KUL-TF-04/29
SU-ITP-04/034
hep-th/0410126

On the fakeness of fake supergravity

Alessio Celi[1], Anna Ceresole[2], Gianguido Dall'Agata[3], Antoine Van Proeyen[1] and Marco Zagermann[4]

[1]Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, Celestijnenlaan 200D B-3001 Leuven, Belgium.

[2] INFN, Sezione di Torino and Dipartimento di Fisica Teorica, Università di Torino Via P. Giuria 1, I-10125 Torino, Italy.

[3] Institut für Physik, Humboldt Universität Newtonstrasse 15, 12489 Adlershof, Berlin, Germany. From 1st October: Theory Unit, CERN, Geneva 23, CH1211, Switzerland

[4] Department of Physics, Stanford University, Varian Building, Stanford, CA 94305-4060, USA.

Abstract

We revisit and complete the study of curved BPS-domain walls in matter-coupled $5D$, $\mathcal{N} = 2$ supergravity and carefully analyse the relation to gravitational theories known as "fake supergravities". We first show that *curved* BPS-domain walls require the presence of non-trivial hypermultiplet scalars, whereas walls that are solely supported by vector multiplet scalars are necessarily *flat*, due to the constraints from very special geometry. We then recover fake supergravity as the effective description of true supergravity where one restricts the attention to the flowing scalar field of a given BPS-domain wall. In general, however, true supergravity can be simulated by fake supergravity at most *locally*, based upon two choices: (i) a suitable adapted coordinate system on the scalar manifold, such that only one scalar field plays a dynamical rôle, and (ii) a gauge fixing of the $SU(2)$ connection on the quaternionic-Kähler manifold, as this connection does not fit the simple formalism of fake supergravity. Employing these gauge and coordinate choices, the BPS-equations for both vector and hypermultiplet scalars become identical to the fake supergravity equations, once the line of flow is determined by the full supergravity equations.

e-mails: Alessio.Celi@fys.kuleuven.ac.be, ceresole@to.infn.it, Gianguido.Dall'Agata@cern.ch, Antoine.VanProeyen@fys.kuleuven.ac.be, zagermann@itp.stanford.edu

Contents

1 Introduction

The study of domain wall solutions of $(d+1)$-dimensional (super-)gravity theories has been an active area of research over the past few years. This research is largely driven by applications in the context of the AdS/CFT correspondence and certain brane world models.

Most of the domain walls that have been studied in this context are "Minkowski–sliced" (or "flat" or "planar") domain walls. That is, having a metric of the form

$$\mathrm{d}s^2 = \mathrm{e}^{2U(r)}\eta_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n + \mathrm{d}r^2 \tag{1.1}$$

with $\eta_{mn} = \mathrm{diag}(-1,1,\ldots,1)$, they preserve the isometries of the d-dimensional Poincaré group. When these domain walls are supersymmetric and non-singular, one expects them to be stable solutions of the underlying supergravity theory, based on standard arguments that involve the existence of Killing spinors and the first-order form of the BPS equations.

The stability arguments for flat BPS domain walls can be formalized and extended to theories that are not necessarily supersymmetric [1,2]. In this approach, the classical stability of a solution is proven by defining a spinor energy along the lines of [3,4] by using some *formal* "transformation laws" that encode the equations of motion in a first-order form. These formal transformation laws have a structure similar to the supersymmetry transformation laws in true supergravity theories. For this to be possible, one needs to find a scalar function

$W(\phi)$ of the scalar fields which is related to the scalar potential $V(\phi)$ in the same way the superpotential is related to the scalar potential in true supergravity. Such a function $W(\phi)$ is often called an *adapted* superpotential. For the stability argument to work, however, this only needs to be a *formal* analogy: the function $W(\phi)$ (provided it exists), need not be a genuine superpotential of a true supergravity theory. In order to emphasize that it is in general only a formal analogy to genuine supergravity, this formalism has been named "fake supergravity" [5].

In [5], an attempt was also made to generalize these fake supergravity arguments for classical stability to "curved", or more precisely, to "AdS_d–sliced" domain walls, i.e., to domain walls of the form

$$\mathrm{d}s^2 = \mathrm{e}^{2U(r)} g_{mn}(x)\, \mathrm{d}x^m\, \mathrm{d}x^n + \mathrm{d}r^2 \tag{1.2}$$

with $g_{mn}(x)$ being a metric of AdS_d with curvature scale L_d. In order to do so, the authors of [5] promoted the scalar function $W(\phi)$ to an $su(2)$-valued (2×2)-matrix $\mathbf{W}(\phi) = W_i{}^j(\phi)$ $(i, j = 1, 2)$, such that the usual formula defining the scalar potential $V(\phi)$ is given by

$$V(\phi) = \frac{2(d-1)^2}{\kappa^2} \left(\frac{1}{2} \operatorname{Tr} \right) \left[\frac{1}{\kappa^2} (\partial_\phi \mathbf{W})^2 - \frac{d}{d-1} \mathbf{W}^2 \right]. \tag{1.3}$$

The matrix $\mathbf{W}$ also substitutes the scalar superpotential W in the corresponding "fake" Killing spinor equations for an $SU(2)$-doublet spinor ϵ:

$$\begin{aligned} \left[\nabla_\mu + \gamma_\mu \mathbf{W}\right] \epsilon &= 0, \\ \left[\gamma^\mu \nabla_\mu \phi - \frac{2(d-1)}{\kappa^2} \partial_\phi \mathbf{W} \right] \epsilon &= 0, \end{aligned} \tag{1.4}$$

where $\nabla_\mu \epsilon = \left(\partial_\mu + \frac{1}{4}\, \omega_\mu{}^{\nu\rho} \gamma_{\nu\rho} \right) \epsilon$.

The idea of introducing such matrix $\mathbf{W}$ was inspired by the earlier work [6–10] on curved BPS domain walls in (genuine) five-dimensional $\mathcal{N} = 2$ gauged supergravity [11–15], where the supersymmetry parameters form a pair of symplectic Majorana spinors, ϵ_i, and the analogue of the superpotential becomes an $su(2)$-valued (2×2)-matrix. This formalism encompasses both curved and flat domain walls, as the latter are retrieved for a diagonal $\mathbf{W}$ matrix.

In this paper, we will refer to *fake supergravities as gravitational theories whose scalar potentials can formally be written in terms of a superpotential-like matrix as in (1.3), such that the equations of motion for domain walls assume a first order form compatible with (1.4).*

The above fake supergravity equations can, in general, capture at most *part* of the structure of generic $5D$, $\mathcal{N} = 2$ gauged supergravity theories. For one thing, the (typically higher than one-dimensional) scalar manifolds of $5D$ vector-, tensor- and hypermultiplets are subject to a variety of strong geometrical constraints, none of which are visible in the single-field formalism of (1.3)–(1.4). In practice, the scalar field geometry requires that the supersymmetry transformation laws and the scalar potential in $5D$, $\mathcal{N}=2$ gauged supergravity may generically contain extra terms that do not immediately fit into the simple fake supergravity set-up.

And yet, this simple set of equations still seems to capture the key aspects of the BPS equations of true supergravity that are relevant for domain-wall stability. Therefore, one might wonder whether there are perhaps some deeper relations between fake and true supergravity that only become apparent if one restricts the attention to the effective dynamics of domain wall solutions. The original motivation of our work was to clarify precisely this point, i.e., to check to what extent the fake supergravity of [5] is really "fake" and under what circumstances it can describe true supergravity in five dimensions. In particular, we wondered under which conditions the single scalar field language of [5] could suffice to encode the distinct geometrical features of the moduli spaces for scalar fields belonging to various kinds of matter multiplets: vector-, tensor- or hypermultiplets.

From the technical point of view, the comparison between fake supergravity and a specific true supergravity model consists essentially in the correct identification of the spinor projector and the superpotential $\mathbf{W}$ from the BPS-equations. One then has to check whether these identifications are compatible with all true BPS-equations and whether the scalar potential agrees with (1.3). As we will see, a crucial commutator constraint on the superpotential, which arises as a consistency condition between specific components of the "fake" Killing spinor equations, will serve as an important test in this analysis.

Interestingly, our attempts to match true and fake supergravity equations along these lines have driven a fruitful re-investigation of the BPS equations of $5D$, $\mathcal{N}=2$ supergravity itself and led us to a number of unexpected and quite general new insights in the context of curved domain walls. In fact, by completing and clarifying previous studies [6–10], we arrive at a remarkably coherent geometrical picture that illustrates the different rôles played by vector-, tensor-, and hypermultiplets. We find that, on a supersymmetric (curved or flat) domain wall solution, the BPS equations and the scalar potential can be locally written in the same form, no matter whether the domain wall is supported by vector- or by hypermultiplet scalars (we show that tensor scalars cannot play any rôle in this set-up). However, despite this formal similarity, the different geometries governing the vector- and hypermultiplet scalar manifolds still leave a strong imprint on the solution spaces of these BPS equations. Indeed, we find that the constraints from the very special geometry forbid a *curved* BPS-domain wall that is supported solely by vector multiplet scalars. By contrast, similar constraints are absent for non-trivial hyper-scalars, either alone or in combination with running vector scalars. These findings are consistent with the examples constructed in [6–10], which always involved at least one running hyper-scalar.

The above results on curved BPS-domain walls in true $5D$, $\mathcal{N} = 2$ supergravity end up having non-trivial consequences also for the comparison with fake supergravity, and even suggest the way to make contact between the two. We show that the BPS-equations and the scalar potentials of vector and hypermultiplets in true supergravity can formally be brought to agreement with the analogous expressions in fake supergravity. However, the impossibility of curved BPS-domain walls supported solely by vector scalars implies that a *curved* BPS-domain wall in true supergravity can be described by fake supergravity only if supported by hyperscalars. It should also be emphasized that any such coincidence between fake and true supergravity is, in general, only valid locally along the flow, as it requires some particular gauge and coordinate choices on the scalar manifolds of $\mathcal{N} = 2$ supergravity that we will precisely identify.

The inverse question, as to whether a given fake supergravity domain wall can be embedded into true supergravity, involves checking various constraints required by quaternionic or very special geometry. But at least for a curved wall, one can immediately rule out that the fake supergravity scalar sits in a vector multiplet.

Among other things, the analysis presented in this paper might finally help in deciding whether solutions such as the Janus solution of [16], whose stability was proven in [5] using fake supergravity, can perhaps be embedded into true five-dimensional supergravity even though it breaks the ten-dimensional Type IIB supersymmetry it descends from [17].

In spite of the focus on $5D$, $\mathcal{N} = 2$ supergravity, we stress that we expect to easily extend our results to $\mathcal{N} = 2$ theories in four and six dimensions as they bare the same quaternionic-Kähler geometry for hypermultiplets as well as the same action of an $SU(2)$ (sub)group of the R–symmetry on the susy spinor. It might also be worthwhile to specialize our results to certain interesting subclasses of theories, such as, e.g., the gauged supergravities that derive from flux compactifications of string theory [18]. Domain wall solutions for this subclass in $4D$ have recently been considered [19, 20].

The organization of the paper is as follows. In Section 2, we briefly review the fake supergravity formalism of [5]. In Section 3, we then describe curved BPS-domain walls in $5D$, $\mathcal{N} = 2$ supergravity based on the earlier work [6–10]. Section 4 constitutes the main part of this paper and is devoted to the comparison between true and fake supergravity. As a by-product, we derive the conditions for a BPS-domain wall of true supergravity to be curved, ruling out the vector scalars as single supporters. In section 5, we show how the use of wisely chosen parameterizations on the scalar manifolds brings the BPS equations and the scalar potentials for vector and hyper-scalars into an identical form if one considers these expressions on a BPS-domain wall solution. We end with some comments in Section 6. Appendix A gives more details about the $SU(2)$ symmetry and its gauge fixing that is performed in order to obtain suitable coordinates.

