized by arclength $s \in \mathbb{R}$. For simplicity, we shall assume that γ is *asymptotically straight*, i.e.,

$$\ddot{\gamma} \in \mathrm{L}^1(\mathrm{d}s) , \tag{6.19}$$

with $\dot{} = \mathrm{d}/\mathrm{d}s$. The *overall bending* of γ is

$$\beta = \int_{-\infty}^{\infty} \ddot{\gamma}(s)\mathrm{d}s ,$$

and takes values in $[-\pi, \pi]$.

Example 1. *If*

$$\beta \neq \pi \ , \tag{6.20}$$

then Ω as described satisfies (GA). In particular, the domains (a, b) in Figure 6 are allowed, but not (c).

Proof. Only part (ii) of (GA) requires proof. Because of $\gamma \in \mathrm{C}^3_{\mathrm{unif}}(\mathbb{R})$ and of (6.19) we also have $\lim_{s\to\pm\infty} \ddot{\gamma}(s) = 0$. Elementary geometric considerations show that, for large enough $r > 0$, the equation $|\gamma(s)| = r$ has exactly two solutions $s = s_\pm(r)$, with $s_\pm(r) \to \pm\infty$, $(r \to \infty)$. We define functions $\varphi_\pm(r)$ through $\gamma(s_\pm(r)) = r(\cos\varphi_\pm(r), \sin\varphi_\pm(r))$ in such a way that $\varphi_\pm(r)$ are continuous and that $\varphi_-(r) - \varphi_+(r) \in (0, 2\pi)$. Their limiting values $\varphi^\infty_\pm = \lim_{r\to\infty} \varphi_\pm(r)$ exist and satisfy $\beta + (\varphi^\infty_- - \varphi^\infty_+) = \pi$. Condition (6.20) implies

$$\varphi^\infty_- - \varphi^\infty_+ > 0 \ . \tag{6.21}$$

Under our assumptions,

$$\frac{\mathrm{d}s_\pm}{\mathrm{d}r} \to \pm 1 \ , \quad \frac{\mathrm{d}^2 s_\pm}{\mathrm{d}r^2} \to 0 \ , \quad r\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}r} \to 0 \ , \quad r\frac{\mathrm{d}^2\varphi_\pm}{\mathrm{d}r^2} \to 0 \ , \tag{6.22}$$

as $r \to \infty$. For $R > 0$ large enough we define $s(x, y)$ on

$$\overline{\Omega} \cap \{(x, y) \mid |(x, y)| > R\} \\ = \{(x, y) = r(\cos\varphi, \sin\varphi) \mid r > R, \ \varphi_+(r) \le \varphi \le \varphi_-(r)\}$$

by linear interpolation along arcs, $|(x, y)| = r$, i.e., by

$$s(x, y) = s_-(r)\frac{\varphi - \varphi_+(r)}{\varphi_-(r) - \varphi_+(r)} + s_+(r)\frac{\varphi_-(r) - \varphi}{\varphi_-(r) - \varphi_+(r)} \ ;$$

we then smoothly extend s further to the compact complement $\overline{\Omega} \cap \{x \mid |x| \le R\}$. Now (6.18) follows from (6.21, 6.22). □

To prove Theorem 3, we will just address the changes required for the generalization from the proof of Theorem 2. For notational simplicity in intermediate results, we first consider the Hamiltonian on Ω. The required scaling $\Omega \to \lambda\Omega$ (see the end of Section 6.1) will be done later.

We define the components of the velocity as $\pi_i = -\mathrm{i}\partial_i - A_i$ $(i = x, y)$ with domain $\mathrm{W}_1(\Omega, A)$; see equation (6.2). We introduce the matrix ε with entries $\varepsilon_{xy} = -\varepsilon_{yx} = -1$, $\varepsilon_{xx} = \varepsilon_{yy} = 0$. which represents a rotation by $\pi/2$, and define the *outer unit normal* $n = n(s) = -\varepsilon\dot{\gamma}(s)$, with components n_i. For $\omega, \rho \in \mathrm{W}_1(\Omega, A)$ we then have the integration by parts identity

$$(\pi_i\omega, \rho) - (\omega, \pi_i\rho) = \mathrm{i}\langle\omega, n_i\rho\rangle \ , \tag{6.23}$$

Also, $H_0\psi = (\pi_x^2 + \pi_y^2)\psi$ for $\psi \in \mathcal{D}(H_0)$.
We next extend the trace theorem (Theorem A) to the present setting.

