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ON THE FELLER PROPERTY OF DIRICHLET FORMS GENERATED BY PSEUDO DIFFERENTIAL OPERATORS

Dedicated to Professor Masatoshi Fukushima on his seventieth birthday

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Abstract. We show that a large class of regular symmetric Dirichlet forms is generated by pseudo differential operators. We calculate the symbols which are closely related to the semimartingale characteristics (Lévy system) of the associated stochastic processes. Using the symbol we obtain estimates for the mean sojourn time of the process for balls. These estimates and a perturbation argument enable us to prove Hölder regularity of the resolvent and semigroup; this entails that the semigroup has the Feller property.

1. Introduction. In the seminal paper *Dirichlet spaces and strong Markov processes* [3] M. Fukushima established a one-to-one correspondence between regular (symmetric) Dirichlet forms and symmetric Hunt processes. Many concrete (jump-type) examples were given by Jacob who used pseudo differential operators to construct stochastic processes and Dirichlet forms, [10] and [11]; the most general results in this direction are to our knowledge due to Hoh [7, 8].

It is not by accident that pseudo differential operators enter the scene. A well-known result by Courrège, cf. [11] for a survey, implies that infinitesimal generators of 'regular' Feller semigroups are pseudo differential operators—by 'regular' we mean that the test functions C_0^∞ are contained in the domain of the generator. Since many Feller processes are Hunt processes, it is possible to study these processes via Dirichlet forms; such Dirichlet forms are, of course, generated by pseudo differential operators. The connection between pseudo differential operators and Dirichlet forms runs, however, deeper. One indication is the Beurling-Deny formula which gives the generator of the form in 'implicit' form, meaning that one should—we take the analogy to the diffusion case—perform some kind of 'integration by parts'. Some sufficient conditions for the generator of a Dirichlet form to be a pseudo differential operator were given in [18].

A drawback of the otherwise very powerful approach via Dirichlet forms is the problem of non-uniqueness in the sense of Theorem 4.2.7 in [6]. That is, any two stochastic processes associated with the same Dirichlet form are equivalent if there exists a common properly exceptional set N such that the transition functions coincide outside of N. This leaves a

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question whether one can find a nice representative for the equivalence class of all associated processes which starts at *every* point in a natural way.

One way to overcome this is to use (r, p)-capacities and refinements, see [5, 12, 4]. A more direct approach can be based on the work of Bass and Levin [2] where they obtained a Harnack inequality for pure jump type integral operators on $\boldsymbol{R}^n$ and showed that the corresponding harmonic functions are Hölder continuous. Note that they *assumed the existence* of a strong Markov process associated with the operator as infinitesimal generator. In [19] Song and Vondraček extended the papers [2, 1] by Bass and co-authors to a larger class of Markov processes.

In this paper we start with a symmetric regular Dirichlet form, i.e., we *know that there exists* a stochastic process associated with the form. To overcome the non-uniqueness we show that the method of Bass *et al.* is applicable at all points outside of the exceptional set N. In particular, we establish a Harnack inequality and Hölder continuity of the harmonic functions associated with the Dirichlet process. This allows us to show that the resolvent and the semigroup of the Dirichlet process can be modified to a Feller resolvent and semigroup. Since we are now dealing with a Feller process, Courrège's theorem implies that the form has been generated by a pseudo differential operator in the first place.

It is therefore natural to consider the generator and to search for an explicit formula for its symbol. This is done in Sections 1 and 2. Having calculated the generator we need to get the method developed by Bass *et al.* to work; this requires further properties of the symbol and, in particular, estimates for the mean sojourn time of the process for small balls. We use a new method using the symbol of the generator to derive such estimates, cf. Section 3. This section is based on results from the paper [17] which was written for Feller processes. As a matter of fact, [17] uses Feller processes *only* to guarantee that the infinitesimal generator is a pseudo differential operator; all other calculations only need this particular form of the generator and strong Markovianity, which means that we can apply these results. The proof of the Harnack inequality and the Hölder estimates is modeled on the papers [2, 1] and [19]. The principal innovation is that we use a perturbation result which allows us to pose only assumptions on the small-jump part and to choose the large-jump part as convenient as necessary. Since we are more interested in the method and do not strive for greatest possible generality in the present paper, we only work out bounded perturbations; more general schemes are outlined in Remark 6.6.

NOTATION. We write $a \vee b$ and $a \wedge b$ for the maximum and minimum of $a, b \in \boldsymbol{R}$ and $L^p(\boldsymbol{R}^n)$ for the usual L^p space with respect to Lebesgue measure dx; C_b^k (resp. C_0^k) denote the k times continuously differentiable functions which are bounded with all their derivatives (resp. with compact support) and by C_∞ we mean the continuous functions vanishing at infinity. The *first entrance time* into a set is denoted by $\tau_U := \inf\{t \geqslant 0 \;:\; X_t \in U\}$ and $\sigma_V := \inf\{t \geqslant 0 \;:\; X_t \notin V\}$ is the *first exit time* from a set V.

2. Generator and symbol of a jump-type Dirichlet form. Throughout this paper we will always consider quadratic forms of the following type:

$$\begin{aligned}\mathcal{E}(u, v) &:= \iint_{x\neq y} (u(x)-u(y))(v(x)-v(y))n(x, y)dxdy \\ \mathcal{D}(\mathcal{E}) &:= \{u \in L^2(\boldsymbol{R}^n) \ : \ \mathcal{E}(u, u) < \infty\}\end{aligned} \tag{1}$$

where $n(x, y)$ is a positive measurable function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$. Denote by $C_0^{0,1}(\boldsymbol{R}^n)$ the set of all *uniformly Lipschitz continuous functions* on $\boldsymbol{R}^n$ with compact support. In [21], see also [20], one of us considered 'stable-like' forms $\mathcal{E}(\bullet, \bullet)$ where $n(x, y) = |x-y|^{-\alpha(x,y)-n}$ and gave conditions on the exponent α which turn $\mathcal{E}(\bullet, \bullet)$ into a Dirichlet form. Since

$$n(x, y) = |x-y|^{-\alpha(x,y)-n} \iff \alpha(x, y) = -\frac{\log n(x, y)}{\log|x-y|} - n\,,$$

these results extend to all quadratic forms of type (1). In the present setting, Theorems 2.1 and 2.2 of [21] become

THEOREM 2.1. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be as above. If the set $\{(x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \ : \ n(x, y) = +\infty\}$ is a Lebesgue null set, then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\boldsymbol{R}^n)$ in the wide sense.*[1] *The domain $\mathcal{D}(\mathcal{E})$ contains the set $C_0^{0,1}(\boldsymbol{R}^n)$ if, and only if, for all compact sets K and all relatively compact open sets $G \supset K$ the following condition is satisfied*

$$\iint_{K\times K} |x-y|^2(n(x, y)+n(y, x))dxdy + \iint_{K\times G^c} (n(x, y)+n(y, x))dxdy < \infty\,.$$

For $\mathcal{F} := \overline{C_0^{0,1}(\boldsymbol{R}^n)}^{\mathcal{E}(\bullet)+\|\bullet\|_{L^2}}$, i.e., the closure of $C_0^{0,1}(\boldsymbol{R}^n)$ under $\mathcal{E}(\bullet, \bullet) + \langle \bullet, \bullet\rangle_{L^2}$, the form $(\mathcal{E}, \mathcal{F})$ becomes a regular (symmetric) Dirichlet form.

