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ON THE EXTREMAL STRUCTURE OF THE UNIT BALLS OF BANACH SPACES OF WEAKLY CONTINUOUS FUNCTIONS AND THEIR DUALS

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ABSTRACT. A sufficient and then a necessary condition are given for a function to be an extreme point of the unit ball of the Banach space $C(K,(X,w))$ of continuous functions, under the supremum norm, from a compact Hausdorff topological space K into a Banach space X equipped with its weak topology w. Strongly extreme points of the unit ball of $C(K,(X,w))$ are characterized as the norm-one functions that are uniformly strongly extreme point valued on a dense subset of K. It is shown that a variety of stronger types of extreme points (e.g. denting points) never exist in the unit ball of $C(K,(X,w))$. Lastly, some naturally arising and previously known extreme points of the unit ball of $C(K,(X,w))^*$ are shown to actually be strongly exposed points.

INTRODUCTION

Several types of extreme points of convex sets in Banach spaces have been invented and studied during the last six decades. For a specific Banach space, it is always of interest to discover whether the closed unit ball has any extreme points and, if so, to exactly describe them; it is of further interest to determine which, if any, of these extreme points are in fact one of the many stronger types of extreme points appearing in the literature. Such discoveries and determinations are the subject matter of this paper.

Let K be an infinite compact Hausdorff topological space, let X be a real Banach space, and let w denote the weak topology on X. Let $C(K,(X,w))$ denote the Banach space of all continuous functions from K into (X,w) equipped with the supremum norm; the geometry of this Banach space has been the subject of several recent investigations (see [ADLR], [C], [DD], and [M]).

In this paper, the extremal structure of the closed unit ball of $C(K,(X,w))$ relative to the extremal structure of the closed unit ball of X is investigated. The rather elusive extreme point structure of the unit ball of $C(K,(X,w))$ is discussed and it is compared to the extreme point structure situation in the unit ball of $C(K,X)$, the space of all continuous functions from K into X equipped with the supremum norm (here X is considered with its norm topology). Next, the strongly extreme points of the unit ball of $C(K,(X,w))$ are characterized in a very natural manner; the strongly extreme point situation here is quite different from that in

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the unit ball of $C(K,X)$ from one point of view and is exactly the same from another point of view. Examples are provided that illustrate this difference, and a proposition is given that witnesses this sameness. Finally, it is shown that the unit ball of $C(K,(X,w))$ never has many of the stronger types of extreme points (for example, denting points).

Also, in this paper, the extremal structure of the unit ball of the dual space $C(K,(X,w))^*$ is examined. The lack of a concrete representation of this dual space (as opposed to the dual space $C(K,X)^*$) considerably clouds the extreme point situation. It is known, from [ADLR], that $C(K,X)^*$ is always complemented in $C(K,(X,w))^*$; hence the extreme points of the unit ball of $C(K,X)^*$, which are well known, have a chance to also be extreme points in $C(K,(X,w))^*$. Several examples of classical Banach spaces are given for which this is the case. The main results here are strengthenings of a theorem of P-K. Lin [L], as illuminated by M. Cambern [C]; these are strengthenings that identify, for the first time, some denting points and some strongly exposed points of the unit ball of $C(K,(X,w))^*$.

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Definitions and Preliminaries

For a real Banach space E, let B_E denote the closed unit ball of E and let S_E denote the unit sphere of E. The unit ball B_E is said to be *stable* provided the mapping from $B_E \times B_E$ into B_E that sends (x,y) to $\frac{1}{2}(x+y)$ is an open mapping. The dual space of a Banach space E is denoted by E^* and the value of x^* in E^* at x in E is denoted by $\langle x^*, x\rangle$. The notation $\|x \pm z\| \leq 1+\delta$ means “$\|x+z\| \leq 1+\delta$ and $\|x-z\| \leq 1+\delta$.”

The following is a well-known list (included here for completeness sake) of geometrical notions.

(i) A point x in B_E is called an *extreme point* of B_E provided x is not the midpoint of any non-trivial line segment lying in B_E.

(ii) A point x in B_E is called a *strongly extreme point* of B_E provided that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\| x \pm z \| \leq 1+\delta$ for z in E implies $\| z \| \leq \varepsilon$.

(iii) A point x in B_E is called a *point of continuity* of B_E provided that whenever $\{x_\lambda\}_{\lambda\in\Lambda}$ is a net in B_E which converges weakly to x, it follows that $\{x_\lambda\}_{\lambda\in\Lambda}$ converges in norm to x. If nets are replaced by sequences in this condition, then x is called a *point of sequential continuity* of B_E.

(iv) A point x in B_E is called a *denting point* of B_E provided x is not an element of the closed convex hull of $\{y \in B_E : \| y - x \| > \varepsilon\}$ for each $\varepsilon > 0$.

(v) A point x in B_E is called a *strongly exposed point* of B_E provided that there exists x^* in B_{E^*} such that $\langle x^*, x\rangle = \| x^* \| = \| x \| = 1$ and whenever $\{x_n\}_{n=1}^\infty$ is a sequence in B_E such that $\lim_{n\to\infty}\langle x^*, x_n\rangle = 1$, it follows that

$$\lim_{n\to\infty} \| x - x_n \| = 0;$$

in this case, the functional x^* is said to *strongly expose* B_E at x.

It is known, from [LLT], that a point in B_E is a denting point of B_E if and only if it is both a point of continuity and an extreme point of B_E. For x^* in E^* and

$\delta > 0$, the *slice* of B_E determined by x^* and δ is the set

$$S(x^*, B_E, \delta) \;=\; \{y \in B_E \;:\; \langle x^*, y\rangle \geq \| \, x^* \, \| - \delta\}.$$

It is straightforward to show that x is a denting point of B_E if and only if for every $\varepsilon > 0$ there exist x^* in E^* and $\delta > 0$ such that $\operatorname{diam} S(x^*, B_E, \delta) < \varepsilon$. Also x is a strongly exposed point of B_E if and only if there exists an element x^* in S_{E^*} such that $\langle x^*, x\rangle \;=\; 1$ and $\lim_{\delta \to 0+} \operatorname{diam} S(x^*, B_E, \delta) = 0$.

If E is a dual space, the notions of a *weak* point of continuity* and a *weak* point of sequential continuity* of B_E are defined as in (iii) above replacing weak convergence by weak* convergence; the notion of a *weak* denting point* of B_E is defined as in (iv) above replacing the closed convex hull by the weak* closed convex hull of the set given there; and the notion of a *weak* strongly exposed point* of B_E is defined as in (v) above insisting that the strongly exposing functional belongs to the predual of E. Statements concerning weak* denting points and weak* strongly exposed points can be made which correspond to the statements made immediately after (v) above about denting points and strongly exposed points.

