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On the Extensions of $\hat{\mathcal{G}}^{(\lambda)}$ by $\hat{\mathbf{G}}_a$ over a $\mathbf{Z}_{(p)}$-Algebra

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Abstract. We will give an explicit description of extensions of the group scheme (resp. the formal group scheme) which gives a deformation of the additive group scheme to the multiplicative group scheme (resp. the additive formal group scheme to the multiplicative formal group scheme) by the additive group scheme (resp. the additive formal group scheme) over an algebra for which all prime numbers except a given prime p is invertible.

Introduction.

Let A be a commutative ring, and let G, H be algebraic groups or formal groups over A. It is an important problem to determine $\mathrm{Hom}_{A-\mathrm{gr}}(G, H)$ and $\mathrm{Ext}^1_A(G, H)$, especially in elementary cases. This was done completely when A is a field (cf. [DG]). Moreover, the calculation was achieved in some cases by Weisfeiler [5] and Sekiguchi-Suwa [2] when A is a discrete valuation ring.

Generally, let A be a ring and $\lambda \in A$. Let $\mathbf{G}_{a,A}$ (resp. $\hat{\mathbf{G}}_{a,A}$) denote the additive group scheme (resp. the additive formal group scheme) over A and $\mathbf{G}_{m,A}$ (resp. $\hat{\mathbf{G}}_{m,A}$) the multiplicative group scheme (resp. the multiplicative formal group scheme) over A. Then the group scheme $\mathcal{G}^{(\lambda)}$ which gives a deformation of $\mathbf{G}_{a,A}$ to $\mathbf{G}_{m,A}$ is defined as follows: $\mathcal{G}^{(\lambda)} = \mathrm{Spec}\, A[T, (1+\lambda T)^{-1}]$. Here, the multiplication of $\mathcal{G}^{(\lambda)}$ is given by $T \mapsto \lambda T \otimes T + T \otimes 1 + 1 \otimes T$. (See 1.1.) Let $\mathrm{Ext}^1_A(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ denote the group formed by the class of commutative extensions of $\mathcal{G}^{(\lambda)}$ by $\mathbf{G}_{a,A}$.

Weisfeiler [5] studied the group $\mathrm{Ext}^1_A(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ when A is an integral domain, and gave an explicit description of the group $\mathrm{Ext}^1_A(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ when A is an integral domain containing a field in the case that $\mathcal{G}^{(\lambda)}$ acts on $\mathbf{G}_{a,A}$ trivially. Sekiguchi-Suwa [2] worked on the subject when A is a discrete valuation ring, including the case that $\mathcal{G}^{(\lambda)}$ acts on $\mathbf{G}_{a,A}$ non-trivially.

On the other hand, structures of $\mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ and $\mathrm{Ext}^1_A(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ are described by Sekiguchi-Suwa [3] when A is a $\mathbf{Z}_{(p)}$-algebra, here p is a fixed prime number. It was crucial there to construct some formal power series which generalize the Artin-Hasse exponential series.

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Our interest is to describe $\mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ and $\mathrm{Ext}^1_A(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ for any $\mathbf{Z}_{(p)}$-algebra A. In fact, we obtain that $\mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $\mathrm{Ext}^1_A(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$) is isomorphic to the kernel (resp. cokernel) of an endomorphism Ψ of $A^{\mathbf{N}}$. (The definition of Ψ is given in 2.1.) It is crucial here to construct some formal power series which is a kind of logarithm.

Now, let $H^2_0(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ (resp. $H^2_0(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$) denote the Hochschild cohology group consisting of symmetric 2-cocycles of $\mathcal{G}^{(\lambda)}$ (resp. $\hat{\mathcal{G}}^{(\lambda)}$) with coefficients in $\mathbf{G}_{a,A}$ (resp. $\hat{\mathbf{G}}_{a,A}$). (See 1.2 and 1.3.) Then our result is stated precisely as follows:

THEOREM (See 2.5.). *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$.*

(1) *The correspondence*

$$(a_r)_{r\geq 0} \mapsto f(T) = \sum_{r=0}^{\infty} a_r \left[\sum_{i=p^r}^{p^{r+1}-1} \frac{p^r}{i}(-\lambda)^{i-p^r} T^i \right]$$

induces isomorphisms

$$\eta^0 : \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \xrightarrow{\sim} \mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}),$$
$$\eta^0 : \mathrm{Ker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \xrightarrow{\sim} \mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A}).$$

(2) *Assume that $\mathcal{G}^{(\lambda)}$ (resp. $\hat{\mathcal{G}}^{(\lambda)}$) acts on $\mathbf{G}_{a,A}$ (resp. $\hat{\mathbf{G}}_{a,A}$) trivially. Then the correspondence*

$$(a_r)_{r\geq 0} \mapsto \sum_{r=0}^{\infty} a_r \tilde{L}_{p,r}(\lambda, X, Y)$$

induces isomorphisms

$$\eta^1 : \mathrm{Coker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \xrightarrow{\sim} H^2_0(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}),$$
$$\eta^1 : \mathrm{Coker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \xrightarrow{\sim} H^2_0(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A}).$$

We will give two proofs of the main theorem.

The first proof is somewhat elementary and self-contained, and we develop an argument in the first three sections. In Section 1 we give some basic definitions and recall on Hochschild 2nd cohomology group. In Section 2 we state the main theorem, and in Section 3 we prove the main result.

The last section is devoted to the second proof. We recall necessary facts on Witt vectors, the Artin-Hasse exponentials and the main result of Sekiguchi-Suwa [3] at the beginning of the section. We need the result of [3] on $\mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ and $H^2_0(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ here. However, it is worthwhile to clarify a relation between results on $\hat{\mathbf{G}}_{m,A}$ and those on $\hat{\mathbf{G}}_{a,A}$, which was suggested in Suwa [4].

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NOTATIONS. Throughout this paper, p denotes a fixed prime number, $\mathbf{Z}_{(p)}$ the localization of $\mathbf{Z}$ at the prime ideal (p) and A a $\mathbf{Z}_{(p)}$-algebra and we put $P = \{p^e; e \geq 0\}$.

$\mathbf{G}_{a,A}$: the additive group scheme over A

$\mathbf{G}_{m,A}$: the multiplicative group scheme over A

$\hat{\mathbf{G}}_{a,A}$: the additive formal group scheme over A

$\hat{\mathbf{G}}_{m,A}$: the multiplicative formal group scheme over A

$H_0^2(G, H)$ denote the Hochschild cohology group consisting of symmetric 2-cocycles of G with coefficients in H. In this paper, we investigate $H_0^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ (resp. $H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$) assuming $\mathcal{G}^{(\lambda)}$ (resp. $\hat{\mathcal{G}}^{(\lambda)}$) acts on $\mathbf{G}_{a,A}$ (resp. $\hat{\mathbf{G}}_{a,A}$) trivially.

1. Some preparations.

First, we give the definition of the group scheme $\mathcal{G}^{(\lambda)}$.

1.1. Let A be a ring and $\lambda \in A$. We define a group scheme $\mathcal{G}^{(\lambda)}$ over A by $\mathcal{G}^{(\lambda)} = \operatorname{Spec} A[T, (1+\lambda T)^{-1}]$ with

(1) the multiplication: $T \mapsto \lambda T \otimes T + T \otimes 1 + 1 \otimes T$,

(2) the unit: $T \mapsto 0$,

(3) the inverse: $T \mapsto -T(1+\lambda T)^{-1}$.

Moreover, we define an A-homomorphism $\alpha^{(\lambda)} : \mathcal{G}^{(\lambda)} \to \mathbf{G}_{m,A}$ by

$$A\left[T, \frac{1}{T}\right] \mapsto A\left[T, \frac{1}{1+\lambda T}\right]; \quad T \mapsto 1 + \lambda T .$$

If λ is invertible in A, $\alpha^{(\lambda)}$ is an A-isomorphism. On the other hand, if $\lambda = 0$, $\mathcal{G}^{(\lambda)}$ is nothing but the additive group scheme $\mathbf{G}_{a,A}$.

Next we recall the definition of Hochschild cohomology. For details, see [DG].

1.2. Let A be a ring and $g(X, Y)$ a formal power series in $A[[X, Y]]$ (resp. a fraction in $A[X, Y, (1+\lambda X)^{-1}, (1+\lambda Y)^{-1}]$). Recall that $g(X, Y)$ is called a symmetric 2-cocycle of $\hat{\mathcal{G}}^{(\lambda)}$ (resp. $\mathcal{G}^{(\lambda)}$) with coefficients in $\hat{\mathbf{G}}_{a,A}$ (resp. $\mathbf{G}_{a,A}$) if $g(X, Y)$ satisfies the following functional equations:

(1) $g(Y, Z) - g(\lambda XY + X + Y, Z) + g(X, \lambda YZ + Y + Z) - g(X, Y) = 0$,

(2) $g(X, Y) = g(Y, X)$.

We denote by $Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$) the additive subgroup of $A[[X, Y]]$ (resp. $A[X, Y, (1+\lambda X)^{-1}, (1+\lambda Y)^{-1}]$) formed by the symmetric 2-cocycles of $\hat{\mathcal{G}}^{(\lambda)}$ (resp. $\mathcal{G}^{(\lambda)}$) with coefficients in $\mathbf{G}_{a,A}$ (resp. $\mathbf{G}_{a,A}$).

Let $f(T)$ be a formal power series in $A[[T]]$ (resp. a fraction in $A[T, (1+\lambda T)^{-1}]$). Then $f(X) + f(Y) - f(\lambda XY + X + Y) \in Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$). We denote by ∂ the correspondence $f(T) \mapsto f(X) + f(Y) - f(\lambda XY + X + Y)$. We denote by $B^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $B^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$) the subgroup of $Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$)

of the symmetric 2-cocycles of the form $f(X)+f(Y)-f(\lambda XY+X+Y)$. Put

$$H_0^2(\hat{\mathcal{G}}^{(\lambda)},\hat{\mathbf{G}}_{a,A})=Z^2(\hat{\mathcal{G}}^{(\lambda)},\hat{\mathbf{G}}_{a,A})/B^2(\hat{\mathcal{G}}^{(\lambda)},\hat{\mathbf{G}}_{a,A})\,,$$
$$H_0^2(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})=Z^2(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})/B^2(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})\,.$$

Since any $\mathbf{G}_a$-torsor over an affine scheme is trivial, we have the following:

LEMMA 1.3. *Let G be an affine A-group, acting on $\mathbf{G}_{a,A}$. Then the canonical map*

$$H_0^2(G,\mathbf{G}_{a,A})\to \mathrm{Ext}_A^1(G,\mathbf{G}_{a,A})$$

is bijective.

On the other hand, $\mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)},\hat{\mathbf{G}}_{a,A})$ (resp. $\mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})$) is isomorphic to the kernel of $\partial: A[[T]]\to Z^2(\hat{\mathcal{G}}^{(\lambda)},\hat{\mathbf{G}}_{a,A})$ (resp. $\partial: A[T,(1+\lambda T)^{-1}]\to Z^2(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})$).

Now we define some polynomials and homomorphisms.

1.4. We define polynomials $C_l(X,Y)$ in $\mathbf{Z}[X,Y]$ by

$$C_l(X,Y)=\begin{cases}X^l+Y^l-(X+Y)^l & (l\notin P)\,,\\ \dfrac{1}{p}\{X^l+Y^l-(X+Y)^l\} & (l\in P)\,.\end{cases}$$

1.5. We define polynomials $L_{p,r}(\Lambda;T)$ in $\mathbf{Z}_{(p)}[\Lambda][T]$ by

$$L_{p,r}(\Lambda;T)=p^r\sum_{k=1}^{p^{r+1}-1}\frac{(-\Lambda)^{k-1}}{k}T^k$$

for all $r\geq 0$.

1.6. We define polynomials $\tilde{L}_{p,r}(\Lambda;X,Y)$ in $\mathbf{Z}_{(p)}[\Lambda,\Lambda^{-1}][X,Y]$ by

$$\tilde{L}_{p,r}(\Lambda;X,Y)=-\frac{1}{(-\Lambda)^{p^{r+1}-1}}\{L_{p,r}(\Lambda;X)+L_{p,r}(\Lambda;Y)-L_{p,r}(\Lambda;\Lambda XY+X+Y)\}$$

for all $r\geq 0$.