According to the general theory of Dirichlet forms, our standard reference is Fukushima, Oshima and Takeda [6], we can associate with every regular Dirichlet form a symmetric Hunt process $\mathbf{M} = (X_t, \boldsymbol{P}_x)$; the family of probability measures $(\boldsymbol{P}_x)$ is uniquely determined only up to a capacity-zero set N of starting points x.

All regular Dirichlet forms can be written in terms of their Beurling-Deny decomposition, see [6, Theorem 3.2.1, Lemma 4.5.4]. In our situation it is easy to see that

$$\mathcal{E}(u, v) = \frac{1}{2}\iint_{x\neq y} (u(x)-u(y))(v(x)-v(y))j(x, y)dxdy\,,$$

$j(x, y) := n(x, y) + n(y, x)$, holds for all $u, v \in C_0^{0,1}(\boldsymbol{R}^n)$ or, more generally, for all quasi-continuous (modifications of) u, v from $\mathcal{F}$. Intuitively, see [6, Theorem 4.5.2] for a precise statement, $j(x, y)$ is the rate at which the paths of the associated Hunt process jump from the current position $X_{t-} = x$ to the point $X_t = y \neq x$. It is sometimes helpful not to look at the rate for the new *position* but at the rate of the jump *size* $X_t - X_{t-} = x - y =: h$. Doing so,

[1] Namely, $\mathcal{E}$ satisfies all requirements for a Dirichlet form except that $\mathcal{D}(\mathcal{E})$ might be not dense in $L^2(\boldsymbol{R}^n)$.

we get

$$\mathcal{E}(u,v) = \frac{1}{2}\iint_{h\neq 0}(u(x+h)-u(x))(v(x+h)-v(x))j(x,x+h)dxdh\,,$$

and the conditions of Theorem 2.1 become

$$\int_{h\neq 0}(|h|^2\wedge 1)j(\bullet,\bullet+h)dh \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)\,. \tag{2}$$

Another way to describe Dirichlet forms is through their L^2-generator $(A,\mathcal{D}(A))$. The connection between form and generator is given by $\mathcal{E}(u,v) = -\langle u, Av\rangle_{L^2}$, $u\in\mathcal{F}$, $v\in\mathcal{D}(A)$. For (jump-type) Dirichlet forms it is, in general, difficult to find a closed expression for A if only the form is known. In the present situation this is, however, possible if we make some more assumptions on the jump density $j(x,y)$.

We call a rotationally invariant, measurable function $\chi : \boldsymbol{R}^n \to [0,1]$ a *centering function* if $\chi(h)$ decays for $|h|\to\infty$ at least as fast as $|h|^{-1}$ and if $\lim_{h\to 0}|\chi(h)h-h|/|h|^2 = 0$. Typical examples are $\chi(h) = (1+|h|^2)^{-1}$ and $\chi(h) = \mathbf{1}_{B_1(0)}(h)$.

THEOREM 2.2. *Let* $(\mathcal{E},\mathcal{D}(E))$ *be given by* (1) *and assume that* (2) *holds. Moreover we assume that the jump density* $j(x,y)$ *satisfies*

$$\int_{|h|\leqslant 1}|h||j(x,x+h)-j(x,x-h)|dh < \infty\,. \tag{3}$$

Then $\mathcal{E}(u,\phi) = -\langle u, A\phi\rangle_{L^2}$ *for all* $u,\phi\in C_0^2(\boldsymbol{R}^n)$ *with the operator A given by*

$$\begin{aligned} A\phi(x) &= \int_{h\neq 0}(\phi(x+h)-\phi(x)-\chi(h)h\cdot\nabla\phi(x))j(x,x+h)dh \\ &\quad + \frac{1}{2}\int_{h\neq 0}\chi(h)h(j(x,x+h)-j(x,x-h))dh\cdot\nabla\phi(x) \end{aligned} \tag{4}$$

$$= \text{p.v.}\int(\phi(x+h)-\phi(x))j(x,x+h)dh. \tag{5}$$

Here $\chi(h)$ *denotes a compactly supported centering function and* p.v.$\int\ldots dh$ *means the Cauchy principal value integral.*

PROOF. Let $u,\phi\in C_0^2(\boldsymbol{R}^n)$ and fix some $\chi(h)$. Since $u(x)-u(y)$, $\phi(x)-\phi(y)$ are of order $O(1)$ as $|x-y|\to\infty$ and $O(|x-y|)$ as $|x-y|\to 0$, we see that

$$\mathcal{E}(u,\phi) = \frac{1}{2}\iint_{h\neq 0}(u(x+h)-u(x))(\phi(x+h)-\phi(x))j(x,x+h)dxdh$$

is well-defined. Since the kernel $j(x,x+h)dxdh$ integrates only $|h|^2$ near the origin, it is not possible to multiply out the product under the integral and to treat the resulting terms separately. Instead we insert the terms

$$-\chi(h)h\cdot\nabla\phi(x+h),\quad -\chi(h)h\cdot\nabla\phi(x)\quad\text{and}\quad \chi(h)h\cdot\nabla\phi(x+h)+\chi(h)h\cdot\nabla\phi(x)$$

and observe that, because of the Taylor formula, the expressions

$$\phi(x+h)-\phi(x)-\chi(h)h\cdot\nabla\phi(x+h)\quad\text{and}\quad\phi(x+h)-\phi(x)-\chi(h)h\cdot\nabla\phi(x)$$

behave like $O(1)$ for $|h| \to \infty$ and like $O(|h|^2)$ for $|h| \to 0$. Using the symmetry $j(x, y) = j(y, x)$ of the kernel, we arrive after some lengthy but elementary calculations at

$$\begin{aligned}
\mathcal{E}(u, \phi) &= -\iint\limits_{h\neq 0} u(x)(\phi(x+h) - \phi(x) - \chi(h)h\nabla\phi(x))j(x, x+h)dxdh \\
&\quad - \frac{1}{2}\iint\limits_{h\neq 0} u(x)\chi(h)h(\nabla\phi(x) - \nabla\phi(x+h))j(x, x+h)dxdh \\
&\quad + \frac{1}{4}\iint\limits_{h\neq 0} (u(x+h) - u(x))\chi(h)h(\nabla\phi(x+h) + \nabla\phi(x))j(x, x+h)dxdh \\
&=: -I_1 - (1/2)I_2 + (1/4)I_3 .
\end{aligned}$$

