Math. Proc. Camb. Phil. Soc. (1985), **98**, 305
Printed in Great Britain

On the facial structure of the unit balls in a GL-space and its dual

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(*Received* 14 *July* 1983; *revised* 23 *October* 1984)

1. *Introduction*

In the early sixties Effros [**9**] and Prosser [**14**] studied, in independent work, the duality of the faces of the positive cones in a von Neumann algebra and its predual space. In an implicit way, this work was generalized to certain ordered Banach spaces in papers of Alfsen and Shultz [**3**] in the seventies, the duality being given in terms of faces of the base of the cone in a base norm space and the faces of the positive cone of the dual space. The present paper is concerned with the facial structure of the unit balls in an ordered Banach space and its dual as well as the duality that reigns between these structures. Specifically, the main results concern the sets of norm-exposed and norm-semi-exposed faces of the unit ball V_1 in a GL-space or complete base norm space V and the sets of weak*-exposed and weak*-semi-exposed faces of the unit ball V_1^* in its dual space V^* which forms a unital GM-space or a complete order unit space.

In § 2 the basic tool which is used in this investigation is introduced. This consists of a pair $E \to E'$, $F \to F_{\prime}$ of mappings, the first sending the set of subsets of the unit ball V_1 in the real Banach space V into the set of weak*-closed faces of the unit ball V_1^* in the dual space V^* of V, and the second sending the set of subsets of V_1^* into the set of norm-closed faces of V_1. Many of the proofs use the order properties of these mappings. In addition the basic results concerning GL-spaces and their duals are listed and the properties of the mappings mentioned above are described when V is a GL-space.

The notion of a P-projection on a GL-space was introduced by Alfsen and Shultz [**3**] and they made a study of a class of GL-spaces having a property which ensures the existence of many such P-projections [**5**]. It is for this class of GL-spaces V that the main two theorems apply. The first describes the set of weak*-semi-exposed faces of the unit ball V_1^* in the dual V^* of V and the set of norm-exposed faces of the unit ball V_1 in V. The second main result shows that the existence of a single weak*-exposed point of the unit ball V_1^* is sufficient to ensure that every weak*-semi-exposed face of V_1^* is weak*-exposed. Using this theorem a criterion is found for deciding when a particular weak*-semi-exposed face of V_1^* is weak*-exposed. These results are proved in § 3.

An example of a GL-space is the predual A_* of a JBW-algebra A. The consequences of the results of § 3 for JBW-algebras are examined in § 4. It is shown that these results can be strengthened considerably in this case. Indeed, it is possible to identify both the set of all weak*-closed faces of the unit ball A_1 in A and the set of all norm-closed faces of the unit ball A_{*1} of A_*. The proofs of these results require rather different techniques.

Finally, in § 5 further applications of the theorems of § 3 are considered.

2. *Generalities*

Let V be a real vector space and let C be a convex subset of V. A convex subset E of C is said to be a *face* of C provided that if x is an element of E such that

$$x = tx_1 + (1-t)x_2,$$

where x_1 and x_2 are elements of C and t is a real number in the open unit interval $(0, 1)$, then x_1 and x_2 are elements of E. Both C and the empty subset $\varnothing$ of C are faces of C. A face of C not equal to one of these is said to be a *proper face* of C. An element x of C is said to be an *extreme point* of C if $\{x\}$ is a face of C.

Let τ be a locally convex Hausdorff topology on V. A subset E of C is said to be a *τ-exposed face* of C provided that there exists a τ-continuous linear functional f on V and a real number t such that, for all elements x in $C\backslash E$,

$$f(x) < t$$

and, for all elements x in E,

$$f(x) = t.$$

Let $E_\tau(C)$ denote the set of τ-exposed faces of C. Both C and $\varnothing$ are elements of $E_\tau(C)$ and the intersection of a finite family of elements of $E_\tau(C)$ again lies in $E_\tau(C)$. The intersection of an arbitrary family of elements of $E_\tau(C)$ is said to be a *τ-semi-exposed face* of C. Let $S_\tau(C)$ denote the set of τ-semi-exposed faces of C. Clearly $E_\tau(C)$ is contained in $S_\tau(C)$ and the intersection of an arbitrary family of elements in $S_\tau(C)$ again lies in $S_\tau(C)$. Hence, with respect to the ordering by set inclusion, $S_\tau(C)$ forms a complete lattice.

When V is a real Banach space with dual space V^* the abbreviations n and w^* will be used for the norm topology of V and the weak* topology of V^* respectively. Then, for each subset E of the unit ball V_1 in V, let E' be the subset of the unit ball V_1^* in V^* defined by

$$E' = \{a \in V_1^* : a(x) = 1, \forall x \in E\}. \tag{2·1}$$

Similarly, for each subset F of V_1^*, let $F_\prime$ be the subset of V_1 defined by

$$F_\prime = \{x \in V_1 : a(x) = 1, \forall a \in F\}. \tag{2·2}$$

The properties of the mappings $E \to E'$ and $F \to F_\prime$ are summarized in the following lemma.

LEMMA 2·1. *Let V be a real Banach space with dual space V^*, let V_1 and V_1^* be the unit balls in V and V^* respectively and let the mappings $E \to E'$ and $F \to F_\prime$ be defined by* (2·1) *and* (2·2) *respectively. Then:*

(i) *For subsets D and E of V_1 and F and G of V_1^*,*

$$(-E)' = -E', \quad (-F)_\prime = -F_\prime,$$
$$E \subseteq (E')_\prime, \qquad F \subseteq (F_\prime)'.$$

If D is contained in E then E' is contained in D', and if F is contained in G then $G_\prime$ is contained in $F_\prime$.

(ii) *A subset E of V_1 is a norm-semi-exposed face if and only if*

$$(E')_, = E$$

and a subset F of V_1^* is a weak-semi-exposed face if and only if*

$$(F_,)' = F.$$

(iii) *The mappings $E \to E'$ and $F \to F_,$ are anti-order isomorphisms between the complete lattices $S_n(V_1)$ of norm-semi-exposed faces of V_1 and $S_{w^*}(V_1^*)$ of weak*-semi-exposed faces of V_1^* and are inverses of each other.*

Proof. The proof of this result is straightforward and will be omitted.