LEMMA 1.7. *Let $r\geq 0$. Then we have followings*:
(1) $\tilde{L}_{p,r}(\Lambda;X,Y)\in\mathbf{Z}_{(p)}[\Lambda][X,Y]$,
(2) $\tilde{L}_{p,r}(\Lambda;X,Y)\equiv C_{p^{r+1}}(X,Y)\bmod\deg(p^{r+1}+1)$.

PROOF. Put

$$f_r(\Lambda;T)=\sum_{k=p^{r+1}}^{\infty}\frac{(-\Lambda)^{k-1}}{k}T^k=(-\Lambda)^{p^{r+1}-1}\sum_{k=p^{r+1}}^{\infty}\frac{(-\Lambda)^{k-p^{r+1}}}{k}T^k\in\mathbf{Q}[[T]]\,.$$

Then we have

$$L_{p,r}(\Lambda;T)=p^r\left\{\frac{1}{\Lambda}\log(1+\Lambda T)-f_r(\Lambda;T)\right\},$$

where

$$\frac{1}{\Lambda}\log(1+\Lambda T)=\sum_{k=1}^{\infty}\frac{(-\Lambda)^{k-1}}{k}T^k\in\mathbf{Q}[[T]]\,.$$

Noting

$$\log(1+\Lambda(\Lambda XY+X+Y)) = \log((1+\Lambda X)(1+\Lambda Y)) = \log(1+\Lambda X)+\log(1+\Lambda Y)$$

and

$$f_r(\Lambda, T) \equiv 0 \mod (\Lambda^{p^{r+1}-1}),$$

we have

$$\begin{aligned} &L_{p,r}(\Lambda; X) + L_{p,r}(\Lambda; Y) - L_{p,r}(\Lambda; \Lambda XY + X + Y) \\ &\quad = -p^r\{f_r(\Lambda; X) + f_r(\Lambda; Y) - f_r(\Lambda; \Lambda XY + X + Y)\} \\ &\quad \equiv 0 \mod (\Lambda^{p^{r+1}-1}). \end{aligned}$$

Moreover, noting

$$f_r(\Lambda; T) \equiv \frac{(-\Lambda)^{p^{r+1}-1}}{p^{r+1}} T^{p^{r+1}} \bmod \deg(p^{r+1}+1),$$

we readily see (2).

By 1.7, we readily see the following:

COROLLARY 1.8. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then $\tilde{L}_{p,r}(\lambda; X, Y) \in Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$.*

1.9. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{(\mathbf{N})}$ (resp. $A^{(\mathbf{N})}$). By 1.8, we can define homomorphisms $\eta^0 : A^{\mathbf{N}} \to A[[T]]$ (resp. $\eta^0 : A^{(\mathbf{N})} \to A[T, (1+\lambda T)^{-1}]$) and $\eta^1 : A^{\mathbf{N}} \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $\eta^1 : A^{\mathbf{N}} \to Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$) by

$$\eta^0 : \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r \left[\sum_{k=p^r}^{p^{r+1}-1} \frac{p^r}{k}(-\lambda)^{k-p^r} T^k\right],$$

$$\eta^1 : \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r \tilde{L}_{p,r}(\lambda; X, Y).$$

2. Statement of the theorem.

2.1. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$ (resp. $A^{(\mathbf{N})}$). We define an endomorphism of $A^{\mathbf{N}}$ (resp. $A^{(\mathbf{N})}$) denoted Ψ (resp. Ψ) by

$$\Psi : \mathbf{a} \mapsto (-(-\lambda)^{p^{i+1}-p^i} a_i + p a_{i+1})_{i \geq 0}.$$

LEMMA 2.2. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then the diagram*

$$\begin{array}{ccc} A^{\mathbf{N}} & \xrightarrow{\eta^0} & A[[T]] \\ \Psi \downarrow & & \downarrow \partial \\ A^{\mathbf{N}} & \xrightarrow{\eta^1} & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) \end{array}$$

is commutative.

PROOF. Note that

$$-(-\Lambda)^{p-1}\left\{-\frac{1}{(-\Lambda)^{p-1}}\sum_{k=1}^{p-1}\frac{(-\Lambda)^{k-1}}{k}T^k\right\}=\sum_{k=1}^{p-1}\frac{(-\Lambda)^{k-1}}{k}T^k$$

and

$$\begin{aligned}&-(-\Lambda)^{p^{r+1}-p^r}\left\{-\frac{1}{(-\Lambda)^{p^{r+1}-1}}\right\}\sum_{k=1}^{p^{r+1}-1}\frac{p^r}{k}(-\Lambda)^{k-1}T^k\\&\quad+p\left\{-\frac{1}{(-\Lambda)^{p^r-1}}\right\}\sum_{k=1}^{p^r-1}\frac{p^{r-1}}{k}(-\Lambda)^{k-1}T^k\\&=\sum_{k=p^r}^{p^{r+1}-1}\frac{p^r}{k}(-\Lambda)^{k-p^r}T^k\end{aligned}$$

for all $r\geq 1$. Then we have

$$-(-\Lambda)^{p-1}\tilde{L}_{p,0}(\Lambda,X,Y)=\sum_{k=1}^{p-1}\frac{(-\Lambda)^{k-1}}{k}\{X^k+Y^k+(X+Y+\Lambda XY)^k\}$$

and

$$\begin{aligned}&-(-\Lambda)^{p^{r+1}-p^r}\tilde{L}_{p,r}(\Lambda;X,Y)+p\tilde{L}_{p,r-1}(\Lambda,X,Y)\\&=\sum_{k=p^r}^{p^{r+1}-1}\frac{p^r}{k}(-\Lambda)^{k-p^r}\{X^k+Y^k+(X+Y+\Lambda XY)^k\}\end{aligned}$$

for all $r\geq 1$.

Now, let $\lambda\in A$ and $\mathbf{a}=(a_0,a_1,a_2,\cdots)\in A^{\mathbf{N}}$. Then

$$\begin{aligned}(\partial\circ\eta^0)(\mathbf{a})&=-(-\lambda)^{p-1}a_0\tilde{L}_{p,0}(\lambda,X,Y)\\&\quad+\sum_{r=1}^{\infty}a_r\{-(-\lambda)^{p^{r+1}-p^r}\tilde{L}_{p,r}(\lambda,X,Y)+p\tilde{L}_{p,r-1}(\lambda,X,Y)\}\\&=\sum_{r=0}^{\infty}\{-(-\lambda)^{p^{r+1}-p^r}a_r+pa_{r+1}\}\tilde{L}_{p,r}(\lambda,X,Y)\\&=(\eta^1\circ\Psi)(\mathbf{a})\,.\end{aligned}$$

COROLLARY 2.2.1. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda\in A$. Then the diagram*

$$\begin{array}{ccc}A^{(\mathbf{N})}&\xrightarrow{\eta^0}&A\left[T,\dfrac{1}{1+\lambda T}\right]\\ \Psi\big\downarrow&&\big\downarrow\partial\\ A^{(\mathbf{N})}&\xrightarrow{\eta^1}&Z^2(\mathcal{G}^{(\lambda)},\mathbf{G}_{a,A})\end{array}$$

is commutative.

LEMMA 2.3. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $f(T) = \sum_{i=1}^{\infty} c_i T^i$ be a formal power series in $A[[T]]$ satisfying*

(a) $c_j = \frac{p^s}{j}(-\lambda)^{j-p^s} c_{p^s}$ *for all s, j with $s \geq 0$, $p^s < j < p^{s+1}$,*

(b) $-(-\lambda)^{p^{s+1}-p^s} c_{p^s} + p c_{p^{s+1}} = 0$ *for all $s \geq 0$.*

Then $f(X) + f(Y) - f(\lambda XY + X + Y) = 0$.

PROOF. Let $s \geq 0$, $p^{s+1} < j < p^{s+2}$ and $1 \leq k \leq j-1$.

The case of $k \leq j - p^{s+1}$:

$$\sum_{i=m(j,k)}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i} c_{j-k+i}$$
$$= \sum_{i=0}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i}\frac{p^{s+1}}{j-i+k}(-\lambda)^{j-k+i-p^{s+1}} c_{p^{s+1}}$$
$$= \left[\sum_{i=0}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^{s+1}}{j-k+i-p^{s+1}}}{\binom{j-k+i}{p^{s+1}}}\right]\lambda^{j-p^{s+1}} c_{p^{s+1}}\,.$$

On the other hand, using Lemma 2.3.1 as $m = j-k$, $n = k$, $l = p^{s+1}$, $t = 0$, we have

$$\sum_{i=0}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^{s+1}}{j-k+i-p^{s+1}}}{\binom{j-k+i}{p^{s+1}}} = 0\,.$$

Hence

$$\sum_{i=m(j,k)}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i} c_{j-k+i} = 0\,.$$

The case of $k > j - p^{s+1}$:

$$\sum_{i=m(j,k)}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i} c_{j-k+i}$$
$$= \sum_{i=0}^{p^{s+1}-j+k-1} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i} c_{j-k+i}$$
$$+ \sum_{i=p^{s+1}-j+k}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i} c_{j-k+i}$$
$$= \sum_{i=p^{s+1}-j+k-1}^{k} \binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i}\frac{p^s}{j-i+k}(-\lambda)^{j-k+i-p^s} c_{p^s}$$

$$
\begin{aligned}
&+\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i}\frac{p^{s+1}}{j-i+k}(-\lambda)^{j-k+i-p^{s+1}}c_{p^{s+1}}\\
&=\left[\sum_{i=p^{s+1}-j+k-1}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^s}{j-k+i-p^s}}{\binom{j-k+i}{p^s}}\right]\lambda^{j-p^s}c_{p^s}\\
&\quad+\left[\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^{s+1}}{j-k+i-p^{s+1}}}{\binom{j-k+i}{p^{s+1}}}\right]\lambda^{j-p^{s+1}}c_{p^{s+1}}\\
&=\lambda^{j-p^{s+1}}\left[-\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^s}{j-k+i-p^s}}{\binom{j-k+i}{p^s}}\lambda^{p^{s+1}-p^s}c_{p^s}\right.\\
&\quad\left.+\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\frac{\binom{-p^{s+1}}{j-k+i-p^{s+1}}}{\binom{j-k+i}{p^{s+1}}}c_{p^{s+1}}\right]\\
&=\lambda^{j-p^{s+1}}\left[-\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}(-1)^{j-k+i-p^s}\frac{-p^s}{j-k+i}\lambda^{p^{s+1}-p^s}c_{p^s}\right.\\
&\quad\left.+\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}(-1)^{j-k+i-p^{s+1}}\frac{-p^{s+1}}{j-k+i}c_{p^{s+1}}\right]\\
&=\left[\lambda^{j-p^{s+1}}\sum_{i=p^{s+1}-j+k}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}(-1)^{j-k+i-p^{s+1}}\frac{-p^{s+1}}{j-k+i}\right]\\
&\quad\times\{-(-\lambda)^{p^{s+1}-p^s}+pc_{p^{s+1}}\}\\
&=0\,.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&f(X)+f(Y)-f(\lambda XY+X+Y)\\
&\qquad=-\sum_{j\geq 2}\sum_{k=1}^{j-1}\sum_{i=m(j,k)}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i}c_{j-k+i}X^kY^{j-k}\\
&\qquad=0\,.
\end{aligned}
$$

LEMMA 2.3.1 ([2, Cor. 2.2.3]). *Let $n, m, l \in \mathbf{N}$ with $m \geq n$ and $m \geq l$. Then*

$$\sum_{i=0}^{n} \binom{m+i}{m-n+2i} \binom{m-n+2i}{i} \frac{\binom{t-l}{m+i-l}}{\binom{m+i}{l}} = \frac{\binom{t-l}{m-l}\binom{t}{n}}{\binom{m}{l}},$$

in $\mathbf{Q}[t]$.