The change of variable $x \rightsquigarrow x - h$ and $h \rightsquigarrow -h$ and the symmetry of $j(x, y)$ show

$$I_2 = -\iint\limits_{h\neq 0} u(x+h)\chi(h)h(\nabla\phi(x+h) - \nabla\phi(x))j(x, x+h)dxdh .$$

Averaging this and the original representation of I_2 yields

$$I_2 = \frac{1}{2}\iint\limits_{h\neq 0} (u(x) + u(x+h))\chi(h)h(\nabla\phi(x) - \nabla\phi(x+h))j(x, x+h)dxdh .$$

Therefore, we have

$$\begin{aligned}
&(1/4)I_3 - (1/2)I_2 \\
&\quad = \frac{1}{2}\,\text{p.v.}\iint \chi(h)(u(x+h)h \cdot \nabla\phi(x+h) - u(x)h \cdot \nabla\phi(x))j(x, x+h)dxdh
\end{aligned}$$

with the principal value integral p.v. $\iint := \lim_{\varepsilon\to 0}\iint_{|h|>\varepsilon}$. Since for all $\varepsilon > 0$

$$\iint\limits_{|h|>\varepsilon} u(x+h)\chi(h)h\cdot\nabla\phi(x+h)j(x, x+h)dhdx = \iint\limits_{|h|>\varepsilon} u(x)\chi(h)h\cdot\nabla\phi(x)j(x, x-h)dhdx ,$$

we get

$$(1/4)I_3 - (1/2)I_2 = \frac{1}{2}\,\text{p.v.}\iint u(x)\chi(h)h \cdot \nabla\phi(x)(j(x, x-h) - j(x, x+h))dhdx ;$$

because of (3) the above principal value integral is absolutely convergent and does not depend on the particular choice of the centering function. Piecing things together we obtain for $u, \phi \in C_0^2(\boldsymbol{R}^n)$ and a fixed centering function $\chi(h)$ formulae (4) and (5). $\square$

COROLLARY 2.3. *Let $(\mathcal{E}, \mathcal{F})$ and A be as in Theorem 2.2. Then A can be extended to the bounded and twice differentiable functions $C_b^2(\boldsymbol{R}^n)$. For $\phi \in C_b^2(\boldsymbol{R}^n)$ we have*

$$\begin{aligned}|A\phi(x)| \leqslant C\Bigg[\int_{h\neq 0} \frac{|h|^2}{1+|h|^2}\, j(x, x+h)dh + \int |h|\chi(h)|j(x, x+h) - j(x, x-h)|dh \Bigg] \\ \times \sum_{|\alpha|\leqslant 2} \|\partial^\alpha \phi\|_\infty\end{aligned} \tag{6}$$

for all $x \in \boldsymbol{R}^n$. Moreover, $A\big|_{C_0^\infty(\boldsymbol{R}^n)}$ is a pseudo differential operator

$$A\phi(x) = -p(x, D)\phi(x) = (2\pi)^{-n/2} \int p(x, \xi)\hat{\phi}(\xi)e^{ix\xi}d\xi \tag{7}$$

($\hat{\phi}(\xi) := (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} e^{-ix\xi}\phi(x)dx$ denotes the Fourier transform) with negative definite symbol $p : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ which is given by the Lévy-Khinchine-type formula

$$\begin{aligned} p(x, \xi) &= \int_{h\neq 0} (1 - \cos h\xi)\, j(x, x + h)dh \\ &\quad - \frac{1}{2} \int_{h\neq 0} i \sin \xi h\, (j(x, x + h) - j(x, x - h))dh \\ &= \text{p.v.} \int (1 - e^{i\xi h})\, j(x, x + h)dh\,. \end{aligned}$$

PROOF. Once we have shown that A can be extended to all C_b^2-functions, we may substitute $\phi(x)$ in (4) for $e_\xi(x) = e^{ix\xi}$. With some routine calculations—see, e.g., Jacob [11]—we then see that $p(x, \xi) = e^{-ix\xi} Ae_\xi(x)$ and that $p(x, \xi)$ is given by

$$\begin{aligned} p(x, \xi) &= \int_{h\neq 0} (1 - e^{ih\xi} + i\xi h\chi(h)l)\, j(x, x + h)dh \\ &\quad - \frac{1}{2} \int_{h\neq 0} i\xi h\chi(h)l(j(x, x+h) - j(x, x-h))dh\,. \end{aligned}$$

Split $p(x, \xi)$ into real and imaginary parts and observe that the functions $1 - \cos h\xi$ and $\chi(h)h\xi - \sin h\xi$ are bounded for large $|h|$ and behave like $O(|h|^2)$ as $h \to 0$. Since $h \mapsto \chi(h)h\xi - \sin h\xi$ is odd, the claimed Lévy-Khinchine-type representation is readily derived.

To see that A extends to $C_b^2(\boldsymbol{R}^n)$ it is clearly enough to prove (6). Using Taylor's formula we get for $\phi \in C_b^2(\boldsymbol{R}^n)$

$$|\phi(x + h) - \phi(x) - \chi(h)h\nabla\phi(x)| \leqslant c(\chi)\, \frac{|h|^2}{1 + |h|^2} \sum_{|\alpha|\leqslant 2} \|\partial^\alpha \phi\|_\infty\,.$$

The estimate (6) follows now from the representation (4) of the operator A. □

In order to identify $A|_{C_0^2(\boldsymbol{R}^n)}$ as the (restriction of the L^2-) generator of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ we have to show that A maps $C_0^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$. For this we have to replace the conditions (2), (3) by the following uniform versions

$$\sup_x \int_{h\neq 0} (|h|^2 \wedge 1) j(x, x+h) dh < \infty, \tag{8}$$

$$\sup_x \int_{|h|\leqslant 1} |h| |j(x, x+h) - j(x, x-h)| dh < \infty. \tag{9}$$