With the obvious abbreviations, the diagram below gives the relative strengths of all these notions.

$$\begin{array}{ccccccc}
w^*\text{-str-exp} & \to & w^*\text{-dent} & \to & w^*\text{-pc} & \to & w^*\text{-psc} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{str-exp} & \to & \text{dent} & \to & \text{pc} & \to & \text{psc} \\
 & & \downarrow & & & & \\
 & & \text{str-ext} & \to & \text{ext} & &
\end{array}$$

The dual space $C(K, X)^*$ can be identified with the space of all countably additive, X^*-valued, Borel measures of bounded variation on K equipped with the variation norm, where the action of μ in $C(K, X)^*$ on f in $C(K, X)$ is given by $\langle \mu, f\rangle \;=\; \int_K f\,d\mu$. The space $C(K, X)$ may be considered as a closed subspace of $C(K, (X, w))$. For f in $C(K, (X, w))$, define $\tilde{f} \;:\; K \times B_{X^*} \to \mathbb{R}$ by

$$\tilde{f}(t, x^*) \;=\; \langle x^*, f(t)\rangle \;\text{ for }\; (t, x^*) \;\text{ in } K \times B_{X^*}.$$

Then $\tilde{f}$ is a continuous function on the completely regular topological space $K \times B_{X^*}$; here B_{X^*} is equipped with its norm topology, and

$$\sup\,\{|\tilde{f}(t, x^*)| \;:\; (t, x^*) \in K \times B_{X^*}\} \;=\; \| \, f \, \| \,.$$

Each $\tilde{f}$ has a unique continuous extension to a function $\bar{f}$ defined on the Stone-Čech compactification Y of $K \times B_{X^*}$. Thus $C(K, (X, w))$ may be considered as a closed subspace of $C(Y) = C(Y, \mathbb{R})$ via the mapping $f \longmapsto \bar{f}$. For g in $C(K)$ and x in X, the symbol $g \otimes x$ denotes the mapping from $K \times B_{X^*}$ into $\mathbb{R}$ given by

$$(g \otimes x)(t, x^*) \;=\; g(t)\langle x^*, x\rangle \;\text{ for }\; (t, x^*) \text{ in } K \times B_{X^*}$$

and $\overline{g \otimes x}$ denotes the continuous extension of $g \otimes x$ to Y.

It should be noted that, in the verification of an example or in the proof of a theorem, consecutive claims and displays are numbered with those numbers referring only to the argument within that verification or proof.

THE SPACE $C(K,(X,w))$

The first result in this section gives a sufficient condition and a necessary condition for an element of $C(K,(X,w))$ to be an extreme point of the unit ball of $C(K,(X,w))$. The proof of assertion (i) is straightforward, and assertion (ii) follows from the fact that, for every element f in $C(K,(X,w))$, the set $\{t \in K : f$ is norm continuous at $t\}$ is a dense G_δ subset of K (see the proof of Theorem 5 below).

Proposition 1. *Let f be an element of $C(K,(X,w))$.*

(i) *If the set of all t in K for which $f(t)$ is an extreme point of B_X is dense in K, then f is an extreme point of $B_{C(K,(X,w))}$.*

(ii) *If f is an extreme point of $B_{C(K,(X,w))}$, then $\| f(t) \| = 1$ for all t in a dense G_δ subset of K and $f(t)$ is an extreme point of B_X for all isolated points t in K.*

Recall, from [BLP], that there is a Banach space X with $\dim X = 4$ and an extreme point f of $B_{C([0,1],X)}$ such that there is no t in $[0,1]$ for which $f(t)$ is an extreme point of B_X. The existence of this example shows that, in general, assertion (ii) of Proposition 1 is as strong as possible and that the converse of assertion (i) is certainly not true. However, if X is strictly convex (that is, every element of S_X is an extreme point of B_X), then the extreme points of $B_{C(K,(X,w))}$ can be nicely characterized.

Corollary 2. *If X is a strictly convex Banach space, then f is an extreme point of $B_{C(K,(X,w))}$ if and only if the set of all t in K for which $f(t)$ is an extreme point of B_X is a dense subset of K.*

Recall (see [G] or [DHS]) that if B_X is stable, then a function f in $C(K,X)$ is an extreme point of $B_{C(K,X)}$ if and only if $f(t)$ is an extreme point of B_X for all t in K. The corresponding statement for f in $C(K,(X,w))$ is false, as the following example shows.

Example 3. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the topology inherited from $\mathbb{R}$ and let $X = \ell^2$ with its usual norm. Define $f : K \to X$ by $f(\frac{1}{n}) = e_n$ for each n in $\mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ is the usual unit vector basis for ℓ^2, and $f(0) = 0$. Then f is in $C(K,(X,w))$ and is an extreme point of $B_{C(K,(X,w))}$ by Proposition 1. Yet f is not everywhere extreme point valued, since $f(0) = 0$, even though B_X is stable.

The function f in Example 3 is not particularly surprising; it exists because the set of extreme points of B_{ℓ^2} is not weakly closed. Note that in a Banach space X for which B_X is stable, the set of extreme points of B_X is norm closed. Since B_X is stable whenever X is strictly convex, it is natural to ask whether the statement of Corollary 2 remains true whenever the hypothesis that X is strictly convex is replaced by the hypothesis that B_X is stable. The authors have no counterexample, but there seems to be little immediate evidence to support an affirmative conjecture.

Definition 4. A subset A of B_X is said to be *uniformly strongly extreme* provided for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all x in A, if $\| x \pm z \| \leq 1 + \delta$ for z in X, then $\| z \| \leq \varepsilon$.

The notion of uniformly strongly extreme subset of B_X permits the very natural and complete characterization of the strongly extreme points of $B_{C(K,(X,w))}$ given in the next theorem. From one point of view, the strongly extreme point situation in $C(K,(X,w))$ is quite different from that in $C(K,X)$, but, from another point of view, it is exactly the same. The difference is illustrated in the examples following the theorem, and the sameness is given in the proposition following the examples. Note that Theorem 5 can be read with or without the parenthetical "and G_δ"; thus it is actually the statement of the equivalence of three assertions.

Theorem 5. *An element f of $C(K,(X,w))$ is a strongly extreme point of $B_{C(K,(X,w))}$ if and only if there exists a dense (and G_δ) subset D of K such that $f(D)$ is a uniformly strongly extreme subset of B_X.*

Proof. Let f be in $C(K,(X,w))$ and suppose there exists a dense subset D of K such that $f(D)$ is a uniformly strongly extreme subset of B_X. Since f is weakly continuous, since D is dense and since B_X is weakly closed, it follows that $\| f(t) \| \leq 1$ for all t in K; that is, $\| f \| \leq 1$. Now, let $\varepsilon > 0$ be given and use the hypothesis on $f(D)$ to produce a corresponding $\delta > 0$. If h is in $C(K,(X,w))$ and is such that $\| f \pm h \| \leq 1+\delta$, then

$$\| f(t) \pm h(t) \| \leq 1+\delta \quad \text{for all } t \text{ in } D,$$

and hence $\| h(t) \| \leq \varepsilon$ for all t in D. As above, it follows that $\| h \| \leq \varepsilon$. This shows that f is a strongly extreme point of $B_{C(K,(X,w))}$.

Conversely, suppose f is a strongly extreme point of $B_{C(K,(X,w))}$. Since f is in $C(K,(X,w))$, it follows that $f(K)$ is weakly compact in X and hence $f(K)$ has the point-of-continuity property. For each n in $\mathbb{N}$, let

$$O_n = \{t \in K : \text{ there exists an open set } V \text{ containing } t \text{ such that } \operatorname{diam} f(V) < \tfrac{1}{n}\}.$$

Note that, by definition, each O_n is open in K and

$$\bigcap_{n=1}^{\infty} O_n = \{t \in K : f \text{ is norm continuous at } t\}.$$

Claim 1. The set O_n is dense in K for each n in $\mathbb{N}$.

To establish this claim, let V be a nonempty, open subset of K and fix n in $\mathbb{N}$. Since $f(V) \subseteq f(K)$ and $f(K)$ has the point-of-continuity property, there exists a weakly open, nonempty set U in X with $U \cap f(V)$ nonempty and $\operatorname{diam}(U \cap f(V)) < \frac{1}{n}$. Let $W = f^{-1}(U) \cap V$. Then W is nonempty, open and a subset of $V \cap O_n$. This establishes the claim.

Claim 2. For every $\varepsilon > 0$, there exists $\delta > 0$ such that the set

$$K_{\varepsilon,\delta} = \{t \in K : \| f(t) \pm z \| \leq 1+\delta \text{ implies } \| z \| \leq \varepsilon\}$$

is dense in K.