2 Curved domain walls in fake supergravity

In this section, we briefly summarize the formalism of fake supergravity developed in [5], to which we refer the reader for further details. Ref. [5] considers scalar gravity actions of the form

$$S = \int d^{d+1}x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \tag{2.1}$$

with a scalar potential $V(\phi)$ given by (1.3). As mentioned in the introduction, $\mathbf{W}(\phi)$ is an $su(2)$-valued (2×2)-matrix, which implies that quadratic expressions such as $\mathbf{W}^2$, $(\partial_\phi\mathbf{W})^2$ or $\{\mathbf{W},\partial_\phi\mathbf{W}\}$ are proportional to the unit matrix. This allows one to write the potential without explicitly taking the trace:

$$V(\phi)\mathbb{1} = \frac{2(d-1)^2}{\kappa^2}\left[\frac{1}{\kappa^2}(\partial_\phi\mathbf{W})^2 - \frac{d}{d-1}\mathbf{W}^2\right], \tag{2.2}$$

which is the form we will use for our later comparison with true supergravity.

The matrix $\mathbf{W}$ also enters some "fake" Killing spinor equations for an $SU(2)$-doublet spinor ϵ as shown in (1.4). Using (1.2) and assuming that the scalar ϕ depends only on the radial coordinate r (which we choose, for all d, to be the fifth coordinate x^5), (1.4) reads

$$\begin{aligned} \left[\nabla_m^{AdS_d} + \gamma_m\left(\frac{1}{2}U'\gamma_5 + \mathbf{W}\right)\right]\epsilon &= 0, && (2.3)\\ \left[\partial_r + \gamma_5\mathbf{W}\right]\epsilon &= 0, && (2.4)\\ \left[\gamma_5\phi' - \frac{2(d-1)}{\kappa^2}\partial_\phi\mathbf{W}\right]\epsilon &= 0, && (2.5) \end{aligned}$$

where $U(r)$ is the warp factor of the metric (1.2), and $\nabla_m^{AdS_d}$ denotes the spacetime covariant derivative with only the spin connection for the AdS_d background metric $g_{mn}(x)$. The prime means[1] a derivative with respect to r.

It is shown in [5] that the system (2.3)-(2.5) reproduces the second order field equations for the warp factor $U(r)$ and the scalar field $\phi(r)$ that follow from (2.1) and (1.3) with

$$\gamma^2(r) \equiv \left(1 - \frac{\mathrm{e}^{-2U(r)}}{2L_d^2\,\mathrm{Tr}\,\mathbf{W}^2(\phi(r))}\right) = \frac{\mathrm{Tr}\{\mathbf{W},\partial_\phi\mathbf{W}\}^2}{2\,\mathrm{Tr}\,\mathbf{W}^2\,\mathrm{Tr}(\partial_\phi\mathbf{W})^2}, \tag{2.6}$$

[1] For clarity, we changed some notation from [5].

(where $L_d^2 = -12/R_{AdS}$ is determined by the scalar curvature of the AdS space) *provided that* the "superpotential" $\mathbf{W}(\phi)$ satisfies the constraint

$$\left[\partial_\phi \mathbf{W}, \frac{d-1}{\kappa^2}\partial_\phi\partial_\phi \mathbf{W} + \mathbf{W}\right] = 0, \tag{2.7}$$

which is a compatibility condition of (2.4) and (2.5).

Eqs. (2.6) and (2.7) have two important consequences: (2.6) implies that a solution where $\mathbf{W}(\phi)$ is proportional to the first derivative $\partial_\phi \mathbf{W}(\phi)$ leads to $\gamma(r) = 1$, i.e., a flat domain wall, as $\gamma(r) = 1$ implies $L_d \to \infty$. Eq. (2.7), on the other hand, implies that the ϕ-dependence of $\mathbf{W}(\phi)$ cannot be arbitrary, but has to satisfy the commutator constraint (2.7). As we will see later, this consistency condition provides an important test on a true supergravity theory in order to fit into the framework of "fake supergravity".

Since we are interested in five-dimensional supergravity, from now on we will specialize the above equations to the case $d = 4$ and set $\kappa^2 = 1$.

3 Curved domain walls in $5D$, $\mathcal{N} = 2$ gauged supergravity

In the previous section, we have summarized the fake supergravity formalism for curved domain walls developed in [5]. In this section, we describe how curved BPS-domain walls arise in true supergravity. In Section 4, we will then compare the results for fake and true supergravity and verify to what extent they can describe the same systems.

3.1 Five-dimensional, $\mathcal{N} = 2$ gauged supergravity

We start by recalling some of the most important features of five-dimensional, $\mathcal{N} = 2$ gauged supergravity theories. Further technical details can be found in the original references [11–15].

The matter multiplets that can be coupled to $5D$, $\mathcal{N} = 2$ supergravity are vector, tensor and hypermultiplets: the scalar ϕ of the previous section could a priori sit in any of these (or even be a combination of different types of scalars, as we will see in section 4.3).

The $(n_V + n_T)$ scalar fields of n_V vector and n_T tensor multiplets parameterize a "very special" real manifold $\mathcal{M}_{\rm VS}$, i.e., an $(n_V + n_T)$–dimensional hypersurface of an auxiliary $(n_V + n_T + 1)$-dimensional space spanned by coordinates $h^{\tilde{I}}$ $(\tilde{I} = 0, 1, \ldots, n_V + n_T + 1)$:

$$\mathcal{M}_{\rm VS} = \{h^{\tilde{I}} \in \mathbb{R}^{(n_V+n_T+1)} : C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1\}, \tag{3.1}$$

where the constants $C_{\tilde{I}\tilde{J}\tilde{K}}$ appear in a Chern-Simons-type coupling of the Lagrangian. The embedding coordinates $h^{\tilde{I}}$ have a natural splitting,

$$h^{\tilde{I}} = (h^I, h^M), \qquad (I = 0, 1, \ldots, n_V), \qquad (M = 1, \ldots, n_T), \tag{3.2}$$

where the h^I are related to the sub-geometry of the n_V vector multiplets, and the h^M refer to the n_T tensor multiplets. On $\mathcal{M}_{\rm VS}$, the $h^{\tilde{I}}$ become functions of the physical scalar fields, φ^x $(x = 1, \ldots, n_V + n_T)$. The metric on the very special manifold is determined via the equations

$$\begin{aligned} &g_{xy} = h^{\tilde{I}}_x\, h_{y\tilde{I}}, \qquad h^{\tilde{I}}_x \equiv -\sqrt{\tfrac{3}{2}}\,\partial_x h^{\tilde{I}}, \qquad h_{\tilde{I}} \equiv C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{J}} h^{\tilde{K}}, \qquad h_{\tilde{I}x} \equiv \sqrt{\tfrac{3}{2}}\,\partial_x h_{\tilde{I}}, \\ &h^{\tilde{I}} h_{\tilde{J}} + h^{\tilde{I}}_x\, g^{xy}\, h_{y\tilde{J}} = \delta^{\tilde{I}}_{\tilde{J}}, \qquad h^{\tilde{I}} h_{\tilde{I}} = 1, \qquad h^{\tilde{I}} h_{\tilde{I}x} = 0. \end{aligned} \tag{3.3}$$

The scalars q^X $(X = 1, \ldots 4n_H)$ of n_H hypermultiplets, on the other hand, take their values in a quaternionic-Kähler manifold $\mathcal{M}_{\rm Q}$ [21], i.e., a manifold of real dimension $4n_H$ with holonomy group contained in $SU(2)\times USp(2n_H)$. We denote the vielbein on this manifold by f^{iA}_X, where $i = 1, 2$ and $A = 1, \ldots, 2n_H$ refer to an adapted $SU(2)\times USp(2n_H)$ decomposition of the tangent space. The hypercomplex structure is (-2) times the curvature of the $SU(2)$ part of the holonomy group[2], denoted as $\mathcal{R}^{rZX}$ $(r = 1, 2, 3)$, so that the quaternionic identity reads

$$\mathcal{R}^r_{XY}\mathcal{R}^{sYZ} = -\tfrac{1}{4}\,\delta^{rs}\,\delta_X{}^Z - \tfrac{1}{2}\,\varepsilon^{rst}\,\mathcal{R}^t_X{}^Z. \tag{3.4}$$

Besides these scalar fields, the bosonic sector of the matter multiplets also contains n_T tensor fields $B^M_{\mu\nu}$ $(M = 1, \ldots, n_T)$ from the n_T tensor multiplets and n_V vector fields from the n_V vector multiplets. Including the graviphoton, we thus have a total of $(n_V + 1)$ vector fields, A^I_μ $(I = 0, 1, \ldots, n_V)$, which can be used to gauge up to $(n_V + 1)$ isometries of the quaternionic-Kähler manifold $\mathcal{M}_{\rm Q}$ (provided such isometries exist). These symmetries act on the vector-tensor multiplets by a representation $t_{I\tilde{J}}{}^{\tilde{K}}$, where in the pure vector multiplet sector $t_{IJ}{}^K = f_{IJ}{}^K$ are the structure constants, and the other components also satisfy some restrictions [13, 15, 22]. The transformations should leave the defining condition in (3.1) invariant, hence

$$t_{I(\tilde{J}}{}^{\tilde{M}} C_{\tilde{K}\tilde{L})\tilde{M}} = 0. \tag{3.5}$$

The very special Kähler target space then has Killing vectors

$$K^x_I(\varphi) = -\sqrt{\tfrac{3}{2}} t_{I\tilde{J}}{}^{\tilde{K}} h^x_{\tilde{K}} h^{\tilde{J}}. \tag{3.6}$$

There may be more Killing vectors, but these are the ones that are gauged using the gauge vectors in the vector multiplets.

[2] In fact, the proportionality factor includes the Planck mass and the metric, which are implicit here.

The quaternionic Killing vectors $K_I^X(q)$ that generate the isometries on $\mathcal{M}_{\rm Q}$ can be expressed in terms of the derivatives of $SU(2)$ triplets of Killing prepotentials $P_I^r(q)$ ($r = 1,2,3$) via

$$D_X P_I^r = \mathcal{R}^r_{XY} K_I^Y , \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rl} K_I^Y &= -\frac{4}{3}\mathcal{R}^{rYX} D_X P_I^r \\ D_X P_I^r &= -\varepsilon^{rst}\mathcal{R}^s_{XY} D^Y P_I^t , \end{array} \right. \tag{3.7}$$

where D_X denotes the $SU(2)$ covariant derivative, which contains an $SU(2)$ connection ω_X^r with curvature $\mathcal{R}^r_{XY}$:

$$D_X P^r = \partial_X P^r + 2\,\varepsilon^{rst}\omega_X^s P^t , \qquad \mathcal{R}^r_{XY} = 2\,\partial_{[X}\omega^r_{Y]} + 2\,\varepsilon^{rst}\omega_X^s\omega_Y^t . \tag{3.8}$$

The prepotentials satisfy the constraint

$$\frac{1}{2}\mathcal{R}^r_{XY} K_I^X K_J^Y - \varepsilon^{rst} P_I^s P_J^t + \frac{1}{2} f_{IJ}{}^K P_K^r = 0, \tag{3.9}$$

where $f_{IJ}{}^K$ are the structure constants of the gauge group.

In the following, we will frequently switch between the above vector notation for $SU(2)$-valued quantities such as P_I^r, and the usual (2×2) matrix notation,

$$\mathbf{P}_I = \left(P_{Ii}{}^j\right) , \qquad P_{Ii}{}^j \equiv {\rm i}\,\sigma_{ri}{}^j P_I^r . \tag{3.10}$$

As in [5], boldface expressions such as $\mathbf{P}_I$ then refer to the (2×2)-matrices with the indices i,j suppressed.