COROLLARY 2.4. *Let* $(\mathcal{E}, \mathcal{F})$ *and* A *be as in Theorem* 2.2. *If* (8), (9) *hold, then the operator* A *has bounded coefficients in the sense that there exist constants* $C, K > 0$ *such that for all* $\phi \in C_b^2(\boldsymbol{R}^n)$ *resp.* $\xi \in \boldsymbol{R}^n$

$$\|A\phi\|_\infty \leqslant C \sum_{|\alpha|\leq 2} \|\partial^\alpha \phi\|_\infty \quad \textit{resp.} \quad \sup_x |p(x, \xi)| \leqslant K\,(1 + |\xi|^2)\,. \tag{10}$$

PROOF. The assumptions guarantee that all integrals appearing in the estimate (6) of $A\phi(x)$ converge uniformly for all x. This proves the first estimate in (10). The second inequality follows immediately from the first since $-p(x, \xi) = e^{-ix\xi} A e_\xi(x)$, $e_\xi(x) := e^{ix\xi}$, see the proof of Corollary 2.3. □

PROPOSITION 2.5. *Let* $(\mathcal{E}, \mathcal{F})$ *and* A *be as in Theorem* 2.2. *If* (8), (9) *hold, then* A *maps* $C_0^2(\boldsymbol{R}^n)$ *to* $L^2(\boldsymbol{R}^n)$. *In particular,* A *coincides on* $C_0^2(\boldsymbol{R}^n)$ *with the generator of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *and* $C_0^2(\boldsymbol{R}^n) \subset \mathcal{D}(A)$.

PROOF. Pick some $\phi \in C_0^2(\boldsymbol{R}^n)$ and choose $r > 0$ so large that $\operatorname{supp}\phi \subset B_r(0)$. Because of (10) we have

$$\begin{aligned} \|A\phi\|_{L^2} &\leqslant \|\mathbf{1}_{B_{2r}(0)} A\phi\|_{L^2} + \|\mathbf{1}_{B_{2r}^c(0)} A\phi\|_{L^2} \\ &\leqslant C\sqrt{\lambda^n(B_{2r}(0))} \sum_{|\alpha|\leqslant 2} \|\partial^\alpha \phi\|_\infty + \|\mathbf{1}_{B_{2r}^c(0)} A\phi\|_{L^2}\,. \end{aligned}$$

To see the finiteness of the second member we observe that (4) reduces for $|x| \geqslant 2r$ to

$$\mathbf{1}_{B_{2r}^c(0)}(x) A\phi(x) = \int_{h\neq 0} \phi(x+h) j(x, x+h) dh\,,$$

and since for $|h| \leqslant r$ and $|x| \geqslant 2r$ we have $|x+h| \geqslant |x| - |h| \geqslant r$ we conclude that

$$\begin{aligned} \|\mathbf{1}_{B_{2r}^c(0)} A\phi\|_{L^2}^2 &= \int_{|x|\geqslant 2r} \left[\int_{|h|>r} \phi(x+h) j(x, x+h) dh \right]^2 dx \\ &\leqslant \|\phi\|_\infty^2 \int_{\boldsymbol{R}^n} \left[\int_{|h|>r} \mathbf{1}_{B_r(0)}(x+h) j(x, x+h) dh \right] \\ &\qquad \times \left[\int_{|h|>r} j(x, x+h) dh \right] dx\,. \end{aligned}$$

For the last estimate we used the Cauchy-Schwarz inequality for the inner integral. We now interchange the order of integration and change variables according to $x \rightsquigarrow y - h$ and then

$h \rightsquigarrow -h$. Since $j(x,z) = j(z,x)$ we arrive at

$$\|\mathbf{1}_{B_{2r}(0)} A\phi\|_{L^2}^2 \leqslant \|\phi\|_\infty^2 \sup_{x \in \boldsymbol{R}^n} \left[\int_{|h|>r} j(x, x+h)dh \right]^2 \int_{\boldsymbol{R}^n} \mathbf{1}_{B_r(0)}(y)dy$$

which is finite under (8), (9). Since $A(C_0^2(\boldsymbol{R}^n)) \subset L^2(\boldsymbol{R}^n)$ Theorem 2.2 shows that A coincides on $C_0^2(\boldsymbol{R}^n)$ with the generator of the Dirichlet form and that $C_0^2(\boldsymbol{R}^n) \subset \mathcal{D}(A)$. □

EXAMPLE 2.6. Here is a simple condition that guarantees (8), (9): if there exist exponents $-\infty < \alpha \leqslant \beta < 2$ and $0 < \gamma \leqslant \infty$ and constants $c, C, K > 0$ such that

$$\frac{c}{|x-y|^{\alpha+n}} \leqslant j(x,y) \leqslant \frac{C}{|x-y|^{\beta+n}} \qquad \text{for all } |x-y| \leqslant 1 \tag{11}$$

$$0 \leqslant j(x,y) \leqslant \frac{K}{|x-y|^{\gamma+n}} \qquad \text{for all } |x-y| > 1 \tag{12}$$

($\gamma = \infty$ means that $j(x,y)$ vanishes if $|x-y| > 1$) then (8) holds. The straightforward calculations are left to the reader.

If we write $j(x,y)$ in the form $|x-y|^{-\alpha(x,y)-n} + |x-y|^{-\alpha(y,x)-n}$ (as, e.g., in [21]), then (11), (12) are essentially equivalent to the conditions

$$-\infty < \alpha \leqslant \alpha(x,y) \leqslant \beta < 2 \qquad \text{for all } |x-y| \leqslant 1 \tag{13}$$

$$0 < \gamma \leqslant \alpha(x,y) \leqslant \infty \qquad \text{for all } |x-y| > 1\,. \tag{14}$$

For (3) resp. (9) we have to make the additional assumption that $\alpha(x,y)$ is Lipschitz continuous for all x, y from any compact set $K \subset \boldsymbol{R}^n$ (resp. globally Lipschitz), i.e., that

$$|\alpha(x,y) - \alpha(x,z)| + |\alpha(y,x) - \alpha(z,x)| \leqslant C_K |y-z|, \quad x, y, z \in K \tag{15}$$

(and that, for (9), the constants C_K are uniformly bounded). If this is the case, we get for $x \in K$ and $|h| < 1$

$$\begin{aligned} &|j(x,x+h) - j(x,x-h)| \\ &\qquad \leq |h|^{-n}(||h|^{-\alpha(x,x+h)} - |h|^{-\alpha(x,x-h)}| + ||h|^{-\alpha(x+h,x)} - |h|^{-\alpha(x-h,x)}|)\,. \end{aligned}$$

The elementary formula $|t^{-a} - t^{-b}| = |\int_a^b t^{-u} \log t\, du|$ shows for $|h| < 1$ that

$$||h|^{-\alpha(x,x+h)} - |h|^{-\alpha(x,x-h)}| = \left| \int_{\alpha(x,x-h)}^{\alpha(x,x+h)} |h|^{-u} \log |h| du \right| \leqslant c_K \log \frac{1}{|h|} |h|^{1-\beta}\,.$$

Thus,

$$\int_{|h|<1} |h||j(x,x+h) - j(x,x-h)|dh \leqslant c_K \int_{|h|<1} |h|^{2-\beta-n} \log \frac{1}{|h|} dh < \infty$$

uniformly for $x \in K$ (resp. uniformly in $x \in \boldsymbol{R}^n$).