To establish this claim, suppose otherwise. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ it is the case that $\overline{K}_{\varepsilon,\delta} \neq K$. Let $G_n = K \setminus \overline{K}_{\varepsilon,\frac{1}{n}}$ for each n in $\mathbb{N}$. Then G_n is nonempty and open, and $G_n \supseteq G_{n+1}$ for each n in $\mathbb{N}$. Consider the following two alternatives:

Case 1. There exists m in $\mathbb{N}$ such that $G_n \subseteq G_m \subseteq \overline{G}_n$ for each $n \geq m$.

In this case, note that since, by Claim 1, the set $O_n \cap G_m$ is open and dense in G_m for each n in $\mathbb{N}$, since G_n is open and dense in G_m for each $n \geq m$, and since G_m is an open (and hence Baire) subset of K, it follows that

$$\left(\bigcap_{n=1}^{\infty} O_n\right) \cap \left(\bigcap_{n=m}^{\infty} G_n\right) \cap G_m \quad \text{is dense in } G_m.$$

Let t_0 be a fixed element of this dense subset of G_m. Then f is norm continuous at t_0, since t_0 is in $\bigcap_{n=1}^{\infty} O_n$, and $f(t_0)$ is not a strongly extreme point of B_X, since t_0 is in $\bigcap_{n=m}^{\infty} G_n$. Now, proceed as in the proof of Theorem 2 of [DHS] to produce a sequence $\{h_n\}_{n=1}^{\infty}$ in $C(K,X)$, and hence in $C(K,(X,w))$, such that $\| f \pm h_n \| \leq 1 + \frac{2}{n}$ and $\| h_n \| \geq \varepsilon$ for each n in $\mathbb{N}$. This shows that f is not a strongly extreme point of $B_{C(K,(X,w))}$, contrary to the hypothesis.

Case 2. For every m in $\mathbb{N}$, there exists $n > m$ such that $G_n \subseteq G_m \not\subseteq \overline{G}_n$.

In this case, there exists an increasing sequence $\{m_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$$A_n = G_{m_n} \setminus \overline{G}_{m_{n+1}} \quad \text{is nonempty for each } n \text{ in } \mathbb{N}.$$

Note that $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence of nonempty, open subsets of K. For each n in $\mathbb{N}$, choose t_n in A_n such that f is norm continuous at t_n; this can be done (see Case 1 above) since A_n is open and hence $A_n \cap \left(\bigcap_{i=1}^{\infty} O_i\right)$ is dense in A_n. To show that this circumstance implies that f is not a strongly extreme point of $B_{C(K,(X,w))}$, which is contrary to the hypothesis, let $\eta > 0$ be given. Choose and fix n in $\mathbb{N}$ such that $m_n > \frac{2}{\eta}$. Since f is norm continuous at t_n, there exists an open subset U containing t_n such that

$$U \subseteq A_n \quad \text{and} \quad \operatorname{diam} f(U) < \frac{\eta}{2}.$$

Now, there exists a continuous function $g : K \to [0,1]$ such that $\operatorname{supp}(g) \subseteq U$ and $g(t_n) = 1$. Since t_n is in A_n, which is a subset of G_{m_n}, it follows that there exists z in X such that

$$\| f(t_n) \pm z \| \leq 1 + \frac{1}{m_n} \quad \text{and} \quad \| z \| > \varepsilon.$$

Define $h : K \to X$ by $h(t) = g(t)z$ for t in K. Then h is in $C(K,(X,w))$ and $\| h \| \geq \| h(t_n) \| = \| z \| > \varepsilon$. For t in $K \setminus U$,

$$\| f(t) \pm h(t) \| = \| f(t) \| \leq 1,$$

while, for t in U,

$$\begin{aligned} \| f(t) \pm h(t) \| &\leq \| f(t) - f(t_n) \| + \| f(t_n) \pm g(t)z \| \\ &< \frac{\eta}{2} + 1 + \frac{1}{m_n} \\ &< 1 + \eta. \end{aligned}$$

Thus $\| f \pm h \| \leq 1 + \eta$ and $\| h \| \geq \varepsilon$. This establishes the claim.

For every n in $\mathbb{N}$, by Claim 2, there exists i_n in $\mathbb{N}$ such that $K_{\frac{1}{n},\frac{1}{i_n}}$ is dense in K. For ease of notation, write $K_{(n,i_n)}$ for this dense set.

Claim 3. The set $O_{2i_n} \subseteq K_{(n,2i_n)}$ for each n in $\mathbb{N}$.

To establish this claim, fix n in $\mathbb{N}$ and let t be in O_{2i_n}. Then there exists an open set V containing t such that $\operatorname{diam} f(V) < \frac{1}{2i_n}$. Since $K_{(n,i_n)}$ is dense in K, there exists a point k in $K_{(n,i_n)} \cap V$. Now, if z is in X such that $\| f(t) \pm z \| \leq 1 + \frac{1}{2i_n}$, then

$$\begin{aligned} \| f(k) \pm z \| &\leq \| f(k) - f(t) \| + \| f(t) \pm z \| \\ &< \frac{1}{2i_n} + 1 + \frac{1}{2i_n} \\ &= 1 + \frac{1}{i_n} \end{aligned}$$

and hence $\| z \| \leq \frac{1}{n}$, since k is in $K_{(n,i_n)}$. This shows t is in $K_{(n,2i_n)}$ and the claim is established.

Finally, let $D = \bigcap_{n=1}^{\infty} O_{2i_n}$. Then D is a G_δ subset of K and D is dense in K, since each O_{2i_n} is open and dense in K. Also, by Claim 3,

$$D \subseteq \bigcap_{n=1}^{\infty} K_{(n,2i_n)},$$

which yields that $f(D)$ is a uniformly strongly extreme subset of B_X. This completes the proof of the theorem. □

The next two examples show that each of the following situations can occur:

(i) there exist an infinite compact Hausdorff space K, a real Banach space X and a strongly extreme point f in $B_{C(K,(X,w))}$ such that not every value of f is a strongly extreme point of B_X; and

(ii) there exist an infinite compact Hausdorff space K, a real Banach space X and f in $B_{C(K,(X,w))}$ such that $f(t)$ is a strongly extreme point of B_X for all t in K yet f is not a strongly extreme point of $B_{C(K,(X,w))}$.

These examples in $C(K,(X,w))$ should be compared to the situation in $C(K,X)$; recall, from [DHS], that f in $C(K,X)$ is a strongly extreme point of $B_{C(K,X)}$ if and only if $f(t)$ is a strongly extreme point of B_X for every t in K.

Example 6. (This is really Example 3 revisited.) Let the topological space K, the Banach space X and the function f be those of Example 3. Then f is a strongly extreme point of $B_{C(K,(X,w))}$, by Theorem 5, since $D = \{\frac{1}{n} : n \in \mathbb{N}\}$ is dense in K and $f(D) = \{e_n : n \in \mathbb{N}\}$ is a uniformly strongly extreme subset of B_X (this is true since X is uniformly rotund). Yet f is not everywhere strongly extreme point valued since $f(0) = 0$.