An important difference in geometrical significance between the very special Killing vectors $K_I^x(\varphi)$ in (3.6) and the quaternionic ones $K_I^X(q)$ in (3.7), is that the former do not arise as derivatives of Killing prepotentials, because there is no natural symplectic structure on the real manifold $\mathcal{M}_{\rm VS}$ that could define a moment map. This feature will also play a rôle in the comparison with fake supergravity.[3]

Turning on only the metric and the scalars, the general Lagrangian of such a gauged supergravity theory is

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{2}g_{xy}\partial_\mu\varphi^x\partial^\mu\varphi^y - \frac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y - g^2\mathcal{V}(\varphi,q), \tag{3.11}$$

whereas the supersymmetry transformation laws of the fermions are given by

$$\delta\psi_{\mu i} = \nabla_\mu\epsilon_i - \omega_{\mu i}{}^j\epsilon_j - \frac{\rm i}{\sqrt{6}}\,g\,\gamma_\mu P_i{}^j\epsilon_j , \tag{3.12}$$

$$\delta\lambda_i^x = -\frac{\rm i}{2}\gamma^\mu(\partial_\mu\varphi^x)\epsilon_i - g\,P_i{}^{jx}\epsilon_j + g\,\mathcal{T}^x\epsilon_i , \tag{3.13}$$

$$\delta\zeta^A = \frac{\rm i}{2} f_X^{iA}\gamma^\mu(\partial_\mu q^X)\epsilon_i - g\,\mathcal{N}^{iA}\epsilon_i . \tag{3.14}$$

[3]The moment maps are related to the fact that the isometries should preserve complex structures. Therefore, they are absent in the real manifold. In 4 dimensions, the scalar manifold of the vector multiplets does have a complex structure. Hence, in that case this sector would also have a moment map structure [23]. This suggests that in four dimensions the same comparison may go along different lines.

Here, ψ_μ^i, λ_i^x, ζ^A are the gravitini, gaugini (tensorini) and hyperini, respectively, g denotes the gauge coupling, the $SU(2)$ connection $\boldsymbol{\omega}_\mu$ is defined as $\omega_{\mu i}{}^j = (\partial_\mu q^X)\omega_{Xi}{}^j$, and

$$
\begin{aligned}
P^r &= h^I(\varphi)P_I^r(q), && (3.15)\\
P_x^r &= -\sqrt{\frac{3}{2}}\partial_x P^r = h_x^I P_I^r, \qquad P^{rx} = g^{xy}P_y^r, && (3.16)\\
\mathcal{N}^{iA} &= \frac{\sqrt{6}}{4} f_X^{iA}(q)h^I(\varphi)K_I^X(q), && (3.17)\\
\mathcal{T}^x &= \frac{\sqrt{6}}{4} h^I(\varphi)K_I^x(\varphi). && (3.18)
\end{aligned}
$$

As a general fact in supergravity, the potential is given by the sum of "squares of the fermionic shifts" (the scalar expressions in the above transformations of the fermions):

$$
\mathcal{V} = -4P^rP^r + 2P_x^rP_y^r g^{xy} + 2\mathcal{N}^{iA}\mathcal{N}^{jB}\varepsilon_{ij}C_{AB} + 2\mathcal{T}^x\mathcal{T}^y g_{xy}, \tag{3.19}
$$

where C_{AB} is the (antisymmetric) symplectic metric of $USp(2n_H)$.

A first glance at (3.12) indicates that, leaving out for the moment the $SU(2)$ connection term $\boldsymbol{\omega}_\mu$, the superpotential matrix $\mathbf{W}$ has to be related to $\mathbf{P}$.

Using the explicit form of the Killing vector, (3.6), in (3.18), one finds that this expression vanishes if the transformation matrix t involves only vector multiplets. This is clear because then $t_{IJ}{}^K = f_{IJ}{}^K$, hence antisymmetric. Therefore, the shift $\mathcal{T}^x$ in the above expressions is non-vanishing only if there are charged tensor multiplets in the theory [4]. Since $\mathcal{T}^x$ appears in (3.13) with the unit matrix in $su(2)$ space, it must vanish on a BPS-domain wall solution for compatibility with the spinor projector (see [24, footnote 8] and [25]). Furthermore, unlike the shifts P_x^r and $\mathcal{N}^{iA}$, $\mathcal{T}^x$ is a purely "D-type" term, in the sense that it is completely unrelated to derivatives of the matrix $\mathbf{P}$. Therefore, it can never fit the pattern (1.4) of the fake supergravity transformations. Thus, neither for BPS-domain walls in $5D$, $\mathcal{N} = 2$ supergravity nor for domain walls in fake supergravity, can non-trivial tensor multiplets play an important rôle, and we can limit our remaining discussion to the case $n_T = 0$, i.e., to supergravity coupled to vector and/or hypermultiplets only. This also means that the index $\tilde{I}$ simply becomes the index I in all previous equations, and the index M disappears.

Using (3.7) and the quaternionic identity (3.4), the scalar potential for vector and hypermultiplets can be written in a form that is somewhat similar to (2.2),

$$
\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - 3(\partial_x\mathbf{P})(\partial^x\mathbf{P}) - (D_X\mathbf{P})(D^X\mathbf{P}). \tag{3.20}
$$

[4] In five dimensions, tensor multiplets that are not charged under some gauge group are equivalent to vector multiplets. We always assume that all uncharged tensor multiplets are converted to vector multiplets.

In Sections 4 and 5, we will elaborate further on the similarities and differences between the true and fake supergravity potentials, as well as on how to remove the asymmetry between the hypermultiplet and the vector multiplet sector in these expressions.

3.2 Curved BPS-domain walls in supergravity

We can now take a closer look at 1/2 supersymmetric, curved domain wall solutions of the above gauged supergravity theories. The careful investigation of this subject was pioneered in [6] and further developed in refs. [7–10], where also some examples were given. Here we mainly review this construction, although in a different language and deriving a new important constraint.

In a curved domain wall background of the form (1.2), when the scalar fields only depend on the radial coordinate, the vanishing of the supersymmetry variations (3.12)-(3.14) implies

$$
\begin{aligned}
\left[\nabla_m^{AdS_4} + \gamma_m\left(\frac{1}{2}U'\gamma_5 - \frac{ig}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon &= 0, &(3.21)\\
\left[D_r + \gamma_5\left(-\frac{\mathrm{i}g}{\sqrt{6}}\mathbf{P}\right)\right]\epsilon &= 0, &(3.22)\\
\left[\gamma_5\varphi^{x\prime} + \mathrm{i}g\,\sqrt{6}\,g^{xy}\partial_y\mathbf{P}\right]\epsilon &= 0, &(3.23)\\
f_X^{iA}\left[\gamma_5 q^{X\prime} - \mathrm{i}g\sqrt{\frac{8}{3}}R^{rXY}D_Y P^r\right]\epsilon_i &= 0, &(3.24)
\end{aligned}
$$

where

$$
D_r\epsilon_i \equiv \partial_r\epsilon_i - q^{X\prime}\omega_{Xi}{}^j\epsilon_j \tag{3.25}
$$

has been introduced.

These equations have a structure similar to (1.4)

$$
\begin{aligned}
\delta\psi_{\mu i} &= \left(\nabla_\mu\delta_i^j - \omega_{\mu i}{}^j - \frac{\mathrm{i}}{\sqrt{6}}\,g\,\gamma_\mu P_i^j\right)\epsilon_j = 0, &(3.26)\\
\delta\lambda_i^x &= \mathcal{O}^x{}_i{}^j\epsilon_j = 0, &(3.27)\\
f_{iA}^X\delta\zeta^A &= \mathcal{O}^X{}_i{}^j\epsilon_j = 0, &(3.28)
\end{aligned}
$$

and it is useful to split the operators $\mathcal{O}_i^j$ into $\mathcal{O}_i^j = \mathcal{O}^0\delta_i^j + \mathrm{i}\mathcal{O}^r\sigma_{ri}{}^j$, whose components in each case can be read off directly from the explicit formulae.

In order to construct solutions to the system (3.21)–(3.24), one usually splits these equations into a projection condition on the supersymmetry parameter ϵ and a system of first order differential equations involving only the scalars and the warp factor. To do so, extending the ideas of [24] to curved domain walls, one chooses a projector on the supersymmetry parameter of the form

$$\epsilon_i = -\mathrm{i}\gamma_5 \Theta_i{}^j \epsilon_j \Leftrightarrow (\mathbb{1} + \mathrm{i}\gamma_5 \mathbf{\Theta})\,\epsilon = 0, \tag{3.29}$$

where $\mathbf{\Theta}^2 = -\mathbb{1} \Leftrightarrow \Theta^r\Theta^r = 1$.

The explicit form of $\mathbf{\Theta}$ can be determined by the solution of the supersymmetry transformations on the matter fields, $\delta\lambda_i^x$ and $\delta\zeta^A$. More precisely, using (3.29), the gaugino transformation (3.23) implies the BPS-equation

$$g_{yx}\,\varphi^{x\prime}\mathbf{\Theta} = \sqrt{6}\,g\,\partial_y \mathbf{P}. \tag{3.30}$$

Note that we can omit ϵ as the only projection on the Killing spinor involves γ_5. Hence [6,7]

$$\mathbf{\Theta} = g\,\sqrt{6}\,\frac{\varphi^{x\prime}\partial_x \mathbf{P}}{\varphi^{y\prime}\varphi^{z\prime}g_{yz}}. \tag{3.31}$$

The hyperino transformation (3.24), on the other hand, implies, after contraction with f_{YjA} and a decomposition into the trace and the traceless part (see [24] for the flat domain wall analogue),

$$\begin{aligned} g_{YX}q^{X\prime}\mathbf{\Theta} + q^{X\prime}[\mathbf{R}_{YX}, \mathbf{\Theta}] - \sqrt{6}\,g\,D_Y\mathbf{P} &= 0, & (3.32)\\ \sqrt{6}\,g\,K_Y - 2\,q^{X\prime}\{\mathbf{R}_{YX}, \mathbf{\Theta}\} &= 0. & (3.33) \end{aligned}$$

These two equations are equivalent and can be converted into one another via contractions with the $SU(2)$ curvature. Another interesting and compact version[5] of the hyperino equation can be obtained by anticommuting (3.32) with $\mathbf{\Theta}$:

$$g_{XY}q^{X\prime} = g\,\sqrt{6}\,\Theta^r D_Y P^r. \tag{3.34}$$

Comparing (3.32) with (3.30), we notice again an obvious asymmetry between the vector and hypermultiplets. Contraction of (3.32) with $q^{X\prime}$ finally yields the expression [8]

$$\mathbf{\Theta} = g\,\sqrt{6}\,\frac{q^{X\prime}D_X\mathbf{P}}{q^{Y\prime}q^{Z\prime}g_{YZ}}. \tag{3.35}$$

When both vector multiplet scalars and hyper-scalars are non-trivial, consistency of (3.35) and (3.31) requires

$$\frac{q^{X\prime}D_X\mathbf{P}}{q^{Y\prime}q^{Z\prime}g_{YZ}} = \frac{\varphi^{x\prime}\partial_x\mathbf{P}}{\varphi^{y\prime}\varphi^{z\prime}g_{yz}}. \tag{3.36}$$

On the other hand, (3.31) and (3.35) also imply that $\mathbf{\Theta}$ is proportional to $D_r\mathbf{P}$:

$$D_r\mathbf{P} \equiv \varphi^{\prime x}\partial_x\mathbf{P} + q^{\prime X}D_X\mathbf{P} = \frac{1}{\sqrt{6}g}g_{\Lambda\Sigma}\phi^{\Lambda\prime}\phi^{\Sigma\prime}\mathbf{\Theta}, \tag{3.37}$$

[5] The relation (3.34) can be directly derived by applying the general analysis of the hyperino equation in [26,27].

where $\phi^\Lambda = \{\varphi^x, q^X\}$.