PROOF. For reader's convenience, we give a short proof. Let $n \in \mathbf{N}$. The next identity is well known:

$$\sum_{i=0}^{n} \binom{t-u}{i} \binom{u}{n-i} = \binom{t}{n}$$

in $\mathbf{Q}[t, u]$. If $n, m \in \mathbf{N}$ with $m \geq n$, we obtain

$$\sum_{i=0}^{n} \binom{m+i}{m-n+2i} \binom{m-n+2i}{i} \binom{t}{m+i} = \binom{t}{m} \binom{t}{n}$$

by the fact above and

$$\binom{m+i}{m-n+2i} \binom{m-n+2i}{i} \binom{t}{m+i} = \binom{u}{n-i} \binom{t}{m} \binom{t-m}{i}.$$

Moreover, let $l \in \mathbf{N}$ with $m \geq l$. Noting

$$\frac{\binom{t-l}{m+i-l}}{\binom{m+i}{l}} = \frac{\binom{t}{m+i}}{\binom{t}{l}},$$

we finally obtain

$$\sum_{i=0}^{n} \binom{m+i}{m-n+2i} \binom{m-n+2i}{i} \frac{\binom{t-l}{m+i-l}}{\binom{m+i}{l}} = \frac{\binom{t}{m}\binom{t}{n}}{\binom{t}{l}} = \frac{\binom{t-l}{m-l}\binom{t}{n}}{\binom{m}{l}}.$$

LEMMA 2.4. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$ (resp. $A^{(\mathbf{N})}$). Then homomorphisms $\eta^0 : A^{\mathbf{N}} \to A[[T]]$ (resp. $\eta^0 : A^{(\mathbf{N})} \to A[T, (1+\lambda T)^{-1}]$) and $\eta^1 : A^{\mathbf{N}} \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ (resp. $\eta^1 : A^{(\mathbf{N})} \to Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$) induces homomorphisms*

$$\eta^0 : \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$$
$$(\textit{resp. } \eta^0 : \mathrm{Ker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A}))$$

and

$$\eta^1 : \mathrm{Coker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$$
$$(\textit{resp. } \eta^1 : \mathrm{Coker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \to H_0^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})).$$

PROOF. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}]$ (resp. $\mathrm{Ker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}]$) and put

$$f(T) = \sum_{r=0}^{\infty} a_r \left[\sum_{i=p^r}^{p^{r+1}-1} \frac{p^r}{i} (-\lambda)^{i-p^r} T^i \right] \in A[[T]] \quad \left(\text{resp. } A[T] \subset A\left[T, \frac{1}{1+\lambda T}\right] \right).$$

Now, let c_j be a coefficient of T^j of $f(T)$. Then we easily see that

(0) $c_{p^s} = a_s$ for all $s \geq 0$,

(1) $c_j = \frac{p^s}{j}(-\lambda)^{j-p^s} c_{p^s}$ for all s, j with $s \geq 0$, $p^s < j < p^{s+1}$,

(2) $-(-\lambda)^{p^{s+1}-p^s} c_{p^s} + p c_{p^{s+1}} = 0$ for all $s \geq 0$.

Hence we obtain $f(X) + f(Y) - f(\lambda XY + X + Y) = 0$, by 2.3.

On the other hand, well-definedness of η^1 is induced immediately from 2.2 (resp. 2.2.1).

Now we can state our main theorems.

THEOREM 2.5. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then*

(1) *the homomorphisms*

$$\eta^0 : \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}),$$
$$\eta^1 : \mathrm{Coker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$$

are bijective,

(2) *the homomorphisms*

$$\eta^0 : \mathrm{Ker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A}),$$
$$\eta^1 : \mathrm{Coker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \to H_0^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$$

are bijective.

3. Proof of the theorem.

Our aim of this section is to give the proof of our main theorems.

LEMMA 3.1. *Let A be a $\mathbf{Z}_{(p)}$-algebra, $\lambda \in A$ and $f(T) \in A[[T]]$. If $f(T)$ satisfies $f(X)+f(Y)-f(\lambda XY+X+Y) = 0$, there exists $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}]$ such that*

$$f(T) = \sum_{r=0}^{\infty} a_r \left[\sum_{i=p^r}^{p^{r+1}-1} \frac{p^r}{i} (-\lambda)^{i-p^r} T^i \right].$$

PROOF. Put $f(T) = \sum_{i=1}^{\infty} c_i T^i$. Consider the Taylor expansion

$$\begin{aligned} 0 &= f(X) + f(Y) - f(\lambda XY + X + Y) \\ &= -\sum_{j \geq 2} \sum_{k=1}^{j-1} \sum_{i=m(j,k)}^{k} \binom{j-k+i}{j-2k+2i} \binom{j-2k+2i}{i} \lambda^{k-i} c_{j-k+i} X^k Y^{j-k}. \end{aligned}$$

Here $m(j,k) = \max\{0, 2k - j\}$. Looking at the coefficient of XY^{j-1}, we see that

$$(j-1)\lambda c_{j-1} + jc_j = 0$$

for all $j \geq 2$. Hence

$$c_j = \frac{p^s}{j}(-\lambda)^{j-p^s} c_{p^s}$$

for all s, j with $s \geq 0$, $(j, p) = 1$.

Here, we claim following statements:

(1) $c_j = \frac{p^s}{j}(-\lambda)^{j-p^s} c_{p^s}$ for all s, j with $s \geq 0$, $p^s < j < p^{s+1}$,

(2) $-(-\lambda)^{p^{s+1}-p^s} c_{p^s} + pc_{p^{s+1}} = 0$ for all $s \geq 0$.

Admit claims above. Putting $a_r = c_{p^r}$ for all $r \geq 0$, we have $\mathbf{a} \in \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}]$ and

$$f(T) = \sum_{r=0}^{\infty} a_r \left[\sum_{i=p^r}^{p^{r+1}-1} \frac{p^r}{i}(-\lambda)^{i-p^r} T^i \right] = \eta^0(\mathbf{a}) \,.$$

Proof of (1). Let $s \geq 0$, $p^s < j < p^{s+1}$ and put $r = \mathrm{ord}_p j$. We prove the assertion by induction on j. Assume that

$$c_l = \frac{p^s}{l}(-\lambda)^{l-p^s} c_{p^s}$$

for all l with $p^s < l < j$.

Consider the above Tayler expansion again. Looking at the coefficient of $X^{p^r} Y^{j-p^r}$, we see that

$$\sum_{i=0}^{p^r-1} \binom{j-p^r+i}{j-2p^r+2i} \binom{j-2p^r+2i}{i} \lambda^{p^r-i} c_{j-p^r+i} + \binom{j}{p^r} c_j = 0 \,.$$

Noting that $\left(\binom{j}{p^r}, p\right) = 1$,

$$c_j = -\frac{1}{\binom{j}{p^r}} \sum_{i=0}^{p^r-1} \binom{j-p^r+i}{j-2p^r+2i} \binom{j-2p^r+2i}{i} \lambda^{p^r-i} c_{j-p^r+i} \,.$$

By the hypothesis of induction,

$$c_j = -\frac{1}{\binom{j}{p^r}} \sum_{i=0}^{p^r-1} \binom{j-p^r+i}{j-2p^r+2i} \binom{j-2p^r+2i}{i} \lambda^{p^r-i} \frac{p^s}{j-p^r+i}(-\lambda)^{j-p^r+i-p^s} c_{p^s} \,.$$

Noting that

$$\frac{\binom{-p^s}{l-p^s}}{\binom{l}{p^s}} = (-1)^{l-p^s} \frac{p^s}{l}$$

if $p^s \le l < p^{s+1}$, we can easily see that

$$c_j = -\frac{1}{\binom{j}{p^r}}\left[\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\frac{\binom{-p^s}{j-p^r+i-p^s}}{\binom{j-p^r+i}{p^s}}\right]\lambda^{j-p^s}c_{p^s}\,.$$

Now, applying 2.3.1 to $m = j - p^r$, $n = p^r$, $l = p^r$, $t = 0$, we have

$$\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\frac{\binom{-p^s}{j-p^r+i-p^s}}{\binom{j-p^r+i}{p^s}} = \binom{j}{p^r}\frac{\binom{-p^s}{j-p^s}}{\binom{j}{p^s}}\,,$$

and therefore

$$c_j = \frac{\binom{-p^s}{j-p^s}}{\binom{j}{p^s}}\lambda^{j-p^s}c_{p^s} = \frac{p^s}{j}(-\lambda)^{j-p^s}c_{p^s}\,.$$

Proof of (2). Let $s \ge 0$. We already have

$$\begin{cases} j\lambda c_j + (j+1)c_{j+1} = 0 & \text{for all } j \ge 1\,,\\ c_j = \dfrac{p^s}{j}(-\lambda)^{j-p^s}c_{p^s} & \text{for all } j \text{ with } p^s < j < p^{s+1}\,. \end{cases}$$

Use these formula as $j = p^{s+1} - 1$, we obtain

$$-(-\lambda)^{p^{s+1}-p^s}c_{p^s} + pc_{p^{s+1}} = 0\,.$$

This completes the proof of 3.1.

3.2. We conclude the bijectivity of $\eta^0 : \mathrm{Ker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ immediately by 3.1. Moreover, we can verify the bijectivity of $\eta^0 : \mathrm{Ker}[\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}] \to \mathrm{Hom}_{A-\mathrm{gr}}(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$ combining 3.3 and 3.1.

LEMMA 3.3. *Let A be a $\mathbf{Z}_{(p)}$-algebra and λ, $a_i \in A$ for all $i \ge 0$. Assume that $-(-\lambda)^{p^{r+1}-p^r}a_r + pa_{r+1} = 0$ for all $r \ge 0$. If*

$$\sum_{r=0}^{\infty}a_r\left[\sum_{i=p^r}^{p^{r+1}-1}\frac{p^r}{i}(-\lambda)^{i-p^r}T^i\right] \in A\left[T, \frac{1}{1+\lambda T}\right],$$

then $a_r = 0$ for almost all r.

PROOF. Put

$$\sum_{r=0}^{\infty}a_r\left[\sum_{i=p^r}^{p^{r+1}-1}\frac{p^r}{i}(-\lambda)^{i-p^r}T^i\right] = \frac{f(T)}{(1+\lambda T)^n}\,,$$

where $f(T) \in A[T]$, $n \geq 0$. Now, we assume $\deg f < p^r$. Looking at the coefficient of T^{p^r} of the formula

$$(1+\lambda T)^n \sum_{r=0}^{\infty} a_r \left[\sum_{i=p^r}^{p^{r+1}-1} \frac{p^r}{i} (-\lambda)^{i-p^r} T^i \right] = f(T)\,,$$

we see that

$$a_r + \sum_{i=1}^{r} c_i (-\lambda)^{p^r - p^{r-i}} a_{r-i} = 0\,,$$

where $c_i \in \mathbf{Z}_{(p)}$ for all i with $1 \leq i \leq r$. Here, by the assumption, we have

$$(-\lambda)^{p^r - p^{r-i}} a_{r-i} = p^i a_r$$

for all i with $1 \leq i \leq r$. Since $1 + \sum_{i=1}^{r} p^i c_i$ is invertible in $\mathbf{Z}_{(p)}$, we have $a_r = 0$ for all r with $\deg f < p^r$.

LEMMA 3.4. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$.* If $g(X,Y) \in Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$, *there exists $\mathbf{b} = (b_0, b_1, b_2, \cdots) \in A^{\mathbf{N}}$ such that $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X, Y)$ is cohomologous to $g(X,Y)$.*

PROOF. Let $g(X,Y) \in A[[X,Y]]$ *and* $g_i(X,Y)$ be the homogeneous part of degree r of $g(X,Y)$. Then there exists $l \geq 2$ such that

$$g(X,Y) = \sum_{r=l}^{\infty} g_r(X,Y)\,.$$

As $g(X,Y)$ satisfies 2-cocycle condition

$$g(Y,Z) - g(\lambda XY + X + Y, Z) + g(X, \lambda YZ + Y + Z) - g(X,Y) = 0\,,$$

we see that

$$g_l(Y,Z) - g_l(X+Y,Z) + g_l(X,Y+Z) - g_l(X,Y) = 0\,.$$

Hence there exists $a_l \in A$ such that

$$g_l(X,Y) = a_l C_l(X,Y)\,,$$

by Lazard's comparison lemma ([1, Lem 3]): Let A be a ring and let $l \geq 2$ and $g(X,Y) \in A[X,Y]$ be a homogeneous polynomial of degree l satisfying

$$\begin{cases} g(Y,Z) - g(X+Y,Z) + g(X,Y+Z) - g(X,Y) = 0 \\ g(X,Y) - g(Y,X) = 0\,. \end{cases}$$

Then there exists $a \in A$ such that $g(X,Y) = aC_l(X,Y)$.