Later on, we will use that $\beta < 2$ also guarantees that

$$\sup_x \int_{|h|<1} |h| \log \frac{1}{|h|} |j(x,x+h) - j(x,x-h)|dh < \infty\,. \tag{16}$$

Exact knowledge of the generator, in particular, the fact that A is a pseudo differential operator with symbol $-p(x,\xi)$, makes simple proofs of (global) properties of the process possible, see [11]; most proofs only use the *existence* of the symbol and the (*strong*) *Markov property* of the underlying process. The following proof is the 'symbolic' version of Oshima's conservativeness criterion [14], see also [9] and [16].

PROPOSITION 2.7. *Let* $(\mathcal{E},\mathcal{F})$ *be as in Theorem* 2.2 *and assume that the conditions* (8), (9) *hold. Then the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *is conservative, i.e., for all* $t>0$ *we have* $T_t 1 = 1$ *Lebesgue a.e., where* $(T_t)_{t\geqslant 0}$ *is the* L^2*-semigroup associated with the form* $(\mathcal{E},\mathcal{F})$.

PROOF. As we have seen in Corollaries 2.3 and 2.4, the generator A of the Dirichlet form is a pseudo differential operator with symbol $-p(x,\xi)$ which has bounded coefficients, i.e., $|p(x,\xi)|\leqslant c(1+|\xi|^2)$. Since $(X_t)_{t\geqslant 0}$ is a Hunt process, we know that for (Lebesgue) almost all starting points x and all $\phi\in\mathcal{D}(A)$

$$T_t\phi(x)-\phi(x)=\boldsymbol{E}^x(\phi(X_t))-\phi(x)=\int_0^t \boldsymbol{E}^x(A\phi(X_s))ds=\int_0^t T_sA\phi(x)ds\,.$$

Pick $\phi\in C_0^\infty(\boldsymbol{R}^n)\subset\mathcal{D}(A)$ satisfying $0\leqslant\phi\leqslant 1$, $\phi(0)=1$, and define $\phi_k(x):=\phi(x/k)$. Noting that $\phi_k(x)\xrightarrow{k\to\infty}1$ and that for the Fourier transform $\widehat{\phi_k}(\xi)=k^n\hat{\phi}(k\xi)$, we see using $A|_{C_c^\infty}=-p(x,D)$,

$$\begin{aligned}
|A\phi_k(x)| &= \left|(2\pi)^{-n/2}\int p(x,\xi)\widehat{\phi_k}(\xi)e^{ix\xi}d\xi\right| \\
&= \left|(2\pi)^{-n/2}\int p(x,\xi/k)\hat{\phi}(\xi)e^{ix\xi/k}d\xi\right| \\
&\leqslant c\,(2\pi)^{-n/2}\int(1+\left|\xi/k\right|^2)|\hat{\phi}(\xi)|d\xi \\
&\leqslant c\,(2\pi)^{-n/2}\int(1+|\xi|^2)|\hat{\phi}(\xi)|d\xi\,.
\end{aligned}\tag{17}$$

The last integral converges absolutely since $\hat{\phi}$ is a rapidly decreasing Schwartz function. Since this estimate is uniform in $k\in\boldsymbol{N}$, we can use dominated convergence in (17), and conclude that $\lim_{k\to\infty}A\phi_k=0$ and $\sup_k\|A\phi_k\|_\infty<\infty$. Therefore, another application of the dominated convergence theorem shows that for almost all x

$$|1-T_t1|=\lim_{k\to\infty}|\phi_k-T_t\phi_k|\leqslant\lim_{k\to\infty}\int_0^t|T_sA\phi_k|ds=\int_0^t\lim_{k\to\infty}|T_sA\phi_k|ds=0\,,$$

whence $T_t1=1$ almost everywhere. □

3. Sojourn times for small balls. In [17] one of us studied the growth behaviour of a class of Feller processes $(X_t)_{t\geqslant 0}$ which are generated by pseudo differential operators. To do so, estimates for the running maxima and the sojourn times $\sigma_r^x:=\inf\{t\geqslant 0 : |X_t-x|\geqslant r\}$, were obtained. Although [17] was written for Feller processes, the necessary input for the estimates to work was that $(X_t)_{t\geqslant 0}$ is a strong Markov process whose infinitesimal generator is a pseudo differential operator $-p(x, D)$—i.e., an operator of the form (7)—with *negative definite symbol* $p(x, \xi)$—i.e., a locally bounded function $p : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ satisfying a Lévy-Khinchine representation

$$p(x, \xi) = a(x) - i\ell(x)\xi + \xi \cdot Q(x)\xi + \int_{h\neq 0}(1 - e^{-ih\xi} - i\chi(h)h\xi)\nu(x, dh)\,,$$

for some centering function $\chi(h)$ and measurable 'coefficients' $a(x) \geqslant 0$, $\ell(x) \in \boldsymbol{R}^n$, the positive semidefinite $Q(x) \in \boldsymbol{R}^{n\times n}$ and the Lévy kernel $\nu(x, dh)$ which is such that $\int_{h\neq 0}(|h|^2 \wedge 1)\nu(x, dh) < \infty$.

In order to use the methods developed in [17] we need to know that the process $(X_t, \boldsymbol{P}_x)$ from Section 2 is a $\boldsymbol{P}_x$-semimartingale for all $x \notin N$ and some capacity-zero set N.