There is, however, one further integrability constraint that was not noticed before. It follows from consistency between (3.29) and (3.22). This relation will play a rôle analogous to the consistency condition (2.7) of the fake supergravity framework. To this end, consider

$$\begin{aligned} 0 &\stackrel{(3.29)}{=} -\mathrm{i}\boldsymbol{\Theta}\gamma_5 D_r[\epsilon + \mathrm{i}\gamma_5\boldsymbol{\Theta}\epsilon] \\ &\stackrel{(3.22)}{=} \frac{g}{\sqrt{6}}\left[\boldsymbol{\Theta}\mathbf{P}\epsilon - \mathrm{i}\gamma_5\mathbf{P}\epsilon\right] + \boldsymbol{\Theta}(D_r\boldsymbol{\Theta})\epsilon \\ &\stackrel{(3.29)}{=} \left[\frac{g}{\sqrt{6}}\left[\boldsymbol{\Theta}, \mathbf{P}\right] + \boldsymbol{\Theta}(D_r\boldsymbol{\Theta})\right]\epsilon. \end{aligned} \tag{3.38}$$

Remembering $D_r(\boldsymbol{\Theta}^2) = 0$, the second term in the last equation can also be written as $\frac{1}{2}[\boldsymbol{\Theta}, D_r\boldsymbol{\Theta}]$, and one finally obtains the consistency condition[6]

$$\left[\boldsymbol{\Theta}\,,\, D_r\boldsymbol{\Theta} + \sqrt{\frac{2}{3}}g\,\mathbf{P}\right] = 0. \tag{3.39}$$

Using (3.37) this leads to the commutator relation

$$\left[D_r\mathbf{P}, D_r D_r\mathbf{P} + \frac{1}{3}g_{\Lambda\Sigma}\phi^{\Lambda\prime}\phi^{\Sigma\prime}\mathbf{P}\right] = 0, \tag{3.40}$$

which is the supergravity version of the equation (2.7) in fake supergravity. The differences between (3.40) and (2.7) are the fact that (3.40) holds with derivatives taken in the (typically higher-dimensional) space of all the scalar fields in the supergravity theory and the consequent appearance of covariant derivatives.

It should be noted that the conditions summarized in this section are just necessary conditions in order to obtain supersymmetric domain wall solutions in five-dimensional supergravity. To verify whether they are also sufficient, one has to further check the equations of motion. These are identically satisfied for $\gamma = 1$, i.e. for flat domain walls, but may give additional constraints in the case of curved ones.

4 True vs. fake supergravity

In this section, we want to find out whether there are supersymmetric *curved* domain walls in true supergravity that can also be described within the simpler framework of fake supergravity. The most obvious obstacle for such a comparison is the number of scalar fields within these two setups. While the fake supergravity formalism of [5] was developed in detail

[6]An analogous equation was independently derived by Klaus Behrndt and Mirjam Cvetič (private communication).

for only one scalar field [7], ϕ, a generic true supergravity theory contains $(n_V + 4n_H)$ scalar fields (we have already discarded the possibility of tensor multiplet scalars) $\phi^\Lambda = (\varphi^x, q^X)$ which exceeds one unless there is precisely one vector and no hypermultiplet. Comparing these two setups is thus only possible if the "superfluous" scalars in true supergravity can somehow be "deactivated". As we will now describe, for any given domain wall solution, one can, in principle, always choose local, adapted coordinate systems on the scalar manifolds of true supergravity such that there is at most one r-dependent scalar field on $\mathcal{M}_{VS}$ and $\mathcal{M}_Q$, and all other scalar fields can effectively be removed from the equations that describe the domain wall. It is in these adapted coordinates that we will be able to investigate the question as to whether fake supergravity also describes some true supergravity systems, as they allow to reduce the discussion essentially to one single scalar.

The usefulness of adapted coordinates goes beyond the comparison of fake and true supergravity. Indeed, as another interesting application, we will show that the differences between vector and hypermultiplets can be switched off along the scalar flow curve when adapted coordinates are used. With this rewriting, we are then able to sharpen the conditions for curved domain walls to exist in true supergravity. In particular, we show that supersymmetric domain walls that are supported by vector scalars only, have to be flat.

4.1 Running vector scalars

To begin with, let us investigate the possibility that the scalar field ϕ of the fake supergravity equations (2.1)-(2.5) belongs to a vector multiplet. For this to be possible, the curved domain wall obviously has to be supported by vector scalars only, i.e., any hyper-scalars (if present) have to stay constant along the flow:

$$q^{X\prime} = 0. \tag{4.1}$$

For this condition to preserve supersymmetry, the hyperino BPS-equation (3.32) implies that

$$D_X \mathbf{P} = 0 \tag{4.2}$$

along the flow.

The constancy of the hyper-scalars also means that the $SU(2)$ connection $q^{X\prime}\omega_{Xi}{}^j$ vanishes along the flow, and the $SU(2)$ covariant derivative D_r defined in (3.25) degenerates to an ordinary derivative, $D_r = \partial_r$.

It is easy to see that the gravitino BPS-equations (3.21)-(3.22) then become equivalent to the fake supersymmetry transformations (2.3)-(2.4), provided we identify

$$\mathbf{W} = -\frac{\mathrm{i}g}{\sqrt{6}}\mathbf{P}. \tag{4.3}$$

[7] A generalization to more scalar fields with particular types of scalar potentials is briefly discussed in [5].

In order to bring the gaugino BPS-equation (3.30) into the form of fake supergravity, one has to get rid of all but one scalar field, which is done by going to a particular, "adapted" coordinate system on the scalar manifold $\mathcal{M}_{\rm VS}$. To this end, recall that, in general, any domain wall solution defines a curve $\phi^{\Lambda}(r)$ on the scalar manifold $\mathcal{M} = \mathcal{M}_{\rm VS} \times \mathcal{M}_Q$. In this subsection, we only consider curves that are trivial on the quaternionic manifold and therefore lie entirely on $\mathcal{M}_{\rm VS}$. As the coordinates φ^x on $\mathcal{M}_{\rm VS}$ can be chosen at will, we can take a basis for these scalars such that only one scalar of the vector multiplets is r-dependent and all the others are constant [8]. We call this single r-dependent scalar field φ. The other scalars, which we call $\varphi^{\hat{x}}$, can then be chosen to be orthogonal to the flow curve (at least on the flow curve itself). Although this last assumption is not strictly necessary to derive most of the results we present here, it is always possible in some patch and therefore we employ it for the sake of simplicity. Along this flow curve, the scalar field metric g_{xy} then takes the form

$$g_{xy} = \begin{pmatrix} g_{\varphi\varphi} & 0 \\ 0 & g_{\hat{x}\hat{y}} \end{pmatrix} . \tag{4.4}$$

Note that in these coordinates, the vanishing of the gaugino transformation (3.23) for the constant scalars $\varphi^{\hat{x}}$ implies

$$\partial_{\hat{x}} \mathbf{P} = 0 \tag{4.5}$$

on the flow curve. The φ-component of (3.23), on the other hand, is now of the form (2.5) given in fake supergravity. Given the identification (4.3) and identifying the scalar of fake supergravity ϕ with φ, it is then easy to see that

- The supersymmetry transformations (3.21)-(3.23) are of the same form as the fake supersymmetry transformation (2.3)-(2.5);

- The consistency condition (3.40) reduces to

$$\left[\partial_{\varphi}\mathbf{P}, \partial_{\varphi}^2\mathbf{P} + \frac{1}{3} g_{\varphi\varphi}\mathbf{P}\right] = 0, \tag{4.6}$$

  which is equivalent to the required compatibility condition (2.7), if one normalizes $g_{\varphi\varphi} = 1$ by an appropriate rescaling of φ;

- Upon this normalization, also the Lagrangian and the scalar potentials agree, see (2.2) versus (3.20) with (4.2) and (4.5).

[8] In practice, this might be a very inconvenient choice to work with, and it might also obscure the one-to-one correspondence between particular scalar fields and certain gauge theory operators in an AdS/CFT context. It is also clear that, by construction, these adapted coordinates are different for different flow solutions and that one might have to use several adapted coordinate patches to cover an entire flow curve, as such a curve might have self-intersections. For our formal arguments, however, this coordinate choice turns out to be convenient and sufficient.

Thus, at first sight, the case of running vector scalars in genuine $5D$, $\mathcal{N}=2$ gauged supergravity automatically seems to fall into the generalized fake supergravity framework of [5]. We will now show that this conclusion is *wrong* when the domain wall is supposed to be *curved*.

The crucial point is the consistency condition (4.6). This equation arose as a consistency condition between the gravitino and the gaugino supersymmetry variations in our desired curved domain wall background. As it stands, it is a constraint on the possible field dependence of the matrix $\mathbf{P}$. Supergravity, on the other hand, already constrains this matrix function independently of any desired background solution. As we will now show, these supergravity constraints are so strong that (4.6) cannot be satisfied non-trivially in a *curved* domain wall background for scalar fields that sit in vector multiplets. This rules out the possibility that the scalar field ϕ of fake supergravity with a genuine matrix superpotential can be a scalar sitting in a vector multiplet. Moreover, it shows that a BPS-domain wall in true supergravity that is supported by vector scalars only can at most be flat.

In order to see this, we observe that the derivatives of $\mathbf{P}$ with respect to the vector scalars φ^x are, from its definition (3.15), determined by the derivatives of h^I:

$$\partial_x \mathbf{P} = -\sqrt{\frac{2}{3}}\mathbf{P}_I h^I_x, \qquad \partial_y\partial_x \mathbf{P} = -\sqrt{\frac{2}{3}}\mathbf{P}_I \partial_y h^I_x. \tag{4.7}$$

From the second line of (3.3), one obtains

$$\begin{aligned} \partial_y h^I_x &= \partial_y h^J_x \left(h_J h^I + h^z_J h^I_z\right) = -\sqrt{\tfrac{2}{3}} h^J_x h_{Jy} h^I + \partial_y h^J_x h^z_J h^I_z, \\ \partial_y \partial_x \mathbf{P} &= \tfrac{2}{3} g_{xy}\mathbf{P} + (\partial_y h^J_x h^z_J)\partial_z \mathbf{P}. \end{aligned} \tag{4.8}$$

In adapted coordinates $(\varphi(r), \varphi^{\hat{x}})$ with (4.5) and metric (4.4), this becomes

$$\partial^2_\varphi \mathbf{P} = \frac{2}{3} g_{\varphi\varphi}\mathbf{P} + \text{ (terms proportional to } \partial_\varphi \mathbf{P}) \tag{4.9}$$

and hence

$$\left[\partial_\varphi \mathbf{P}, \partial^2_\varphi \mathbf{P} - \frac{2}{3} g_{\varphi\varphi}\mathbf{P}\right] = 0, \tag{4.10}$$

which differs from the desired relation (4.6). The only way, (4.10) and (4.6) could be reconciled would be to demand that, along the flow curve, $[\partial_\varphi \mathbf{P}, \mathbf{P}] = 0$, or equivalently, $\partial_\varphi \mathbf{W} = f(\varphi)\mathbf{W}$, with some function $f(\varphi)$, which, however, would then imply $\gamma(r) = 1$ via (2.6), i.e. a flat domain wall[9]. Thus, we conclude that a BPS domain wall in $5D$, $\mathcal{N}=2$ supergravity that is supported by vector scalars only, can at most be flat. Therefore, the

[9]Because of $\partial_{\hat{x}}\mathbf{P} = 0$ in adapted coordinates, this is nothing but the condition $\partial_x Q^s = 0$ for flat domain walls of [24], where Q^s denotes the phase of P^s [cf. (5.13)].

curved domain walls of [5] cannot be the ones described by true supergravity where only the scalars of vector multiplets are running.