Recall that a GL-*space* [**18**] (or *complete base norm space* [**1**]) V is a real Banach space partially ordered by a norm-closed cone V^+ such that the norm is additive on V^+ and the unit ball V_1 in V coincides with the convex hull $\operatorname{conv}((V^+ \cap V_1) \cup ((-V^+) \cap V_1))$ of the set $(V^+ \cap V_1) \cup ((-V^+) \cap V_1))$. Then the set K of elements of V^+ of norm one forms a base for V^+ such that

$$V_1 = \operatorname{conv}(K \cup (-K)). \tag{2·3}$$

A GM-*space* [**18**] A is a real Banach space partially ordered by a norm-closed cone A^+ such that the open unit ball in A is upward filtering and the unit ball A_1 in A coincides with the set $(A_1 + A^+) \cap (A_1 - A^+)$. If A_1 possesses a greatest element e then A is said to be a *unital* GM-*space* (or *complete order unit space* [**1**]). Then

$$A_1 = [-e, e], \tag{2·4}$$

where, for each pair a, b of elements of A, $[a, b]$ denotes the order interval

$$\{c \in A : a \leqslant c,\, c \leqslant b\}.$$

When endowed with the dual cone V^{*+} the dual space V^* of a GL-space V is a unital GM-space, the order unit e being defined, for each element x in V^+, by

$$e(x) = \|x\|.$$

For further information on GL-spaces, and GM-spaces the reader is referred to [**1**], [**2**], [**6**], [**10**] and [**13**].

Lemma 2·2. *Let V be a* GL-*space having unit ball V_1, cone V^+ with base K consisting of elements of norm one and let V^* be the unital* GM-*space which is the dual of V having unit ball V_1^* and order unit e. If the mappings $E \to E'$ and $F \to F_,$ are defined by* (2·1) *and* (2·2) *respectively, then:*

(i) *The set K is a norm-exposed face of V_1 and the set $\{e\}$ is a weak*-semi-exposed face of V_1^* such that*

$$K' = \{e\}, \quad \{e\}_, = K.$$

(ii) *The set $E_n(K)$ of norm-exposed faces of K is contained in the set $E_n(V_1)$ of norm-exposed faces of V_1 and the set $S_n(K)$ of norm-semi-exposed faces of K is contained in the set $S_n(V_1)$ of norm-semi-exposed faces of V_1.*

(iii) *For each face E of V_1, the set* $\operatorname{conv}((E \cap K) \cup (E \cap (-K)))$ *coincides with E and, if E is an element of the complete lattice $S_n(V_1)$, then*

$$E = (E \wedge K) \vee (E \wedge (-K)).$$

(iv) *For each weak*-semi-exposed face F of V_1^* there exist weak*-semi-exposed faces G and H of V_1^*, each containing e, such that F coincides with the set $G \cap (-H)$.*

Proof. (i) This is immediate since K coincides with the set $e^{-1}(\{1\}) \cap V^+$.

(ii) For each element E of $E_n(K)$ different from K, there exists an element a in V^* and a real number t such that, for each element x in $K \backslash E$,

$$a(x) < t$$

and, for each element x in E,

$$a(x) = t.$$

If t is equal to 0 then $-a$ is a non-zero element of V^{*+}. In this case $e + a/\|a\|$ lies in $[0, e]$ and, for each element x in E,

$$(e + (a/\|a\|))(x) = 1,$$

which shows that E is contained in the set $\{e + (a/\|a\|)\}_{,}$. Conversely, if x is an element of $\{e + (a/\|a\|)\}_{,}$, then x is an element of K and

$$e(x) + a(x)/\|a\| = 1.$$

This implies that

$$a(x) = 0$$

and hence that x is an element of E. Therefore, E coincides with the set $\{e + (a/\|a\|)\}_{,}$ and, by Lemma 2·1, E is contained in $E_n(V_1)$. If t is non-zero a similar argument shows that E coincides with the set $\{e + (a - te)/\|a - te\|\}_{,}$. Therefore, $E_n(K)$ is contained in $E_n(V_1)$ and, since every element of $S_n(K)$ is the intersection of a family of elements of $E_n(K)$, it follows that $S_n(K)$ is contained in $S_n(V_1)$.

(iii) For each face E of V_1 the set $\operatorname{conv}((E \cap K) \cup (E \cap (-K)))$ is clearly contained in E and the reverse inclusion follows from (2·3). If, now, E is an element of $S_n(V_1)$ it follows that

$$E = \operatorname{conv}((E \cap K) \cup (E \cap (-K))) \subseteq (E \wedge K) \vee (E \wedge (-K)) \subseteq E$$

as required.

(iv) Using (i) and (iii) it follows from Lemma 2·1 that

$$\begin{aligned} F = (F_{,})' &= (((F_{,}) \wedge K) \vee (F_{,} \wedge (-K)))' \\ &= (F_{,} \wedge K)' \wedge (F_{,} \wedge (-K))' \\ &= (F \vee \{e\}) \wedge (F \vee \{-e\}). \end{aligned}$$

The proof is completed by choosing G and H to be the elements $F \vee \{e\}$ and $-(F \vee \{-e\})$ of $S_{w^*}(V_1^*)$ respectively.

With reference to Lemma 2·2 (iii), notice that it is not necessarily true that, for norm-semi-exposed faces E_1 and E_2 of V_1, the set $\operatorname{conv}(E_1 \cup E_2)$ is a face of V_1. Therefore, in general, $\operatorname{conv}(E_1 \cup E_2)$ does not coincide with $E_1 \vee E_2$.

3. *Main results*

Let V be a GL-space having unit ball V_1, cone V^+ with base K consisting of elements of norm one and let V^* be the unital GM-space which is the dual of V having unit ball V_1^*, cone V^{*+} and order unit e. For a positive linear projection P on V, let $\operatorname{im}^+ P$ and $\ker^+ P$ respectively denote the intersections with V^+ of the range $\operatorname{im} P$ and the kernel $\ker P$ of P. Recall that such a positive projection P of norm one is said to be a

P-projection if there exists a (necessarily unique) positive projection $P^{\#}$ of norm one such that

$$\left.\begin{aligned} \operatorname{im}^+ P = \ker^+ P^{\#}, \quad \operatorname{im}^+ P^* = \ker^+ P^{\#*}, \\ \ker^+ P = \operatorname{im}^+ P^{\#}, \quad \ker^+ P^* = \operatorname{im}^+ P^{\#*}, \end{aligned}\right\} \tag{3·1}$$

where P^* and $P^{\#*}$ respectively denote the adjoint projections on V^* of P and $P^{\#}$. For a pair P, Q of elements of the set $P(V)$ of P-projections on V, write $P \leqslant Q$ when $\operatorname{im} P$ is contained in $\operatorname{im} Q$. This defines a partial ordering on $P(V)$ such that, for all elements P in $P(V)$,

$$0 \leqslant P \leqslant I,$$

where 0 and I denote the zero and identity mappings on V respectively. Moreover, the mapping $P \to P^{\#}$ is an orthocomplementation on $P(V)$ since it enjoys the properties that, for all elements P in $P(V)$,

$$P^{\#\#} = P$$

and I is the supremum of $\{P, P^{\#}\}$ and if P and Q are elements of $P(V)$ with

$$P \leqslant Q$$

then $$Q^{\#} \leqslant P^{\#}.$$

Hence $P(V)$ forms an orthocomplemented partially ordered set.