Theorem C. *The map*

$$L: \ \mathrm{W}^2(\Omega, A) \to \mathrm{W}^{3/2}(\partial\Omega, A\cdot\dot\gamma) \times \mathrm{W}^{1/2}(\partial\Omega, A\cdot\dot\gamma) \tag{6.24}$$

$$\psi \mapsto (\psi \restriction \partial\Omega, \pi\cdot n\psi \restriction \partial\Omega) \tag{6.25}$$

is continuous, has $\ker L = \mathrm{W}^2_0(\Omega, A)$*, and admits a continuous right inverse R:* $LR = 1$*. The norms of L and R depend only on the* C^3*–regularity data of* Ω*.*

Proof. The uniform C^3–property is associated with a cover $\{U_j\}$ of $\partial\Omega$ and corresponding diffeomorphisms (of class C^3) $\Psi_j: B = \{(x,y)\in\mathbb{R}^2 \mid |(x,y)| < 1\} \to U_j$, such that $\Psi_j(B\cap\{y<0\}) = \Omega\cap U_j$; see [26], Section 4.6 for details. The statement for $A = 0$ is [26], Theorem 7.53 (see also Section 4.29 allowing for unbounded $\partial\Omega$; or [27], Theorem 5.9). The proof makes use of a partition of unity for $\partial\Omega$ subordinate to $\{U_j\}$, with the effect of reducing the statement to the analogous restriction property from $B_- := B\cap\{y<0\}$ to $B_0 := B\cap\{y=0\}$. Similarly, in our case ($A\neq 0$) matters are reduced to the same statement, (6.25), with the replacements $\Omega\to B_-$, $\partial\Omega\to B_0$, $A\to\tilde A = \Psi_j^* A$ and $A\cdot\dot\gamma\to\tilde A_x$, where $\tilde A$ is the pull–back of A under Ψ_j, in the sense of 1–forms. The claim so left to prove is gauge covariant, and we may choose the special gauge

$$\tilde A(x,y) = \Big(-\int_0^y \tilde\omega(x,\xi)\mathrm{d}\xi, 0\Big),$$

where $\tilde\omega(x,y)\mathrm{d}x\wedge\mathrm{d}y = \Psi_j^*(\mathrm{d}x\wedge\mathrm{d}y)$ is the pull–back of the area 2–form on U_j. We note that $\tilde\omega, \tilde A\in\mathrm{C}^2(B_-)$, with bounded derivatives. Moreover, $\tilde A = 0$ on B_0. Now the claim again follows from the case $\tilde A = 0$, since the norms of $\mathrm{W}^2(B_-)$ and of $\mathrm{W}^2(B_-, \tilde A)$ are equivalent. □

Remark. As in the proof of Lemma 2, we shall also need the boundedness of the map $\mathrm{W}^{5/2}(\Omega, A)\to\mathrm{W}^1(\partial\Omega, A\cdot\dot\gamma)$, $\psi\mapsto\pi\cdot n\psi\restriction\partial\Omega$. The same argument applies and, in fact, it is only here that a uniform C^2–property does not suffice.

In the context of Theorem 3, a formal candidate for a conjugate operator is $\Pi = s(Z)$, where s is arclength as given by (GA), and Z is the guiding center, which we can write as $\vec Z = \vec x - \epsilon\vec\pi$. This definition is unsuitable, because $p = -\mathrm{i}\nabla$ and hence Z are not self–adjoint (vector) operators, and because the two components of Z do not commute. Instead, we formally linearize $s(Z)$ in $\vec Z - \vec x$ and set

$$\Pi = s(\vec x) - \nabla s\cdot\varepsilon\vec\pi = s(\vec x) - \varepsilon\vec\pi\cdot\nabla s\ , \tag{6.26}$$

which is a well–defined (non self–adjoint) operator. The second expression follows from $\varepsilon_{ij}[\pi_j, \partial_i s] = -\mathrm{i}\varepsilon_{ij}\partial_j\partial_i s = 0$. Here, and in the following, the summation convention is understood.

Lemma 4. *Let s satisfy (6.18). For $\varphi, \psi \in \mathcal{D}(H_0) \cap \mathcal{D}(s)$ we have*

$$\mathrm{i}[(H_0\varphi, \Pi\psi) - (\Pi\varphi, H_0\psi)] = (\varphi, \pi_i(m_{ij} + m_{ji})\pi_j\psi) + \langle \pi_i\varphi, (\dot\gamma_j \partial_j s)\pi_i\psi\rangle \ , \quad (6.27)$$

where $m_{ij}(\vec{x}) = \varepsilon_{jk}\partial_k\partial_i s$.