However, for the flat domain walls we find indeed agreement as $[\partial_\varphi \mathbf{P}, \mathbf{P}] = 0$ is always satisfied. This can be proven as follows. We assume no running hyper-scalars (or the situation without hyper-scalars), i.e. the q^X sit at a critical point q_0^X, such that due to (3.33),

$$h^I(\varphi) K_I^X(q_0) = 0. \tag{4.11}$$

Then (3.9) contracted with h_x^I and h^J implies

$$[\partial_x \mathbf{P}, \mathbf{P}] = 0. \tag{4.12}$$

The proof is obvious in the case of an Abelian gauge group, but holds also in the non-Abelian case, making use of (3.3), the invariance requirement on the coefficients C_{IJK} leading to $f_{IJ}{}^K h_K h^J = 0$ [12], and (4.5). This reconciles clearly (4.6) with (4.10).

4.2 Running hyper-scalars

In this section, we will consider curved BPS-domain walls that are supported by hyper-scalars only, i.e., we will assume that all potential vector scalars are constant:

$$\varphi^{x\prime} = 0. \tag{4.13}$$

The possibility that both vector scalars and hyper-scalars are running will be considered in section 4.3. The gaugino BPS-equations (3.23) now imply

$$\partial_x \mathbf{P} = 0 \tag{4.14}$$

for consistency.

We now turn to the other BPS equations. If we again make the identification (4.3), it is easy to see that, modulo the $SU(2)$-connection $q^{X\prime}\omega_{Xi}{}^j$, the gravitino BPS-equations (3.21)-(3.22) are again of the same form as the corresponding equations (2.3)-(2.4) of fake supergravity. We are thus led to the question as to whether the $SU(2)$-connection can be gauge fixed in such a way as to reproduce exactly equations (2.3)-(2.4). To answer this question, note that we only need the vanishing of this $SU(2)$ connection in one direction (the one of $q^{X\prime}$). Thus, if one can achieve

$$SU(2) \text{ gauge choice:} \qquad q^{X\prime}\omega_X^r = 0, \tag{4.15}$$

the gravitino equations in fake and true supergravity with running hyper-scalars, locally, agree. However, on a sufficiently short segment of the flow curve, this gauge can always be achieved by simply taking the relevant gauge transformation equal to the inverse of the Wilson line of the original $SU(2)$-connection along that curve segment. This is further explained and formalized in Appendix A.

Before we proceed, we would like to comment on the validity of the local $SU(2)$-symmetry that underlies this gauge choice. Geometrically, the local $SU(2) \times USp(2n_H)$ invariance is the part of the naive $SO(4n_H)$ tangent space group of the target manifold $\mathcal{M}_Q$ that survives the supersymmetric coupling to the fermions. As such, this local composite invariance should not interfere with the gaugings of isometries of the target space metric g_{XY}, as the latter is manifestly invariant under the $SU(2) \times USp(2n_H)$ reparametrizations of the quaternionic vielbeins f_X^{iA}. And indeed, as can be read off explicitly from the expressions in [14, 15], the gauged Lagrangian and supersymmetry transformations are still manifestly invariant and covariant, respectively, with respect to $SU(2)$ (and $USp(2n_H)$). The BPS-equations for domain wall solutions, in which the vector fields and fermions are set to zero, also inherit this $SU(2)$ covariance, i.e., any BPS-domain wall is part of an $SU(2)$ orbit of gauge equivalent solutions, and we are free to partially fix that gauge symmetry in the way we do above and in Appendix A.

Such a gauge choice thus restricts the form of the quaternionic vielbeins, but not the form of the metric. As an example of such a gauge choice for a curved domain wall, consider Model II in [8]. As the flow is along constant σ and $\theta = c\tau$ for constant c, the expression $q^{X\prime}\omega_X^r$ has components $q^{X\prime}\omega_X^2 = cq^{X\prime}\omega_X^1$ and $q^{X\prime}\omega_X^3 = 0$. Hence it points only in one direction, and though it is a complicated expression, an $SU(2)$ gauge transformation in that one direction can annihilate $q^{X\prime}\omega_X^r$. In the equations below, we will not explicitly use this $SU(2)$ gauge choice. However, to reproduce the formulae of fake supergravity such a gauge choice has to be assumed.

It remains to check the hyperino equation (3.24), which we already transformed to (3.32). Just as for the vector scalars in the previous section, we can now again choose adapted coordinates on $\mathcal{M}_{\rm Q}$ such that only one of the scalars q^X has a non-trivial r-dependence. We choose to call this scalar field q, and denote the orthogonal, constant, scalars by $q^{\hat{X}}$:

$$g_{XY} = \begin{pmatrix} g_{qq} & 0 \\ 0 & g_{\hat{X}\hat{Y}} \end{pmatrix}, \tag{4.16}$$

The supersymmetry condition (3.32) now splits into two equations:

$$\begin{aligned} q'\mathbf{\Theta} - g\sqrt{6}D_q\mathbf{P} &= 0, & (4.17)\\ q'[\mathbf{R}_{\hat{X}q}, \mathbf{\Theta}] - g\sqrt{6}D_{\hat{X}}\mathbf{P} &= 0. & (4.18) \end{aligned}$$

In view of (3.29), the first equation (4.17) is easily seen to be equivalent to the fake supergravity equation (2.5), *provided* the $SU(2)$ gauge (4.15) has been imposed.

The second equation (4.18), on the other hand, plays a somewhat different rôle. First note that it is different from the corresponding equation (4.5) in Section 4.1, where we had only non-trivial vector scalars. In that case, the derivative of $\mathbf{P}$ with respect to the orthogonal, constant scalars $\varphi^{\hat{x}}$ had to vanish, whereas in the case of running hyper-scalars, (4.18) no longer implies the independence of $\mathbf{P}$ of the orthogonal scalars $q^{\hat{X}}$. In fact, squaring (4.18) and using (4.17) and the quaternionic identity (3.4), one obtains, on a supersymmetric flow solution,

$$D_{\hat{X}}\mathbf{P}D^{\hat{X}}\mathbf{P} = 2D_q\mathbf{P}D^q\mathbf{P}. \tag{4.19}$$

This equation shows that at least some of the $D_{\hat{X}}\mathbf{P}$ have to be non-zero and illustrates the meaning of (4.18), which can be thought of as a constraint on the hatted derivatives of $\mathbf{P}$ that allows one to effectively eliminate the dependence of the equations on the constant scalars $q^{\hat{X}}$. The fact that this "elimination" of the $q^{\hat{X}}$ proceeds in a much less trivial way than for the vector scalars $\varphi^{\hat{x}}$, has also important consequences for the scalar potential. Recalling that $\partial_x\mathbf{P} = 0$, the potential (3.20) is

$$\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - (D_X\mathbf{P})(D^X\mathbf{P}). \tag{4.20}$$

Naively, this seems to have the wrong prefactor (-1) instead of (-3) in front of $(D_X\mathbf{P})(D^X\mathbf{P})$ in order to be identifiable with the scalar potential (2.2) of fake supergravity. However, we can at most expect to identify these two expressions after we expressed everything in terms of the only running scalar q, and, indeed, (4.19) precisely corrects the prefactor (-1) to (-3):

$$\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - 3(D_q\mathbf{P})(D^q\mathbf{P}). \tag{4.21}$$

Thus, in adapted coordinates, the supersymmetry conditions and the scalar potential agree with the corresponding expressions in fake supergravity, *provided* the gauge choice (4.15) is taken. As the $SU(2)$ curvature is proportional to the hypercomplex structure, and hence non-degenerate, $\partial_q\omega^r_{\hat{X}}$ has to be non-zero on the flow curve in the gauge where $\omega^r_q = 0$. These non-vanishing components are important for the consistency of (4.18) with (4.19).

As for the consistency condition (3.40), which for hypers only reads

$$\left[q^{X\prime}D_X\mathbf{P}\,,\, q^{X\prime}D_Xq^{Y\prime}D_Y\mathbf{P} + \frac{1}{3}q^{Y\prime}q^{Z\prime}g_{YZ}\mathbf{P}\right] = 0, \tag{4.22}$$

the use of adapted coordinates yields

$$\left[D_q\mathbf{P}, D_qD_q\mathbf{P} + \frac{1}{3}g_{qq}\mathbf{P}\right] = 0. \tag{4.23}$$

Again, this is equivalent to the fake supergravity equation (2.7) *provided* the $SU(2)$ gauge (4.15) is adopted. Note also that in contrast to the vector scalars, the hyper-scalars do not, in general, have to satisfy an analogue of the very special geometric identity (4.9) that could render the compatibility condition (4.23) automatically inconsistent for curved domain walls. In fact, it is known that curved BPS-domain walls supported by hyper-scalars exist [7,8].

To sum up, we have shown that a curved BPS domain wall supported by a hyper-scalar falls into the framework of fake supergravity, *provided that* the $SU(2)$ gauge (4.15) is imposed.

4.3 Running vector- and hyper-scalars

We conclude the comparison between the supergravity and the fake formalism by studying the case of non-trivial vector- and hyper-scalars. Applying the experience gained in the previous sections, we will show that also this general case can, at least locally, be included in the formalism of fake supergravity. This requires the choice of the adapted coordinates in two steps. First, we move to a coordinate system in which just one scalar of the vector multiplet and one hyper-scalar are running, namely φ and q. According to the results of the previous section, in this step it is necessary to adopt the $SU(2)$ gauge (4.15) that removes the $SU(2)$ spin-connection from the expression D_r, and to cast the hyperini equation (as well as its corresponding contribution to the potential) in the same form as the gaugini equation (and its corresponding contribution to the potential). In this way, the two sectors become in many aspects analogous, as will be explained more thoroughly in the next section. Here, we only focus on the commutator constraint (3.40), which, in these adapted coordinates on $\mathcal{M}_{\rm VS}$ and $\mathcal{M}_{\rm Q}$ reduces to

$$\left[\partial_r \mathbf{P}\,,\, \partial_r\partial_r \mathbf{P} + \frac{1}{3}\left(g_{qq}(q')^2 + g_{\varphi\varphi}(\varphi')^2\right)\mathbf{P}\right] = 0, \tag{4.24}$$

where $\partial_r = q'\partial_q + \varphi'\partial_\varphi$. Note that there is no mixing of kinetic terms of vector- and hypermultiplets, hence $g_{\varphi q} = 0$.

We can now perform a second change of coordinates in order to have just one scalar flowing, which is a combination of the scalars of the two sectors. Normally, coordinate transformations that mix vector and hypermultiplet scalars completely obscure the supersymmetry of a supergravity theory. In our reduced and gauge fixed system of equations, however, both types of scalars enter symmetrically, and one can consider non-trivial coordinate transformations in the plane (φ, q). We can then take a new, "total", adapted coordinate system, in which only one scalar field $\phi(r)$ is running, whereas the other one, which we will call $\hat{\phi}$, is constant and orthogonal to ϕ, at least along the flow. Thus, we use the coordinate transformation

$$(\varphi(r), q(r)) \to (\phi(r), \hat{\phi}), \tag{4.25}$$

with $\partial_r = \phi' \partial_\phi$. Dropping the vanishing terms and the overall factors in the commutator as in section 4.1, we end up with

$$\left[\partial_\phi \mathbf{P} \, , \, \partial_\phi \partial_\phi \mathbf{P} + \frac{1}{3} g_{\phi\phi} \mathbf{P} \right] = 0. \tag{4.26}$$

Now, setting $g_{\phi\phi} = 1$ by rescaling ϕ, the above commutator reduces to the corresponding expression (2.7) of fake supergravity.