Example 7. For each i in $\mathbb{N}$, let

$$B_i = co\Big\{ \big(\pm 1, 0\big),\ \big(0, \pm 1\big),\ \big(\pm \frac{i}{i+1},\ \pm \frac{i}{i+1}\big) \Big\},$$

the "octagon" ball in $\mathbb{R}^2$, and let $|\cdot|_i$ be the Minkowski functional of B_i. Then let $X_i = \mathbb{R}^3$ with the norm $|||\cdot|||_i$ given by

$$|||\,(\alpha,\beta,\gamma)\,|||_i = (|\alpha|^2 + |(\beta,\gamma)|_i^2)^{\frac{1}{2}},$$

that is, $X_i = \mathbb{R} \oplus_2 (\mathbb{R}^2, |\cdot|_i)$ for each i in $\mathbb{N}$. Finally, let $X = \ell^\infty(X_i)$, the Banach sequence space, equipped with the usual supremum norm. Now, let

$$e_0 = (1,0,0) \text{ and } e_1 = (0,1,0)$$

in X_i and then let

$$x_0 = (e_0, e_0, e_0, \dots)$$

and

$$x_n = (\underbrace{e_0, e_0, \dots, e_0}_{n-1}, e_1, e_0, e_0, \dots)$$

for each n in $\mathbb{N}$. Note $\{x_n\}_{n=1}^{\infty}$ and x_0 are in X and $\| x_n \| = \| x_0 \| = 1$.

Claim 1. Each x_n and x_0 are strongly extreme points of B_X.

To establish this claim, let $|\cdot|_\infty$ denote the supremum norm on $\mathbb{R}^2$ and then let $X_0 = \mathbb{R}^3$ with the norm $|||\cdot|||$ given by

$$|||\,(\alpha,\beta,\gamma)\,||| = (|\alpha|^2 + |(\beta,\gamma)|_\infty^2)^{\frac{1}{2}},$$

that is, $X_0 = \mathbb{R} \oplus_2 (\mathbb{R}^2, |\cdot|_\infty)$. Since $B_{X_i} \subseteq B_{X_0}$ and $B_{(\mathbb{R}^2,|\cdot|_\infty)} \subseteq 2B_{(\mathbb{R}^2,|\cdot|_i)}$ for each i in $\mathbb{N}$, it follows that

$$|||\,z\,||| \leq |||\,z\,|||_i \leq 2|||\,z\,||| \tag{1}$$

for all z in $\mathbb{R}^3$. It is easily seen that e_0 is a strongly extreme point of B_{X_0}; in fact, e_0 is a strongly exposed point of B_{X_0} exposed by the functional $(1,0,0)$ in X_0^*. Using this fact combined with (1), it follows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all i in $\mathbb{N}$,

$$|||\,e_0 \pm z\,|||_i \leq 1+\delta \quad \text{implies} \quad |||\,z\,|||_i \leq 2\varepsilon \tag{2}$$

for all z in $\mathbb{R}^3$. To show x_0 is a strongly extreme point of B_X, let $\varepsilon > 0$ be given and suppose $y = (y(1), y(2), \dots)$ is in X such that $\| x_0 \pm y \| \leq 1+\delta$, where δ is given as in (2). Then $|||\,e_0 + y(i)\,|||_i \leq 1+\delta$ and hence $|||\,y(i)\,|||_i \leq 2\varepsilon$ for each i in $\mathbb{N}$. Thus $\| y \| \leq 2\varepsilon$. This shows x_0 is a strongly extreme point of B_X. Now, fix n in $\mathbb{N}$; to show x_n is a strongly extreme point of B_X, let $\varepsilon > 0$ be given. Consider the unit ball B_{X_n} and note that e_1 is a strongly extreme point of B_{X_n}; in fact, e_1 is a strongly exposed point of B_{X_n} exposed by the functional $(0,1,0)$ in X_n^*. Use this fact in combination with (2) to produce $\delta > 0$ such that

$$|||\,e_1 \pm w\,|||_n \leq 1+\delta \quad \text{implies} \quad |||\,w\,|||_n \leq \varepsilon$$

for all w in $\mathbb{R}^3$ and, for all i in $\mathbb{N}$,

$$|||\,e_0 \pm z\,|||_i \leq 1+\delta \quad \text{implies} \quad |||\,z\,|||_i \leq 2\varepsilon$$

for all z in $\mathbb{R}^3$. Suppose $y = (y(1), y(2), \dots)$ is in X such that $\| x_n \pm y \| \leq 1+\delta$. Then

$$|||\,e_1 \pm y(n)\,|||_n \leq 1+\delta \quad \text{and} \quad |||\,e_0 \pm y(i)\,|||_i \leq 1+\delta$$

for all $i \neq n$ in $\mathbb{N}$, and hence

$$|||\,y(n)\,|||_n \leq \varepsilon \quad \text{and} \quad |||\,y(i)\,|||_i \leq 2\varepsilon$$

for all $i \neq n$ in $\mathbb{N}$. Thus $\| y \| \leq 2\varepsilon$. This shows that x_n is a strongly extreme point of B_X and Claim 1 is established.

Claim 2. The sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_0 in X.

To establish this claim, note that

$$x_n - x_0 \ = \ (\underbrace{0, 0, ..., 0}_{n-1}, e_1 - e_0, 0, 0, ...)$$

for each n in $\mathbb{N}$ and hence the sequence $\{x_n - x_0\}_{n=1}^{\infty}$ is actually in $c_0(X_i)$. Recall that the dual of $c_0(X_i)$ is $\ell^1(X_i^*)$ and so if $x^* = (x^*(1), x^*(2), ...)$ is in $\ell^1(X_i^*)$, then

$$\left| \ \langle x^*, x_n - x_0 \rangle \ \right| \ = \ \left| \ \langle x^*(n), e_1 - e_0 \rangle \ \right| \ \leq \ 2||| \ x^*(n) \ |||_n$$

for each n in $\mathbb{N}$. But $\lim_{n\to\infty} ||| \ x^*(n) \ |||_n = 0$ since $\| \ x^* \ \| = \ \sum_{n=1}^{\infty} ||| \ x^*(n) \ |||_n$. This shows that $\{x_n - x_0\}_{n=1}^{\infty}$ converges weakly to 0 in $c_0(X_i)$ and hence converges weakly to 0 in $X = \ell^{\infty}(X_i)$. Thus Claim 2 is established.

Let $K = \{\frac{1}{n} : \ n \in \mathbb{N}\} \cup \{0\}$ with the topology inherited from $\mathbb{R}$, and then define $f \ : \ K \to X$ by $f(\frac{1}{n}) \ = \ x_n$, for each n in $\mathbb{N}$, and $f(0) = x_0$. Then f is in $C(K, (X, w))$ by Claim 2, and note $\| \ f \ \| = \ 1$. Let $D = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$ and observe that D is the only proper, dense subset of K and that $f(D)$ is not a uniformly strongly extreme subset of B_X; this last observation follows by examining the point e_1 in the unit ball B_{X_i} as i increases. It follows, by Theorem 5, that f is not a strongly extreme point of $B_{C(K,(X,w))}$; yet, by Claim 1, every value of f is a strongly extreme point of B_X. This completes Example 7.

As mentioned above, the last two examples show that the strongly extreme point situation in the unit ball of $C(K, (X, w))$ is quite different from that in the unit ball $C(K, X)$ when viewed from the perspective of an element f being *strongly extreme point valued at every t in K.* However, from the perspective of an element f being *uniformly strongly extreme point valued on a dense subset of K,* the situations in $C(K, (X, w))$ and $C(K, X)$ are exactly the same, as is seen from Theorem 5 and part (iv) of the following proposition (which may be read with or without the parenthetical "and G_δ").

