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ON THE FAMILY OF ANALYTIC MAPPINGS BETWEEN TWO ULTRAHYPERELLIPTIC SURFACES

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§ 1. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions each of which has no zero other than an infinite number of simple zeros. We denote by $\mathfrak{A}(R,S)$ the family of non-trivial analytic mappings φ of R into S. It follows from Ozawa's theorem [5] that *for every* $\varphi\in\mathfrak{A}(R,S)$ *there exists a non-constant entire function* $h(z)$ *satisfying the equation*

$$f(z)^2G(z)=g\circ h(z)$$

with a suitable entire function $f(z)$. Then we shall call $h(z)$ the *projection* of the analytic mapping φ (cf. Ozawa [6]). We denote by $\mathfrak{H}(R,S)$ the family of projections of analytic mappings belonging to $\mathfrak{A}(R,S)$. Let ρ_f be the order of the referred function f.

From now on we may suppose that G (or g) is always expressed as the canonical product having the same zeros of the original function G (or g) when the order $\rho_{N(r,0,G)}$ (or $\rho_{N(r,0,g)}$) is finite.

§ 2. Theorem 1 in Hiromi-Mutō [2] may be stated as in the following form:

THEOREM A. *If* $\rho_G<+\infty$, $0<\rho_g<+\infty$ *and* $\mathfrak{A}(R,S)$ *is not empty, then every element* $h(z)$ *belonging to* $\mathfrak{H}(R,S)$ *is a polynomial of same degree* p.

In this paper we shall prove the following theorems:

THEOREM 1. *Assume that* $\rho_g<+\infty$ *and there exists a polynomial* $h_p(z)$ *of degree* p *belonging to* $\mathfrak{H}(R,S)$. *Then every element* $h(z)$ *belonging to* $\mathfrak{H}(R,S)$ *is a polynomial of the same degree* p.

And further if $\rho_g>0$, *or if* p *is odd, then we have* $|a_p|=|b_p|$, *where* $h_p(z)=a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0$ $(a_p\neq 0)$ *and* $h(z)=b_pz^p+b_{p-1}z^{p-1}+\cdots+b_0$ $(b_p\neq 0)$.

The last assertion of this Theorem 1 is best possible. This fact will be shown by an example in § 6.

THEOREM 2. *Let* R *and* S *be two ultrahyperelliptic surfaces with* $P(R)=4$ *and* $P(S)=4$, *respectively. If there exists a polynomial* $h_p(z)$ *of degree* p *belonging to* $\mathfrak{H}(R,S)$, *then every element* $h(z)$ *belonging to* $\mathfrak{H}(R,S)$ *is a polynomial of the same*

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degree p. *And further, we have* $|a_p|=|b_p|$, *where* $h_p(z)=a_p z^p+a_{p-1}z^{p-1}+\cdots+a_0$ $(a_p\neq 0)$ *and* $h(z)=b_p z^p+b_{p-1}z^{p-1}+\cdots+b_0$ $(b_p\neq 0)$.

In general, a study of these theorems suggests the following problem which we have been unable to solve:

For every pair $h_1(z)$ *and* $h_2(z)$ *belonging to* $\mathfrak{H}(R, S)$, *is there a polynomial* $F_{h_1,h_2}(x, y)$ *of* x *and* y *such that* $F_{h_1,h_2}(h_1(z), h_2(z))\equiv 0$?

§ **3.** In the first place we shall prove the following lemmas:

LEMMA 1. *If* $g(z)$ *and* $h(z)$ *are transcendental entire functions and* $h_p(z)$ *is a polynomial of degree* $p\geqq 1$, *then we have*

$$\lim_{r\to\infty}\frac{T(r, g\circ h_p)}{T(r, g\circ h)}=0.$$

Proof. Since $h(z)$ is a transcendental entire function, by Hayman [1, p. 51], we have for any fixed $N>p$ and sufficiently large r,

$$T(r, g\circ h)\geqq\frac{1}{3}T(r^{N+1}, g).$$

On the other hand, we set $h_p(z)=a_p z^p+a_{p-1}z^{p-1}+\cdots+a_1 z+a_0$ $(a_p\neq 0)$. Since $|h_p(z)|\leqq|a_p|r^p(1+\varepsilon)$ for sufficiently large $|z|=r$, we have

$$\begin{aligned}T(r, g\circ h_p)&\leqq\log M_{g\circ h_p}(r)\leqq\log M_g(M_{h_p}(r))\leqq\log M_g(|a_p|r^p(1+\varepsilon))\\&\leqq 3T(2|a_p|r^p(1+\varepsilon), g).\end{aligned}$$