LEMMA 3.1. *Let* $(\mathcal{E}, \mathcal{F})$ *and* A *be as in Theorem* 2.2 *and assume that* (8), (9) *hold true.* (*In particular, the symbol* $p(x, \xi)$ *has bounded coefficients in the sense of* (10).) *Denote by* N' *the exceptional set outside of which the process* $(X_t, \boldsymbol{P}_x)$, *which is properly associated with* $(\mathcal{E}, \mathcal{F})$, *is uniquely defined. Then* $(X_t, \boldsymbol{P}_x)$ *is a* $\boldsymbol{P}_x$*-semimartingale for all* x *outside some* (*possibly larger*) *exceptional set* $N \supset N'$.

PROOF. The proof is similar to the argument from [17, Lemma 3.2] and we only sketch the differences. Recall that our assumptions imply $C_0^2(\boldsymbol{R}^n) \subset \mathcal{D}(A)$. Pick $\phi_k^j \in C_0^2(\boldsymbol{R}^n)$ such that $\phi_k^j(x) = x_j$ on $B_k(0)$ and $\phi_k^j(x) = 0$ on $B_{2k}^c(0)$, $j = 1, 2, \ldots, n$, and $k \in \boldsymbol{N}$. Because of the Fukushima decomposition of additive functionals, see [6, Theorem 5.2.2],

$$M_t^{[u]} := u(X_t) - u(X_0) - \int_0^t Au(X_s)ds\,, \qquad u \in \mathcal{D}(A)\,,$$

is a $\boldsymbol{P}_x$-martingale for all $x \in \boldsymbol{R}^n \setminus M$ where M is an exceptional set that may depend on u; thus, $M_t := M_t^{[u]}$, $u = \phi_k^j(\cdot - x)$, is a $\boldsymbol{P}_x$-martingale for all x outside some exceptional set $M_{j,k}$. We set $N := \bigcup_{j,k} M_{j,k} \cup N'$ which is again a capacity-zero set. We may now literally follow the argument of [17]. □

Since all proofs of [17] only involve stopping techniques (at a sequence of countably many stopping times), we can use all arguments of that paper in the present situation, possibly at the expense of a larger exceptional set.

THEOREM 3.2. *Let* $\{(X_t, \boldsymbol{P}_x),\ t \geqslant 0,\ x \in \boldsymbol{R}^n \setminus N\}$ *be the Hunt process associated with the regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *and assume that the generator of* $\mathcal{E}$ *is* (*on the test functions* $C_0^\infty(\boldsymbol{R}^n)$) *a pseudo differential operator* $-p(x, D)$ *with negative definite symbol* $-p(x, \xi)$. *If the symbol has bounded coefficients in the sense that* $\sup_x |p(x, \xi)| \leqslant K(1 + |\xi|^2)$ *for all* $\xi \in \boldsymbol{R}^n$ *and if it satisfies a sector condition* $|\operatorname{Im} p(x, \xi)| \leqslant c_0 \operatorname{Re} p(x, \xi)$, $x \in \boldsymbol{R}^n$, $|\xi| > \rho$, *with absolute constants* $c_0, \rho > 0$, *then we have for all* $x \in \boldsymbol{R}^n \setminus N$, $t \geqslant 0$ *and* $r > 0$

$$\boldsymbol{P}_x\left(\sup_{s\leqslant t}|X_s - x| \geqslant r\right) \leqslant c_n\, t \sup_{|x-y|<2r}\ \sup_{|\mathbf{e}|\leqslant 1} \operatorname{Re} p(y, \mathbf{e}/r) \tag{18}$$

and for all $x \in \boldsymbol{R}^n \setminus N$, $t \geqslant 0$ *and* $0 < r < \rho^{-1}$

$$\boldsymbol{P}_x\left(\sup_{s\leqslant t}|X_s - x| < \frac{r}{m}\right) \leqslant \frac{c_\kappa^m}{t^m \inf\limits_{|x-y|<2r}\sup\limits_{|\mathbf{e}|=1} \operatorname{Re} p(y, \mathbf{e}/4\kappa r)}, \qquad m = 1, 2 \tag{19}$$

with absolute constants $\kappa^{-1} := 4\arctan(1/2c_0)$ *and* $c_n, c_\kappa > 0$.

Since $\{\sup_{s\leqslant t}|X_s - x| < r\} \subset \{\sigma_r^x > t\} \subset \{\sup_{s\leqslant t}|X_s - x| \leqslant r\}$, for all $r > 0$ and $t > 0$, Theorem 3.2 gives also upper bounds for the sojourn probabilities $\boldsymbol{P}_x(\sigma_r^x > t)$ and $\boldsymbol{P}_x(\sigma_r^x \leqslant t)$. As in [17, Theorem 4.7, Remark 4.8] these lead to the following estimates for the mean sojourn time for $x \in \boldsymbol{R}^n \setminus N$ and small $r < \rho^{-1}$,

$$\frac{c_n}{\sup\limits_{|x-y|<2r}\sup\limits_{|\mathbf{e}|\leqslant 1} \operatorname{Re} p(y, \mathbf{e}/r)} \leqslant \boldsymbol{E}_x(\sigma_r^x) \leqslant \frac{C_\kappa}{\inf\limits_{|x-y|<2r}\sup\limits_{|\mathbf{e}|=1} \operatorname{Re} p(y, \mathbf{e}/4c_\kappa r)}, \tag{20}$$

with absolute constants $\kappa^{-1} := 4\arctan(1/2c_0)$ and $c_n, C_\kappa > 0$. In [17] we assumed the sector condition of Theorem 3.2 for *all* $\xi \in \boldsymbol{R}^n$ and obtained estimates for *all* $r > 0$. If it holds only for $|\xi| > \rho$ for some fixed $\rho > 0$, the argument [17, p. 607, line 6 from below] shows that the estimates are still valid if we restrict ourselves to small values $r < \rho^{-1}$.

If $z \in B_{r/2}(x)$, then it is clear that $B_{r/2}(z) \subset B_r(x) \subset B_{r/2+r}(z)$. Hence, $\sigma_{r/2}^z \leqslant \sigma_r^x \leqslant \sigma_{3r/2}^z$. This gives

COROLLARY 3.3. *Under the assumptions of Theorem* 3.2 *we have*

$$\frac{c_n}{\sup\limits_{|x-y|<3r/2}\sup\limits_{|\mathbf{e}|\leqslant 1} \operatorname{Re} p(y, 2\mathbf{e}/r)} \leqslant \boldsymbol{E}_z(\sigma_r^x) \leqslant \frac{C_\kappa}{\inf\limits_{|x-y|<7r/2}\sup\limits_{|\mathbf{e}|=1} \operatorname{Re} p(y, \mathbf{e}/6c_\kappa r)} \tag{21}$$

for all $x \in \boldsymbol{R}^n$, $r < \rho^{-1}$ *and all* $z \in B_{r/2}(x) \setminus N$.