The last term in (6.27) can also be written as $\langle(\pi \cdot n)\varphi, (\dot\gamma \cdot \nabla s)(\pi \cdot n)\psi\rangle$, since $\pi_i\psi = n_i(\pi \cdot n)\psi$ on γ for $\psi \in \mathcal{D}(H_0)$. If, as in (GA), s is equal to arclength on γ we have $\dot\gamma \cdot \nabla s = 1$ by definition.

Proof. We first claim that for $\varphi \in \mathcal{D}(H_0) \cap \mathcal{D}(s)$, $\rho \in \mathrm{W}_1(\Omega, A) \cap \mathcal{D}(s)$ we have

$$\mathrm{i}[(\pi_i\varphi, \Pi\rho) - (\Pi\varphi, \pi_i\rho)] = (\pi_j\varphi, m_{ij}\rho) \ . \quad (6.28)$$

Indeed, the contribution from s in (6.26) is

$$\mathrm{i}[(\pi_i\varphi, s\rho) - (s\varphi, \pi_i\rho)] = \mathrm{i}(\varphi, [\pi_i, s]\rho) = (\varphi, \partial_i s\rho) \ , \quad (6.29)$$

by using (6.23) and $\varphi \in \mathcal{D}(H_0)$, which makes the boundary term vanish. To compute the other contribution we note that

$$\begin{aligned}\mathrm{i}[(\pi_i\varphi, \pi_j\rho) - (\pi_j\varphi, \pi_i\rho)] &= -\mathrm{i}([\pi_i, \pi_j]\varphi, \rho) + \langle\pi_i\varphi, n_j\rho\rangle - \langle\pi_j\varphi, n_i\rho\rangle \\ &= \varepsilon_{ij}(\varphi, \rho) \ ,\end{aligned}$$

since $\pi_i\varphi = n_i(\pi \cdot n)\varphi$ on $\partial\Omega$. Hence, for $f_j = \varepsilon_{kj}\partial_k s$,

$$\begin{aligned}&\mathrm{i}[(\pi_i\varphi, f_j\pi_j\rho) - (f_j\pi_j\varphi, \pi_i\rho)] \\ &\qquad = \mathrm{i}[(\pi_i\varphi, \pi_j f_j\rho) - (\pi_j\varphi, \pi_i f_j\rho)] - (\pi_i\varphi, \partial_j f_j\rho) + (\pi_j\varphi, \partial_i f_j\rho) \\ &\qquad = (\varphi, \partial_i s\rho) - (\pi_j\varphi, m_{ij}\rho) \ ,\end{aligned} \quad (6.30)$$

because of $\varepsilon_{ij}f_j = \partial_i s$, $\partial_j f_j = 0$ and of $\partial_i f_j = -m_{ij}$. Subtracting (6.30) from (6.29) yields (6.28), which can also be written as

$$\mathrm{i}[(\pi_i\omega, \Pi\psi) - (\Pi\omega, \pi_i\psi)] = (\omega, m_{ij}\pi_j\psi) \ , \quad (6.31)$$

for $\omega \in \mathrm{W}_1(\Omega, A) \cap \mathcal{D}(s)$. Moreover, (6.23) yields

$$\mathrm{i}[(\Pi\omega, \rho) - (\omega, \Pi\rho)] = \langle\omega, \varepsilon_{ji}\partial_j s n_i\rho\rangle = \langle\omega, \dot\gamma_i\partial_i s\rho\rangle \ . \quad (6.32)$$

Writing the left hand side of (6.27) as

$$\begin{aligned}&\mathrm{i}[(\pi_i^2\varphi, \Pi\psi) - (\Pi\pi_i\varphi, \pi_i\psi) \ + \ (\Pi\pi_i\varphi, \pi_i\psi) - (\pi_i\varphi, \Pi\pi_i\psi) \\ &\qquad\qquad + (\pi_i\varphi, \Pi\pi_i\psi) - (\Pi\varphi, \pi_i^2\psi)] \ .\end{aligned}$$

and using (6.31, 6.32, 6.28) with $\omega = \pi_i\varphi$ and $\rho = \pi_i\psi$, respectively, concludes the proof if $\omega, \rho \in \mathcal{D}(s)$. By density this suffices. □