And we know that $T(r, g)$ is an increasing convex function of $\log r$, so that $T(r, g)/\log r$ is finally increasing and hence

$$\frac{T(2|a_p|r^p(1+\varepsilon), g)}{\log 2|a_p|r^p(1+\varepsilon)}\leqq\frac{T(r^{N+1}, g)}{\log r^{N+1}},$$

that is,

$$\frac{T(2|a_p|r^p(1+\varepsilon), g)}{T(r^{N+1}, g)}\leqq\frac{p\log r+\log 2|a_p|(1+\varepsilon)}{(N+1)\log r}\to\frac{p}{N+1}\qquad\text{as}\quad r\to+\infty.$$

Thus we obtain

$$\varlimsup_{r\to\infty}\frac{T(r, g\circ h_p)}{T(r, g\circ h)}\leqq\varlimsup_{r\to\infty}\frac{3T(2|a_p|r^p(1+\varepsilon), g)}{(1/3)T(r^{N+1}, g)}\leqq\frac{9p}{N+1},$$

and this proves Lemma 1. q.e.d.

LEMMA 2. *Let* $g(z)$ *be an entire function and* $h_1(z)$ *and* $h_2(z)$ *be two polynomials of the form* $a_p z^p+a_{p-1}z^{p-1}+\cdots+a_0$ $(a_p\neq 0)$ *and* $b_p z^p+b_{p-1}z^{p-1}+\cdots+b_0$ $(b_p\neq 0)$, *respectively. Then we have*

$$\lim_{r\to\infty}\frac{M_{g\circ h_1}(r)}{M_{g\circ h_2}(r)}=\begin{cases}(|a_p|/|b_p|)^q, & \textit{if } g(z) \textit{ is a polynomial of degree } q,\\ 0 & \textit{if } g(z) \textit{ is transcendental and } |a_p|<|b_p|,\\ +\infty & \textit{if } g(z) \textit{ is transcendental and } |a_p|>|b_p|.\end{cases}$$

Proof of Lemma 2. The result is clearly true in the case where $g(z)$ is a polynomial of degree q.

Suppose that $g(z)$ is transcendental and $|a_p|<|b_p|$. Then for $\varepsilon>0$ satisfying $|b_p|(1-\varepsilon)>|a_p|(1+\varepsilon)$, there exists $r_1>0$ such that $|h_1(z)|\leqq|a_p|r^p(1+\varepsilon)$ and $|h_2(z)|\geqq|b_p|r^p(1-\varepsilon)$ are valid for all $r>r_1$, $r=|z|$. Putting $m_{h_2}(r)=\min_{|z|=r}|h_2(z)|$, we have for $r>r_1$,

$$M_{g\circ h_1}(r)\leqq M_g(M_{h_1}(r))\leqq M_g(|a_p|r^p(1+\varepsilon))$$

and

$$M_{g\circ h_2}(r)\geqq M_g(m_{h_2}(r))\geqq M_g(|b_p|r^p(1-\varepsilon)).$$

It is well known from Hadamard's three circle theorem that $\log M_g(r)$ is an increasing convex function of $\log r$, so that $\log M_g(r)/\log r$ is finally increasing and tends to infinite as $r\to+\infty$. Hence we have for $r>r_2>r_1$,

$$\frac{\log M_g(|a_q|r^p(1+\varepsilon))}{\log|a_p|r^p(1+\varepsilon)}\leqq\frac{\log M_g(|b_p|r^p(1-\varepsilon))}{\log|b_p|r^p(1-\varepsilon)},$$

and for any fixed N and $r>r_3>r_1$,

$$M_g(|b_p|r^p(1-\varepsilon))\geqq(|b_p|r^p(1-\varepsilon))^N.$$

Therefore we deduce for all $r>\max(r_2, r_3)$,

$$\begin{aligned}\frac{M_{g\circ h_1}(r)}{M_{g\circ h_2}(r)}&\leqq\frac{M_g(|a_p|r^p(1+\varepsilon))}{M_g(|b_p|r^p(1-\varepsilon))}\\ &\leqq M_g(|b_p|r^p(1-\varepsilon))^{-(\log|b_p|(1-\varepsilon)-\log|a_p|(1+\varepsilon))/\log|b_p|r^p(1-\varepsilon)}\\ &\leqq(|b_p|r^p(1-\varepsilon))^{-N(\log|b_p|(1-\varepsilon)-\log|a_p|(1+\varepsilon))/\log|b_p|r^p(1-\varepsilon)}\\ &=\exp\left(-N\log\frac{|b_p|(1-\varepsilon)}{|a_p|(1+\varepsilon)}\right)=\left(\frac{|b_p|(1-\varepsilon)}{|a_p|(1+\varepsilon)}\right)^{-N}.\end{aligned}$$