In order to make Theorem 3.2 and Corollary 3.3 work in the setting of Section 2, we have to verify the conditions on the symbol $p(x, \xi)$. From Theorem 2.2 and Corollary 2.4 we know already that (8), (9) imply that a symbol exists and that $\sup_x |p(x, \xi)| \leqslant K\,(1 + |\xi|^2)$. For the sector condition of Theorem 3.2 we need some preparations.

LEMMA 3.4. *Let* $(\mathcal{E}, \mathcal{F})$ *be as in Theorem* 2.1 *and assume that* (8), (9) *and* (16) *hold. If for all* $|h| < 1$

$$|j(x, x+h) - j(x, x-h)| \leqslant c\,|h| \log\frac{1}{|h|}\,(j(x, x+h) + j(x, x-h)), \tag{22}$$

then the symbol $p(x, \xi)$ *satisfies the estimate* $|\operatorname{Im} p(x, \xi)| \leqslant c\sqrt{\operatorname{Re} p(x, \xi)}$ *for all* $x, \xi \in \boldsymbol{R}^n$.

PROOF. Obviously, (16) is stronger than (9) which means that the form $\mathcal{E}$ is generated by a pseudo differential operator with symbol $p(x, \xi)$. Define $\theta(h) := (|h| \wedge 1)\log(e/(|h| \wedge 1))$,

$e = 2.71828\ldots$. Then we see using the Cauchy-Schwarz inequality

$$
\begin{aligned}
|\mathrm{Im}\, p(x,\xi)| &= \left| \frac{1}{2} \int_{h\neq 0} \sin(h\xi)(j(x,x+h) - j(x,x-h))dh \right| \\
&\leqslant \left(\int_{h\neq 0} \sin^2(h\xi)\, \frac{|j(x,x+h) - j(x,x-h)|}{\theta(h)}\, dh \right)^{1/2} \\
&\quad \times \left(\int_{h\neq 0} \theta(h)\, |j(x,x+h) - j(x,x-h)| dh \right)^{1/2}.
\end{aligned}
$$

Because of (8) and (16) the second factor is uniformly bounded for all x; using (22) and the elementary estimate $\sin^2 t \leqslant 2(1-\cos t)$ the claim follows. □

LEMMA 3.5. *Let $p(x,\xi)$ be as in Corollary 2.3. If for some absolute constant $c > 0$*

$$
\liminf_{|\xi|\to\infty} \frac{j(x, x + h/|\xi|)}{|\xi|^n} \geqslant c > 0\,, \quad x \in \boldsymbol{R}^n,\ |h| < 1\,, \tag{23}
$$

then $\liminf_{|\xi|\to\infty} |p(x,\xi)| \geqslant (c/12)\, \pi^{n/2}/\Gamma(n/2+2)$.

PROOF. Since $1 - \cos t \geqslant t^2/3$ for $|t| \leqslant 1$ and since $j(x, x+h/|\xi|)|\xi|^{-n} \geqslant c/2$ for large values of $|\xi|$, we find for $|\xi| \gg 1$

$$
\begin{aligned}
|p(x,\xi)| &\geqslant \int_{h\neq 0} (1 - \cos(h\cdot\xi)) j(x,x+h) dh \geqslant \frac{1}{3} \int_{|h||\xi|\leqslant 1} (h\cdot\xi)^2 j(x,x+h) dh \\
&= \frac{1}{3} \int_{|y|\leqslant 1} (y\cdot\xi/|\xi|)^2 j(x, x + y/|\xi|)\, \frac{dy}{|\xi|^n} \\
&\geqslant \frac{c}{6} \int_{|y|\leqslant 1} (y\cdot\xi/|\xi|)^2 dy = \frac{c}{12}\, \frac{\pi^{n/2}}{\Gamma(n/2+2)}
\end{aligned}
$$

where we used Sonin's formula $\int_{|y|\leqslant 1} (y\cdot a)^2\, dy = (1/2)(\pi^{n/2}/\Gamma(n/2+2))|a|^2$. Since the right-hand side is independent of ξ, the claim follows as $|\xi| \to \infty$. □

If $\mathcal{E}$ is as in Section 2 we have the alternative representation $j(x,y) = |x-y|^{\alpha(x,y)} + |x-y|^{\alpha(y,x)}$ for the jump density.

PROPOSITION 3.6. *Let $(\mathcal{E},\mathcal{F})$ be as in Theorem 2.1, assume that (8), (9), (16) and (23) hold and that $\alpha(x,y)$ is Lipschitz continuous. Then $p(x,\xi)$ satisfies the sector condition of Theorem 3.2 for large $|\xi|$.*

PROOF. The assertion follows directly from Lemmata 3.4 and 3.5. Only (22) needs proof. For this we use the elementary formula

$$
|t^{-a} - t^{-b}| = \left| \int_a^b t^{-u} \log t\, du \right| \leqslant |b-a| |\log t| (t^{-a} + t^{-b})
$$

with $a = \alpha(x, x+h)$ resp. $\alpha(x+h, x)$, $b = \alpha(x, x-h)$ resp. $\alpha(x-h, x)$ and $t = |h|$. □

EXAMPLE 3.7. We have seen in Example 2.6 that (11), (12), and (15) imply (8), (9) and even (16). If we also assume that $0 \leqslant \alpha \leqslant \beta < 2$, we find for $|\xi| > 1$ and $|h| < 1$

$$\frac{j(x, x + h/|\xi|)}{|\xi|^n} \geqslant \frac{c}{|h|^{n+\alpha}} |\xi|^{n+\alpha} |\xi|^{-n} = c \frac{|\xi|^\alpha}{|h|^{n+\alpha}} \geqslant \frac{c}{|h|^{n+\alpha}} \geqslant c$$

which shows that condition (23) from Lemma 3.5 is satisfied. Thus, Proposition 3.6 holds.