Now, we put

$$t_r(X,Y) = X^r + Y^r - (\lambda XY + X + Y)^r \in B^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$$

for all $r \geq 1$. Noting that

$$t_l(X,Y) \equiv X^l + Y^l - (X+Y)^l \bmod \deg(l+1)\,,$$

if l is not a p-power,

$$g(X,Y) - a_l t_l(X,Y) \equiv 0 \mod \deg(l+1)\,.$$

As $g(X,Y)$ is cohomologous to $g(X,Y) - a_l t_l(X,Y)$, we may replace $g(X,Y)$ by $g(X,Y) - a_l t_l(X,Y)$ and

$$g(X,Y) = \sum_{r=l+1}^{\infty} g_r(X,Y)\,.$$

Repeating the same argument, we may assume l is a p-power and

$$g(X,Y) = \sum_{r=l}^{\infty} g_r(X,Y)\,.$$

Namely there exists $e \geq 1$ such that

$$g(X,Y) \equiv a_{p^e} C_{p^e}(X,Y) \mod \deg(p^e+1)\,.$$

Next, put $h(X,Y) = g(X,Y) - a_{p^e} \tilde{L}_{p,e}(\lambda; X, Y)$. Recall that

$$\tilde{L}_{p,e}(\lambda; X,Y) \equiv C_{p^e}(X,Y) \mod \deg(p^e+1)\,,$$
$$h(X,Y) \equiv 0 \mod \deg(p^e+1)\,.$$

Hence we can write

$$h(X,Y) = \sum_{r=p^e+1}^{\infty} h_r(X,Y)\,,$$

here $h_r(X,Y)$ is the homogeneous part of degree r of $h(X,Y)$. Repeating the above argument, we may assume

$$h(X,Y) = \sum_{r=p^{e+1}}^{\infty} h_r(X,Y)\,.$$

Namely

$$h(X,Y) \equiv a_{p^{e+1}} C_{p^{e+1}}(X,Y) \mod \deg(p^{e+1}+1)\,.$$

Repeating the above processes, we finally get

$$g(X,Y) - \sum_{\substack{r\geq l\\ r\notin P}} a_r t_r(X,Y) - \sum_{r=e}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X,Y) = 0$$

in $A[[X,Y]]$, here above $g(X,Y)$ is the original $g(X,Y)$ and we replaced a_{p^r} by b_r. Hence, $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X,Y)$ is cohomologous to $g(X,Y)$ as elements in $Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$.

COROLLARY 3.5. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. If $g(X,Y) \in Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$, there exists $\mathbf{b} = (b_0, b_1, b_2, \cdots) \in A^{(\mathbf{N})}$ such that $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X,Y)$ is cohomologous to $g(X,Y)$.*

PROOF. Let $g(X,Y)$ be a polynomial in $A[X,Y]$ of degree m. Take $N \geq 1$ such that $p^{N-1} \leq m \leq p^N$. As $g(X,Y)$ is a polynomial,

$$g(X,Y) = \sum_{r=e}^{N} b_r \tilde{L}_{p,r}(\lambda; X,Y) + \sum_{\substack{r=l \\ r \notin P}}^{p^N-1} a_r t_r(X,Y) + \sum_{r=p^N+1}^{2p^N-2} [a_r t_r(X,Y) \bmod \deg(2p^N-1)]$$

in $A[X,Y]$. Now, we put

$$U(X,Y) = \sum_{r=p^N+1}^{2p^N-2} a_r [t_r(X,Y) - [t_r(X,Y) \bmod \deg(2p^N-1)]] .$$

Noting that $g(X,Y), \tilde{L}_{p,r}(\lambda; X,Y), t_r(X,Y) \in Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$, we have $U(X,Y) \in Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$. Hence

$$\begin{aligned} 0 &= U(Y,Z) - U(\lambda XY + X + Y, Z) + U(X, \lambda YZ + Y + Z) - U(X,Y) \\ &\equiv \sum_{j=p^N+1}^{2p^N-2} \Bigg[-a_j \lambda^{2p^N-1-j} \binom{j}{2j-2p^N+1} Y^{2p^N-1-j} \sum_{i=1}^{2j-2p^N+1} \binom{2j-2p^N+1}{i} \\ &\quad \times \{X^{2p^N-1-j} Z^i (X+Y)^{2j-2p^N+1-i} - Z^{2p^N-1-j} X^i (Y+Z)^{2j-2p^N+1-i}\} \Bigg] \bmod \deg(2p^N) . \end{aligned}$$

Looking at the coefficient of $Y^{2p^N-1-j} X^{2p^N-1-j} Z^{2j-2p^N+1}$, we see that

$$a_j \lambda^{2p^N-1-j} \binom{j}{2j-2p^N+1} = 0$$

for all j with $p^N + 1 \leq j \leq 2p^N - 2$. Noting that

$$t_i(X,Y) - [t_i(X,Y) \bmod \deg(2p^N-1)] \in (\lambda^{2p^N-1-i})$$

for all i with $p^N + 1 \leq i \leq 2p^N - 2$, we obtain $U(X,Y) = 0$. Hence

$$\sum_{i=p^N+1}^{2p^N-2} [a_i t_i(X,Y) \bmod \deg(2p^N-1)] = \sum_{i=p^N+1}^{2p^N-2} a_i t_i(X,Y) \in B^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A}) .$$

Thus, we conclude that $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\Lambda, X, Y)$ is cohomologous to $g(X,Y)$ as elements in $Z^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{a,A})$.

This completes the proof of 3.5.

LEMMA 3.6. *Let A be a $\mathbf{Z}_{(p)}$-algebra, $\lambda \in A$ and $\mathbf{b} = (b_0, b_1, b_2, \cdots) \in A^{\mathbf{N}}$. Put $g(X,Y) = \sum_{r=1}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X,Y) \in A[[X,Y]]$. Assume that there exists $f(T) = \sum_{i=1}^{\infty} c_i T^i \in A[[T]]$ such that*

$$g(X,Y) = f(X) + f(Y) - f(\lambda XY + X + Y) . \tag{1}$$

Then there exists $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$ such that

$$b_r = -(-\lambda)^{p^r - p^{r-1}} a_{r-1} + p a_r$$

for all $r \geq 1$.

PROOF. Recall that the Taylor expansion of $f(X)+f(Y)-f(\lambda XY+X+Y)$. As $g(X,Y)\equiv 0 \bmod \deg p$,

$$\sum_{k=1}^{j-1}\sum_{i=m(j,k)}^{k}\binom{j-k+i}{j-2k+2i}\binom{j-2k+2i}{i}\lambda^{k-i}c_{j-k+i}=0$$

for all j with $2\leq j\leq p-1$. Hence we obtain

$$c_j=\frac{1}{j}(-\lambda)^{j-1}c_1$$

for all j with $2\leq j\leq p-1$. (cf. Proof of 3.1.)

The degree p part of the left-hand side of (1) is

$$-\frac{b_1}{p}\sum_{i=1}^{p-1}\binom{p}{i}X^iY^{p-i}\,.$$

The coefficient of XY^{p-1} of the right-hand side of (1) is

$$-\{(p-1)\lambda c_{p-1}+pc_p\}\,.$$

Hence

$$b_1=(p-1)\lambda c_{p-1}+pc_p=-(-\lambda)^{p-1}c_1+pc_p\,.$$

Generally, let $e\geq 1$. The degree p^e part of the left-hand side of (1) is

$$-\frac{b_e}{p}\sum_{i=1}^{p^e-1}\binom{p^e}{i}X^iY^{p^e-i}\,.$$

The coefficient of $X^{p^{e-1}}Y^{p^e}$ of the right-hand side of (1) is

$$-\sum_{i=0}^{p^{e-1}}\binom{p^e-p^{e-1}+i}{p^e-2p^{e-1}+2i}\binom{p^e-2p^{e-1}+2i}{i}\lambda^{p^{e-1}-i}c_{p^e-p^{e-1}+i}\,.$$

Hence

$$\frac{b_e}{p}\binom{p^e}{p^{e-1}}=\sum_{i=0}^{p^{e-1}}\binom{p^e-p^{e-1}+i}{p^e-2p^{e-1}+2i}\binom{p^e-2p^{e-1}+2i}{i}\lambda^{p^{e-1}-i}c_{p^e-p^{e-1}+i}\,.$$

Noting that

$$\frac{1}{p}\binom{p^e}{p^{e-1}}\equiv 1 \mod p\,,$$

$$b_e = \frac{1}{\frac{1}{p}\binom{p^e}{p^{e-1}}} \sum_{i=0}^{p^{e-1}} \binom{p^e - p^{e-1} + i}{p^e - 2p^{e-1} + 2i} \binom{p^e - 2p^{e-1} + 2i}{i} \lambda^{p^{e-1}-i} c_{p^e - p^{e-1}+i}$$

$$= \frac{1}{\frac{1}{p}\binom{p^e}{p^{e-1}}} \left[\sum_{i=0}^{p^{e-1}-1} \binom{p^e - p^{e-1} + i}{p^e - 2p^{e-1} + 2i} \binom{p^e - 2p^{e-1} + 2i}{i} \right.$$

$$\left. \times \frac{p^{e-1}}{p^e - p^{e-1} + i} (-\lambda)^{p^e - 2p^{e-1}+i} c_{p^{e-1}} + \binom{p^e}{p^{e-1}} c_{p^e} \right].$$

Noting that

$$\frac{\binom{-p^{e-1}}{l - p^{e-1}}}{\binom{l}{p^{e-1}}} = (-1)^{l-p^{e-1}} \frac{p^{e-1}}{l},$$

if $p^{e-1} \leq l < p^e$, we can easily see that

$$b_e = \frac{1}{\frac{1}{p}\binom{p^e}{p^{e-1}}}$$

$$\times \left[\sum_{i=0}^{p^{e-1}-1} \binom{p^e - p^{e-1} + i}{j - 2p^{e-1} + 2i} \binom{p^e - 2p^{e-1} + 2i}{i} \frac{\binom{-p^{e-1}}{p^e - 2p^{e-1} + i}}{\binom{p^e - p^{e-1} + i}{p^{e-1}}} \right] \lambda^{p^e - p^{e-1}} c_{p^{e-1}}$$

$$+ p c_{p^e}.$$

Now, applying 2.3 to $m = p^e - p^{e-1}$, $n = l = p^{e-1}$, $t = 0$, we have

$$\sum_{i=0}^{p^{e-1}-1} \binom{p^e - p^{e-1} + i}{p^e - 2p^{e-1} + 2i} \binom{p^e - 2p^{e-1} + 2i}{i} \frac{\binom{-p^{e-1}}{p^e - 2p^{e-1} + i}}{\binom{p^e - p^{e-1} + i}{p^{e-1}}} = -\binom{-p^{e-1}}{p^e - p^{e-1}},$$

and therefore

$$b_e = -\frac{\binom{-p^{e-1}}{p^e - p^{e-1}}}{\frac{1}{p}\binom{p^e}{p^{e-1}}} \lambda^{p^e - p^{e-1}} c_{p^{e-1}} + p c_{p^e}.$$

Noting that

$$\frac{\binom{-p^{e-1}}{p^e - p^{e-1}}}{\frac{1}{p}\binom{p^e}{p^{e-1}}} = (-1)^{p^e - p^{e-1}},$$

we obtain

$$b_e = -(-\lambda)^{p^e - p^{e-1}} c_{p^{e-1}} + p c_{p^e} .$$

Taking c_{p^i} as a_i for all $i \geq 0$, this completes the proof of 3.6.

3.7. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Now we prove the bijectivity of η^1 : Coker$[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$.

For $[g(X, Y)] \in H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$, there exist $\mathbf{b} = (b_0, b_1, b_2, \cdots) \in A^{\mathbf{N}}$ such that $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X, Y)$ is cohomologous to $g(X, Y)$ by 3.4. This shows that

$$\eta^1 : \text{Coker}[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) ; \quad \mathbf{b} \mapsto \sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X, Y)$$

is surjective.

On the other hand, if $\sum_{r=0}^{\infty} b_r \tilde{L}_{p,r}(\lambda; X, Y) \in B^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$, $\mathbf{b} = (b_0, b_1, b_2, \cdots) \in$ Im$[\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}]$ by 3.6. Thus, we also see the injectively of η^1.

EXAMPLE 3.8. We regain following well-known facts by 2.5.