Proof of Theorem 3. Statement and proof of Lemma 2 hold true with the replacement $p_y \to \pi \cdot n$, with Δ, α depending only on the regularity data for Ω. These data are inherited by $\lambda\Omega$ with $\lambda \geq 1$. On the other hand $\lambda\Omega$ satisfies part (ii) of (GA) with $s_\lambda(\vec{x}) = \lambda s(\vec{x}/\lambda)$, and correspondingly $m_\lambda(\vec{x}) = \lambda^{-1}m(\vec{x}/\lambda)$ in equation (6.27). There, the first term on the right hand side is then estimated in absolute value for $\varphi = \psi = E_\Delta(H_\lambda)\psi$ by a constant times

$$\lambda^{-1}\|\pi E_\Delta(H_\lambda)\psi\|^2 \leq \lambda^{-1}(|E| + |\Delta|)^{1/2}\|\psi\|^2 \, .$$

As a result, the positivity of the commutator (6.27) on $E_\Delta(H_\lambda)$ obtains for large $\lambda > 0$. As to the regularity assumption, the conjugate operator $A = s(x)$ gives rise to the analytic family

$$H(\tau) = \mathrm{e}^{\mathrm{i}\tau s} H \mathrm{e}^{-\mathrm{i}\tau s} = H - \tau(\nabla s \cdot \pi + \pi \cdot \nabla s) + \tau^2(\nabla s)^2 \, . \tag{6.33}$$

Statement and proof of Lemma 3 are changed accordingly, with $\Pi - s(x) = -\varepsilon\pi\cdot\nabla s$ replacing p_y. In (6.33) we have suppressed the subscript λ. One should, however, notice that the relative bound of $H'(0) = \nabla s \cdot \pi + \pi \cdot \nabla s$ with respect to H is independent of λ. □

We finally come to the

Proof of Theorem 4. We set $s(x, y) = -x$, i.e., $\Pi = -\pi_y - x$. Then, in (6.27), $m_{ij} = 0$ and $\dot{\gamma} \cdot \nabla s = (1 + f'(x)^2)^{-1/2} \geq \delta$ for some $\delta > 0$ and all $x \in \mathbb{R}$. Upon scaling, $f'_\lambda(x) = f'(x/\lambda)$ has $\mathrm{C}^2_{\mathrm{unif}}(\mathbb{R})$–norm which can be bounded independently of $\lambda \geq \lambda_0$. In particular, bounds on the norms associated with (6.25), as well as δ, are independent of $\lambda \geq \lambda_0$. □

Acknowledgment

We thank the referees for useful comments that helped clarify the argument in section 2, and improve the presentation in section 6.

A Corbino disc geometry

In this appendix, we adapt the arguments of section 4 to the Corbino disc geometry (see figure 7).

In polar coordinates and with a suitable gauge, the Hamiltonian is

$$H = \frac{1}{2m}\left(-\partial_r^2 - \frac{1}{r}\partial_r + \left(\frac{1}{ir}\partial_\varphi - e\left(\frac{Br}{2} + \frac{\Phi}{2\pi r}\right)\right)^2\right) + V(r, \varphi). \tag{A.1}$$

For $V = 0$, the spectrum and eigenfunctions of H can be obtained by elementary methods. For $\Phi = 0$, the spectrum consists only of the Landau levels, with energy $(n + 1/2)\omega_c$. In contrast to the case of the cylinder, there is here a