We have here identified the commutator relation of fake supergravity, which is a consistency condition of the BPS equations and the potential. Our task of the next section will be to identify these BPS equations and to show how the potential of true supergravity reduces to the one of fake supergravity such that the identification of this commutator relation can be understood.

We like to complete our discussion emphasizing that there is no obstruction to the existence of curved domain walls in the presence of non-trivial hypermultiplets and vector multiplets. In section 4.1 we showed that there can be no *curved* BPS domain walls that are supported solely by vector scalars. On the other hand, there are known examples of AdS-sliced domain walls that are supported by both vector and hyper-scalars [9]. One might therefore wonder what exactly it is that the hypermultiplets do in order to circumvent the "no-go theorem" for the vector multiplets. The material we have accumulated in the previous sections allows us to give a simple answer to this question.

In (3.39) we have now

$$D_r \mathbf{\Theta} \equiv [\varphi^{x\prime} \partial_x + q^{X\prime} D_X] \mathbf{\Theta}. \tag{4.27}$$

Inserting (3.31) for $\mathbf{\Theta}$ into (3.39) and dropping all terms that do not contribute to the commutator, one derives

$$\begin{aligned} 0 &= \left[\varphi^{x\prime} \partial_x \mathbf{P}, D_r(\varphi^{x\prime} \partial_x \mathbf{P}) + \frac{1}{3} \varphi^{y\prime} \varphi^{z\prime} g_{yz} \mathbf{P} \right] \\ &= \left[\varphi^{x\prime} \partial_x \mathbf{P}, \varphi^{x\prime\prime} \partial_x \mathbf{P} + \varphi^{x\prime} \varphi^{y\prime} \partial_y \partial_x \mathbf{P} + \varphi^{x\prime} q^{X\prime} \partial_x D_X \mathbf{P} + \frac{1}{3} \varphi^{y\prime} \varphi^{z\prime} g_{yz} \mathbf{P} \right] \end{aligned} \tag{4.28}$$

where $[\partial_x, D_X] = 0$ has been used.

Choosing again adapted coordinates φ^x and q^X such that only one component of the φ^x (which we call φ) and one component of q^X (which we call q) depends on r, the above commutator simplifies to

$$\left[\partial_\varphi \mathbf{P}, \partial_\varphi \partial_\varphi \mathbf{P} + \frac{q'}{\varphi'} \partial_\varphi D_q \mathbf{P} + \frac{1}{3} g_{\varphi\varphi} \mathbf{P} \right] = 0. \tag{4.29}$$

Equation (3.36) also simplifies to

$$\frac{D_q \mathbf{P}}{q' g_{qq}} = \frac{\partial_\varphi \mathbf{P}}{\varphi' g_{\varphi\varphi}}. \tag{4.30}$$

One might now be tempted to use (4.30) to re-express $D_q\mathbf{P}$ in terms of $\partial_\varphi\mathbf{P}$ in the commutator equation (4.29). Just as in section 4.1, one would then again conclude that the only way to satisfy that constraint would be by $[\partial_\varphi\mathbf{P},\mathbf{P}] = 0$, which would then forbid curved domain walls.

However, there is a flaw in this argument: (4.30) is valid only on the chosen flow curve, as it is based on a coordinate choice that is adapted to that particular curve. Differentiating this equation with respect to φ, however, probes this relation in a direction which is not tangential to the curve, because we also have running hyper-scalars. Away from the curve, however, (4.30) is in general no longer valid. Thus, it is illegitimate to transform the mixed derivative in (4.29) into a pure φ-derivative using a φ-derivative of (4.30). What circumvents the "no-go theorem" for vector scalars, is thus the presence of the mixed derivative in (4.29) and therefore this is how hyper-scalars cure the incompatibility between curved walls and running vector scalars.

5 Similarities between vector and hypermultiplets

For a generic field configuration of $5D$ supergravity, the scalars of vector and hypermultiplets enter the field equations and the supersymmetry transformation rules in a rather different way, due to the distinct geometric structures of the corresponding scalar manifolds. This is also true for curved BPS-domain wall solutions when a generic coordinate system (φ^x, q^X) of the scalar manifold is used. Indeed, the original papers on curved domain walls in $5D$ supergravity [6–10] find visibly different BPS equations for vector and hypermultiplet scalars, and also the scalar potential does not appear "symmetric" with respect to vector and hyper scalars, as happens for flat domain walls. In sections 4.1–4.2, on the other hand, we have seen that the use of *adapted* coordinates $\varphi^x = (\varphi, \varphi^{\hat{x}})$ and $q^X = (q, q^{\hat{X}})$ and the gauge fixing of the $SU(2)$ connection formally lead to the same expressions for both types of scalars in a BPS-domain wall background. As this is an interesting result in its own right, we devote this extra section to this observation and show explicitly how the adapted coordinates lead to the same equations for both types of scalars also in the formulation of [7,9,10], where the BPS equations are not expressed in the $SU(2)$ matrix-valued form of (3.30)–(3.34).

The expressions in true supergravity that we are interested in, are the scalar potential

$$\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - 3(\partial_x\mathbf{P})(\partial^x\mathbf{P}) - (D_X\mathbf{P})(D^X\mathbf{P}), \tag{5.1}$$

and the matter BPS-equations (3.30), (3.32),

$$g_{yx}\varphi^{x\prime}\mathbf{\Theta} - \sqrt{6}g\partial_y\mathbf{P} = 0, \tag{5.2}$$
$$g_{YX}q^{X\prime}\mathbf{\Theta} + q^{X\prime}[\mathbf{R}_{YX},\mathbf{\Theta}] - \sqrt{6}gD_Y\mathbf{P} = 0. \tag{5.3}$$

Obviously, these expressions treat the vector- and the hyper-scalars differently. On the other hand, from the results of the previous section, we should be able to express them in a more symmetric form.

Let us first see, how the similarity between vector- and hyper-scalars arises at the level of the BPS-equations. As seen in Sections 4.1 and 4.2, using adapted coordinates, the BPS-equations (5.2) and (5.3) simplify to

$$\begin{aligned} \varphi' \mathbf{\Theta} &= \sqrt{6}\, g\, g^{\varphi\varphi} \partial_\varphi \mathbf{P}, && (5.4) \\ 0 &= \sqrt{6}\, g\, \partial_{\hat{x}} \mathbf{P}, && (5.5) \\ q' \mathbf{\Theta} &= \sqrt{6}\, g\, g^{qq} D_q \mathbf{P}, && (5.6) \\ q' [\mathbf{R}_{\hat{Y}q}, \mathbf{\Theta}] &= \sqrt{6}\, g\, D_{\hat{Y}} \mathbf{P}. && (5.7) \end{aligned}$$

Modulo the $SU(2)$ connection, which can be gauged away along the flow curve, (5.4) and (5.6) have the same form. Moreover, after the transformation (4.25) only one scalar is flowing and, using the gauge (4.15), we have the new BPS-equations

$$\begin{aligned} \phi' \mathbf{\Theta} &= \sqrt{6} g g^{\phi\phi} \partial_\phi \mathbf{P} \\ 0 &= \sqrt{6} g g^{\hat{\phi}\hat{\phi}} \partial_{\hat{\phi}} \mathbf{P}. \end{aligned} \qquad (5.8)$$

In this form, the BPS equation of the flowing scalar is the same as in the fake supergravity theory.

The scalar potential, *on a BPS-domain wall*, can also be made symmetric between vector and hyper-scalars. The restriction to BPS-domain walls is crucial here, because proving these statements requires using the information encoded in the orthogonal BPS equations (5.5) and (5.7). Indeed, as we saw in sections 4.1 and 4.2, eqs. (5.5) and (5.7) are constraints that allow one to eliminate the hatted derivatives of $\mathbf{P}$ in the scalar potential (5.1) to obtain the symmetric form

$$\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - 3g^{\varphi\varphi}(\partial_\varphi \mathbf{P})^2 - 3g^{qq}(D_q \mathbf{P})^2. \qquad (5.9)$$

The gauge fixing of the $SU(2)$ connection and the transformation (4.25) further simplify this to [using $\partial_{\hat{\phi}} \mathbf{P} = 0$ from (5.8)]

$$\mathcal{V}\mathbb{1} = 4\mathbf{P}^2 - 3g^{\phi\phi}(\partial_\phi \mathbf{P})^2, \qquad (5.10)$$

which reproduces (2.2) of fake supergravity upon the normalization $g^{\phi\phi} = 1$.

We have now shown that the BPS equations and scalar potential can be put in a form which treats symmetrically vector- and hyper-scalars when using $SU(2)$ matrix-valued expressions. In what follows we want to show that one can obtain more from the choice of adapted coordinates and put also the BPS equations and potential provided in [7, 9, 10] in a symmetric form. When this happens, we expect the BPS equations and potential to match those of fake supergravity in [28].

Using the norm of $\mathbf{W}$, defined as[10] $\mathbf{W}^2 = \frac{1}{4}\, g^2\, W^2\, \mathbb{1}$, the potential and first order equations of fake supergravity become those of [28]. More precisely, the potential reads

$$V = g^2\, \mathcal{V}, \qquad \mathcal{V} = -6\, W^2 + \frac{9}{2}\, \gamma^{-2}\, \partial_\phi W \partial_\phi W, \tag{5.11}$$

with γ as in (2.6), and the warp factor and scalar field satisfy the first order equations

$$\begin{aligned} U' &= \pm g\, \gamma\, W, \\ \phi' &= \mp 3g\, \gamma^{-1}\, \partial_\phi W. \end{aligned} \tag{5.12}$$

Trying to mimic this form in real supergravity and following the ideas in [24], the authors of [7, 9, 10] split the prepotential $\mathbf{P}$ in norm $W(\varphi, q)$ and phase $\mathbf{Q}(\varphi, q)$

$$P^r = \sqrt{\tfrac{3}{2}} W Q^r, \qquad Q^r Q^r = 1, \qquad \text{i.e. } \mathbf{Q}^2 = -\mathbb{1}. \tag{5.13}$$

By doing so [6, 7, 24], the potential gets closer to the fake supergravity one of (5.11):

$$\mathcal{V} = -6\, W^2 + \frac{9}{2}\, \Gamma^{-2}\, g^{xy} \partial_x W \partial_y W \; + \frac{9}{2}\, g^{XY}\, \partial_X W \partial_Y W. \tag{5.14}$$

Here[11]

$$\Gamma^{-2}(\varphi, q) \equiv 1 + W^2\, \frac{g^{xy}(\partial_x Q^s)(\partial_y Q^s)}{g^{xy}\partial_x W \partial_y W}. \tag{5.15}$$

Also the BPS equations can be extracted from the $SU(2)$-valued form by applying the above decomposition of $\mathbf{P}$ and by using the projector [6]

$$\mathrm{i}\gamma_5 \epsilon = [A(r)\; \mathbf{Q} + B(r)\; \mathbf{M}]\, \epsilon, \tag{5.16}$$

where, $\mathbf{M}$ is a field-dependent phase orthogonal to $\mathbf{Q}$ (i.e. $\mathbf{M}^2 = -\mathbb{1}$ and $\{\mathbf{Q}, \mathbf{M}\} = 0$) and A and B are functions of r, which satisfy the consistency requirement $A^2(r) + B^2(r) = 1$. This is obviously related to the projector (3.29) introduced in section 3 as

$$\mathbf{\Theta} = A\, \mathbf{Q} + B\, \mathbf{M}, \tag{5.17}$$

[10] The unnatural factor in this equation is due to the merging of the matrix notation in [5] and the notations in previous papers on true $5D$ gauged supergravity, where g denotes the gauge coupling.