Proposition 8. *Let f be an element of $C(K, X)$. The following assertions are equivalent:*

(i) *f is a strongly extreme point of $B_{C(K,X)}$.*
(ii) *$f(t)$ is a strongly extreme point of B_X for each t in K.*
(iii) *$f(K)$ is a uniformly strongly extreme subset of B_X.*
(iv) *there exists a dense (and G_δ) subset D of K such that $f(D)$ is a uniformly strongly extreme subset of B_X.*

Proof. The equivalence of (i) and (ii) is from [DHS]. Suppose (ii) is true for f in $C(K, X)$. If (iii) fails for f, then, following a line of argument from [DHS], there exist $\varepsilon > 0$ and sequences $\{t_n\}$ in K and $\{z_n\}$ in X such that, for all n in $\mathbb{N}$,

$$\|f(t_n) \pm z_n\| \ \leq \ \ 1 + \frac{1}{n} \ \text{ and } \ \|z_n\| \ > \ \varepsilon.$$

Since K is compact, there exist a subnet $\{t_{n_\lambda}\}$ of $\{t_n\}$ and t_0 in K such that $\lim_\lambda t_{n_\lambda} = t_0$. Since $f(t_0)$ is a strongly extreme point of B_X, there exists n_0 in $\mathbb{N}$ such that $\|f(t_0) \pm z\| \leq 1 + \frac{1}{n_0}$ for z in X; then $\|z\| \leq \varepsilon$. Using the continuity of f at t_0, choose λ_0 such that

$$\|f(t_{n_{\lambda_0}}) - f(t_0)\| < \frac{1}{2n_0} \quad \text{and} \quad n_{\lambda_0} > 2n_0.$$

Then

$$\begin{aligned}\|f(t_0) \pm z_{n_{\lambda_0}}\| &\leq \|f(t_0) - f(t_{n_{\lambda_0}})\| + \|f(t_{n_{\lambda_0}}) \pm z_{n_{\lambda_0}}\| \\ &< \frac{1}{2n_0} + 1 + \frac{1}{n_{\lambda_0}} \\ &< 1 + \frac{1}{n_0},\end{aligned}$$

yet $\|z_{n_{\lambda_0}}\| > \varepsilon$, contrary to the choice of n_0. This shows that (ii) implies (iii). That (iii) implies (iv) is immediate. Finally, if (iv) is true for f in $C(K, X)$, then Theorem 5 yields that f is a strongly extreme point of $B_{C(K,(X,w))}$, and hence, since $C(K, X)$ is a subspace of $C(K,(X,w))$, assertion (i) follows. This completes the proof. □

An immediate corollary of Theorem 5 and Proposition 8 is formally stated below. Although the corresponding statement for extreme points is unlikely to hold, the authors know of no counterexample.

Corollary 9. *If f is a strongly extreme point of the unit ball of $C(K, X)$, then f is also a strongly extreme point of the unit ball of $C(K,(X,w))$.*

As an immediate consequence of the next theorem, it follows that the unit ball of $C(K,(X,w))$ contains neither denting points nor strongly exposed points.

Theorem 10. *The unit ball of $C(K,(X,w))$ contains no point of sequential continuity.*

Proof. Let f be in $B_{C(K,(X,w))}$ and set $D = \{t \in K \ : \ f \text{ is norm continuous at } t\}$. Recall that D is dense in K. Since K is infinite, there exists a pairwise disjoint sequence $\{V_n\}_{n=1}^{\infty}$ of nonempty, open subsets of K. For each n in $\mathbb{N}$, choose a point t_n in $V_n \cap D$ and use the norm continuity of f at t_n to obtain an open subset U_n containing t_n such that $U_n \subseteq V_n$ and $\operatorname{diam} f(U_n) < \frac{1}{6}$. Since $\| f(t_n) \| \leq 1$ for each n in $\mathbb{N}$, choose x_n in X such that $\| x_n \| = \frac{1}{6}$ and $\| f(t_n) + x_n \| \leq \frac{5}{6}$; it follows that, for all t in U_n,

$$\begin{aligned}\| f(t) + x_n \| &\leq \| f(t) - f(t_n) \| + \| f(t_n) + x_n \| \\ &\leq \frac{1}{6} + \frac{5}{6} \\ &= 1.\end{aligned}$$

For each n in $\mathbb{N}$, choose a continuous function $g_n : K \to [0,1]$ such that $\operatorname{supp}(g_n) \subseteq U_n$ and $g_n(t_n) = 1$, and then define $h_n : K \to X$ by $h_n(t) = g_n(t)x_n$ for t in K. Note that h_n is in $C(K, X)$ and $\| h_n \| = \frac{1}{6}$ for each n in $\mathbb{N}$.

Claim. The sequence $\{h_n\}_{n=1}^{\infty}$ converges weakly to 0 in $C(K, X)$.

To establish this claim, let μ be in $C(K,X)^*$. Since $\operatorname{supp}(h_n) \subseteq U_n$, it follows that

$$\begin{aligned} |\ \langle \mu, h_n \rangle\ | &= \left| \int_K h_n \ d\mu \right| \\ &= \left| \int_{U_n} g_n(t) x_n \ d\mu(t) \right| \\ &\leq \int_{U_n} |g_n(t)| \parallel x_n \parallel d|\mu|(t) \\ &\leq |\mu|(U_n) \end{aligned}$$

for each n in $\mathbb{N}$. Since the sequence $\{U_n\}_{n=1}^{\infty}$ is pairwise disjoint, it follows that $|\mu|(U_n) \to 0$. This combined with the last inequality establishes the claim.

Now, for each n in $\mathbb{N}$, let $f_n = f + h_n$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ is in $C(K,(X,w))$. Note that if t is in $K \setminus \operatorname{supp}(g_n)$, then

$$\parallel f_n(t) \parallel \ = \ \parallel f(t) \parallel \ \leq \ 1,$$

and if t is in $\operatorname{supp}(g_n)$, then, since t is in U_n,

$$\begin{aligned} \parallel f_n(t) \parallel &= \parallel f(t) + h_n(t) \parallel \\ &\leq (1 - g_n(t)) \parallel f(t) \parallel \ + \ g_n(t) \parallel f(t) \ + \ x_n \parallel \\ &\leq (1 - g_n(t)) \ + \ g_n(t) \\ &= 1. \end{aligned}$$

Thus $\{f_n\}_{n=1}^{\infty}$ is in $B_{C(K,(X,w))}$. Finally, note that $\{f_n\}_{n=1}^{\infty}$ converges weakly to f by the claim above, but $\{f_n\}$ does not converge in norm to f since $\parallel f_n - f \parallel = \parallel h_n \parallel = \frac{1}{6}$ for each n in $\mathbb{N}$. This shows that f is not a point of sequential continuity of $B_{C(K,(X,w))}$, and the proof is complete. □

The space $C(K,(X,w))^*$

The absence of a concrete representation of the space $C(K,(X,w))^*$ makes an investigation of the extremal structure of its unit ball difficult. From [ADLR], it is known that $C(K,X)^*$ is a complemented subspace of $C(K,(X,w))^*$ for any Banach space X and any compact Hausdorff space K. Indeed,

$$(*) \qquad C(K,(X,w))^* \ = \ C(K,X)^* \ \oplus \ C(K,X)^{\perp}.$$

Thus an extreme point of $B_{C(K,X)^*}$ has a chance to be an extreme point of $B_{C(K,(X,w))^*}$. Recall the classical result, from [S], that an element of $C(K,X)^*$ is an extreme point of $B_{C(K,X)^*}$ if and only if it has the form L_{k,x^*} where k is in K and x^* is an extreme point of B_{X^*}, with the action of this functional on an f in $C(K,X)$ given by

$$\langle L_{k,x^*}, f \rangle \ = \ \langle x^*, f(k) \rangle.$$

So one question is: for which (if any) extreme points x^* of B_{X^*}, are the functionals L_{k,x^*} extreme points of $B_{C(K,(X,w))^*}$? From [ADLR], every extreme point F^* of $B_{C(K,(X,w))^*}$ is a so-called *point functional*, that is, there exists a point y in Y, the Stone-Čech compactification of $K \times B_{X^*}$, such that $F^* = \delta_y|_{C(K,(X,w))}$, where δ_y denotes the functional in $C(Y)^*$ given by point mass measure at y. Here $C(K,(X,w))$ is considered to be a subspace of $C(Y)$ as described in the Definitions and Preliminaries section. Note that if $y = (k, x^*)$ is in $K \times B_{X^*}$, then

$\delta_y|_{C(K,(X,w))} = L_{k,x^*}$. Also, from [ADLR], every point functional is supported at a point in K, that is, for y in Y there is a point k in K, an element x^* in X^* and $F^\perp$ in $C(K,X)^\perp$ such that

$$\delta_y|_{C(K,(X,w))} \quad = \quad L_{k,x^*} \quad + \quad F^\perp \; ;$$

thus every extreme point F^* of $B_{C(K,(X,w))^*}$ can be written in the form $F^* = L_{k,x^*} + F^\perp$. So another question is: are there extreme points of $B_{C(K,(X,w))^*}$ where $F^\perp$ is nonzero?