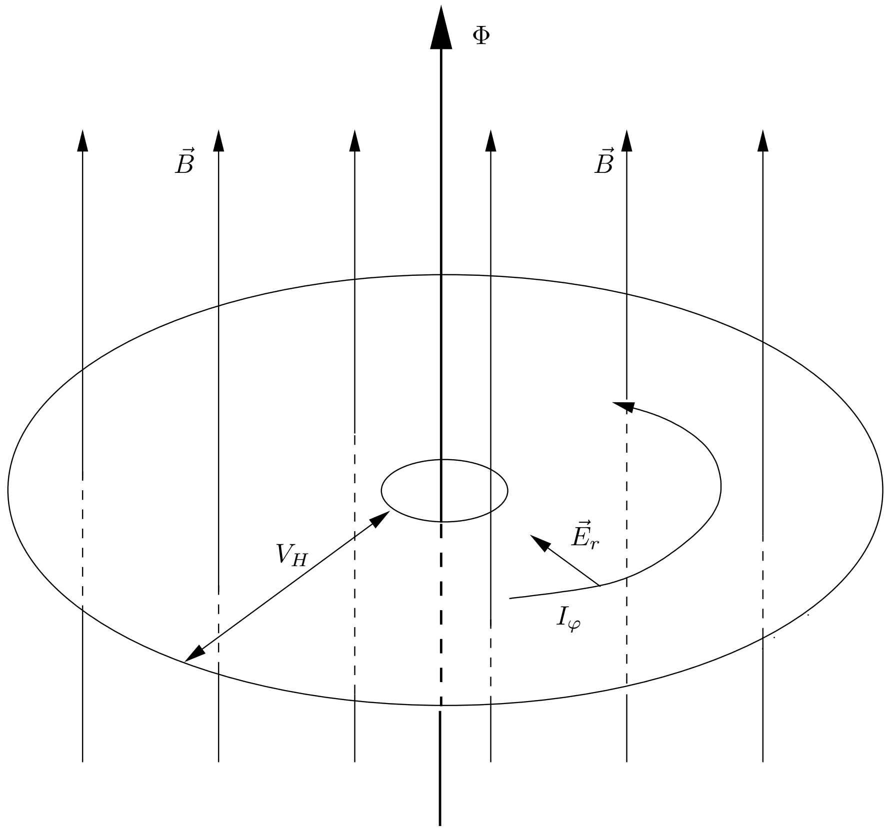

Figure 7: Corbino disc geometry

restriction on the angular momentum, $l \geq -n$. The states are localized in radial direction near $r_0(l) = \sqrt{2\,|l|\,/eB}$. If $\Phi \neq 0$, $0 < \Phi \leq 2\pi/e$, the localization of the states is shifted from $r_0(l)$ to $r_0(l - e\Phi/2\pi)$. The energy of the states with $l - e\Phi/2\pi \geq 0$ remains unchanged, but the energy of the states with $l - e\Phi/2\pi < 0$ is changed to $(n + e\Phi/2\pi + 1/2)\omega_c$. It is convenient to change the definition of the index n and to introduce functions $E_{n,l}(\Phi)$ in such a way that the spectral flow can be written as in equation (2.2),

$$E_{n,l}(\Phi + 2\pi/e) = E_{n,l-1}(\Phi). \tag{A.2}$$

The energies are $E_{n,l}(\Phi) = (n + 1/2)\omega_c$ if $l - e\Phi/2\pi \geq 0$ and $E_{n,l}(\Phi) = (n - l + e\Phi/2\pi + 1/2)\omega_c$ if $l - e\Phi/2\pi < 0$. With this definition of n, l is unrestricted, but the bands for fixed n are bent upwards when $l - e\Phi/2\pi$ is negative. Thus, the flux Φ has effects similar to an edge at the center of the disc. Besides considering only one edge, we also need to restrict the analysis to $\Phi = 0$ in order to avoid resonances.

The discussion with included edge potential, V_0, and weak disorder potential, gV_d, is completely parallel to the case of the cylinder geometry and we do not repeat it here.

We now derive the expression for the azimuthal current corresponding to equation (4.3) of section 4. In the Corbino disc geometry, the azimuthal current carried by a state, ψ, is

$$\begin{aligned} I_\varphi &= -\frac{\partial E}{\partial \Phi} = \Big(\psi, \frac{2}{2\pi r}\Big(\frac{\partial_\varphi}{ir} - \frac{\Phi}{2\pi r} - \frac{Br}{2}\Big)\psi\Big) \\ &= \big(\psi, \frac{\hat{B}\times\vec{r}}{2\pi r^2}[H, i\vec{r}]\psi\big). \end{aligned} \tag{A.3}$$

Note here that $\hat{B}\times\vec{r}/r$ is a unit vector pointing in the azimuthal direction, and that the commutator $[H, i\vec{r}]$ gives the velocity ($m = 1/2$ and $e = 1$). Now replace $\vec{r}$ in the commutator with the guiding center, $\vec{r} = \vec{Z} - (\vec{p} - \vec{A})\times\hat{B}/B$, and use the equation of motion for $\vec{Z}$, $[H, i\vec{Z}] = \hat{B}/B \times \operatorname{grad} V$. This yields

$$I_\varphi = \big(\psi, \frac{1}{2\pi B r}\partial_r V\psi\big) - \big(\psi, \frac{\hat{B}\times\vec{r}}{2\pi B r^2}[H, i(\vec{p}-\vec{A})\times\hat{B}]\psi\big). \tag{A.4}$$

Using the fact that the expectation value of a commutator with H in an energy eigenstate vanishes, the second term in (A.4) is equal to

$$\big(\psi, \frac{1}{2\pi B}\big[H, i\frac{\hat{B}\times\vec{r}}{r^2}\big](\vec{p}-\vec{A})\times\hat{B}\psi\big)\,. \tag{A.5}$$