(1) Assume that $\lambda = 1$. Then $\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}$ is bijective. Hence we have

$$\text{Hom}_{A-\text{gr}}(\mathbf{G}_{m,A}, \mathbf{G}_{a,A}) = 0 , \quad H_0^2(\mathbf{G}_{m,A}, \mathbf{G}_{a,A}) = 0 .$$

(2) Assume that $\lambda = 0$. If A is of characteristic 0, we see that $\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}$ is surjective. Hence $H_0^2(\mathbf{G}_{a,A}, \mathbf{G}_{a,A}) = 0$, and we obtain

$$\text{Hom}_{A-\text{gr}}(\mathbf{G}_{a,A}, \mathbf{G}_{a,A}) \simeq \{aT; a \in A\} \simeq A .$$

On the other hand, if A is of characteristic p, we see that $\Psi : A^{(\mathbf{N})} \to A^{(\mathbf{N})}$ is a zero homomorphism. Hence we have

$$\text{Hom}_{A-\text{gr}}(\mathbf{G}_{a,A}, \mathbf{G}_{a,A}) \simeq \left\{ \sum_{r=0}^{\infty} a_r T^{p^r}; (a_0, a_1, \cdots) \in A^{(\mathbf{N})} \right\} \simeq A^{(\mathbf{N})} ,$$

$$H_0^2(\mathbf{G}_{a,A}, \mathbf{G}_{a,A}) \simeq \left\{ \sum_{r=0}^{\infty} a_r \left[\frac{X^{p^{r+1}} + Y^{p^{r+1}} - (X+Y)^{p^{r+1}}}{p} \right]; (a_0, a_1, \cdots) \in A^{(\mathbf{N})} \right\} \simeq A^{(\mathbf{N})} .$$

4. Relation with $\text{Hom}_{A-\text{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ and $\text{Ext}_A^1(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$.

We start with reviewing necessary facts on Witt vectors. For details, see [DG, Chap. V] or [HZ, Chap. III].

4.1. For each $r \geq 0$, we denote by $\Phi_r(\mathbf{T}) = \Phi_r(T_0, T_1, \cdots, T_r)$ the so-called Witt polynomial

$$\Phi_r(\mathbf{T}) = T_0^{p^r} + pT_1^{p^{r-1}} + \cdots + p^r T_r$$

in $\mathbf{Z}[\mathbf{T}] = \mathbf{Z}[T_0, T_1, \cdots]$. We define polynomials

$$S_r(\mathbf{X}, \mathbf{Y}) = S_r(X_0, \cdots, X_r, Y_0, \cdots, Y_r),$$
$$P_r(\mathbf{X}, \mathbf{Y}) = P_r(X_0, \cdots, X_r, Y_0, \cdots, Y_r)$$

in $\mathbf{Z}[\mathbf{X}, \mathbf{Y}] = \mathbf{Z}[X_0, X_1, \cdots, Y_0, Y_1, \cdots]$ inductively by

$$\Phi_r(S_0(\mathbf{X}, \mathbf{Y}), S_1(\mathbf{X}, \mathbf{Y}), \cdots, S_r(\mathbf{X}, \mathbf{Y})) = \Phi_r(\mathbf{X}) + \Phi_r(\mathbf{Y}),$$
$$\Phi_r(P_0(\mathbf{X}, \mathbf{Y}), P_1(\mathbf{X}, \mathbf{Y}), \cdots, P_r(\mathbf{X}, \mathbf{Y})) = \Phi_r(\mathbf{X})\Phi_r(\mathbf{Y}).$$

Then as is well known, the ring structure of the scheme of Witt vectors

$$W_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[T_0, T_1, T_2, \cdots]$$

is given by the addition

$$T_0 \mapsto S_0(\mathbf{X}, \mathbf{Y}), T_1 \mapsto S_1(\mathbf{X}, \mathbf{Y}), T_2 \mapsto S_2(\mathbf{X}, \mathbf{Y}), \cdots$$

and the multiplication

$$T_0 \mapsto P_0(\mathbf{X}, \mathbf{Y}), T_1 \mapsto P_1(\mathbf{X}, \mathbf{Y}), T_2 \mapsto P_2(\mathbf{X}, \mathbf{Y}), \cdots.$$

4.2. Define now polynomials

$$F_r(\mathbf{T}) = F_r(T_0, \cdots, T_r, T_{r+1}) \in \mathbf{Q}[T_0, \cdots, T_r, T_{r+1}]$$

inductively by

$$\Phi_r(F_0(\mathbf{T}, \cdots, F_r(\mathbf{T})) = \Phi_{r+1}(T_0, \cdots, T_r, T_{r+1})$$

for $r \geq 0$. Then

$$F_r(\mathbf{T}) = F_r(T_0, \cdots, T_r, T_{r+1}) \in \mathbf{Z}[T_0, \cdots, T_r, T_{r+1}]$$

for $r \geq 0$. We denote by $F : W_{\mathbf{Z}} \to W_{\mathbf{Z}}$ the morphism defined by

$$T_0 \mapsto F_0(\mathbf{T}), T_1 \mapsto F_1(\mathbf{T}), T_2 \mapsto F_2(\mathbf{T}), \cdots.$$

Then it is verified without difficulty that F is a homomorphism of ring schemes. If A is an $\mathbf{F}_p$-algebra, $F : W_A \to W_A$ is nothing but the usual Frobenius endomorphism.

Next we recall a formal power series which is a generalization of the Aritn-Hasse exponential series. For details, see Sekiguchi-Suwa [3, Sec 2].

4.3. We define a formal power series $E_p(U, \Lambda; T)$ in $\mathbf{Q}[U, \Lambda][[T]]$ by

$$E_p(U, \Lambda; T) = (1 + \Lambda T)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{\frac{1}{p^k}\left\{\left(\frac{U}{\Lambda}\right)^{p^k} - \left(\frac{U}{\Lambda}\right)^{p^{k-1}}\right\}}.$$

Recall now the definition of the Artin-Hasse exponential series

$$E_p(T) = \exp\left(\sum_{r \geq 0} \frac{T^{p^r}}{p^r}\right) \in \mathbf{Z}_{(p)}[[T]].$$

Then we can verify that $E_p(U, \Lambda; T) \in \mathbf{Z}_{(p)}[U, \Lambda][[T]]$ and that $E_p(1, 0; T) = E_p(T)$ and $E_p(\Lambda, \Lambda; T) = 1 + \Lambda T$ ([3, 2.5, 2.6]).

Let $\mathbf{U} = (U_0, U_1, U_2, \cdots)$. We define a formal power series $E_p(\mathbf{U}, \Lambda; T)$ in $\mathbf{Z}_{(p)}[U_0, U_1, U_2, \cdots, \Lambda][[T]]$ by

$$E_p(\mathbf{U}, \Lambda; T) = \prod_{k=0}^{\infty} E_p(U_k, \Lambda^{p^k}; T^{p^k}) .$$

Then the equality

$$E_p(\mathbf{U}, \Lambda; T) = (1 + \Lambda T)^{\frac{\Phi_0(\mathbf{U})}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{\frac{1}{p^k \Lambda^{p^k}}\{\Phi_k(\mathbf{U}) - \Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\mathbf{U})\}}$$

is verified.

We define a formal power series $F_p(\mathbf{U}, \Lambda; X, Y)$ in $\mathbf{Q}[U_0, U_1, U_2, \cdots, \Lambda][[X, Y]]$ by

$$F_p(\mathbf{U}, \Lambda; X, Y) = \prod_{k=1}^{\infty} \left[\frac{(1 + \Lambda^{p^k} X^{p^k})(1 + \Lambda^{p^k} Y^{p^k})}{1 + \Lambda^{p^k}(X + Y + \Lambda XY)^{p^k}} \right]^{\Phi_{k-1}(\mathbf{U})/p^k \Lambda^{p^k}} .$$

It is known that $F_p(\mathbf{U}, \Lambda; X, Y) \in \mathbf{Z}_{(p)}[U_0, U_1, U_2, \cdots, \Lambda][[X, Y]]$.

4.4. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. There are defined homomorphisms $\xi^0 : W(A) \to A[[T]]^\times$ and $\xi^1 : W(A) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ by $\mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; T)$ and $\mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y)$, respectively ([3, Sec 2]). On the other hand, we define a cocycle map $\partial : A[[T]]^\times \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ by $f(T) \mapsto f(X)f(Y)f(\lambda XY + X + Y)^{-1}$. Then the diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\xi^0} & A[[T]]^\times \\ {\scriptstyle F - [\lambda^{p-1}]}\downarrow & & \downarrow{\scriptstyle \partial} \\ W(A) & \xrightarrow{\xi^1} & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) \end{array}$$

commutes, where $[\lambda^{p-1}] = (\lambda^{p-1}, 0, 0, \cdots) \in W(A)$. Moreover, ξ^0 and ξ^1 induce homomorphisms

$$\xi^0 : \mathrm{Ker}[F - [\lambda^{p-1}] : W(A) \to W(A)] \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) ; \quad \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; T) ,$$
$$\xi^1 : \mathrm{Coker}[F - [\lambda^{p-1}] : W(A) \to W(A)] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) ; \quad \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y) ,$$

and the following theorem is proved in [3, Sec 3]: Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then homomorphisms

$$\xi^0 : \mathrm{Ker}[F - [\lambda^{p-1}] : W(A) \to W(A)] \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) ,$$
$$\xi^1 : \mathrm{Coker}[F - [\lambda^{p-1}] : W(A) \to W(A)] \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$$

are bijective.

Now we give another proof of 2.5, basing our arguments on the above result. We begin with establishing some formalisms on Witt vectors and defining formal power series.

LEMMA 4.5. *Put* $\mathbf{X} = (X_0, X_1, X_2, \cdots)$, $\mathbf{Y} = (Y_0, Y_1, Y_2, \cdots)$ *and* $\mathbf{T} = (T_0, T_1, T_2, \cdots)$. *Then we have followings*:

(1) $S_r(\mathbf{X}, \mathbf{Y}) \equiv X_r + Y_r \mod (X_0, \cdots, X_{r-1}, Y_0, \cdots, Y_{r-1})^2$ *for all* $r \geq 1$,

(2) $P_r(\mathbf{X},\mathbf{Y}) \equiv \Phi_r(\mathbf{X})Y_r \mod (Y_0,\cdots,Y_{r-1})^2$ *for all* $r \geq 1$,

(3) $F_{r-1}(\mathbf{T}) \equiv pT_r \mod (T_0,\cdots,T_{r-1})^2$ *for all* $r \geq 1$.

PROOF. We prove the assertion by the induction on r. We can easily see that

$$S_1(\mathbf{X},\mathbf{Y}) = X_1 + Y_1 + \frac{X_0^p + Y_0^p - (X_0+Y_0)^p}{p},$$

$$P_1(\mathbf{X},\mathbf{Y}) = \Phi_1(\mathbf{X})Y_1 + X_1 Y_0^p,$$

$$F_0(\mathbf{T}) = T_0^p + pT_1 \equiv pT_1 \mod (T_0)^2,$$

respectively. Assume that

$$S_k(\mathbf{X},\mathbf{Y}) \equiv X_k + Y_k \mod (X_0,\cdots,X_{k-1},Y_0,\cdots,Y_{k-1})^2,$$

$$P_k(\mathbf{X},\mathbf{Y}) \equiv \Phi_k(\mathbf{X})Y_k \mod (Y_0,\cdots,Y_{k-1})^2,$$

$$F_{k-1}(\mathbf{T}) \equiv pT_k \mod (T_0,\cdots,T_{k-1})^2$$

for $k = 1,\cdots,r-1$, respectively. Then we have

$$S_k(\mathbf{X},\mathbf{Y})^{p^{r-k}} \equiv 0 \mod (X_0,\cdots,X_{r-1},Y_0,\cdots,Y_{r-1})^2,$$

$$P_k(\mathbf{X},\mathbf{Y})^{p^{r-k}} \equiv 0 \mod (Y_0,\cdots,Y_{r-1})^2,$$

$$F_{k-1}(\mathbf{T})^{p^{r-k}} \equiv 0 \mod (T_0,\cdots,T_{r-1})^2$$

for $k = 1,\cdots,r-1$, respectively. Hence we obtain

$$\begin{aligned}
&\Phi_r(S_0(\mathbf{X},\mathbf{Y}), S_1(\mathbf{X},\mathbf{Y}),\cdots,S_r(\mathbf{X},\mathbf{Y}))\\
&\quad = S_0(\mathbf{X},\mathbf{Y})^{p^r} + pS_1(\mathbf{X},\mathbf{Y})^{p^{r-1}} + \cdots + p^{r-1}S_{r-1}(\mathbf{X},\mathbf{Y})^p + p^r S_r(\mathbf{X},\mathbf{Y})\\
&\quad \equiv p^r S_r(\mathbf{X},\mathbf{Y}) \mod (X_0,\cdots,X_{r-1},Y_0,\cdots,Y_{r-1})^2,
\end{aligned}$$