With regard to the first question, from [C], the functional L_{k,x^*} need not be an extreme point of $B_{C(K,(X,w))^*}$ whenever x^* is an extreme point of B_{X^*}. This is in marked contrast to the situation in $C(K,X)^*$ as given above. However, from [L] as illuminated in [C], if x^* is a weak* strongly exposed point of B_{X^*} and k is in K, then L_{k,x^*} is an extreme point of $B_{C(K,(X,w))^*}$. The results given below, Theorem 14 and Theorem 16, are considerable strengthenings of this last statement. Indeed, if x^* is a weak* strongly exposed point of B_{X^*} and k is in K, then (Theorem 14) L_{k,x^*} is a strongly exposed point of $B_{C(K,(X,w))^*}$. If the hypothesis on x^* is relaxed to that of being a weak* denting point of B_{X^*}, then it follows (Theorem 16) that L_{k,x^*} is a denting point of $B_{C(K,(X,w))^*}$; in particular, L_{k,x^*} is a point of continuity of $B_{C(K,(X,w))^*}$. In both of these results the conclusion for L_{k,x^*} is much stronger than that of just being an extreme point of $B_{C(K,(X,w))^*}$, but the hypothesis on x^* is also much stronger than that of just being an extreme pont of B_{X^*}. There are some situations, as illustrated by Proposition 11 below, in which L_{k,x^*} is an extreme point of $B_{C(K,(X,w))^*}$ whenever x^* is an extreme point of B_{X^*}. The proof of Proposition 11 is straightforward since under its hypothesis the sum in $(*)$ is an ℓ^1-sum and so the classical result from [S], mentioned above, can be applied in the case of an extreme point, a result from [HS] can be applied in the case of a strongly extreme point, and results from [RS] can be applied in the cases of a denting point or a strongly exposed point.

Proposition 11. *Let X be a Banach space such that $C(K,X)$ is an M-ideal in $C(K,(X,w))$. Then L_{k,x^*} is an extreme point (respectively, strongly extreme point, denting point, strongly exposed point) of $B_{C(K,(X,w))^*}$ whenever k is in K and x^* is an extreme point (respectively, strongly extreme point, denting point, strongly exposed point) of B_{X^*}.*

It should be noted, from [ADLR], that $C(K,X)$ is an M-ideal in $C(K,(X,w))$ whenever X has the Schur approximation property. Also, from [ADLR], the space c_0, the space $c_0(\Gamma)$ and c_0-sums of spaces with the Schur property have the Schur approximation property.

With regard to the second question, there are spaces X for which there are extreme points of $B_{C(K,(X,w))^*}$ other than those of the form L_{k,x^*}. For example, let $X = c_0$. Then the sum in $(*)$ is an ℓ^1-sum and so any extreme point of $B_{C(K,X)^\perp}$ will also be an extreme point of $B_{C(K,(X,w))^*}$. That $B_{C(K,X)^\perp}$ has extreme points follows by the Krein-Milman theorem since $B_{C(K,X)^\perp}$ is weak* compact and convex.

Since the next two theorems involve the functional L_{k,x^*} for x^* in B_{X^*} with $\| x^* \| = 1$, in the proofs Y will denote the Stone-Čech compactification of $K \times S_{X^*}$ and, in a manner completely analogous to that given in the Definitions and Preliminaries section (just replace B_{X^*} by S_{X^*}), the space $C(K,(X,w))$ will be considered as a subspace of $C(Y)$. Also, for $\varepsilon > 0$, let

$$S_\varepsilon^+ = \{z^* \in S_{X^*} : \| z^* - x^* \| < \varepsilon\}$$

and

$$S_\varepsilon^- = \{z^* \in S_{X^*} : \| z^* - (-x^*) \| < \varepsilon\};$$

and then let $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$ and $S_\varepsilon^c = S_{X^*} \setminus S_\varepsilon$.

The proofs of Theorem 14 and Theorem 16 require a great deal of work but the rewards are great as well since, for the first time, some strongly exposed points and some denting points of the unit ball of $C(K,(X,w))^*$ are identified without restriction on X or K. The proofs of both theorems will be facilitated by the following two technical results.

Lemma 12. *Let ε and δ be positive numbers each less than $\frac{1}{2}$ and let x in S_X and x^* in S_{X^*} be such that $\langle x^*, x\rangle > 1-\delta$ and $diam S(x, B_{X^*}, \delta) < \varepsilon$. Let k be a point of K, let V be an open subset of K that contains k and let $g : K \to [0,1]$ be a continuous function such that $g(k) = 1$ and $supp(g) \subseteq V$. Let Y be the Stone-Čech compactification of $K \times S_{X^*}$. If μ in $C(Y)^*$ is such that $\| \mu \| \leq 1$ and*

$$\int_Y \overline{g \otimes x}\, d\mu \quad > \quad 1 \; - \; \delta\varepsilon,$$

then

$$\mu^+\left(\overline{V \times S_\varepsilon^+}\right) \quad + \quad \mu^-\left(\overline{V \times S_\varepsilon^-}\right) \quad > \quad 1 \; - \; \varepsilon.$$

Proof. If z^* is in S_ε^+, then $\| z^* - x^* \| < \varepsilon$. This combined with $\langle x^*, x\rangle > 1-\delta$ and $\varepsilon + \delta < 1$ yields that $\langle z^*, x\rangle \geq 0$. Thus $g \otimes x \geq 0$ on $V \times S_\varepsilon^+$, and hence it follows that $\overline{g \otimes x} \geq 0$ on $\overline{V \times S_\varepsilon^+}$, where the closure is in Y. Similarly, it can be shown that $\overline{g \otimes x} \leq 0$ on $\overline{V \times S_\varepsilon^-}$. Thus

$$\int_{\overline{V\times S_\varepsilon^+}} \overline{g \otimes x}\, d\mu^- \quad \geq \quad 0 \quad \text{and} \quad \int_{\overline{V\times S_\varepsilon^-}} \overline{g \otimes x}\, d\mu^+ \quad \leq \quad 0.$$

From these inequalities and the fact that $\| \overline{g \otimes x} \| \leq 1$, it follows that

(1)
$$\begin{aligned}
\int_{\overline{V\times S_\varepsilon}} \overline{g \otimes x}\, d\mu \; &= \; \int_{\overline{V\times S_\varepsilon^+}} \overline{g \otimes x}\, d\mu^+ \; - \; \int_{\overline{V\times S_\varepsilon^+}} \overline{g \otimes x}\, d\mu^- \\
&\qquad + \int_{\overline{V\times S_\varepsilon^-}} \overline{g \otimes x}\, d\mu^+ \; - \; \int_{\overline{V\times S_\varepsilon^-}} \overline{g \otimes x}\, d\mu^- \\
&\leq \; \int_{\overline{V\times S_\varepsilon^+}} \overline{g \otimes x}\, d\mu^+ \; - \; \int_{\overline{V\times S_\varepsilon^-}} \overline{g \otimes x}\, d\mu^- \\
&\leq \; \mu^+\left(\overline{V \times S_\varepsilon^+}\right) \; + \; \mu^-\left(\overline{V \times S_\varepsilon^-}\right).
\end{aligned}$$