A face F of K is said to be *projective* if there exists an element P of $P(V)$ such that

$$F = \operatorname{im} P \cap K.$$

The mapping $$P \to F_P = \operatorname{im} P \cap K \tag{3·2}$$

is an order isomorphism from $P(V)$ onto the set $F(K)$ of projective faces of K partially ordered by set inclusion. The mapping $F \to F^{\#}$ defined by

$$(\operatorname{im} P \cap K)^{\#} = \operatorname{im} P^{\#} \cap K$$

is then an orthocomplementation on $F(K)$. A point p of the order interval $[0, e]$ is said to be a *projective unit* if there exists an element P in $P(V)$ such that

$$p = P^*e.$$

The mapping $$P \to p = P^*e \tag{3·3}$$

is an order isomorphism from $P(V)$ onto the set $U(V^*)$ of projective units endowed with the partial ordering inherited from V^*. Moreover, since

$$P^{\#*}e = e - P^*e,$$

the mapping $$p \to e - p$$

is an orthocomplementation on $U(V^*)$.

For details of the results quoted above the reader is referred to [3].

LEMMA 3·1. *Under the conditions of Lemma* 2·2, *suppose that P is a P-projection on V with corresponding projective face F_P of K and corresponding projective unit p defined by* (3·2) *and* (3·3) *respectively. Then*

$$F_P' = [2p - e, e].$$

Proof. Since p lies in the interval $[0, e]$, it follows from (2·4) that $2p-e$ is an element of V_1^*. Let a be an element of $[2p-e, e]$. Then, for each element x in F_P,

$$1 = 2e(Px)-e(x) = (2p-e)(x) \leqslant a(x) \leqslant e(x) = 1$$

and it follows that a is an element of F_P'. Conversely, if a is an element of F_P' then, for each element x in V^+,

$$a(Px) = e(Px)$$

and it follows that $\frac{1}{2}(e-a)$ is an element of the set $\ker^+ P^* \cap [0, e]$. Using (3·1) and [3], proposition 2·11,

$$0 \leqslant \tfrac{1}{2}(e-a) \leqslant P^{\#*}e = e-p,$$

which implies that a is contained in the order interval $[2p-e, e]$. This completes the proof of the lemma.

It is clear that, under the conditions of Lemma 3·1, the set $F(K)$ of projective faces of K is contained in the set $E_n(K)$ of norm-exposed faces of K. In [5] the properties of a GL-space V satisfying the condition that the sets $F(K)$ and $E_n(K)$ coincide were studied. In that case the orthocomplemented partially ordered sets $P(V)$, $F(K)$ and $U(V^*)$ are complete orthomodular lattices and $F(K)$ coincides with the complete lattice $S_n(K)$ of norm-semi-exposed faces of K.

The first main result of the paper follows.

THEOREM 3·2. *Let V be a* GL-*space having unit ball V_1, cone V^+ with base K consisting of elements of norm one and let V^* be the unital* GM-*space which is the dual of V having unit ball V_1^* and order unit e. Suppose that V has the property that every norm-exposed face of K is projective. Then:*

(i) *For each pair p, q of projective units in V^* the order interval $[2p-e, 2q-e]$ is a weak*-semi-exposed face of V_1^*.*

(ii) *Every weak*-semi-exposed face of V_1^* is of the form $[2p-e, 2q-e]$ for projective units p and q.*

(iii) *An element s in V_1^* is a weak*-semi-exposed point of V_1^* if and only if $\frac{1}{2}(e+s)$ is a projective unit.*

Proof. (i) Using the mappings $P \to F_P$ and $P \to p$ defined by (3·2) and (3·3) respectively it follows from Lemma 3·1 that

$$\begin{aligned}[2p-e, 2q-e] &= [2p-e, e] \cap [-e, 2q-e]\\ &= [2p-e, e] \cap -[2(e-q)-e, e]\\ &= (F_P' \wedge -(F_{Q\#}'))\\ &= (F_P \vee (-F_{Q\#}))'\end{aligned}$$

which, by Lemma 2·1, is an element of $S_{w^*}(V_1^*)$.

(ii) Let F be an element of $S_{w^*}(V_1^*)$. Then, by Lemma 2·2 (iv) there exist elements G and H of $S_{w^*}(V_1^*)$ containing e such that F coincides with $G \cap (-H)$. It follows that $G_,$ and $H_,$ are elements of $S_n(V_1)$ contained in K and therefore that there exist P-projections P and Q on V such that $G_,$ and $H_,$ coincide with F_P and $F_{Q\#}$ respectively. Therefore, by Lemma 3·1,

$$G = (G_,)' = F_P' = [2p-e, e]$$

and $$-H = -(H_,)' = -F_{Q\#}' = -[2(e-q)-e, e] = [-e, 2q-e].$$

Finally, $$F = G \cap (-H) = [2p-e, 2q-e],$$
as required.

(iii) This follows immediately from (i) and (ii).

Recall that a pair P, Q of P-projections on V is said to be *orthogonal* if

$$P \leqslant Q^{\#}.$$

Corresponding definitions exist for pairs of projective faces of K and pairs of projective units. The following corollary describes the set of norm-semi-exposed faces of the unit ball V_1 in a GL-space satisfying the conditions of Theorem 3·2.

COROLLARY 3·3. *Let V be the* GL-*space described in Theorem* 3·2. *Then:*

(i) *If P and Q are orthogonal P-projections on V with corresponding projective faces F_P and F_Q respectively then* $\operatorname{conv}(F_P \cup (-F_Q))$ *is a norm-semi-exposed face of V_1.*

(ii) *If E is a norm semi-exposed face of V_1 different from V_1 there exists a pair P, Q of orthogonal P-projections on V such that E coincides with* $\operatorname{conv}(F_P \cup (-F_Q))$.

Proof. (i) Let p and q be the elements of $U(V^*)$ corresponding to P and Q respectively. Then it follows from the proof of Theorem 3·2 (i) that

$$\operatorname{conv}(F_P \cup (-F_Q)) \subseteq F_P \vee (-F_Q) = [2p-e, -(2q-e)]_,.$$

Conversely, if x and y are elements of K and t is a real number in the open unit interval $(0, 1)$ such that $tx-(1-t)y$ lies in $[2p-e, -(2q-e)]_,$, then

$$t(2p-e)(x)-(1-t)(2p-e)(y) = 1$$

and $$t(2q-e)(x)-(1-t)(2q-e)(y) = -1.$$

It follows that

$$p(x) = q(y) = 1, \quad p(y) = q(x) = 0.$$

Therefore, x is an element of F_P and y is an element of F_Q, which implies that $tx-(1-t)y$ lies in $\operatorname{conv}(F_P \cup (-F_Q))$. If x is an element of $K \cap [2p-e, -(2q-e)]$, then

$$p(x) = 1$$

and x lies in F_P. Similarly every element of $-K \cap [2p-e, -(2q-e)]$, lies in $-F_Q$ and it follows that $[2p-e, -(2q-e)]$, is contained in $\operatorname{conv}(F_P \cup (-F_Q))$ as required.