$$\begin{aligned}
&\Phi_r(P_0(\mathbf{X},\mathbf{Y}), P_1(\mathbf{X},\mathbf{Y}),\cdots,P_r(\mathbf{X},\mathbf{Y}))\\
&\quad = P_0(\mathbf{X},\mathbf{Y})^{p^r} + pP_1(\mathbf{X},\mathbf{Y})^{p^{r-1}} + \cdots + p^{r-1}P_{r-1}(\mathbf{X},\mathbf{Y})^p + p^r P_r(\mathbf{X},\mathbf{Y})\\
&\quad \equiv p^r P_r(\mathbf{X},\mathbf{Y}) \mod (Y_0,\cdots,Y_{r-1})^2,
\end{aligned}$$

$$\begin{aligned}
&\Phi_{r-1}(F_0(\mathbf{T}), F_1(\mathbf{T}),\cdots,F_{r-1}(\mathbf{T}))\\
&\quad = F_0(\mathbf{T})^{p^{r-1}} + pF_1(\mathbf{T})^{p^{r-2}} + \cdots + p^{r-2}F_{r-2}(\mathbf{T})^p + p^{r-1}F_{r-1}(\mathbf{T})\\
&\quad \equiv p^{r-1}F_{r-1}(\mathbf{T}) \mod (T_0,\cdots,T_{r-1})^2,
\end{aligned}$$

respectively. On the other hand, we have

$$\Phi_r(\mathbf{X}) + \Phi_r(\mathbf{Y}) \equiv p^r(X_r + Y_r) \mod (X_0,\cdots,X_{r-1},Y_0,\cdots,Y_{r-1})^2,$$

$$\Phi_r(\mathbf{X})\Phi_r(\mathbf{Y}) \equiv \Phi_r(\mathbf{X}) \cdot p^r Y_r \mod (Y_0,\cdots,Y_{r-1})^2,$$

$$\Phi_r(\mathbf{T}) \equiv p^r T_r \mod (T_0,\cdots,T_{r-1})^2,$$

respectively. Hence we hold the claim.

Let A be a commutative ring with the unit element, and $A[\varepsilon]$ be a ring of dual numbers, that is $\varepsilon^2 = 0$. Recall now that the augmentation homomorphism $W(A[\varepsilon]) \to W(A)$ is defined by $(a_0 + b_0\varepsilon, a_1 + b_1\varepsilon, a_2 + b_2\varepsilon, \cdots) \mapsto (a_0, a_1, a_2, \cdots)$. Then

$$\mathrm{Ker}[W(A[\varepsilon]) \to W(A)] = \{(b_0\varepsilon, b_1\varepsilon, b_2\varepsilon, \cdots);\ b_i \in A \text{ for all } i \geq 0\}.$$

PROPOSITION 4.6. $\mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$ *is a square null ideal of* $W(A[\varepsilon])$ *and is isomorphic to* $A^{\mathbf{N}}$ *as additive group.*

PROOF. Let $\mathbf{b} = (b_0\varepsilon, b_1\varepsilon, b_2\varepsilon, \cdots)$, $\mathbf{c} = (c_0\varepsilon, c_1\varepsilon, c_2\varepsilon, \cdots) \in \mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$. Then we see by 4.5 that $P_r(\mathbf{b}, \mathbf{c}) = 0$ and $S_r(\mathbf{b}, \mathbf{c}) = (b_r + c_r)\varepsilon$, for all $r \geq 0$. Hence $\mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$ is square null ideal of $A[\varepsilon]$ and

$$\mathrm{Ker}[W(A[\varepsilon]) \to W(A)] \to A^{\mathbf{N}}; \quad (a_0\varepsilon, a_1\varepsilon, a_2\varepsilon, \cdots) \mapsto (a_0, a_1, a_2, \cdots)$$

is an isomorphism as additive group.

COROLLARY 4.7. $F : W(A[\varepsilon]) \to W(A[\varepsilon])$ *induces an A-endomorphism* $(a_0, a_1, a_2, \cdots) \mapsto (pa_1, pa_2, pa_3, \cdots)$ *on* $A^{\mathbf{N}}$.

PROOF. Let $\mathbf{a} = (a_0\varepsilon, a_1\varepsilon, a_2\varepsilon, \cdots) \in \mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$. Then we see by 4.5 that $F_r(\mathbf{a}) = pa_{r+1}\varepsilon$ for all $r \geq 0$. Hence $F : W(A[\varepsilon]) \to W(A[\varepsilon])$ induces an A-endomorphism $F : \mathbf{a} \mapsto (pa_1\varepsilon, pa_2\varepsilon, pa_3\varepsilon, \cdots)$ on $\mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$.

COROLLARY 4.8. *Let* $\mathbf{c} = (c_0, c_1, c_2, \cdots) \in W(A)$. *Then the homothety on* $W(A[\varepsilon])$ *by* $\mathbf{c}$ *induces an A-endomorphism* $(a_0, a_1, a_2, \cdots) \mapsto (\Phi_0(\mathbf{c})a_0, \Phi_1(\mathbf{c})a_1, \Phi_2(\mathbf{c})a_2, \cdots)$ *on* $A^{\mathbf{N}}$.

PROOF. Let $\mathbf{c} = (c_0, c_1, c_2, \cdots) \in W(A)$ and $\mathbf{a} = (a_0\varepsilon, a_1\varepsilon, a_2\varepsilon, \cdots) \in \mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$. Then we see by 4.5 that $P_r(\mathbf{c}, \mathbf{a}) = \Phi_r(\mathbf{c})a_r\varepsilon$ for all $r \geq 0$. Hence the homothety on $W(A[\varepsilon])$ by $\mathbf{c}$ induces an A-endomorphism $\mathbf{a} \mapsto (\Phi_0(\mathbf{c})a_0\varepsilon, \Phi_1(\mathbf{c})a_1\varepsilon, \Phi_2(\mathbf{c})a_2\varepsilon, \cdots)$ on $\mathrm{Ker}[W(A[\varepsilon]) \to W(A)]$.

Combining 4.7 and 4.8, we have the following:

COROLLARY 4.9. *Let* $\mathbf{c} = (c_0, c_1, c_2, \cdots) \in W(A)$. *Then* $F - \mathbf{c} : W(A[\varepsilon]) \to W(A[\varepsilon])$ *induces an A-endomorphism* $(a_0, a_1, a_2, \cdots) \mapsto (pa_1 - \Phi_0(\mathbf{c})a_0, pa_2 - \Phi_1(\mathbf{c})a_1, pa_3 - \Phi_2(\mathbf{c})a_2, \cdots)$ *on* $A^{\mathbf{N}}$.

4.10. Let A be a $\mathbf{Z}_{(p)}$-algebra and $A[\varepsilon]$ be a ring of dual numbers. We define a group homomorphism $A[[T]] \to (A[\varepsilon][[T]])^\times$ by $f(T) \mapsto 1 + \varepsilon f(T)$. Since

$$(A[\varepsilon][[T]])^\times = \{f(T) + \varepsilon g(T);\ f(T) \in A[[T]]^\times,\ g(T) \in A[[T]]\},$$

we can also define a group homomorphism $(A[\varepsilon][[T]])^\times \to A[[T]]^\times$ by $f(T) + \varepsilon g(T) \mapsto f(T)$. Then we can easily see that

$$0 \to A[[T]] \to (A[\varepsilon][[T]])^\times \to A[[T]]^\times \to 0$$

is an exact sequence of groups.

We define a group homomorphism $Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]})$ by $f(X,Y) \mapsto 1 + \varepsilon f(X,Y)$. This is well defined, since we can easily obtain equalities

$$\begin{aligned}&\{1+\varepsilon f(Y,Z)\}\{1+\varepsilon f(X,\lambda YZ+Y+Z)\}\{1+\varepsilon f(X,Y)\}^{-1}\{1+\varepsilon f(\lambda XY+X+Y,Z)\}^{-1}\\ &\quad = 1+\varepsilon\{f(Y,Z)+f(X,\lambda YZ+Y+Z)-f(X,Y)-f(\lambda XY+X+Y,Z)\}\end{aligned}$$

and

$$\{1+\varepsilon f(X,Y)\}\{1+\varepsilon g(X,Y)\} = 1+\varepsilon\{f(X,Y)+g(X,Y)\}$$

for $f(X,Y), g(X,Y) \in A[[X,Y]]$. We can also define a homomorphism of multiplicative groups $Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]}) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A})$ by $f(X,Y) + \varepsilon g(X,Y) \mapsto f(X,Y)$, where $f(X,Y) \in A[[X,Y]]^\times$ and $g(X,Y) \in A[[X,Y]]$. Then we can easily see that

$$0 \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]}) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) \to 0$$

is an exact sequence of groups.

4.11. We define formal power series $L_p(\Lambda; T)$ by

$$L_p(\Lambda;T) = \frac{1}{\Lambda}\left\{\log(1+\Lambda T) - \frac{1}{p}\log(1+\Lambda^p T^p)\right\}$$

in $\mathbf{Q}[\Lambda][[T]]$.

LEMMA 4.12. *We have*

$$L_p(\Lambda;T) = \begin{cases} \displaystyle\sum_{(k,p)=1} \frac{(-\Lambda)^{k-1}}{k} T^k & (p>2)\,, \\ \displaystyle\sum_{(k,2)=1} \frac{\Lambda^{k-1}}{k} T^k - \sum_{(k,2)=1} \frac{\Lambda^{2k-1}}{k} T^{2k} & (p=2)\,. \end{cases}$$

PROOF. Let $p > 2$. Note that

$$\frac{1}{\Lambda}\log(1+\Lambda T) = \sum_{k=1}^{\infty} \frac{(-\Lambda)^{k-1}}{k} T^k\,,$$

$$\frac{1}{p\Lambda}\log(1+\Lambda^p T^p) = \frac{1}{p}\sum_{k=1}^{\infty} \frac{(-\Lambda)^{kp-1}}{k} T^{kp} = \sum_{(k,p)\neq 1} \frac{(-\Lambda)^{k-1}}{k} T^k\,.$$

Hence we immediately obtain

$$L_p(\Lambda;T) = \sum_{(k,p)=1} \frac{(-\Lambda)^{k-1}}{k} T^k\,.$$

Let $p = 2$. Note that

$$\begin{aligned}\frac{1}{\Lambda}\log(1+\Lambda T) &= \sum_{(k,2)=1} \frac{\Lambda^{k-1}}{k} T^k + \sum_{(k,2)\neq 1} \frac{(-\Lambda)^{k-1}}{k} T^k \\ &= \sum_{(k,2)=1} \frac{\Lambda^{k-1}}{k} T^k - \sum_{(k,2)=1} \frac{\Lambda^{2k-1}}{2k} T^{2k} - \sum_{(k,2)\neq 1} \frac{\Lambda^{2k-1}}{2k} T^{2k}\,,\end{aligned}$$

$$\frac{1}{2\Lambda}\log(1+\Lambda^2T^2)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Lambda^{2k-1}}{2k}T^{2k}=\sum_{(k,2)=1}\frac{\Lambda^{2k-1}}{2k}T^{2k}-\sum_{(k,2)\neq 1}\frac{\Lambda^{2k-1}}{2k}T^{2k}.$$

Hence we immediately obtain

$$L_2(\Lambda;T)=\sum_{(k,2)=1}\frac{\Lambda^{k-1}}{k}T^k-\sum_{(k,2)=1}\frac{\Lambda^{2k-1}}{k}T^{2k}.$$

By 4.12, we readily see the following:

COROLLARY 4.13. $L_p(\Lambda;T)\in\mathbf{Z}_{(p)}[\Lambda][[T]]$.

LEMMA 4.14. *Let* $\mathbf{U}=(U_0,U_1,U_2,\cdots)$. *Then we have followings*:
(1) $\log E_p(\mathbf{U},\Lambda;T)=\sum_{r=0}^{\infty}\frac{\Phi_r(\mathbf{U})}{p^r}L_p(\Lambda^{p^r};T^{p^r})$,
(2) $\log E_p(\mathbf{U},\Lambda;T)\equiv\sum_{r=0}^{\infty}U_rL_p(\Lambda^{p^r};T^{p^r}) \mod (U_0,U_1,U_2,\cdots)^2$,
(3) $E_p(\mathbf{U},\Lambda;T)\equiv 1+\sum_{r=0}^{\infty}U_rL_p(\Lambda^{p^r};T^{p^r}) \mod (U_0,U_1,U_2,\cdots)^2$.