Now, for z^* in S_ε^c, it is the case that

$$\| z^* - x^* \| \;\geq\; \varepsilon \quad \text{and} \quad \| z^* - (-x^*) \| \;\geq\; \varepsilon,$$

and hence, since $\text{diam}S(x, B_{X^*}, \delta) < \varepsilon$, it follows that z^* is in neither $S(x, B_{X^*}, \delta)$ nor $S(-x, B_{X^*}, \delta)$. This yields $\langle z^*, x\rangle \leq 1-\delta$ and $\langle z^*, -x\rangle \leq 1-\delta$, that is, $|\langle z^*, x\rangle| \leq 1-\delta$. Thus, for (t, z^*) in $V \times S_\varepsilon^c$,

$$|(g\otimes x)(t,z^*)| \quad \leq \quad |\langle z^*, x\rangle| \quad \leq \quad 1-\delta,$$

and hence $|\overline{g\otimes x}| \leq 1-\delta$ on $\overline{V \times S_\varepsilon^c}$. Also, $\overline{g\otimes x} = 0$ on $\overline{V^c \times S_{X^*}}$ since $\text{supp}(g) \subseteq V$. Now it follows that

$$\int_{Y\setminus \overline{V\times S_\varepsilon}} \overline{g\otimes x}\; d\mu \quad \leq \quad (1-\delta)|\mu|(Y\setminus \overline{V\times S_\varepsilon}). \tag{2}$$

Combining (1) and (2) yields

$$\begin{aligned}
1-\delta\varepsilon \quad &< \quad \int_Y \overline{g\otimes x}\; d\mu \\
&= \quad \int_{\overline{V\times S_\varepsilon}} \overline{g\otimes x}\; d\mu \quad + \quad \int_{Y\setminus\overline{V\times S_\varepsilon}} \overline{g\otimes x}\; d\mu \\
&\leq \quad \mu^+\left(\overline{V\times S_\varepsilon^+}\right) \quad + \quad \mu^-\left(\overline{V\times S_\varepsilon^-}\right) \\
&\qquad\qquad + \quad (1-\delta)|\mu|\left(Y\setminus\overline{V\times S_\varepsilon}\right) \\
&\leq \quad |\mu|(Y) \quad - \quad \delta|\mu|\left(Y\setminus\overline{V\times S_\varepsilon}\right).
\end{aligned} \tag{3}$$

From this and the fact that $\|\mu\| \leq 1$, it follows that $|\mu|(Y\setminus\overline{V\times S_\varepsilon}) < \varepsilon$. Now, using this and the first two inequalities listed in (3) yields

$$\begin{aligned}
\mu^+\left(\overline{V\times S_\varepsilon^+}\right) \quad + \quad \mu^-\left(\overline{V\times S_\varepsilon^-}\right) \quad &\geq \quad 1-\delta\varepsilon \quad - \quad (1-\delta)|\mu|\left(Y\setminus\overline{V\times S_\varepsilon}\right) \\
&> \quad 1-\delta\varepsilon \quad - \quad (1-\delta)\varepsilon \\
&= \quad 1 \quad - \quad \varepsilon.
\end{aligned}$$

This completes the proof. □

Lemma 13. *Let ε and δ be positive numbers each less than $\frac{1}{2}$ and let x in S_X and x^* in S_{X^*} be such that $\langle x^*, x\rangle > 1-\delta\varepsilon$ and $\text{diam}S(x, B_{X^*}, \delta) < \varepsilon$. Let k be in K and let $N(k)$ denote the collection of all open subsets of K which contain the point k. For each V in $N(k)$, choose a continuous function $g_V : K \to [0,1]$ such that $g_V(k) = 1$ and $\text{supp}(g_V) \subseteq V$. Then $\{g_V \otimes x\}_{V\in N(k)}$ forms a net (ordered by inclusion) in $B_{C(K,(X,w))}$ which can be considered as a subset of $B_{C(K,(X,w))^{**}}$. Let T be a weak* cluster point of this net. Then*

(i) *$\langle T, L_{k,x^*}\rangle = \langle x^*, x\rangle > 1-\delta\varepsilon$ and $\|T\| \leq 1$,*
(ii) *if $\langle T, F^*\rangle > 1-\delta\varepsilon$, where F^* is in $B_{C(K,(X,w))^*}$, it follows that $\|F^* - L_{k,x^*}\| \leq 4\varepsilon$.*

Proof. Note that since

$$\langle\, L_{k,x^*}\;,\; g_V\otimes x\,\rangle \quad = \quad \langle\, x^*, x\,\rangle \quad > \quad 1-\delta\varepsilon$$

for every V in $N(k)$, it follows that $\langle\, T, L_{k,x^*}\,\rangle = \langle x^*, x\rangle > 1-\delta\varepsilon$. It is clear that $\|\,T\,\| \leq 1$. Suppose F^* is in $B_{C(K,(X,w))^*}$ such that $\langle\, T, F^*\,\rangle > 1-\delta\varepsilon$. Let Y be the Stone-Čech compactification of $K \times S_{X^*}$ and consider $C(K,(X,w))$ as a subspace of $C(Y)$. Apply the Hahn-Banach theorem to obtain μ in $C(Y)^*$ such that μ extends F^* while $\| \mu \| = \| F^* \|$; thus

$$(1) \qquad \langle F^*, f \rangle = \int_Y \bar{f} \, d\mu \quad \text{for all } f \text{ in } C(K,(X,w)).$$

Claim. For every U in $N(k)$,

$$\mu^+ \left(\overline{U \times S_\varepsilon^+} \right) + \mu^- \left(\overline{U \times S_\varepsilon^-} \right) > 1 - \varepsilon.$$

Since T is a weak* cluster point of $\{g_V \otimes x\}_{V \in N(k)}$ in $C(K,(X,w))^{**}$, from the hypothesis on F^*, there exists V in $N(k)$ such that $V \subset U$ and $\langle F^*, g_V \otimes x \rangle > 1 - \delta\varepsilon$. Hence, by (1),

$$\int_Y \overline{g_V \otimes x} \, d\mu > 1 - \delta\varepsilon.$$

Lemma 12 is applicable, and it yields that

$$\begin{aligned} \mu^+ \left(\overline{U \times S_\varepsilon^+} \right) + \mu^- \left(\overline{U \times S_\varepsilon^-} \right) &\geq \mu^+ \left(\overline{V \times S_\varepsilon^+} \right) + \mu^- \left(\overline{V \times S_\varepsilon^-} \right) \\ &> 1 - \varepsilon. \end{aligned}$$

This establishes the Claim.