[11] Defining the analogous object Γ_H^{-2} with derivatives to the scalars q^X rather than to φ^x would lead to $\Gamma_H^{-2} = 3$ by using (3.7), which in the present notations implies $W D_X Q^r = 2\varepsilon^{rst} Q^s \mathcal{R}^t_{XY} \partial^Y W$. This gives an understanding of the difference between (5.14) and (3.20).

and the components A, B and $\mathbf{M}$ can be read off from (3.31) and (3.35) by simple projections. An alternative way of fixing these functions is via the consistency conditions that follow from the integrability equations of the gravitino variation [6–8]. For instance, an interesting relation that specifies A in terms of a function of the cosmological constant on the domain wall follows from the integrability of $\delta\psi_{m\,i}$:

$$A = \mp\gamma(r) \equiv \sqrt{1 + \frac{\mathrm{e}^{-2U}}{L_d^2 g^2 W^2}}. \tag{5.18}$$

A further expression for A may be obtained by the projection of (3.30) on $\mathbf{Q}$, which results in

$$g_{xy}\varphi^{x\prime} = 3gA^{-1}\partial_y W. \tag{5.19}$$

The consistency of the square of (5.19) with the square of (3.30) then yields

$$A^{-2} = \frac{2}{3}\frac{\partial_x P^r \partial^x P^r}{\partial_y W \partial^y W} = 1 + W^2\,\frac{g^{xy}(\partial_x Q^s)(\partial_y Q^s)}{g^{xy}\partial_x W \partial_y W} = \Gamma^{-2}, \tag{5.20}$$

which further implies that (5.15) must also satisfy $\Gamma = \mp\gamma$ (so far, Γ was only defined up to a sign). At this point the other integrability conditions following from the gravitino transformations are identically satisfied and one can write the BPS equations of the system in terms of the scalar function W [7, 10]:

$$U' = \pm g\,\gamma\,W, \tag{5.21}$$

$$\phi^{\Lambda\prime} = \mp 3g\,G^{\Lambda\Sigma}\partial_\Sigma W, \tag{5.22}$$

where $G^{\Lambda\Sigma}$ is defined by

$$G^{xy} = \gamma^{-1}\,g^{xy}, \tag{5.23}$$

$$G^{XY} = \gamma\,g^{XY} + 2\sqrt{1-\gamma^2}\,\varepsilon^{rst}M^r Q^s R^{tXY}, \tag{5.24}$$

$$G^{xY} = G^{Xy} = 0. \tag{5.25}$$

Notice that when the domain wall becomes flat, i.e. when $\gamma = 1$, the projector reduces to $\mathbf{\Theta} = \mathbf{Q}$, $G^{\Lambda\Sigma} = g^{\Lambda\Sigma}$ and we recover the BPS equations of [24].

Eq. (5.14) and (5.23)–(5.25) show explicitly the afore-mentioned asymmetry between vector and hypermultiplets which appears in the formulation of [7,9,10] (This is encoded for instance in the different expressions for G^{xy} and G^{XY}). However, we are now in a position to show that this apparent asymmetry disappears when one uses the adapted coordinates.

Let us start from the hyperino BPS equation. Contracting (5.6) with $\mathbf{\Theta}$ and using the decomposition in norm and phase of $\mathbf{P}$, one can write

$$q' = 3g\, g^{qq}\Theta^r \left[W D_q Q^r + (\partial_q W) Q^r\right]. \tag{5.26}$$

The last term can be simplified by using $\Theta^r Q^r = \mp\gamma$ [see (5.17) and (5.18)] while $\Theta^r D_q Q^r$ can be determined from equation (28) of [7], which reads

$$q' B M^r D_q Q^r = \mp \left(\frac{1-\gamma^2}{\gamma W}\right) q' \partial_q W\,. \tag{5.27}$$

Adding AQ^r to the left-hand side and recalling that $Q^r D_q Q^r = 0$, this actually becomes

$$\Theta^r D_q Q^r = \mp \left(\frac{1-\gamma^2}{\gamma W}\right) \partial_q W. \tag{5.28}$$

Substituting these expressions for the projections $\Theta^r Q^r$ and $\Theta^r D_q Q^r$ into (5.26) we finally obtain

$$\begin{aligned} g_{qq} q' &= \mp 3\, g \left(\frac{1-\gamma^2}{\gamma}\right) \partial_q W \mp 3\, g\, \gamma \partial_q W \\ &= \mp 3 g \frac{1}{\gamma} \partial_q W. \end{aligned} \tag{5.29}$$

Using the inverse metric we finally get that (5.29) takes the same form [12] as (5.23) and that both look like (5.12). This shows that, in adapted coordinates, (5.22) implies the same form (5.12) for both vector and hyper-scalars despite the apparent asymmetry encoded in the matrix $G^{\Lambda\Sigma}$.

Also the potential (5.14) gets now a symmetric form using the adapted coordinates. In order to show this, one uses the fact that the non-vanishing of $D_{\hat{X}}\mathbf{P}$, implied by (4.19), has some important consequences for the derivatives of $W = \sqrt{\frac{2}{3} P^r P^r}$. Indeed, $D_{\hat{X}}\mathbf{P} \neq 0$ in general implies that $\partial_{\hat{X}} W \neq 0$, and the true supergravity potential becomes

$$\begin{aligned} \mathcal{V} &= -6W^2 + \frac{9}{2}\Gamma^{-2}(\partial_x W)(\partial^x W) + \frac{9}{2}(\partial_X W)(\partial^X W) \\ &= -6W^2 + \frac{9}{2}\Gamma^{-2}(\partial_\varphi W)(\partial^\varphi W) + \frac{9}{2}\tilde{\Gamma}^{-2}(\partial_q W)(\partial^q W), \end{aligned} \tag{5.30}$$

where

$$\Gamma^{-2} = 1 + W^2 \frac{g^{xy}(\partial_x Q^r)(\partial_y Q^r)}{g^{xy}(\partial_x W)(\partial_y W)} = 1 + W^2 \frac{(\partial_\varphi Q^r)(\partial^\varphi Q^r)}{(\partial_\varphi W)(\partial^\varphi W)}, \tag{5.31}$$

$$\tilde{\Gamma}^{-2} = 1 + \frac{\partial_{\hat{X}} W \partial^{\hat{X}} W}{(\partial_q W)(\partial^q W)}, \tag{5.32}$$

[12]Eq. (26) in [7] already proved the contracted version of (5.29).

with the last term in (5.32) possibly non-zero.

To increase the similarity between these formulae, one recalls from (4.19) that

$$\begin{aligned} 2\,\mathcal{N}_{iA}\mathcal{N}^{iA} &= \frac{9}{2}\,\partial_X W \partial^X W = \frac{9}{2}\,(\partial_q W)\,(\partial^q W) + \frac{9}{2}\,(\partial_{\hat{X}} W)(\partial^{\hat{X}} W) \\ &= (D_X P^r)(D^X P^r) = 3(D_q P^r)(D^q P^r) \\ &= \frac{9}{2}\,\left[(\partial_q W)(\partial^q W) + W^2 (D_q Q^r)(D^q Q^r)\right], \end{aligned} \tag{5.33}$$

so that

$$\partial_{\hat{X}} W \partial^{\hat{X}} W = W^2 D_q Q^r D^q Q^r, \tag{5.34}$$

and thus

$$\tilde{\Gamma}^{-2} = 1 + W^2\,\frac{D_q Q^r D^q Q^r}{(\partial_q W)(\partial^q W)}, \tag{5.35}$$

which is then exactly as for the vector scalars in (5.31). Hence, again, we see that the similarity with the vector scalars only appears *after* the "deactivated" hatted hyper-scalars have been properly taken care of.

Using (5.35) in (5.30) one gets a perfectly symmetric form between vector- and hyper-scalars, but the potential is not yet exactly in the form of (5.11)

$$\mathcal{V} = -6\,W^2 + \frac{9}{2}\,\gamma^{-2}\,(\partial_\phi W)^2, \tag{5.36}$$

which contains only *one* scalar field ϕ. In order to reproduce (5.36), one first recalls that $\Gamma^2 = \gamma^2$. A similar relation can also be derived for the hypermultiplet analogue $\tilde{\Gamma}$, by projecting (5.6) on $\mathbf{Q}$, which gives

$$q' = 3gA^{-1} g^{qq} \partial_q W. \tag{5.37}$$

The consistency of the square of (5.37) and the square of (5.6) then implies

$$A^{-2} = 1 + W^2\,\frac{D_q Q^r D^q Q^r}{(\partial_q W)(\partial^q W)} = \tilde{\Gamma}^{-2} \tag{5.38}$$

and hence $\tilde{\Gamma}^2 = \gamma^2$ via (5.18). Thus, (5.30) becomes

$$\mathcal{V} = -6\,W^2 + \frac{9}{2}\,\gamma^{-2}\,[(\partial_\varphi W)(\partial^\varphi W) + (\partial_q W)(\partial^q W)]. \tag{5.39}$$

If the domain wall is supported by vector scalars ($\phi = \varphi$) or by hyperscalars ($\phi = q$), we have $\partial_q W = 0$ or $\partial_\varphi W = 0$, respectively, and (5.36) is reproduced. In the mixed case, one has to go to the total adapted coordinates $(\phi(r), \hat{\phi})$, and also obtains (5.36). Note that in this formulation with γ instead of Γ and $\tilde{\Gamma}$, the explicit dependence on the $SU(2)$ connection has disappeared from the scalar potential and the BPS equations. It only reenters upon the identification of γ^2 with $\tilde{\Gamma}^2$.

Finally, we can also read off the conditions for a BPS domain wall to be curved. If the domain wall is supported by vector scalars only, a domain wall would be curved if $\Gamma \neq 1 \Leftrightarrow \partial_\varphi Q^r \neq 0$. As we saw, however, this is incompatible with the constraints imposed by very special geometry.

A BPS domain wall of true supergravity that is supported by hyper-scalars only, by contrast, is curved if any of the following equivalent conditions is satisfied (they are equivalent on a BPS-domain wall solution):

$$\partial_{\hat{X}} W \neq 0 \Leftrightarrow D_q Q^r \neq 0. \tag{5.40}$$

As there are examples of such domain walls in true supergravity, these conditions, as well as the commutator constraint

$$\left[D_q \mathbf{P}, D_q D_q \mathbf{P} + \frac{1}{3} g_{qq} \mathbf{P} \right] = 0, \tag{5.41}$$

have solutions in quaternionic geometry.

6 Conclusions

Our original motivation to find the relation between fake supergravity and genuine supergravity partially evolved into an independent and insightful general study of curved BPS-domain walls in $5D$, $\mathcal{N} = 2$ gauged supergravity. Completing and clarifying previous work, we have derived several results that deserve interest in their own right. Most importantly, we showed that curved BPS-domain walls in true supergravity require non-trivial profiles of scalars that sit in hypermultiplets. It is interesting to notice that a similar outcome was obtained in order to construct domain–wall solutions interpolating between minima of the scalar potential as argued in [29] and then proved in [24, 30, 31]. This result is of general validity and is independent of the relation to fake supergravity.

In order to make contact with fake supergravity, a true supergravity theory has to be subjected to two types of gauge fixings. The first one has to do with the above-mentioned observation that a supersymmetric curved domain wall must involve non-trivial hyper-scalars. These in turn introduce the $SU(2)$-connection, where $SU(2)$ is the factor of the holonomy group of quaternionic-Kähler manifolds. This $SU(2)$-connection is absent in the fake supergravity framework of [5]. The equations of fake supergravity can thus only be reproduced if a particular $SU(2)$ gauge, (4.15), is chosen. We showed that, locally, this is always possible.

The second type of gauge fixing is a partial fixing of the coordinates on the scalar manifold of true supergravity. That is, one has to use "adapted coordinates", in which only one scalar field is flowing in order to make contact with the one-scalar formalism of [5]. Clearly, the identification of this scalar and hence the adapted coordinate system depend on the particular domain wall one is considering, and is in general only a local coordinate choice on the scalar manifold. It may even be a local choice for part of the flow only. Indeed, a flow line in the complete scalar manifold may return to the same point of the manifold but flowing in a different direction. This implies that at this later stage of the flow, one has to use different adapted coordinates, although one is describing the same region on the scalar manifold.