This implies

$$\overline{\lim_{r\to\infty}}\frac{M_{g\circ h_1}(r)}{M_{g\circ h_2}(r)}\leqq\left(\frac{|b_p|(1-\varepsilon)}{|a_p|(1+\varepsilon)}\right)^{-N}.$$

Since N can be chosen as large as we please, we obtain

$$\lim_{r\to\infty}\frac{M_{g\circ h_1}(r)}{M_{g\circ h_2}(r)}=0.$$

The last assertion of the lemma is clearly deduced from the above argument.
q.e.d.

§ 4. *Proof of Theorem* 1. Our assumption implies that with a suitable entire function $f_p(z)$, the equation

$$f_p(z)^2G(z)=g\circ h_p(z) \tag{4. 1}$$

is valid. And for $h(z)$ belonging to $\mathfrak{H}(R, S)$, there exists a suitable entire function $f(z)$ satisfying the equation

$$f(z)^2G(z)=g\circ h(z). \tag{4. 2}$$

In the first place we shall prove that every element $h(z)$ of $\mathfrak{H}(R, S)$ is a polynomial of degree p. To this end, we shall consider two cases according as $\rho_g>0$ or $\rho_g=0$.

CASE $0<\rho_g<+\infty$. If ρ_g is finite, so is $\rho_{g\circ h_p}$, for $h_p(z)$ is a polynomial. From the equation (4. 1) we deduce that

$$N(r, 0, G)\leqq N(r, 0, g\circ h_p). \tag{4. 3}$$

Hence $\rho_{N(r,0,G)}$, that is, ρ_G is finite. Therefore it follows from Theorem A that every element $h(z)$ of $\mathfrak{H}(R, S)$ is a polynomial of degree p.

CASE $\rho_g=0$. If ρ_g is zero, so is $\rho_{g\circ h_p}$. Then (4. 3) yields that $\rho_{N(r,0,G)}=0$, that is, $\rho_G=0$. Hence by (4. 1) we have $\rho_{f_p}=0$. Since $f_p(z)$ has only at most $p-1$ zero points where $h_p'(z)$ vanishes, $f_p(z)$ is a polynomial of degree at most $p-1$.

We contrarily assume that $h(z)$ is a transcendental entire function. Then using the reasoning of Hiromi-Mutō [2, pp. 239-240], we deduce that $h(z)$ is of finite order and

$$\lim_{r\to\infty}\frac{T(r, h)}{N_2(r, 0, g\circ h)}=0, \qquad \lim_{r\to\infty}\frac{N(r, 0, g\circ h)}{N_2(r, 0, g\circ h)}=1, \tag{4. 4}$$

where $N_2(r, 0, f)$ is the counting function of simple zeros of the referred function f. Using (4. 4) together with $N(r, 0, G)\geqq N_2(r, 0, g\circ h)$ and $\rho_G=0$, we have $\rho_h=0$. It follows from (4. 1), (4. 2) and (4. 4) that

$$N(r, 0, g\circ h_p)\geqq N(r, 0, G)\geqq N_2(r, 0, g\circ h_p)=N(r, 0, g\circ h_p)+O(\log r)$$

and

$$N(r, 0, g\circ h)\geqq N(r, 0, G)\geqq N_2(r, 0, g\circ h)=N(r, 0, g\circ h)+o(N_2(r, 0, g\circ h)).$$

Hence we have

$$\lim_{r\to\infty}\frac{N(r,0,g\circ h_p)}{N(r,0,g\circ h)}=1. \tag{4.5}$$

Using Lemma 1 and (4.5) we have

$$\varlimsup_{r\to\infty}\frac{N(r,0,g\circ h)}{T(r,g\circ h)}\leqq\varlimsup_{r\to\infty}\frac{T(r,g\circ h_p)}{T(r,g\circ h)}\varlimsup_{r\to\infty}\frac{N(r,0,g\circ h_p)}{T(r,g\circ h_p)}\varlimsup_{r\to\infty}\frac{N(r,0,g\circ h)}{N(r,0,g\circ h_p)}=0,$$

that is, $\delta(0,g\circ h)=1$.