Let f be any element of $B_{C(K,(X,w))}$. By the weak continuity of f at k, there exists V in $N(k)$ such that

$$(2) \qquad | \langle x^*, f(t) - f(k) \rangle | < \varepsilon \quad \text{for all } t \text{ in } V.$$

If (t, z^*) is in $V \times S_\varepsilon^+$, then $\| z^* - x^* \| < \varepsilon$ and hence, by (2),

$$\begin{aligned} | \tilde{f}(t,z^*) - \langle x^*, f(k) \rangle | &= | \langle z^*, f(t) \rangle - \langle x^*, f(k) \rangle | \\ &\leq | \langle z^* - x^*, f(t) \rangle | + | \langle x^*, f(t) - f(k) \rangle | \\ &< \varepsilon + \varepsilon. \end{aligned}$$

This yields that

$$(3) \qquad | \bar{f} - \langle x^*, f(k) \rangle | \leq 2\varepsilon \quad \text{on } \overline{V \times S_\varepsilon^+}.$$

Similarly, it can be shown that

$$(4) \qquad | \bar{f} + \langle x^*, f(k) \rangle | \leq 2\varepsilon \quad \text{on } \overline{V \times S_\varepsilon^-}.$$

From the Claim and the fact that $\| \mu \| = |\mu|(Y) \leq 1$, it follows that

$$\mu^+ \left(\overline{V \times S_\varepsilon^-} \right) + \mu^- \left(\overline{V \times S_\varepsilon^+} \right) + |\mu| \left(Y \setminus \overline{V \times S_\varepsilon} \right) < \varepsilon.$$

Combining this with (1), (3), (4), the Claim, and the fact that $\| f \| \leq 1$, it follows that

$$\begin{aligned}
&| \langle F^* - L_{k,x^*}, f \rangle | \\
&\quad = \left| \int_Y \bar{f}\, d\mu - \langle x^*, f(k) \rangle \right| \\
&\quad \leq \left| \int_{\overline{V\times S_\varepsilon^+}} \bar{f}\, d\mu^+ - \int_{\overline{V\times S_\varepsilon^-}} \bar{f}\, d\mu^- - \langle x^*, f(k) \rangle \right| \\
&\qquad + \left| \int_{\overline{V\times S_\varepsilon^-}} \bar{f}\, d\mu^+ - \int_{\overline{V\times S_\varepsilon^+}} \bar{f}\, d\mu^- + \int_{Y\setminus\overline{V\times S_\varepsilon}} \bar{f}\, d\mu \right| \\
&\quad \leq \left| \int_{\overline{V\times S_\varepsilon^+}} \left(\bar{f} - \langle x^*, f(k) \rangle\right) d\mu^+ - \int_{\overline{V\times S_\varepsilon^-}} \left(\bar{f} + \langle x^*, f(k) \rangle\right) d\mu^- \right. \\
&\qquad\qquad \left. - \langle x^*, f(k) \rangle \left[1 - \mu^+\left(\overline{V\times S_\varepsilon^+}\right) - \mu^-\left(\overline{V\times S_\varepsilon^-}\right)\right] \right| \\
&\qquad + \mu^+\left(\overline{V\times S_\varepsilon^-}\right) + \mu^-\left(\overline{V\times S_\varepsilon^+}\right) + |\mu|\left(Y\setminus\overline{V\times S_\varepsilon}\right) \\
&\quad \leq 2\varepsilon\, \mu^+\left(\overline{V\times S_\varepsilon^+}\right) + 2\varepsilon\, \mu^-\left(\overline{V\times S_\varepsilon^-}\right) \\
&\qquad + \left| 1 - \mu^+\left(\overline{V\times S_\varepsilon^+}\right) - \mu^-\left(\overline{V\times S_\varepsilon^-}\right) \right| \\
&\qquad + \varepsilon \\
&\quad < 2\varepsilon\, |\mu|\left(\overline{V\times S_\varepsilon}\right) + \varepsilon + \varepsilon \\
&\quad \leq 4\varepsilon.
\end{aligned}$$

Since f was arbitrary, it follows that $\| F^* - L_{k,x^*} \| \leq 4\varepsilon$. This completes the proof. □

Theorem 14. *If k is in K and x^* is a weak* strongly exposed point of B_{X^*}, then L_{k,x^*} is a strongly exposed point of $B_{C(K,(X,w))^*}$.*

Proof. Let x in S_X be such that x weak* strongly exposes B_{X^*} at x^*. Let $N(k)$ denote the collection of all open subsets of K which contain the point k. For each V in $N(k)$, choose a continuous function $g_V : K \to [0,1]$ such that $g_V(k) = 1$ and $\mathrm{supp}(g_V) \subseteq V$. Then $\{g_V \otimes x\}_{V\in N(k)}$ forms a net (ordered by inclusion) in $B_{C(K,(X,w))}$ which can be considered as a subset of $B_{C(K,(X,w))^{**}}$. Let T be a weak* cluster point of this net. It will be shown that T strongly exposes $B_{C(K,(X,w))^*}$ at L_{k,x^*}. Lemma 13 yields that $\langle T, L_{k,x^*}\rangle = \langle x^*, x\rangle = 1$ and $\|T\| \leq 1$; hence $\| T \| = 1$. Suppose $\{F_n^*\}_{n=1}^\infty$ is a sequence in $B_{C(K,(X,w))^*}$ such that $\lim_{n\to\infty}\langle T, F_n^* \rangle = 1$; the proof will be complete once it is shown that $\lim_{n\to\infty} \| F_n^* - L_{k,x^*} \| = 0$. Let ε be such that $0 < \varepsilon < \frac{1}{2}$. Choose δ such that $0 < \delta < \frac{1}{2}$ and $\mathrm{diam} S(x, B_{X^*}, \delta) < \varepsilon$. Since $\lim_{n\to\infty}\langle T, F_n^*\rangle = 1$, Lemma 13 (ii) applies and yields that $\|F_n^* - L_{k,x^*}\| \leq 4\varepsilon$ for all sufficiently large n in $\mathbb{N}$. This completes the proof. □

The proof of the following corollary is immediate, since its hypothesis implies that every x^* in S_{X^*} is a weak* strongly exposed point of B_{X^*}.

Corollary 15. *If X is a reflexive Banach space with Fréchet differentiable norm (for example, if X is any of the spaces $\ell^p(\Gamma)$ or $L^p(\mu)$ for $1 < p < \infty$), then L_{k,x^*} is a strongly exposed point of $B_{C(K,(X,w))^*}$ whenever k is in K and x^* is in S_{X^*}.*

The next result should be compared with Theorem 14; it applies to a wider variety of extreme points but, as expected, delivers a little less in its conclusion. Nevertheless, it does yield denting points of $B_{C(K,(X,w))^*}$ with no restrictions on K or X.

Theorem 16. *If k is in K and x^* is a weak* denting point of B_{X^*}, then L_{k,x^*} is a denting point of $B_{C(K,(X,w))^*}$.*

Proof. Let k and x^* be as in the hypothesis. Let ε be such that $0 < \varepsilon < \frac{1}{2}$. By Lemma 5 in [HL], there exist x in S_X and $\delta > 0$ such that

$$\langle x^*, x\rangle > 1 - \delta\varepsilon \quad \text{and} \quad \operatorname{diam} S(x, B_{X^*}, \delta) < \varepsilon.$$

Note that $\delta \leq \varepsilon < \frac{1}{2}$. Now, let T be a weak* cluster point of the net considered in the statement of Lemma 13. Let $\delta_1 = \|T\| - (1 - \delta\varepsilon)$. By Lemma 13 (i), it follows that $\delta_1 > 0$ and L_{k,x^*} is in $S(T, B_{C(K,(X,w))^*}, \delta_1)$. Let F^* be an element of $S(T, B_{C(K,(X,w))^*}, \delta_1)$. Then

$$\langle T, F^*\rangle \; > \; \|T\| - \delta_1 \;\; = \;\; 1 - \delta\varepsilon$$

and hence, by Lemma 13 (ii),

$$\|F^* - L_{k,x^*}\| \;\; \leq \;\; 4\varepsilon.$$

Thus

$$\operatorname{diam} S(T, B_{C(K,(X,w))^*}, \delta_1) \leq 8\varepsilon.$$

This shows that L_{k,x^*} is a denting point of $B_{C(K,(X,w))^*}$, and the proof is complete. ☐

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