(ii) Since $E \cap K$ and $-(E \cap (-K))$ are norm-semi-exposed faces of K, there exist P-projections P and Q on V such that these faces coincide with F_P and F_Q respectively. Then it follows from Lemma 2·2 (iii) that

$$E = \operatorname{conv}(F_P \cup (-F_Q)) = F_P \vee (-F_Q).$$

Therefore, $$E' = F_P' \cap (-F_Q)' = [2p-e, 2(e-q)-e].$$

Since E is different from V_1, it follows that E' is non-empty and therefore that

$$p \leqslant e-q,$$

which implies that the pair P, Q of P-projections is orthogonal.

Although this result describes the elements of the complete lattice $S_n(V_1)$ of norm-semi-exposed faces of the unit ball V_1 in the GL-space V, the lattice itself possesses some unusual properties one of which is described below.

COROLLARY 3·4. *Under the conditions of Theorem* 3·2, *every non-maximal proper norm-semi-exposed face of* V_1 *is the intersection of two distinct maximal proper norm-semi-exposed faces of* V_1.

Proof. By Theorem 3·2 the set of minimal non-zero elements in the complete lattice $S_{w^*}(V_1^*)$ is the set $\{\{s\}: \frac{1}{2}(e+s) \in U(V^*)\}$. Therefore, by Lemma 2·1, the set of maximal proper elements in $S_n(V_1)$ coincides with the set $\{\{s\}_{,}: \frac{1}{2}(e+s) \in U(V^*)\}$. If E is a non-maximal proper norm-semi-exposed face of V_1 then, by Theorem 3·2, there exist distinct elements s and t of V_1^* such that $\frac{1}{2}(e+s)$ and $\frac{1}{2}(e+t)$ lie in $U(V^*)$ and

$$E' = [s, t].$$

In addition, there exist elements u and v in V_1^* such that $\frac{1}{2}(e+u)$ and $\frac{1}{2}(e+v)$ lie in $U(V^*)$ and

$$[u, v] = \{s\} \vee \{t\} \subseteq [s, t] \subseteq [u, v]$$

since both s and t are elements of $[u, v]$. It follows that u and v coincide with s and t respectively and hence that

$$E = (E')_{,} = \{s\}_{,} \cap \{t\}_{,}$$

as required.

COROLLARY 3·5. *Under the conditions of Theorem* 3·2 *every norm-semi-exposed face of* V_1 *is norm-exposed*.

Proof. This follows immediately from the proof of Corollary 3·4 since the faces $\{s\}_{,}$ and $\{t\}_{,}$ of V_1 are norm-exposed.

Recall that a family $(P_j)_{j \in \Lambda}$ of non-zero P-projections on the GL-space V is said to be *orthogonal* if each pair of elements of the family is orthogonal. Corresponding definitions hold for families of non-empty projective faces and non-zero projective units. The GL-space V is said to be *σ-finite* (or *countably generated*) if every orthogonal family of non-zero P-projections is at most countable.

Suppose now that the GL-space V satisfies the condition that every norm-exposed face of K is projective. For each element x in V^+ define

$$S(x) = \bigwedge\{P \in P(V): Px = x\}.$$

Then $S(x)$ is said to be the *support P-projection* of x. The corresponding projective face face $F_{S(x)}$ of K and projective unit $s(x)$ are said to be the *support projective face* of x and the *support projective unit of* x respectively.

The next theorem is the second main result of the paper.

THEOREM 3·6. *Let V be a* GL-*space having unit ball* V_1, *cone* V^+ *with base K consisting of elements of norm one and let* V^* *be the unital* GM-*space which is the dual of* V, *having unit ball* V_1^* *and order unit e. Suppose that V has the property that every norm-exposed face of K is projective. Then the following conditions are equivalent:*

(i) *Every weak*-semi-exposed face of* V_1^* *is weak*-exposed.*
(ii) *There exists a weak*-exposed point of* V_1^*.
(iii) *The order unit e is a weak*-exposed point of* V_1^*.
(iv) *The* GL-*space V is σ-finite.*

Proof. (i) ⇒ (ii). This is clear since $\{e\}$ is a weak*-semi-exposed face of V_1^*.

(ii) ⇒ (iii). Let s be a weak*-exposed point of V_1^*. Then, as in the proof of Lemma 2·1, using (2·4), there exist elements y and z in K and a real number t in the closed unit interval $[0, 1]$ such that

$$\{ty-(1-t)z\}' = \{s\}.$$

If t is equal to 1 then

$$\{e\} = K' \subseteq \{y\}' = \{s\}$$

and the proof is complete. Similarly, if t is equal to 0 then $-s$, which is also a weak*-exposed point of V_1^*, coincides with e and again the proof is complete. Therefore, suppose that t is contained in the open unit interval $(0, 1)$. For each element a in V_1^*,

$$a(ty+(1-t)(-z)) = 1$$

if and only if

$$a(y) = a(-z) = 1.$$

It follows that

$$\{s\} = \{y\}' \cap \{-z\}'.$$

By Theorem 3·2, there exist weak*-semi-exposed points u and v of V_1^* such that

$$\{y\}' = [u, e], \quad \{z\}' = [-v, e].$$

Therefore,

$$\{s\} = \{y\}' \cap \{-z\}' = [u, v],$$

from which it follows that

$$s = u = v.$$

Replacing z by $-z$ in the above argument, it is clear that

$$\{ty+(1-t)z\}' = \{y\}' \cap \{z\}' = [s, e] \cap [-s, e].$$

By Theorem 3·2 (iii) there exists a projective unit p such that s is equal to $2p-e$. Since the mapping $b \to \frac{1}{2}(e+b)$ is an affine order isomorphism from V_1^* onto $[0, e]$ and the mapping $b \to e-b$ is an affine anti-order automorphism of $[0, e]$, it follows that a is an element of $[s, e] \cap [-s, e]$ if and only if $\frac{1}{2}(e-a)$ is an element of $[0, p] \cap [0, e-p]$, which, by [**3**], proposition 2·11, is the set $\{0\}$. It follows that

$$\{ty+(1-t)z\}' = \{e\},$$

which shows that e is a weak*-exposed point of V_1^*.