A straightforward calculation in polar coordinates then shows that the azimuthal current carried by an eigenstate of H can be written as

$$\begin{aligned} I_\varphi &= \big(\psi, \frac{2}{2\pi r}(\vec{p}-\vec{A})_\varphi\psi\big) \\ &= \big(\psi, \frac{1}{2\pi B r}\partial_r V\psi\big) + \frac{2}{2\pi B}\big(\frac{1}{r}\partial_r\psi, \frac{1}{r}\partial_r\psi\big) - \frac{1}{2\pi B}\big(\psi, \frac{2}{r^2}(\vec{p}-\vec{A})^2_\varphi\psi\big). \end{aligned} \tag{A.6}$$

This expression is similar to (4.3), with correction terms due to the circular geometry. The first term in (A.6) is positive for weak disorder, and decays inversely proportional to the size R of the sample. The second term is always positive, while the third term is negative, but bounded, and decays as $1/R^2$. Thus $I_\varphi|_{\Phi=0}$ is positive for an eigenstate of H with energy between Landau levels, if the sample is large, and the disorder weak.

The rest of this appendix is devoted to making the estimates in this argument rigorous. To deal with the singularity at $r = 0$, it will be useful to know that edge states eigenfunctions are exponentially small near the origin. We start by considering the spatial decay of the free resolvent $(E - H_0)^{-1}$ (for $\Phi = 0$). This free resolvent for a homogeneous magnetic field Hamiltonian in two dimensions can be calculated using an expansion in eigenfunctions of H_0, which are known explicitly in terms of Laguerre polynomials. See [21] and [22] for more details. The result contains the confluent hypergeometric function called Ψ in the notation of [29],

$$\begin{aligned} R_0(x, y; E) &= (E - H_0)^{-1}(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) \\ &= -\frac{1}{4\pi} e^{i\pi\zeta} \Gamma(-\zeta) e^{i r_1 r_2 \sin(\varphi_1 - \varphi_2)} e^{-\frac{B}{4}|x-y|^2} \Psi\big(-\zeta, 1; \frac{B}{4}|x-y|^2\big). \end{aligned} \tag{A.7}$$

Here, ζ is related to the energy by $E = (2\zeta + 1)B$, so that R_0 has singularities at Landau band energies. $\Psi(-\zeta, 1; z)$ has a logarithmic singularity at $z = 0$ and behaves for large $|z|$ as $|z|^\zeta$. This implies a Gaussian decay for large $|x - y|$. We will, however, only use an estimate of the form

$$|R_0(x, y; E)| \leq C e^{-|x-y|/\xi} \left|\ln(|x - y|/\xi)\right|, \tag{A.8}$$

with a decay length scale ξ on the order of the magnetic length $\sqrt{B}^{-1}$.

We now include the disorder potential and claim the following

Lemma 5. *Let V_0 be a not too fast increasing edge potential that describes the Corbino disc with varying sample size R. Let further V_d be bounded by a constant sufficiently small compared to the magnetic field. Let finally Δ be an energy interval in the spectral gaps of $H_0 + V_d$. Whenever $a < R$, there are positive constants C and λ, such that an eigenfunction ψ of $H = H_0 + V_0 + V_d$ with energy $E \in \Delta$ satisfies*

$$|\psi(x)| \leq C e^{-(R-a)/\lambda} \|\psi\| \tag{A.9}$$

for $|x| \leq a$, uniformly in the sample size R and the energy $E \in \Delta$.

Proof. Use the equation

$$\psi(x) = \int (E - H_0 - V_d)^{-1}(x, y) V_0(y) \psi(y) dy \tag{A.10}$$

and expand the resolvent in a Neumann-series.

$$(E - H_0 - V_d)^{-1} = \sum_{n=0}^{\infty} \left((E - H_0)^{-1} V_d\right)^n (E - H_0)^{-1}. \tag{A.11}$$

Consider a fixed n, and use the estimate (A.8) for each free resolvent. This results in the following integrals to be estimated:

$$\begin{aligned} \int \Big(\prod_{i=1}^{n} C \left|\ln(|z_{i-1} - z_i|/\xi)\right| e^{-|z_{i-1}-z_i|/\xi} V_d(z_i)\Big) \times \\ C \left|\ln(|z_n - y|/\xi)\right| e^{-|z_n - y|/\xi} V_0(y) \left|\psi(y)\right| dz_1 \dots dz_n dy, \end{aligned} \tag{A.12}$$

with $z_0 = x$, $z_{n+1} = y$. Taking out V_d out of the integrals, and splitting the exponentials in 3, this is estimated by