On the other hand (4.5) together with $\rho_{g\circ h_p}=0$ yields $\rho_{N(r,0,g\circ h)}=0$. Combining $\rho_{N(r,0,g\circ h)}=0$ and $\rho_g=\rho_h=0$, we obtain $\rho_{g\circ h}=0$. In fact, let $\{w_\mu\}$ be the set of zeros of $g(w)$ and $\{z_{\mu\nu}\}$ be the set of w_μ-points of $h(z)$. If $g(0)=A\neq 0$ and $g(h(0))\neq 0$, then, taking $\rho_g=\rho_h=0$ into account, we have

$$g(w)=A\prod_{\mu=1}^{\infty}\left(1-\frac{w}{w_\mu}\right),\qquad w_\mu\neq 0, \tag{4.6}$$

and

$$1-\frac{h(z)}{w_\mu}=\left(1-\frac{h(0)}{w_\mu}\right)\prod_\nu\left(1-\frac{z}{z_{\mu\nu}}\right),\qquad z_{\mu\nu}\neq 0. \tag{4.7}$$

Since $\rho_{N(r,0,g\circ h)}=0$, the product

$$\prod_{\mu,\nu}\left(1-\frac{z}{z_{\mu\nu}}\right) \tag{4.8}$$

converges uniformly in any bounded circle. Therefore by (4.6), (4.7) and (4.8) we have

$$g\circ h(z)=A\prod_{\mu=1}^{\infty}\left(1-\frac{h(0)}{w_\mu}\right)\prod_{\mu=1}\prod_{\nu}\left(1-\frac{z}{z_{\mu\nu}}\right)$$
$$=g\circ h(0)\prod_{\mu,\nu}\left(1-\frac{z}{z_{\mu\nu}}\right).$$

Thus we have $\rho_{g\circ h}=0$ when $g(0)\neq 0$, $g(h(0))\neq 0$. In the other cases we similarly deduce $\rho_{g\circ h}=0$.

Since an entire function of order zero has no deficient value, we have a desired contradictory fact, $\rho_{g\circ h}=0$ and $\delta(0,g\circ h)=1$. Hence $h(z)$ must be a polynomial.

Next we assume that $h_p(z)=a_pz^p+\cdots+a_1z+a_0$ $(a_p\neq 0)$, $h(z)=b_qz^q+\cdots+b_1z+b_0$ $(b_q\neq 0)$ and $q>p$. Then we have, for any ε with $0<\varepsilon<1$ and for any sufficiently large r,

$$N(r,0,g\circ h)\geqq N(|b_q|r^q(1-\varepsilon),0,g)+O(\log r)$$

and

$$N(r,0,g\circ h_p)\leqq N(|a_p|r^p(1+\varepsilon),0,g)+O(\log r)$$

And we know that $N(r, 0, g)$ is an increasing convex function of $\log r$, so that $N(r, 0, g)/\log r$ is finally increasing and hence

$$\frac{N(r, 0, g\circ h)}{N(r, 0, g\circ h_p)} \geqq \frac{N(|b_q|r^q(1-\varepsilon), 0, g)+O(\log r)}{N(|a_p|r^p(1+\varepsilon), 0, g)+O(\log r)}$$

$$\sim \frac{N(|b_q|r^q(1-\varepsilon), 0, g)}{N(|a_p|r^p(1+\varepsilon), 0, g)} \geqq \frac{q \log r+\log |b_q|(1-\varepsilon)}{p \log r+\log |a_p|(1+\varepsilon)}$$

$$\rightarrow \frac{q}{p} > 1 \quad \text{as} \quad r\rightarrow\infty.$$

This contradicts (4. 5). Similarly we have also a contradiction when $q<p$. Therefore we have $q=p$, that is, $h(z)$ is a polynomial of degree p.

§ **5.** In order to complete our proof we shall prove that if $\rho_g>0$, or if p is odd, then $|a_p|=|b_p|$.