(iii) ⇒ (iv). Let x be an element of K such that

$$\{x\}' = \{e\}$$

and let $(p_j)_{j \in \Lambda}$ be an orthogonal family of non-zero projective units. By [**3**], proposition 4·4, the net $(\Sigma_{j \in \Lambda'} p_j)$ where Λ' ranges over all finite subsets of Λ is monotone and bounded by e. Hence, $(\Sigma_{j \in \Lambda'} p_j(x))$ is a bounded monotone increasing real net, which therefore converges. It follows that there exists a countable subset Λ_0 of Λ such that, for all elements j of $\Lambda \backslash \Lambda_0$,

$$p_j(x) = 0.$$

Hence, for each element j in $\Lambda \backslash \Lambda_0$, the element $e-p_j$ lies in the set $\{x\}'$ and it follows that p_j is equal to 0. Since, for every element j of Λ, p_j is non-zero, the set Λ_0 coincides with the set Λ, which is therefore countable.

(iv) ⇒ (i). Let P be a non-zero P-projection having corresponding projective face F_P and projective unit p. Let $(x_j)_{j \in \Lambda}$ be a family of elements of F_P, the family $(S(x_j))_{j \in \Lambda}$

of supports of which forms a maximal orthogonal family of P-projections on V. The set Λ is at most countable and, in the complete lattice $P(V)$,

$$R = \bigvee_{j \in \Lambda} S(x_j) \leqslant P.$$

If y is an element of the projective face $F_P \wedge F_{R^\#}$ then it follows that

$$S(y) \leqslant P \wedge R^\#,$$

which contradicts the maximality of $(S(x_j))_{j \in \Lambda}$. Therefore, using the orthomodularity of $P(V)$, it follows that P and R coincide. If Λ is infinite, identifying Λ with the set of natural numbers, let x_P be the norm limit of the increasing sequence $(\Sigma_{j=1}^{n} 2^{-j} x_j)$ of elements of V^+. Then clearly x_P is an element of F_P and so

$$F'_P \subseteq \{x_P\}'.$$

Conversely, if a is an element of $\{x_P\}'$, then

$$a(x_j) = 1, \quad j = 1, 2, \ldots,$$

and it follows that x_j is an element of the norm-exposed face $\{a\}_, \cap K$ of K. Since every norm-exposed face of K is projective,

$$F_{S(x_j)} \subseteq \{a\}_, \cap K, \quad j = 1, 2, \ldots,$$

and hence, in the complete lattice $F(K)$,

$$F_P = \bigvee_{j=1}^{\infty} F_{S(x_j)} \subseteq \{a\}_, \cap K.$$

Therefore, $$\{a\} \subseteq (\{a\}_,)' \vee \{e\} = (\{a\}_, \cap K)' \subseteq F'_P.$$

Hence the set $\{x_P\}'$ coincides with F'_P which, by Lemma 3·1, is the order interval $[2p-e, e]$. If Λ is finite and identified with the set $\{1, 2, \ldots, n\}$ then the same result can be obtained by choosing

$$x_P = n^{-1} \sum_{j=1}^{n} x_j.$$

If F is a weak*-semi-exposed face of V_1^* then, by Theorem 3·2, there exist P-projections P and Q on V with corresponding projective units p and q respectively, such that F coincides with the order interval $[2p-e, 2q-e]$. Let x_P and $x_{Q^\#}$ be the elements of K constructed as above. Then

$$F = [2p-e, 2q-e] = [2p-e, e] \cap -[2(e-q)-e, e] = \{x_P\}' \cap -\{x_{Q^\#}\}'$$

which, being the intersection of two weak*-exposed faces of V_1^*, is itself weak*-exposed.

Let V be a GL-space having unit ball V_1 and cone V^+ with base K consisting of elements of norm one. If P is a P-projection on V then, with respect to the norm inherited from V and the cone $\mathrm{im}^+ P$, the Banach space $\mathrm{im}\, P$ is a GL-space. Moreover, the base of $\mathrm{im}^+ P$ consisting of elements of norm one is the projective face F_P corresponding to P. If the GL-space V has the property that every norm-exposed face of K is projective then, by [**5**], proposition 1·10, the same holds for the GL-space $\mathrm{im}\, P$.

Corollary 3·7. *Under the conditions of Theorem* 3·2, *let F be a non-empty weak*-semi-exposed face of V_1^* and let P and Q be the P-projections on V, with corresponding projective units p and q respectively, such that F coincides with the order interval* $[2p-e, 2q-e]$. *Then F is weak*-exposed if and only if the* GL-*spaces* $\mathrm{im}\, P$ *and* $\mathrm{im}\, Q^\#$ *are σ-finite.*

Proof. Suppose that $\operatorname{im} P$ and $\operatorname{im} Q^{\#}$ are σ-finite. As in the proof of Theorem 3·6, there exist elements x_P and $x_{Q\#}$ in K such that F'_P and $F'_{Q\#}$ coincide with $\{x_P\}'$ and $\{x_{Q\#}\}'$ respectively. Continuing as in that proof, the set F is the intersection of the weak*-exposed faces $\{x_P\}'$ and $-\{x_{Q\#}\}'$ of V_1^* and hence is itself weak*-exposed.

Conversely, if F is weak*-exposed then there exist elements y and z in K and a real number t in the closed unit interval $[0, 1]$ such that

$$\{ty-(1-t)z\}' = [2p-e, 2q-e].$$

If t is equal to 1 then

$$\{e\} = K' \subseteq \{y\}' = [2p-e, 2q-e],$$

and hence q is equal to e. Therefore,

$$\{y\} \subseteq (\{y\}')_, = [2p-e, e]_, = F_P.$$

By [3], proposition 2·11, the unit ball $(\operatorname{im} P)_1^*$ in the dual space $(\operatorname{im} P)^*$ of the GL-space $\operatorname{im} P$ may be identified with the order interval $[-p, p]$. Therefore,

$$\{y\}' \cap [-p, p] = [2p-e, e] \cap [-p, p] = \{p\}.$$

Hence the unit ball $(\operatorname{im} P)_1^*$ possesses a weak*-exposed point and, by Theorem 3·6, the GL-space $\operatorname{im} P$ is σ-finite. If t is equal to 0 then

$$\{e\} = K' \subseteq \{z\}' = -\{-z\}' = [-2q+e, -2p+e],$$

and it follows that p is equal to 0. Therefore,

$$\{z\} \subseteq (\{z\}')_, = [2(e-q)-e, e]_, = F_{Q\#}$$

and a similar argument to that above shows that the GL-space $\operatorname{im} Q^{\#}$ is σ-finite. Finally, suppose that t is an element of the open unit interval $(0, 1)$ and let a be an element of V_1^*. Then

$$a(ty+(1-t)(-z)) = 1$$

if and only if

$$a(y) = a(-z) = 1.$$

It follows that

$$\{y\}' \cap \{-z\}' = [2p-e, 2q-e].$$

By Theorem 3·2, there exist projective units l and m such that

$$[2p-e, 2q-e] = [2l-e, e] \cap [-e, 2m-e] = [2l-e, 2m-e],$$

which implies that l and m coincide with p and q respectively. Therefore,

$$\{y\} \subseteq (\{y\}')_, = [2p-e, e]_, = F_P$$

and, as in the case when t is equal to 1, the GL-space $\operatorname{im} P$ is σ-finite. Similarly, z is an element of $F_{Q\#}$ and, as in the case when t is equal to 0, the GL-space $\operatorname{im} Q^{\#}$ is σ-finite.