The adapted coordinates can be viewed as an analogue of the "free fall" reference frame of a freely falling body, which, as in general relativity, is certainly somewhat unsatisfactory due to the breaking of the general coordinate invariance. However, as a technical device, this coordinate choice is essential to make contact with the one-field formalism of [5].

The identification of true and fake supergravity applies only on the line of flow of a chosen domain wall in the scalar manifold. The formulae for the corresponding potentials can be made equal due to a relation (4.19) between derivatives of the prepotential in directions along and orthogonal to the line. This equation is a consequence of the BPS equations of scalars not considered in the one-scalar formalism of fake supergravity [5]. This is nothing but an illustration of the obvious further richness of ordinary supergravity, which contains more equations than what can be encoded into its 'fake' counterpart. These extra equations should determine how the line of flow is embedded into the larger scalar manifold of the full supergravity theory. The equivalence indeed only holds along such a line determined by the true supergravity equations. Furthermore, the full supergravity theory gives the expression for the triplet superpotential $\mathbf{W}$. This object is not specified in fake supergravity, and an arbitrary expression for $\mathbf{W}$ cannot necessarily arise from a true supergravity.

Finally, as a further interesting observation whose relevance goes beyond the comparison with fake supergravity, we have shown that the careful choice of "adapted coordinates" and the fine tuning of the $SU(2)$ connection allows to describe BPS flows in a formalism that treats symmetrically vector and hypermultiplet scalars, both with respect to their equations of motion and the prefactors in the potential (5.9). In this way, at least for the purposes of this paper, a lot of information can be encoded in the dynamics of a single flowing (possibly "hybrid") scalar field.

Acknowledgments

We thank Klaus Behrndt, Gabriel Lopes Cardoso, Mirjam Cvetič, Riccardo D'Auria, Sergio Ferrara, Dan Freedman, Renata Kallosh, Jan Louis and Martin Schnabl for interesting discussions. Anna C. and G.D. are grateful to K.U. Leuven for hospitality during the early stages of this project.

This work is supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime. The work of A.Celi and A.V.P. is supported in part by the Federal Office for Scientific, Technical and Cultural Affairs through the "Interuniversity Attraction Poles Programme – Belgian Science Policy" P5/27. The work of M.Z. is supported by an Emmy-Noether-Fellowship of the German Research Foundation (DFG) (ZA 279/1-1).

A The SU(2) gauge choice

The geometric structure of quaternionic manifolds is determined by complex structures. The 3 complex structures $(J^r)_X{}^Y$ form a span, which means that one can rotate them locally in the manifold, i.e. depending on local functions $l^r(q)$, without changing the geometry:

$$\delta_l (J^r)_X{}^Y = \varepsilon^{rst} l^s (J^t)_X{}^Y. \tag{A.1}$$

Also other vector quantities, such as the moment maps, rotate in the same way under these $su(2)$-reparametrizations. The gauge field is the connection ω_X^r:

$$\delta_l \omega_X^r = -\tfrac{1}{2}\partial_X l^r + \varepsilon^{rst} l^s \omega_X^t. \tag{A.2}$$

As mentioned in section 3.1, the curvature of this gauge field $\mathcal{R}^r_{XY}$ is proportional to the complex structure multiplied by the quaternionic-Kähler metric. Killing spinors transform in the doublet representation, i.e.

$$\delta_l \epsilon_i = l^r (\sigma^r)_i{}^j \epsilon_j. \tag{A.3}$$

Notice that these are not local spacetime gauge transformations, but transformations on the description of the quaternionic structures, local in the quaternionic-Kähler manifold. This is the gauge freedom that we are fixing with the choice (4.15). Note that these gauge transformations leave the quaternionic metric g_{XY} invariant and are thus compatible with the adapted coordinate choice (4.16). More details can be found in [32, 33].

We now consider the finite transformations rather than the infinitesimal ones mentioned above. These transform the doublet spinors as $\epsilon_i \to V_i{}^j \epsilon_j$ and the $SU(2)$ connection transforms as

$$\boldsymbol{\omega}_X \to \tilde{\boldsymbol{\omega}}_X = -\mathbf{V}(\partial_X - \boldsymbol{\omega}_X)\mathbf{V}^{-1}. \tag{A.4}$$

Let $q^X(r)$ be a curve on $\mathcal{M}_Q$ with starting point $q_0^X = q^X(0)$. The path-ordered exponential ("Wilson line")

$$\mathbf{U}(q^X(r), q_0^X) \equiv \mathcal{P}\left\{\exp\left[\int_0^r \boldsymbol{\omega}_X(q^X(\bar r)) q^{X\prime}(\bar r) \mathrm{d}\bar r\right]\right\} \tag{A.5}$$

satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\mathbf{U}(q^X(r), q_0^X) &= q^{X\prime}(r)\boldsymbol{\omega}_X(q^X(r))\mathbf{U}(q^X(r), q_0^X) \\ \Leftrightarrow q^{X\prime}(\partial_X - \boldsymbol{\omega}_X)\mathbf{U} &= 0. \end{aligned} \tag{A.6}$$

$\mathbf{U}$ has been defined only on the curve, but there should be some analytic continuation of that function, at least in a neighborhood of the curve. If we now choose the $SU(2)$ transformation

$$\mathbf{V} = \mathbf{U}^{-1}, \tag{A.7}$$

on this neighborhood, the tangential component of the new, gauge transformed, $SU(2)$ connection (A.4) becomes, remembering (A.6),

$$q^{X\prime}\tilde{\boldsymbol{\omega}}_X = -q^{X\prime}\mathbf{U}^{-1}(\partial_X - \boldsymbol{\omega}_X)\mathbf{U} = 0. \tag{A.8}$$

In other words, the component of the $SU(2)$ connection $\boldsymbol{\omega}_X$ tangential to the flow curve can always be gauged away for sufficiently short curve segments.

Using these notations, one may also rephrase the procedure of adopting the gauge choice (4.15) as replacing the identification (4.3) with

$$\mathbf{W} = -\frac{\mathrm{i}g}{\sqrt{6}}\mathbf{U}^{-1}\mathbf{P}\mathbf{U}, \tag{A.9}$$

in a patch where the flows do not intersect and starting from some q_0^X on any line of flow.

References

[1] P. K. Townsend, *Positive energy and the scalar potential in higher dimensional (super)gravity theories*, Phys. Lett. **B148** (1984) 55

[2] K. Skenderis and P. K. Townsend, *Gravitational stability and renormalization-group flow*, Phys. Lett. **B468** (1999) 46–51, `hep-th/9909070`

[3] E. Witten, *A simple proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981) 381

[4] J. M. Nester, *A new gravitational expression with a simple positivity proof*, Phys. Lett. **83A** (1981) 241

[5] D. Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, Phys. Rev. **D69** (2004) 104027, `hep-th/0312055`

[6] G. Lopes Cardoso, G. Dall'Agata and D. Lüst, *Curved BPS domain wall solutions in five-dimensional gauged supergravity*, JHEP **07** (2001) 026, `hep-th/0104156`

[7] G. L. Cardoso, G. Dall'Agata and D. Lüst, *Curved BPS domain walls and RG flow in five dimensions*, JHEP **03** (2002) 044, `hep-th/0201270`

[8] K. Behrndt and M. Cvetič, *Bent BPS domain walls of $D = 5$ $N = 2$ gauged supergravity coupled to hypermultiplets*, Phys. Rev. **D65** (2002) 126007, `hep-th/0201272`

[9] G. L. Cardoso and D. Lüst, *The holographic RG flow in a field theory on a curved background*, JHEP **09** (2002) 028, `hep-th/0207024`

[10] A. H. Chamseddine and W. A. Sabra, *Einstein brane-worlds in* $5D$ *gauged supergravity*, Phys. Lett. **B517** (2001) 184–190, `hep-th/0106092`, Erratum-ibid. **B537** (2002) 353

[11] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of* $N = 2$ *Maxwell–Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984) 244

[12] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the* $d = 5$ *Maxwell–Einstein supergravity theories: more on Jordan algebras*, Nucl. Phys. **B253** (1985) 573

[13] M. Günaydin and M. Zagermann, *The gauging of five-dimensional,* $N = 2$ *Maxwell–Einstein supergravity theories coupled to tensor multiplets*, Nucl. Phys. **B572** (2000) 131–150, `hep-th/9912027`

[14] A. Ceresole and G. Dall'Agata, *General matter coupled* $\mathcal{N} = 2$, $D = 5$ *gauged supergravity*, Nucl. Phys. **B585** (2000) 143–170, `hep-th/0004111`

[15] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, $N = 2$ *supergravity in five dimensions revisited*, Class. Quant. Grav. **21** (2004) 3015–3041, `hep-th/0403045`

[16] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of* AdS_5 *and its field theory dual*, JHEP **05** (2003) 072, `hep-th/0304129`

[17] A. Clark and A. Karch, in preparation

[18] G. Dall'Agata, *String compactifications with fluxes*, Class. Quant. Grav. **21** (2004) S1479–S1499

[19] C. Mayer and T. Mohaupt, *Domain walls, Hitchin's flow equations and* G_2*-manifolds*, `hep-th/0407198`

[20] J. Louis and S. Vaulà, in preparation

[21] J. Bagger and E. Witten, *Matter couplings in* $N = 2$ *supergravity*, Nucl. Phys. **B222** (1983) 1

[22] J. R. Ellis, M. Günaydin and M. Zagermann, *Options for gauge groups in five-dimensional supergravity*, JHEP **11** (2001) 024, `hep-th/0108094`

[23] R. D'Auria, S. Ferrara and P. Frè, *Special and quaternionic isometries: General couplings in $N = 2$ supergravity and the scalar potential*, Nucl. Phys. **B359** (1991) 705–740

[24] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. **D64** (2001) 104006, `hep-th/0104056`

[25] A. Ceresole and G. Dall'Agata, *Five-dimensional gauged supergravity and the brane world*, Nucl. Phys. Proc. Suppl. **102** (2001) 65–70

[26] A. Celi, *Toward the classification of BPS solutions of $\mathcal{N} = 2$, $d = 5$ gauged supergravity with matter couplings*, `hep-th/0405283`, Ph.D. thesis, Milano

[27] S. L. Cacciatori and A. Celi, *On the general properties of the BPS equations in $\mathcal{N} = 2$, $d = 5$ supergravity with matter*, in preparation

[28] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, Phys. Rev. **D62** (2000) 046008, `hep-th/9909134`

[29] K. Behrndt, C. Herrmann, J. Louis and S. Thomas, *Domain walls in five dimensional supergravity with non-trivial hypermultiplets*, JHEP **01** (2001) 011, `hep-th/0008112`

[30] D. V. Alekseevsky, V. Cortés, C. Devchand and A. Van Proeyen, *Flows on quaternionic-Kähler and very special real manifolds*, Commun. Math. Phys. **238** (2003) 525–543, `hep-th/0109094`

[31] K. Behrndt and G. Dall'Agata, *Vacua of $N = 2$ gauged supergravity derived from non-homogeneous quaternionic spaces*, Nucl. Phys. **B627** (2002) 357–380, `hep-th/0112136`

[32] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *The map between conformal hypercomplex/hyperkähler and quaternionic(-Kähler) geometry*, in preparation

[33] A. Van Proeyen, *Supergravity with Fayet-Iliopoulos terms and R-symmetry*, `hep-th/0410053`, proceedings of the EC-RTN Workshop 'The quantum structure of spacetime and the geometric nature of fundamental interactions', Kolymbari, Crete, 5-10/9/2004, to be published in Fortschritte der Physik