$$
\begin{aligned}
&(C\left\|V_d\right\|_\infty)^n \sup_{z_1,\dots z_{n+1}} \exp(-\sum_{i=1}^{n+1}\left|z_{i-1}-z_i\right|/3\xi) \times \\
&\quad \Big(\int e^{-|w|/3\xi}\left|\ln(|w|/\xi)\right|\Big)^n \sup_{z_1,\dots z_n}\int e^{-|z_n-y|/3\xi}\left|\ln(|z_n-y|/\xi)\right| \times \\
&\qquad \exp(-\sum_{i=1}^{n+1}\left|z_{i-1}-z_i\right|/3\xi)V_0(y)\left|\psi(y)\right| dy. \qquad (A.13)
\end{aligned}
$$

Now $\inf_{z_1,\dots z_n}\sum_{i=1}^{n+1}|z_{i-1}-z_i| \geq |x-y|$ and, applying the Schwarz inequality to the last integral, the bound moves to

$$
\begin{aligned}
&(\tilde{C}\left\|V_d\right\|_\infty)^n e^{-|x-R|/3\xi}\Big(\int e^{-2|w|/3\xi}\left|\ln(|w|/\xi)\right|^2 dw\Big)^{1/2} \times \\
&\qquad \Big(\int e^{-2|x-y|/3\xi}(V_0(y))^2\left|\psi(y)\right|^2 dy\Big)^{1/2}. \qquad (A.14)
\end{aligned}
$$

If the potential does not increase too fast, the integral containing V_0 converges and is bounded by $E\left\|\psi\right\|$, and after summing over n, making the bound on the disorder small enough, the claim follows. □

With the help of Lemma 5, we now prove the positivity of the current from the expression (A.6),

$$
\begin{aligned}
I_\varphi &= \big(\psi, \frac{2}{2\pi r}(\vec{p}-\vec{A})_\varphi\psi\big) \\
&= \big(\psi, \frac{1}{2\pi B r}\partial_r V\psi\big) + \frac{2}{2\pi B}\big(\frac{1}{r}\partial_r\psi, \frac{1}{r}\partial_r\psi\big) - \frac{1}{2\pi B}\big(\psi, \frac{2}{r^2}(\vec{p}-\vec{A})_\varphi^2\psi\big). \qquad (A.15)
\end{aligned}
$$

The first step is to eliminate the singularity at $r = 0$ by replacing ψ with $j\psi$, where j is a cutoff at radius $a > 0$ near the origin. We want to show that the error introduced by this replacement,

$$
\begin{aligned}
&\big(\psi, \frac{1}{r}(p-A)_\varphi\psi\big) - \big(j\psi, \frac{1}{r}(p-A)_\varphi j\psi\big) = \\
&\quad \big((1-j)\psi, \frac{1}{r}(p-A)_\varphi j\psi\big) + \big(j\psi, \frac{1}{r}(p-A)_\varphi(1-j)\psi\big) + \\
&\qquad \big((1-j)\psi, \frac{1}{r}(p-A)_\varphi(1-j)\psi\big), \qquad (A.16)
\end{aligned}
$$

is small for large R. Consider bounding the term

$$\begin{aligned}\Big|\big((1-j)\psi, \frac{1}{r}(p-A)_\varphi(1-j)\psi\big)\Big| &\le \|\frac{1}{r^{3/4}}\psi\|_a \, \|\frac{1}{r^{1/2}}(p-A)_\varphi\psi\|_a \\ &\le Ce^{-R/\lambda}a^{1/4}\,\|\psi\|\,\|\frac{1}{r}\psi\|_a^{1/2}\,\big\|(p-A)_\varphi^2\psi\big\|_a^{1/2} \\ &\le Ce^{-3R/2\lambda}a^{1/2}\,\|\psi\|^2 \end{aligned} \tag{A.17}$$

where $\|\cdot\|_a$ has its obvious meaning, and we have taken advantage of the fact the r commutes with $(p-A)_\varphi$ and that $\|(\vec{p}-\vec{A})_\varphi^2\psi\|$ can be bounded by the energy[‡]. The other terms in (A.16) similarly decay exponentially with the sample size.