COROLLARY 3·8. *Let the* GL-*space V described in Theorem* 3·2 *be separable. Then the equivalent conditions* (i)–(iv) *of Theorem* 3·6 *hold.*

Proof. If V is separable then the weak* topology of the unit ball V_1^* in V^* is metrizable. Therefore, every closed subset of the weak* compact set V_1^* is a G_δ. By [1], proposition II.5·16, every weak*-semi-exposed face of V_1^* is weak*-exposed and the result follows.

4. *Applications to* JBW-*algebras*

A real Jordan algebra A which is also the dual of a real Banach space A_* with the property that the dual norm on A satisfies the conditions that, for all elements a and b in A,

$$\|a^2\| = \|a\|^2$$

and

$$\|a^2 - b^2\| \leqslant \max\{\|a^2\|, \|b^2\|\}$$

is said to be a JBW-*algebra*. Examples of JBW-algebras are all formally real finite-dimensional Jordan algebras [**12**], the self-adjoint parts of W^*-algebras and the weakly closed Jordan subalgebras of the Jordan algebra of bounded self-adjoint operators on complex Hilbert spaces, or JW-algebras [**17**].

The set A^+ consisting of squares of elements of A forms a weak*-closed cone in A and, with respect to the associated partial ordering, A is monotone complete. It follows that A possesses a multiplicative unit e. An element p in A is called an *idempotent* if

$$p^2 = p.$$

A pair of idempotents p, q is said to be *orthogonal* if

$$p \circ q = 0.$$

An element s in A is said to be a *symmetry* if

$$s^2 = e.$$

The set of symmetries in A coincides with the set $\{2p - e : p \text{ an idempotent}\}$. Also notice that the symmetries are precisely the extreme points of the unit ball A_1 in A.

Let A_*^+ denote the norm-closed cone in A_* which is predual to the cone A^+ and let K denote the set of elements of A_*^+ of norm one. The elements of K are said to be *normal states* of A. The set K is a base for the cone A_*^+ and the unit ball A_{*1} in A_* coincides with the convex hull $\operatorname{conv}(K \cup -K)$ of the set $K \cup -K$. Therefore A_* is a GL-space with respect to the cone A_*^+, and it follows that A together with its cone A^+ forms a unital GM-space, the order unit in A being its multiplicative unit e.

For each element a in A, the weak*-continuous linear mappings L_a and U_a on A are defined, for each element b in A, by

$$L_a b = a \circ b,$$

$$U_a b = \{aba\}$$

where, for elements a, b and c in A, the Jordan triple product $\{abc\}$ is defined by

$$\{abc\} = a \circ (b \circ c) - b \circ (c \circ a) + c \circ (a \circ b).$$

The mapping U_a^* on A_* is defined by

$$b(U_a^* x) = U_a b(x)$$

for all elements b in A and x in A_*.

The P-projections on A_* are precisely the mappings U_p^* for p an idempotent element in A. The set $U(A)$ of projective units coincides with the set of idempotents in A.

Moreover, each norm-exposed face of the base K in A_* is projective [5]. Notice that a pair p, q of idempotents in A is orthogonal if and only if the P-projections U_p^*, U_q^* form an orthogonal pair.

For each element a in A, the support $r(a)$ of a is defined by

$$r(a) = \bigwedge\{p \in U(A): U_p a = a\}.$$

Notice that $r(a)$ is the unit element in the smallest JBW-subalgebra $M(A)$ of A containing a. Hence, when a is an element of the order interval $[0, e]$, the support $r(a)$ is the least upper bound of the increasing sequence $(e-(e-a)^n)$ in $M(a)$.

A pair y, z of elements of A_*^+ is said to be *orthogonal* if their support projective units $s(y)$, $s(z)$ form an orthogonal pair of idempotents. Each element x in A_* has a unique decomposition

$$x = y - z,$$

where y and z are elements of A_*^+ such that

$$\|x\| = \|y\| + \|z\|.$$

It follows that the pair y, z is orthogonal. The decomposition above is said to be the *orthogonal decomposition of* x.

For details of these and other properties of JBW-algebras the reader is referred to [**3**], [**4**], [**7**], [**8**] and [**16**].

It is now possible to prove the first main result of this paragraph.

THEOREM 4·1. *Let A be a* JBW*-algebra with unit ball A_1 and unit e. Then*

(i) *For each pair s, t of symmetries in A, the order interval $[s, t]$ is a weak*-closed face of A_1.*

(ii) *For each weak*-closed face F of A_1 there exists a pair s, t of symmetries in A such that F coincides with $[s, t]$.*

Proof. (i) This is an immediate consequence of Theorem 3·2 (i).

(ii) Since the mapping $a \to 2a-e$ is a weak*-homeomorphic affine order isomorphism from the order interval $[0, e]$ onto A_1, it is sufficient to show that every weak*-closed face of $[0, e]$ is of the form $[p, q]$ for idempotents p and q in A.

Suppose that F is a non-empty weak*-closed face of $[0, e]$. Let a be an element of F and let face (a) denote the smallest face of $[0, e]$ containing a. Let the sequence (a_n) of elements of A be defined by

$$a_n = e-(e-a)^n.$$

Then (a_n) is a monotone increasing sequence in the smallest JBW-subalgebra $M(a)$ of A containing a and has the least upper bound $r(a)$. Moreover, using spectral theory, it follows that for each integer n greater than one

$$\|a_n\| \leqslant 1,$$

$$0 \leqslant a_n \leqslant na$$

and

$$\|na - a_n\| \leqslant n-1.$$

Moreover,

$$a = (1/n)\,a_n + ((n-1)/n)\,((na-a_n)/(n-1)).$$

Hence the sequence (a_n) is contained in face (a). It follows that the support $r(a)$ of a is contained in F and, by (i), is an upper bound for face (a).

Notice that, since, for each pair a, b of elements of F, the faces face (a) and face (b) are contained in face $((a+b)/2)$, the support $r((a+b)/2)$ of $(a+b)/2$ majorizes both a and b. Therefore the face F is directed and, being weak*-compact, has a largest element q which is an idempotent since its support belongs to F. Since the mapping $c \to e-c$ is a weak*-homeomorphic affine anti-order-automorphism of $[0, e]$, we conclude that F contains a least element p, an idempotent. Hence F is contained in the order interval $[p, q]$.