PROOF. By the definition of $E_p(\mathbf{U},\Lambda;T)$, we have

$$\begin{aligned}
&\log E_p(\mathbf{U},\Lambda;T)\\
&=\log\left[(1+\Lambda T)^{\frac{\Phi_0(\mathbf{U})}{\Lambda}}\prod_{r=1}^{\infty}(1+\Lambda^{p^r}T^{p^r})^{\frac{1}{p^r\Lambda^{p^r}}\{\Phi_r(\mathbf{U})-\Lambda^{p^{r-1}(p-1)}\Phi_{r-1}(\mathbf{U})\}}\right]\\
&=\frac{\Phi_0(\mathbf{U})}{\Lambda}\log(1+\Lambda T)+\sum_{r=1}^{\infty}\frac{\Phi_r(\mathbf{U})-\Lambda^{p^{r-1}(p-1)}\Phi_{r-1}(\mathbf{U})}{p^r\Lambda^{p^r}}\log(1+\Lambda^{p^r}T^{p^r})\\
&=\sum_{r=0}^{\infty}\frac{\Phi_r(\mathbf{U})}{p^r\Lambda^{p^r}}\left\{\log(1+\Lambda^{p^r}T^{p^r})-\frac{1}{p}\log(1+\Lambda^{p^{r+1}}T^{p^{r+1}})\right\}\\
&=\sum_{r=0}^{\infty}\frac{\Phi_r(\mathbf{U})}{p^r}L_p(\Lambda^{p^r};T^{p^r}).
\end{aligned}$$

Moreover, noting that

$$\Phi_r(\mathbf{U})\equiv p^rU_r \mod (U_0,U_1,U_2,\cdots)^2 \quad\text{and}\quad \exp(T)=\sum_{n=0}^{\infty}\frac{T^n}{n!},$$

we immediately have (2) and (3).

4.15. Let p be a prime number. We define formal power series $L_{p,r}(\Lambda;X,Y)$ by

$$L_{p,r}(\Lambda;X,Y)=\frac{1}{p\Lambda^{p^{r+1}}}\log\frac{(1+\Lambda^{p^{r+1}}X^{p^{r+1}})(1+\Lambda^{p^{r+1}}Y^{p^{r+1}})}{1+\Lambda^{p^{r+1}}(X+Y+\Lambda XY)^{p^{r+1}}}$$

in $\mathbf{Q}[\Lambda][[X,Y]]$.

LEMMA 4.16. $L_{p,r}(\Lambda;X,Y)\in\mathbf{Z}_{(p)}[\Lambda][[X,Y]]$.

PROOF. Noting that

$$\frac{(1+\Lambda^{p^{r+1}}X^{p^{r+1}})(1+\Lambda^{p^{r+1}}Y^{p^{r+1}})}{1+\Lambda^{p^{r+1}}(X+Y+\Lambda XY)^{p^{r+1}}}\equiv 1 \mod p,$$

we easily see the fact.

LEMMA 4.17. *Let* $\mathbf{U} = (U_0, U_1, U_2, \cdots)$. *Then we have followings*:
(1) $\log F_p(\mathbf{U}, \Lambda; X, Y) = \sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{U})}{p^r} L_{p,r}(\Lambda; X, Y)$,
(2) $\log F_p(\mathbf{U}, \Lambda; X, Y) \equiv \sum_{r=0}^{\infty} U_r L_{p,r}(\Lambda; X, Y) \mod (U_0, U_1, U_2, \cdots)^2$,
(3) $F_p(\mathbf{U}, \Lambda; X, Y) \equiv 1 + \sum_{r=0}^{\infty} U_r L_{p,r}(\Lambda; X, Y) \mod (U_0, U_1, U_2, \cdots)^2$.

PROOF. By the definition of $F_p(\mathbf{U}, \Lambda; X, Y)$, we have

$$\begin{aligned}
\log F_p(\mathbf{U}, \Lambda; X, Y) &= \log \prod_{r=1}^{\infty} \left[\frac{(1 + \Lambda^{p^r} X^{p^r})(1 + \Lambda^{p^r} Y^{p^r})}{1 + \Lambda^{p^r}(X + Y + \Lambda XY)^{p^r}} \right]^{\Phi_{r-1}(\mathbf{U})/p^r \Lambda^{p^r}} \\
&= \sum_{r=1}^{\infty} \frac{\Phi_{r-1}(\mathbf{U})}{p^r \Lambda^{p^r}} \log \frac{(1 + \Lambda^{p^r} X^{p^r})(1 + \Lambda^{p^r} Y^{p^r})}{1 + \Lambda^{p^r}(X + Y + \Lambda XY)^{p^r}} \\
&= \sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{U})}{p^{r+1} \Lambda^{p^{r+1}}} \log \frac{(1 + \Lambda^{p^{r+1}} X^{p^{r+1}})(1 + \Lambda^{p^{r+1}} Y^{p^{r+1}})}{1 + \Lambda^{p^{r+1}}(X + Y + \Lambda XY)^{p^{r+1}}} \\
&= \sum_{r=0}^{\infty} \frac{\Phi_r(\mathbf{U})}{p^r} L_{p,r}(\Lambda; X, Y) .
\end{aligned}$$

Moreover, noting that

$$\Phi_r(\mathbf{U}) \equiv p^r U_r \mod (U_0, U_1, U_2, \cdots)^2 \quad \text{and} \quad \exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!} ,$$

we immediately have (2) and (3).

4.18. Let A be a $\mathbf{Z}_{(p)}$-algebra, $A[\varepsilon]$ a ring of dual numbers and $\lambda \in A$. We define homomorphisms $\xi^0 : A^{\mathbf{N}} \to A[[T]]$ and $\xi^0 : W(A[\varepsilon]) \to (A[\varepsilon][[T]])^\times$ by

$$\xi^0 : A^{\mathbf{N}} \to A[[T]] ; \quad \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r L_p(\lambda^{p^r}; T^{p^r}) ,$$

$$\xi^0 : W(A[\varepsilon]) \to (A[\varepsilon][[T]])^\times ; \quad \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; T) ,$$

respectively.

LEMMA 4.19. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{\mathbf{N}} & \longrightarrow & W(A[\varepsilon]) & \longrightarrow & W(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi^0} & & \downarrow{\scriptstyle \xi^0} & & \downarrow{\scriptstyle \xi^0} & & \\
0 & \longrightarrow & A[[T]] & \longrightarrow & (A[\varepsilon][[T]])^\times & \longrightarrow & A[[T]]^\times & \longrightarrow & 0 .
\end{array}$$

PROOF. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$. By 4.14,

$$\prod_{r=0}^{\infty} E_p(a_r \varepsilon, \lambda^{p^r}; T^{p^r}) = 1 + \sum_{r=0}^{\infty} (a_r \varepsilon) L_p(\lambda^{p^r}; T^{p^r}) = 1 + \varepsilon \sum_{r=0}^{\infty} a_r L_p(\lambda^{p^r}; T^{p^r}) .$$

4.20. Let A be a $\mathbf{Z}_{(p)}$-algebra, $A[\varepsilon]$ a ring of dual numbers and $\lambda \in A$. We define homomorphisms $\xi^1 : A^{\mathbf{N}} \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ and $\xi^1 : W(A[\varepsilon]) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]})$ by

$$\xi^1 : A^{\mathbf{N}} \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})\,; \quad \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r L_{p,r}(\lambda; X, Y)\,,$$
$$\xi^1 : W(A[\varepsilon]) \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]})\,; \quad \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y)\,,$$

respectively.

LEMMA 4.21. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\mathbf{N}} & \longrightarrow & W(A[\varepsilon]) & \longrightarrow & W(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi^1} & & \downarrow{\scriptstyle \xi^1} & & \downarrow{\scriptstyle \xi^1} & & \\ 0 & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]}) & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0\,. \end{array}$$

PROOF. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$ and put $\mathbf{b} = (a_0\varepsilon, a_1\varepsilon, a_2\varepsilon, \cdots)$. By 4.17,

$$F_p(\mathbf{b}, \Lambda; X, Y) = 1 + \sum_{r=0}^{\infty} (a_r\varepsilon) L_{p,r}(\Lambda; X, Y) = 1 + \varepsilon \sum_{r=0}^{\infty} a_r L_{p,r}(\Lambda; X, Y)\,.$$

The following lemma is easily verified so we ommit the proof.

LEMMA 4.22. *Homomorphisms $\xi^0 : A^{\mathbf{N}} \to A[[T]]$ and $\xi^1 : A^{\mathbf{N}} \to Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ are A-homomorphisms.*

4.23. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$. We define a homomorphism

$$\Phi : A^{\mathbf{N}} \to A^{\mathbf{N}}\,; \quad \mathbf{a} \mapsto (-\lambda^{p^{r+1} - p^r} a_r + p a_{r+1})_{r \geq 0}\,.$$

LEMMA 4.24. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\mathbf{N}} & \longrightarrow & W(A[\varepsilon]) & \longrightarrow & W(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle F - [\lambda^{p-1}]} & & \downarrow{\scriptstyle F - [\lambda^{p-1}]} & & \\ 0 & \longrightarrow & A^{\mathbf{N}} & \longrightarrow & W(A[\varepsilon]) & \longrightarrow & W(A) & \longrightarrow & 0\,. \end{array}$$

PROOF. Note that $\Phi_r([\lambda^{p-1}]) = \lambda^{p^r(p-1)}$ for all $r \geq 0$. Then $F - [\lambda^{p-1}] : W(A[\varepsilon]) \to W(A[\varepsilon])$ induces A-homomorphism

$$\Phi : A^{\mathbf{N}} \to A^{\mathbf{N}}\,; \quad \mathbf{a} \mapsto (pa_1 - \lambda^{p^r(p-1)} a_0, pa_2 - \lambda^{p^r(p-1)} a_1, pa_3 - \lambda^{p^r(p-1)} a_2, \cdots)$$

by 4.9.

LEMMA 4.25. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then we have a commutative diagram with exact rows*

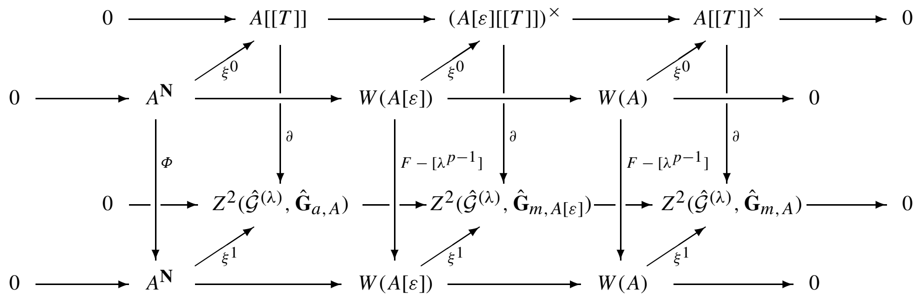
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A[[T]] & \longrightarrow & (A[\varepsilon][[T]])^{\times} & \longrightarrow & A[[T]]^{\times} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A[\varepsilon]}) & \longrightarrow & Z^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0\,.
\end{array}$$

PROOF. Let $f(T) \in A[[T]]$. Noting $\{1+\varepsilon f(T)\}^{-1} = 1-\varepsilon f(T)$, we obtain an equality

$$\begin{aligned}
&\{1+\varepsilon f(X)\}\{1+\varepsilon f(Y)\}\{1+\varepsilon f(X+Y+\lambda XY)\}^{-1} \\
&\qquad = 1+\varepsilon\{f(X)+f(Y)-f(X+Y+\lambda XY)\}\,.
\end{aligned}$$

By 4.4, 4.19, 4.21, 4.24, and 4.25, we readily see the following:

LEMMA 4.26. *Let A be a $\mathbf{Z}_{(p)}$-algebra, $\lambda \in A$ and $\varepsilon^2 = 0$. Then we have a following commutative diagram with exact rows*:



COROLLARY 4.27. *Homomorphisms* $\xi^0 : \operatorname{Ker}\Phi \to \operatorname{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ *and* $\xi^1 : \operatorname{Coker}\Phi \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ *are bijective.*

PROOF. Put $B = A[\varepsilon]$. Then the correspondences $a \mapsto 1+a\varepsilon$ and $a+b\varepsilon \mapsto a$ induce following split exact sequence of formal group schemes:

$$0 \to \hat{\mathbf{G}}_{a,A} \to \Big[\prod_{B/A} \mathbf{G}_{m,B}\Big]^{\wedge} \to \hat{\mathbf{G}}_{m,A} \to 0\,.$$