The replacement of ψ with $j\psi$ introduces additional terms, because in the derivation of (A.6), we used that ψ was an eigenstate of H, whereas $j\psi$ is not. Those additional terms are

$$\begin{aligned}&\big(j\psi, \frac{1}{2\pi B}i\frac{\hat{B}\times\vec{r}}{r^2}(\vec{p}-\vec{A})\times\hat{B}[H,j]\psi\big) + \big(\psi,[H,j]\frac{1}{2\pi B}i\frac{\hat{B}\times\vec{r}}{r^2}(\vec{p}-\vec{A})\times\hat{B}j\psi\big) \\ &\qquad = \big(j\psi, \frac{1}{2\pi Br}(-\partial_r)[H,j]\psi\big) + \big(\psi,[H,j]\frac{1}{2\pi Br}(-\partial_r)j\psi\big)\end{aligned} \tag{A.18}$$

$[H,j] = -2j'\partial_r - j'' - j'/r$ has support for r near a. Organize the various terms so that either no radial derivative is acting on ψ, or one radial derivative, or the expression $\frac{1}{r}\partial_r r\partial_r$. Terms with one radial derivative are estimated, for example, as

$$\begin{aligned}\Big|\big(j\psi, j''\frac{1}{r}\partial_r j\psi\big)\Big| &\le \|j''\frac{1}{r}j'\psi\|_a\,\|\partial_r\psi\| \\ &\le \frac{C}{a^4}e^{-(R-a)/\lambda}\,\|\psi\|^2\,,\end{aligned} \tag{A.19}$$

where C contains the energy as bound for $\|\partial_r\psi\|$. The terms with two radial derivatives, such as

$$\big(j'j\frac{1}{r}\psi, \frac{1}{r}\partial_r r\partial_r\psi\big), \tag{A.20}$$

are estimated similarly. Terms without radial derivative acting on ψ are even easier.

Cutting off with j thus introduces error terms that decay exponentially with the sample size.

From (A.6) with $j\psi$ instead of ψ, we now estimate

$$\Big|\big(j\psi, \frac{1}{r^2}(\vec{p}-\vec{A})_\varphi^2 j\psi\big)\Big| \le \|\frac{1}{r^2}j\psi\|\,\Big\|j(\vec{p}-\vec{A})_\varphi^2\psi\Big\|\,. \tag{A.21}$$

[‡]This Kato estimate for $(\vec{p}-\vec{A})_\varphi^2$ is not trivial, since radial and azimuthal part of the kinetic energy do not commute. The same kind of estimate in the half-plane geometry was needed in section 6.

The second factor is bounded by the energy. The first is split into a part with the radial coordinate between a and $R/2$, so that it decays as $e^{-R/2\lambda}/a^2$, and a part from $R/2$ to ∞ which decays as $1/R^2$. The positivity of

$$\big(j\psi, \frac{1}{r}\partial_r V j\psi\big) \tag{A.22}$$

is proved in a fashion very similar to the one in section 5 by introducing another cutoff function at the edge $r = R$[§]. The lower bound for the expression decays as $1/R$. If the sample is large and the disorder weak, the current is positive.

B Random potentials and almost sure spectrum

Consider a Hamiltonian H_1 on $\mathcal{H} = L_2(\mathbb{R}^2)$ which is invariant in x-direction. Let $\sigma(H_1)$ be the spectrum of H_1. Add to H_1 a disorder in the form of a random potential $V_{d,\omega}$, where $\omega \in \Omega$, and (Ω, P) is a probability space. Assume three things about (Ω, P):

(i) For every $\omega \in \Omega$, $V_{d,\omega}$ is bounded by a constant δ which is independent of ω.

(ii) The group $G^{(x)}$ of translations in x-direction acts measure-preserving and ergodically on (Ω, P). This allows one to speak of an almost sure (a.s.) spectrum of $H_\omega = H_1 + V_{d,\omega}$, denoted by $\Sigma(H_\omega)$.

(iii) For every measurable compact set $\Lambda \subset \mathbb{R}^2$ and every $\epsilon > 0$ the probability

$$P\big\{\omega;\, |V_{d,\omega}(x,y)| < \epsilon \;\; \forall (x,y) \in \Lambda\big\} \tag{B.1}$$

is positive.

Those assumptions are, for example, satisfied in an Anderson model for the disorder. Assumption (i) is added for consistency with the proofs in sections 5 and 6. As mentioned in section 3, it can very likely be replaced by boundedness of the variance of $V_{d,\omega}$. Assumptions (ii) and (iii) allow the proof of the following:

Lemma 6. *Let notation be as introduced and assumptions (i) to (iii) be satisfied. Then*

$$\sigma(H_1) \subset \Sigma(H_\omega) \subset \sigma(H_1) + [-\delta, \delta]\,. \tag{B.2}$$

For a proof, see [24].

[§]The cutoff at $r = a$ does not significantly perturb the argument of section 5 because ψ is exponentially small near the center.

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