Let a be an element of $[p, q]$. Then the element $p+q-a$ belongs to $[p, q]$ and

$$a/2+(p+q-a)/2 = (p+q)/2.$$

Therefore, a is an element of F.

Theorem 3·2 leads immediately to the following corollary.

COROLLARY 4·2. *Every weak*-closed face of the unit ball A_1 in a* JBW*-algebra A is weak*-semi-exposed.*

LEMMA 4·3. *Let A be a* JBW*-algebra with predual A_*, let A_*^+ be the cone in the* GL-*space A_*. For every non-empty norm-closed face G of A_*^+ there exists an idempotent p in A such that G coincides with $U_p^* A_*^+$.*

Proof. For each element x in A_*^+, let face(x) denote the smallest face of A_*^+ containing x and let $\overline{\text{face}\,(x)}$ denote its norm closure. Then, by [**11**], appendix 2, lemma 9, the set $\overline{\text{face}\,(x)}$ coincides with the face $U_{s(x)}^* A_*^+$. When ordered by set inclusion, the set $\{\text{face}(x)\colon x \in G\}$ is directed, and if x and y are elements of G such that face(x) is contained in face(y) then

$$s(x) \leqslant s(y).$$

Moreover, if face(x) and face(y) coincide so also do $s(x)$ and $s(y)$. Therefore, $(s(x)\colon \text{face}\,(x) \subseteq G)$ forms an increasing net in $U(A)$, which therefore converges in the weak*-topology to its least upper bound p. Since, for each element a in A, the linear operator L_a on A is weak*-continuous, for each element y in A_* the net $(U_{s(x)}^* y\colon \text{face}\,(x) \subseteq G)$ converges weakly to the element $U_p^* y$ in A_*. In particular, if y is an element of the norm-closed face $U_p^* A_*^+$ of A_*^+, it follows that y is contained in the weak closure $\bar{G}_0^w$ of the set G_0 defined by

$$G_0 = \bigcup \{U_{s(x)}^* A_*^+\colon x \in G\} = \bigcup \{\overline{\text{face}\,(x)}\colon x \in G\}.$$

Notice that if Y is an element of G_0 then, for some element x in G, y is contained in the set $\overline{\text{face}\,(x)}$ and hence in G. Clearly, G is contained in G_0 and therefore the two sets coincide. Hence

$$U_p^* A_*^+ \subseteq \bar{G}_0^w = \bar{G}^w = G,$$

since the norm and weak closure of convex subsets of A_* coincide. However, if x is an element of G then

$$s(x) \leqslant p,$$

and hence

$$U_p^* x = x,$$

which implies that x lies in $U_p^* A_*^+$. This completes the proof of the lemma.

This result was proved for von Neumann algebras by Effros[**9**] and Prosser[**14**]. Their proofs, however, do not easily generalize to this context. The second main result of this paragraph depends heavily upon the lemma above.

THEOREM 4·4. *Let A be a* JBW*-algebra with predual A_*, let A_{*1} be the unit ball in A_* and let K be the set of normal states on A. Then*

(i) *For each pair p, q of orthogonal idempotents in A, the set*

$$\mathrm{conv}\,((U_p^* A_* \cap K) \cup (U_q^* A_* \cap -K))$$

is a norm-closed face of A_{*1}.

(ii) *For each norm-closed face E of A_{*1} different from A_{*1}, there exists a pair p, q of orthogonal idempotents in A such that E coincides with the set*

$$\mathrm{conv}\,((U_p^* A_* \cap K) \cup (U_p^* A_* \cap -K)).$$

Proof. (i) This follows from Corollary 3·3 (i).

(ii) By Lemma 2·2 (iii), the face E of A_{*1} coincides with the set

$$\mathrm{conv}\,((E \cap K) \cup (E \cap -K)).$$

Notice that the mapping $G \to G \cap K$ is an order isomorphism from the complete lattice of non-empty norm-closed faces of A_*^+ onto the complete lattice of norm-closed faces of K. Since $E \cap K$ and $-(E \cap -K)$ are norm-closed faces of K it follows from Lemma 4·3 that there exist idempotents p and q in A such that E coincides with the set $\mathrm{conv}\,((U_p^* A_* \cap K) \cup (U_q^* A_* \cap -K))$. If $E \cap K$ or $E \cap -K$ is empty then clearly p and q are orthogonal. Otherwise, let y and z be elements of $U_p^* A_* \cap K$ and $U_q^* A_* \cap K$ respectively and let t be a real number in the open interval $(0, 1)$. Then, the element x defined by

$$x = ty - (1-t)z$$

is an element of E. Since E is a face of A_{*1} different from A_{*1} itself all elements of E are of norm one. Therefore

$$1 = \|x\| \leqslant t\|y\| + (1-t)\|z\| = 1$$

and it follows that the decomposition of x above is precisely its orthogonal decomposition. Therefore, the support projective units $s(y), s(z)$ form an orthogonal pair of idempotents. However, from the proof of Lemma 4·3 it can be seen that p and q are the weak*-limits of the increasing nets

$$(s(y)\colon \mathrm{face}\,(y) \subseteq U_p^* A^* \cap K) \quad \text{and} \quad (s(z)\colon \mathrm{face}\,(z) \subseteq U_q^* A_* \cap K)$$

respectively. Since multiplication by elements of A is weak*-continuous on A it follows that p, q is an orthogonal pair of idempotents as required.

The following result is an immediate consequence of Theorem 4·4, Corollaries 3·3 and 3·5.

COROLLARY 4·5. *Every norm-closed face of the unit ball A_{*1} in the predual of a* JBW-*algebra A is norm-exposed.*

The JBW-algebra A is said to be *σ-finite* if every family $(p_j)_{j \in \Lambda}$ of non-zero pairwise orthogonal idempotents in A is at most countable. Clearly, a JBW-algebra A is σ-finite if and only if the GL-space A_* is σ-finite. A normal state x of A is said to be *faithful* if its support projective unit $s(x)$ coincides with the unit e in A. Notice that a normal state x is faithful if and only if the set of elements a in A^+ on which x vanishes is $\{0\}$.

THEOREM 4·6. *Let A be a* JBW*-algebra with unit ball A_1. Then the following conditions are equivalent.*

(i) *Every weak*-closed face of A_1 is weak*-exposed.*
(ii) *There exists a weak*-exposed point in A_1.*
(iii) *The unit element e in A is a weak*-exposed point of A_1.*
(iv) *There exists a faithful normal state on A.*
(v) *The* JBW*-algebra A is σ-finite.*

Proof. All except the equivalence of condition (iv) with the other conditions follow from Theorems 4·1 and 3·6. Suppose that (iv) holds and that x is a faithful normal state of A. If a is an element of A_1 such that

$$a(x) = 1,$$

then $e-a$ is an element of A^+ on which x vanishes and it follows that a is equal to e. Lemma 2·1 now shows that (iii) holds.