We contrarily suppose that $|a_p|<|b_p|$. For $\varepsilon>0$ satisfying $|b_p|(1-\varepsilon)^3>|a_p|(1+\varepsilon)^3$, there exists $r_1>0$ such that $|a_p|r^p(1-\varepsilon)<|h_p(z)|<|a_p|r^p(1+\varepsilon)$ and $|b_p|r^p(1-\varepsilon)<|h(z)|<|b_p|r^p(1+\varepsilon)$ are valid for all $r\geqq r_1$, $r=|z|$. It follows from (4. 1) and (4. 2) that

$$n(r, 0, G)\leqq n(r, 0, g\circ h_p)\leqq pn(|a_p|r^p(1+\varepsilon), 0, g)$$

and

$$n(r, 0, G)\geqq n(r, 0, g\circ h)-2(p-1)\geqq pn(|b_p|r^p(1-\varepsilon), 0, g)-2(p-1),$$

for all $r\geqq r_1$. Hence we obtain, for all $r\geqq r_1$,

$$p(n(|a_p|r^p(1+\varepsilon), 0, g)-n(|b_p|r^p(1-\varepsilon), 0, g)+2)\geqq 2,$$

that is, for all $r>r_1$,

$$n(|b_p|r^p(1-\varepsilon), 0, g)-n(|a_p|r^p(1+\varepsilon), 0, g)=0 \quad \text{or } 1. \tag{5. 1}$$

Let $\{w_j\}_{j=1}^{\infty}$ be the set of zeros of $g(w)$ satisfying $|w_j|>|b_p|r_1^p(1+\varepsilon)$, and suppose that $|w_1|\leqq|w_2|\leqq\cdots$. From (5. 1) we deduce, for all $j\geqq 1$,

$$0<\left|\frac{w_j}{w_{j+1}}\right|\leqq\frac{|a_p|(1+\varepsilon)}{|b_p|(1-\varepsilon)}<1. \tag{5. 2}$$

Therefore the exponent of convergence of the sequence $\{w_j\}$ is zero. Hence $\rho_{N(r,0,g)}=0$, that is $\rho_g=0$.

Next, if $\rho_g=0$, then $\rho_{g\circ h_p}=\rho_{g\circ h}=\rho_G=0$. Hence $f_p(z)$ and $f(z)$ must be polynomials of degree at most $p-1$. We denote by μ and ν the degrees of $f_p(z)$ and $f(z)$, respectively. If $\mu=\nu$, then it follows from equations (4. 1) and (4. 2) that

$$M_{g\circ h_p}(r)=M_{f_p^2 G}(r)\geqq m_{f_p^2}(r)M_G(r)$$

and

$$M_{g\circ h}(r)=M_{f^2G}(r)\leqq M_{f^2}(r)M_G(r).$$

Hence we have

$$\varliminf_{r\to\infty}\frac{M_{g\circ h_p}(r)}{M_{g\circ h}(r)}\geqq\varliminf_{r\to\infty}\frac{m_{f_p^2}(r)M_G(r)}{M_{f^2}(r)M_G(r)}>0.$$

However, using Lemma 2 and noting $|a_p|<|b_p|$, we have

$$\lim_{r\to\infty}\frac{M_{g\circ h_p}(r)}{M_{g\circ h}(r)}=0,$$

which is a contradiction. Therefore, noting Lemma 2, we obtain $\nu>\mu$.

From the equations (4. 1) and (4. 2) we deduce that

$$2n(r,0,f_p)+n(r,0,G)=n(r,0,g\circ h_p)$$

and

$$2n(r,0,f)+n(r,0,G)=n(r,0,g\circ h),$$

that is, for all $r>r_2>r_1$,

$$(5.3)\qquad 2(\nu-\mu)=2(n(r,0,f)-n(r,0,f_p))=n(r,0,g\circ h)-n(r,0,g\circ h_p)>0.$$

Let w_j be an element of $\{w_j\}$ satisfying the inequality $|w_j|>|b_p|r_2^p(1+\varepsilon)$. We put $r_j'=(|w_{j+1}|/(|b_p|(1-\varepsilon)))^{1/p}$, $r_j''=(|w_j|/(|a_p|(1-\varepsilon)))^{1/p}$ and $r_j=\max(r_j',r_j'')$ $(>r_2)$. Then, using (5. 2), $|a_p|(1+\varepsilon)^3<|b_p|(1-\varepsilon)^3$, $|a_p|r^p(1-\varepsilon)<|h_p(z)|<|a_p|r^p(1+\varepsilon)$ and $|b_p|r^p(1-\varepsilon)<|h(z)|<|b_p|r^p(1+\varepsilon)$, we obtain

$$|w_{j+1}|\frac{|a_p|}{|b_p|}<\min_{|z|=r_j'}|h_p(z)|\leqq\max_{|z|=r_j'}|h_p(z)|<|w_{j+1}|,$$

$$|w_j|<\min_{|z|=r_j''}|h_p(z)|\leqq\max_{|z|=r_j''}|h_p(z)|<|w_{j+1}|,$$

$$|w_{j+1}|<\min_{|z|=r_j'}|h(z)|\leqq\max_{|z|=r_j'}|h(z)|<|w_{j+2}|$$

and

$$|w_j|\frac{|b_p|}{|a_p|}<\min_{|z|=r_j''}|h(z)|\leqq\max_{|z|=r_j''}|h(z)|<|w_{j+2}|.$$