Here, $\prod_{B/A}$ denotes the Weil restriction functor. Noting that

$$\operatorname{Hom}_{A-\mathrm{gr}}\Big(\hat{\mathcal{G}}^{(\lambda)}, \Big[\prod_{B/A} \mathbf{G}_{m,B}\Big]^{\wedge}\Big) \simeq \operatorname{Hom}_{B-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,B})\,,$$

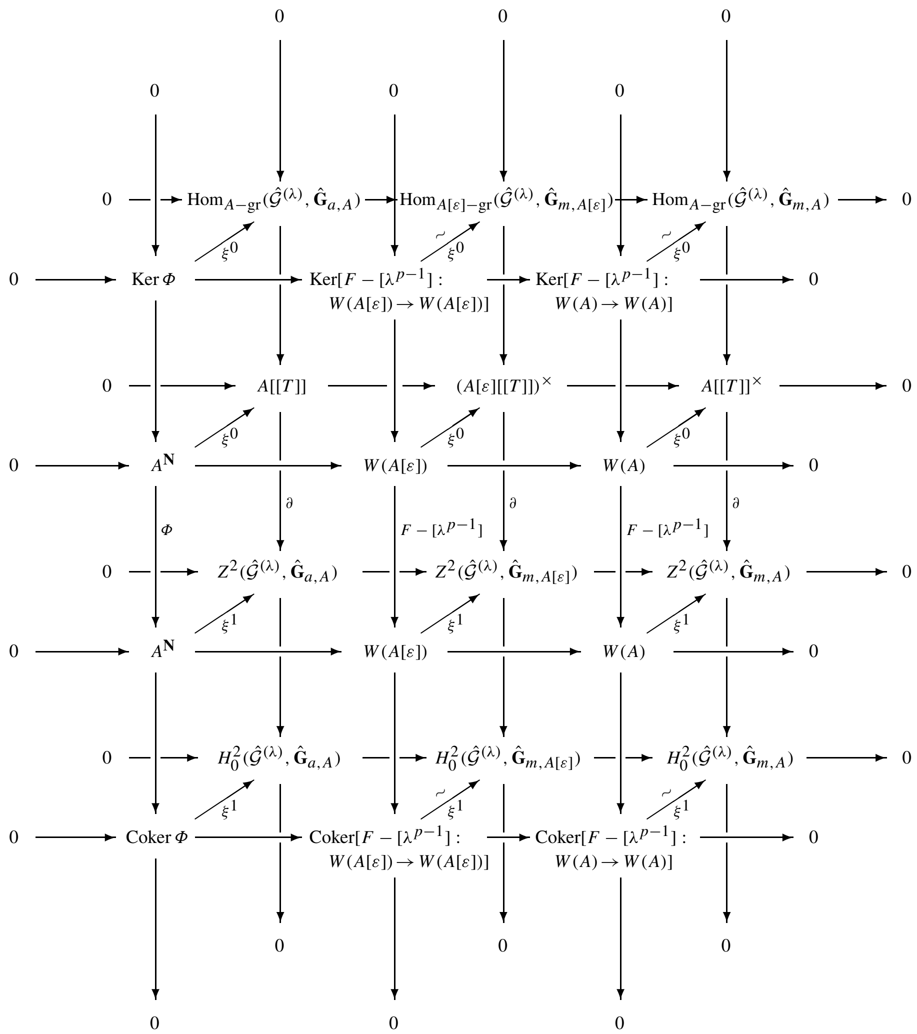
$$H_0^2\Big(\hat{\mathcal{G}}^{(\lambda)}, \Big[\prod_{B/A} \mathbf{G}_{m,B}\Big]^{\wedge}\Big) \simeq H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,B})\,,$$

we obtain long exact sequences

$$0 \to \operatorname{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) \to \operatorname{Hom}_{B-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,B}) \to \operatorname{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) \to 0\,,$$

$$0 \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,B}) \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{m,A}) \to 0\,.$$

Hence, we obtain following commutative diagram with exact sequences by 4.26. Moreover, combining the result of the argument of 4.4, we immediately have the bijectivity of ξ^0 : $\operatorname{Ker}\Phi \to \operatorname{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$ and $\xi^1 : \operatorname{Coker}\Phi \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$.



4.28. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Let $\mathbf{a} = (a_0, a_1, a_2, \cdots) \in A^{\mathbf{N}}$. We define a homomorphism

$$\Psi : A^{\mathbf{N}} \to A^{\mathbf{N}}; \quad \mathbf{a} \mapsto (-(-\lambda)^{p^{r+1}-p^r} a_r + p a_{r+1})_{r \geq 0}.$$

If $p > 2$, $\Psi = \Phi$. Let $p = 2$. We define an automorphism ι of $A^{\mathbf{N}}$ by $(a_0, a_1, a_2, \cdots) \mapsto (-a_0, a_1, a_2, \cdots)$. Then $\Psi = \Phi \circ \iota$.

LEMMA 4.29. *Let A be a $\mathbf{Z}_{(p)}$-algebra. Let $\lambda, a_0, a_1, a_2, \cdots \in A$. Assume that $\lambda^{p^{r+1}-p^r} a_r = pa_{r+1}$ for all $r \geq 0$. Then*

$$\sum_{r=0}^{\infty} a_r L_p(\lambda^{p^r}; T^{p^r}) = \begin{cases} \displaystyle\sum_{r=0}^{\infty} a_r \Big[\sum_{k=p^r}^{p^{r+1}-1} \frac{p^r}{k} (-\lambda)^{k-p^r} T^k \Big] & (p > 2)\,, \\ \displaystyle a_0 T - \sum_{r=1}^{\infty} a_r \Big[\sum_{k=2^r}^{2^{r+1}-1} \frac{2^r}{k} (-\lambda)^{k-2^r} T^k \Big] & (p = 2)\,. \end{cases}$$

PROOF. Let $p > 2$, $(k, p) = 1$ and $p^s \leq k < p^{s+1}$. Then the coefficient of $T^{p^r k}$ of the right-hand side is

$$\frac{(-\lambda)^{p^r k - p^{s+r}} p^{r+s}}{p^r k} a_{r+s}\,.$$

Here, since $pa_{r+s-j} = \lambda^{p^{r+s-j-1}(p-1)} a_{r+s-j-1}$ for all j with $0 \leq j < s$,

$$p^s a_{r+s} = \lambda^{p^{r+s-1}(p-1)} \lambda^{p^{r+s-2}(p-1)} \cdots \lambda^{p^r(p-1)} a_r = \lambda^{p^{r+s}-p^r} a_r = (-\lambda)^{p^{r+s}-p^r} a_r\,.$$

Hence

$$\begin{aligned} \frac{(-\lambda)^{p^r k - p^{r+s}} p^{r+s}}{p^r k} a_{r+s} &= \frac{(-\lambda)^{p^r k - p^{r+s}} (-\lambda)^{p^{r+s}-p^r} p^r}{p^r k} a_r = \frac{(-\lambda)^{p^r k - p^r}}{k} a_r \\ &= \frac{(-\lambda^{p^r})^{k-1}}{k} a_r\,. \end{aligned}$$

On the other hand,

$$\sum_{r=0}^{\infty} a_r L_p(\lambda^{p^r}; T^{p^r}) = \sum_{r=0}^{\infty} a_r \Big[\sum_{(k,p)=1} \frac{(-\lambda^{p^r})^{k-1}}{k} T^{p^r k} \Big]$$

by 4.12.

Let $p = 2$, $(k, 2) = 1$ and $2^s \leq k < 2^{s+1}$. When $s \geq 1$, the coefficient of T^k of the right-hand side is

$$-\frac{(-\lambda)^{k-2^s} 2^s}{k} a_s\,.$$

Here, since $2a_{s-j} = \lambda^{2^{s-j-1}} a_{s-j-1}$ for all j with $0 \leq j < s$,

$$2^s a_s = \lambda^{2^{s-1}} \lambda^{2^{s-2}} \cdots \lambda^2 \lambda a_0 = \lambda^{2^s-1} a_0\,.$$

Hence

$$-\frac{(-\lambda)^{k-2^s} 2^s}{k} a_s = -\frac{(-\lambda)^{k-2^s} \lambda^{2^s-1}}{k} a_0 = \frac{\lambda^{k-2^r} \lambda^{2^s-1}}{k} a_0 = \frac{\lambda^{k-1}}{k} a_0\,.$$

Let $p = 2$, $(k, 2) = 1$ and $2^s \leq k < 2^{s+1}$. When $s \geq 0$ and $r \geq 1$, the coefficient of $T^{2^r k}$ of the right-hand side is

$$-\frac{(-\lambda)^{2^r k - 2^{r+s}} 2^{r+s}}{2^r k} a_{r+s}\,.$$

Here, since $2a_{s-j} = \lambda^{2^{s-j-1}} a_{s-j-1}$ for all j with $0 \leq j < s$,

$$2^{r+s} a_{r+s} = \lambda^{2^{r+s-1}} \lambda^{2^{r+s-2}} \cdots \lambda^{2^r} a_r = \lambda^{p^{r+s}-p^r} a_r = \lambda^{2^{r+s}-2^r} a_r .$$

Hence

$$-\frac{(-\lambda)^{2^r k - 2^{r+s}} 2^{r+s}}{2^r k} a_s = -\frac{(-\lambda)^{2^r k - 2^{r+s}} \lambda^{2^{r+s}-2^r} 2^r}{2^r k} a_r = -\frac{\lambda^{2^r k - 2^{r+s}} \lambda^{2^{r+s}-2^r}}{k} a_r$$
$$= -\frac{\lambda^{2^r (k-1)}}{k} a_r .$$

On the other hand,

$$\sum_{r=0}^{\infty} a_r L_p(\lambda^{p^r}; T^{p^r}) = \sum_{r=0}^{\infty} a_r \Bigg[\sum_{(k,2)=1} \frac{\lambda^{2^r (k-1)}}{k} T^{2^r k} - \sum_{(k,2)=1} \frac{\lambda^{2^r (2k-1)}}{k} T^{2^{r+1} k} \Bigg]$$

by 4.12. The coefficient of $T^{2^r k}$ of the right-hand side is

$$\frac{\lambda^{2^r (k-1)}}{k} a_r - \frac{\lambda^{2^{r-1}(2k-1)}}{k} a_{r-1} .$$

Since $2a_r = \lambda^{2^{r-1}} a_{r-1}$,

$$\frac{\lambda^{2^r (k-1)}}{k} a_r - \frac{\lambda^{2^{r-1}(2k-1)}}{k} a_{r-1} = \frac{\lambda^{2^r (k-1)}}{k} a_r - \frac{\lambda^{2^r (k-1)}}{k} (2a_r) = -\frac{\lambda^{2^r (k-1)}}{k} a_r .$$

Now we attain our goal.

COROLLARY 4.30. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then the homomorphism*

$$\eta^0 : \operatorname{Ker} \Psi \to \operatorname{Hom}_{A-\mathrm{gr}}(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) ; \quad \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r \Bigg[\sum_{k=p^r}^{p^{r+1}-1} \frac{p^r}{k} (-\lambda)^{k-p^r} T^k \Bigg]$$

is bijective.

4.31. Put

$$\tilde{L}_{p,r}(\Lambda; X, Y) = \frac{p^r}{(-\Lambda)^{p^{r+1}-1}} \sum_{k=1}^{p^{r+1}-1} \frac{(-\Lambda)^{k-1}}{k} \{X^k + Y^k - (X + Y + \Lambda XY)^k\}$$

for all $r \geq 0$.

COROLLARY 4.32. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then*

$$[L_{p,r}(\lambda; X, Y)] = \begin{cases} [\tilde{L}_{p,r}(\lambda; X, Y)] & (p > 2) , \\ -[\tilde{L}_{p,r}(\lambda; X, Y)] & (p = 2) \end{cases}$$

in $H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A})$.

COROLLARY 4.33. *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then the homomorphism*

$$\eta^1 : \operatorname{Coker} \Psi \to H_0^2(\hat{\mathcal{G}}^{(\lambda)}, \hat{\mathbf{G}}_{a,A}) ; \quad \mathbf{a} \mapsto \sum_{r=0}^{\infty} a_r \tilde{L}_{p,r}(\lambda; X, Y)$$

is bijective.

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