Conversely, suppose that (iii) holds and, again using Lemma 2·1, let x be an element of A_{*1} such that the set $\{a \in A_1 : a(x) = 1\}$ coincides with $\{e\}$. There exist normal states y and z of A and a real number t in the closed interval $[0, 1]$ such that

$$x = ty-(1-t)z.$$

Therefore,
$$1 = e(x) = te(x)-(1-t)e(z) = 2t-1,$$

and it follows that x and y coincide. Hence x is a normal state of A; a reversal of the argument above shows that x is faithful.

The next result follows from Theorem 4·1 and Corollary 3·7.

THEOREM 4·7. *Let A be a* JBW*-algebra with unit ball A_1 and unit e. Let p and q be idempotents in A with p majorized by q. Then the weak*-closed face $[2p-e, 2q-e]$ of A_1 is weak*-exposed if and only if the* JBW*-algebras $U_p A$ and $U_{e-q} A$ are σ-finite.*

5. *Further examples*

(1) *Finite-dimensional spaces*

Let V be a finite-dimensional GL-space which satisfies the condition that every norm-exposed face of the set K of positive elements of V of norm one is projective. Then, by [5], proposition 2·5, and [3], proposition 8·7, it follows that the set $U(V^*)$ of projective units in the dual space V^* of V coincides with the set of extreme points of the order interval $[0, e]$ in V^*. Moreover, in finite-dimensional spaces the notions of exposure and semi-exposure coincide. Consequently the only result of interest in this example is Theorem 3·2, which leads to the following

THEOREM 5·1. *Let V be a finite-dimensional* GL*-space with dual unital* GM*-space V^*. Let V_1^* be the unit ball in V^* and suppose that the set K of positive elements of V of norm one has the property that every norm-exposed face of K is projective. Then:*

(i) *For each pair s, t of extreme points of V_1^*, the order interval $[s, t]$ is a weak-exposed face of V_1^*.*

(ii) *Every weak-exposed face of V_1^* is of the form $[s, t]$ for extreme points s and t or V_1^*.*

(2) GL-*spaces with smooth strictly convex bases*

Let V be a GL-space with dual GM-space V^* and let K be the set ot positive elements of V of norm one. Recall that K is said to be *strictly convex* if every proper face of K is of the form $\{x\}$ for some point x of K and is said to be V^*-*smooth* if for every extreme point x of K there exists a unique element u_x in V^* of norm one such that

$$u_x(x) = 1.$$

THEOREM 5·2. *Let V be a* GL-*space with dual V^* and let V_1^* be the unit ball in V^*. Suppose that the set K of positive elements of norm one in V is strictly convex, V^*-smooth and weakly compact. Then:*

(i) *Every weak-semi-exposed face of V_1^* is weak-exposed.*

(ii) *Every extreme point of V_1^* is weak-exposed.*

(iii) *Every proper non-singleton weak-exposed face of V_1^* is one-dimensional and contains either $+e$ or $-e$.*

Proof. It follows from [**3**], theorem 10·5, that every norm-exposed face of K is projective and that every face of K is norm-exposed. Moreover, an orthogonal family of non-empty projective faces has at most two elements. Therefore, V is σ-finite and (i) follows from Theorem 3·6. Furthermore, the set of projective units in V^* coincides with the set of extreme points of the order interval $[0, e]$. Hence (ii) holds true.

By (i), Theorem 3·2 (ii) and the orthomodularity of $U(V^*)$, it follows that a proper non-singleton weak-exposed face of V_1^* is of the form $\pm[2p-e, e]$ for some projective unit p different from e and 0. It follows from (ii), Theorem 3·2 (iii) and again the orthomodularity of $U(V^*)$ that the elements $2p-e$ and e are the only extreme points of the face $[2p-e, e]$. This proves (iii).

This research was supported by Schweizerischer Nationalfonds/Fonds national suisse under grant numbers 2.021-0.81 and 2.478-0.82.

REFERENCES

[**1**] E. M. ALFSEN. *Compact Convex Sets and Boundary Integrals* (Springer-Verlag, 1971).

[**2**] E. M. ALFSEN and E. G. EFFROS. Structure in real Banach spaces II. *Ann. Math.* **96** (1972), 129–173.

[**3**] E. M. ALFSEN and F. W. SHULTZ. Non-commutative spectral theory for affine function spaces on convex sets. *Mem. Amer. Math. Soc.* **172** (1976).

[**4**] E. M. ALFSEN, F. W. SHULTZ and E. STORMER. A Gelfand–Naimark theorem for Jordan algebras. *Adv. in Math.* **28** (1978), 11–56.

[**5**] E. M. ALFSEN and F. W. SHULTZ. On non-commutative spectral theory and Jordan algebras. *Proc. London Math. Soc.* **38** (1979), 497–516.

[**6**] L. ASIMOV and A. J. ELLIS. *Convexity Theory and its Applications in Functional Analysis* (Academic Press, 1980).

[**7**] C. M. EDWARDS. Ideal theory in JB-algebras. *J. London Math. Soc.* **16** (1977), 507–513.

[**8**] C. M. EDWARDS. On the facial structure of a JB-algebra. *J. London Math. Soc.* **19** (1979), 335–344.

[**9**] E. G. EFFROS. Order ideals in a C^*-algebra and its dual. *Duke Math. J.* **30** (1963), 391–411.

[**10**] A. J. ELLIS. Linear operators in partially ordered normed vector spaces. *J. London Math. Soc.* **41** (1966), 323–332.

[**11**] B. IOCHUM. *Cônes autopolaires et algèbres de Jordan.* Lecture Notes in Math. vol. 1049 (Springer-Verlag, 1984).

[**12**] P. JORDAN, J. VON NEUMANN and E. WIGNER. On an algebraic generalization of the quantum mechanical formalism. *Ann. Math.* **35** (1934), 29–64.

[13] K.-F. NG. The duality of partially ordered Banach spaces. *Proc. London Math. Soc.* **19**, (1969), 269–288.
[14] R. T. PROSSER. On the ideal structure of operator algebras. *Mem. Amer. Math. Soc.* **45** (1963).
[15] S. SAKAI. *C*-Algebras and W*-Algebras* (Springer-Verlag, 1971).
[16] F. W. SHULTZ. On normed Jordan algebras which are Banach dual spaces. *J. Funct. Anal.* **31** (1979), 360–376.
[17] D. M. TOPPING. Jordan algebras of self-adjoint operators. *Mem. Amer. Math. Soc.* **53** (1965).
[18] W. WILS. The ideal center of partially ordered vector spaces. *Acta Math.* **127** (1971), 41–77.