Noting that if $r'_j \geqq r''_j$, then $|w_j| \leqq |w_{j+1}|\, |a_p|/|b_p|$ and if $r'_j \leqq r''_j$, then $|w_{j+1}| \leqq |w_j|\, |b_p|/|a_p|$, we find

$$|w_j| < \min_{|z|=r_j} |h_p(z)| \leqq \max_{|z|=r_j} |h_p(z)| < |w_{j+1}|$$

and

$$|w_{j+1}| < \min_{|z|=r_j} |h(z)| \leqq \max_{|z|=r_j} |h(z)| < |w_{j+2}|.$$

Therefore we deduce

$$n(r_j, 0, g\circ h_p) = pn(|w_j|, 0, g)$$

and

$$n(r_j, 0, g\circ h) = pn(|w_{j+1}|, 0, g\circ h),$$

that is,

$$n(r_j, 0, g\circ h) - n(r_j, 0, g\circ h_p) = p.$$

From (5. 3), we have $2(\nu-\mu)=p$. This implies that p is even. Similarly we have the same result when $|a_p| > |b_p|$.

Therefore we obtain the desired result that if $\rho_g = 0$ or if p is odd, then we have $|a_p| = |b_p|$. This completes the proof of Theorem 1. q.e.d.

REMARK. It is worth while to be remarked that our argument in this section remains valid when $\rho_g = +\infty$.

§ 6. The last assertion of our Theorem 1 is best possible. Let R be an ultrahyperelliptic surface defined by $y^2 = G(z)$,

$$G(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^p}{(a^n-1)/(a-1)}\right), \qquad a>1 \text{ and } p \text{ is even.}$$

Let S be an ultrahyperelliptic surface defined by $u^2 = g(w)$,

$$g(w) = w\prod_{n=1}^{\infty}\left(1 - \frac{w}{(a^n-1)/(a-1)}\right).$$

Then it is clear that $\rho_g = 0$. $h_p(z) = (1/a)(z^p - 1)$ and $h(z) = z^p$ belong to $\mathfrak{H}(R, S)$. For, setting

$$f_p(z)^2 = -\frac{1}{a}\prod_{n=1}^{\infty}\left(1 + \frac{a-1}{a(a^n-1)}\right)$$

and $f(z) = z^{p/2}$, we have

$$f_p(z)^2 G(z) = g\circ h_p(z) \qquad \text{and} \qquad f(z)^2 G(z) = g\circ h(z).$$

§7. *Proof of Theorem* 2. Let R and S be two ultrahyperelliptic surfaces with $P(R)=P(S)=4$ defined by the equation $y^2=G(z)$ and $u^2=g(w)$, respectively. Then by a result in [4], we have

$$F(z)^2G(z)=(e^{H(z)}-\alpha)(e^{H(z)}-\beta), \qquad \alpha\beta(\alpha-\beta)\neq 0, \qquad H(0)=0,$$

where $F(z)$ is a suitable entire function and $H(z)$ is a non-constant entire function and

$$f(w)^2g(w)=(e^{L(w)}-\gamma)(e^{L(w)}-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0, \qquad L(0)=0$$

where $f(w)$ is a suitable entire function and $L(w)$ is a non-constant entire function.

Hiromi-Ozawa [3] implies that for $h_p(z)\in\mathfrak{H}(R,S)$ one of two equations

$$(7.1) \qquad H(z)=L\circ h_p(z)-L\circ h_p(0) \quad \text{and} \quad H(z)=-L\circ h_p(z)+L\circ h_p(0),$$

and for $h(z)\in\mathfrak{H}(R,S)$ one of two equations

$$(7.2) \qquad H(z)=L\circ h(z)-L\circ h(0) \quad \text{and} \quad H(z)=-L\circ h(z)+L\circ h(0)$$

are valid. Since $h_p(z)$ is a polynomial of degree p, using Lemma 1 and Lemma 2 together with their proof, the equations (7.1) and (7.2) imply that $h(z)$ must be a polynomial of degree p and further $|a_p|=|b_p|$. q.e.d.

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