The above assumptions are more far-reaching. Let $p(x, \xi)$ and $j(x, y)$ be as before and define a new symbol by

$$p_1(x, \xi) := \text{p.v.}\int_{h \neq 0} (1 - e^{ih\xi}) j_1(x, x + h) dh\,, \tag{24}$$

with the modified jump measure

$$j_1(x, y) := j(x, y) \mathbf{1}_{B_1(0)}(x - y) + e^{-|x-y|} \mathbf{1}_{B_1^c(0)}(x - y)\,. \tag{25}$$

The corresponding pseudo differential operator can then be written as

$$\begin{aligned} -p_1(x, D)u(x) = \text{p.v.}&\int_{0<|h|<1} (u(x + h) - u(x)) j(x, x + h) dh \\ &+ \int_{|h| \geqslant 1} (u(x + h) - u(x)) e^{-|h|}\, dh\,. \end{aligned} \tag{26}$$

Note that j_1 has the same small jumps as j but that large jumps occur at a different rate than before. A short calculation using (11) shows that for suitable constants c_β, $c_{\alpha,\rho}$ not depending on x and $|\xi| > \rho$

$$c_{\alpha,\rho} |\xi|^\alpha \leqslant \operatorname{Re} p_1(x, \xi) \leqslant c_\beta |\xi|^\beta\,, \qquad x \in \boldsymbol{R}^n\,,\ |\xi| \geqslant \rho\,. \tag{27}$$

Using (8) it is now easy to see that $\operatorname{Re} p(x, \xi) \sim \operatorname{Re} p_1(x, \xi)$ for large $|\xi|$ and that

$$c_{\alpha,\rho} |\xi|^\alpha \leqslant \operatorname{Re} p(x, \xi) \leqslant c_{\beta,\rho} |\xi|^\beta\,, \qquad x \in \boldsymbol{R}^n\,,\ |\xi| \geqslant \rho \tag{28}$$

for some $c_{\beta,\rho}$. Substituting (28) into (18), (19) and (21) yields

COROLLARY 3.8. *Let $p(x, \xi)$ and $p_1(x, \xi)$ be as in Example* 3.7, *i.e., satisfying* (13)–(15) *and take ρ as in* (27), (28). *Then the following estimates hold for the sojourn time σ_r^x, $x \in \boldsymbol{R}^n$, $r < 1$, of the process belonging to either symbol*:

$$\boldsymbol{P}_x(\sigma_r^x \leqslant t) \leqslant c_n\, t\, r^{-\beta}\,, \quad \text{for all } r < 1\,, \quad \boldsymbol{P}_x(\sigma_r^x > t) \leqslant c_\kappa\, t^{-1} r^\alpha\,, \quad \text{for all } r < 1/\rho\,,$$
$$c_{n,\beta,\rho}\, r^\beta \leqslant \boldsymbol{E}_z \sigma_r^x \leqslant c_{\kappa,\alpha,\rho}\, r^\alpha, \quad \text{for all } r < 1/\rho,\ z \in B_{r/2}(x)\,.$$

REMARK 3.9. The arguments used in Example 3.7 show that we can replace in Theorem 3.2 and Corollary 3.3 the symbol $\operatorname{Re} p$ by the modification $\operatorname{Re} p_1$ if $\liminf_{|\xi| \to \infty} \operatorname{Re} p(x, \xi) \geqslant c > 0$ and $\sup_x \sup_\xi |p(x, \xi) - p_1(x, \xi)| < \infty$; the latter is always implied by conditions of the type (8).

4. A perturbation result. In Example 3.7 we have modified the (large-jump part of the) symbol $p(x,\xi)$ from Corollary 2.3 to become

$$p_1(x,\xi) = \text{p.v.}\int_{|h|<1} (1-e^{ih\xi}) j(x,x+h)dh + \int_{|h|\geqslant 1} (1-e^{ih\xi})\, e^{-|h|}\, dh\,.$$

The jump measure $j(x,y)$ was assumed to satisfy the conditions (11)–(15) which meant, in particular, that $-p(x,D)$ and $-p_1(x,D)$ (defined on, say, $C_0^\infty(\boldsymbol{R}^n)$), have extensions A and A_1 to generators of Dirichlet forms and that for the corresponding stochastic processes all results of Sections 1–3 hold true. *We will now always make these assumptions.*

For $q_1(x,\xi) := p(x,\xi) - p_1(x,\xi)$ we have $q_1(x,\xi) = \int_{|h|\geqslant 1}(1-e^{ih\xi})(j(x,x+h) - e^{-|h|})dh$. It is easy to see that under (8)

$$\| -q_1(\bullet, D)u\|_\infty \leqslant 2\Big(\sup_x \int_{|h|\geqslant 1} j(x,x+h)dh + c_n\Big)\|u\|_\infty\,, \tag{29}$$

which shows that $-q_1(x,D)$ extends naturally to a continuous operator B on $L^\infty(\boldsymbol{R}^n)$, resp., $B_b(\boldsymbol{R}^n)$. From $A\big|_{C_0^\infty(\boldsymbol{R}^n)} = -p(x,D) = -p_1(x,D) - q_1(x,D)$, we see that $-p_1(x,D)$ has an extension A_1 on $\mathcal{D}(A)$.

Since A and A_1 generate Dirichlet forms, we can associate with both of them sub-Markovian semigroups and resolvent operators on $L^2(\boldsymbol{R}^n)$ which we will denote by T_t^A, $T_t^{A_1}$ and R_λ^A, $R_\lambda^{A_1}$, respectively.

Recall that a sub-Markovian operator T is said to be *Feller*, if T maps the set $C_\infty(\boldsymbol{R}^n)$ into itself; T is called a *strong Feller* operator, if T maps $B_b(\boldsymbol{R}^n)$ into $C_b(\boldsymbol{R}^n)$. Note that a sub-Markovian operator that is defined on $L^2(\boldsymbol{R}^n)$ has a canonical extension onto $L^\infty(\boldsymbol{R}^n)$, while a Feller operator can be canonically extended to $B_b(\boldsymbol{R}^n)$, see the proof of Lemma 1.6.4 in Fukushima *et al.* [6].

LEMMA 4.1. *Let $T : L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$ be a continuous operator which* (*has an extension which*) *is strongly Fellerian. Then T maps $L^\infty(\boldsymbol{R}^n)$ into $C_b(\boldsymbol{R}^n)$.*

PROOF. The point is to show that any two representatives $f,\phi \in B_b(\boldsymbol{R}^n)$ of some equivalence class $[f] \in L^\infty(\boldsymbol{R}^n)$ have the same image under T. By assumption, $N := \{x : f(x) \neq \phi(x)\}$ is a Lebesgue null set and, therefore, $f-\phi \in L^2(\boldsymbol{R}^n)$ and so $\|T(f-\phi)\|_{L^2} = 0$. Thus, $Tf = T\phi$ almost everywhere, hence, everywhere since $Tf, T\phi \in C_b(\boldsymbol{R}^n)$. □

PROPOSITION 4.2. *Let A, A_1 be infinitesimal generators of sub-Markovian semigroups on $L^2(\boldsymbol{R}^n)$ such that $B := A - A_1$ is a bounded operator on $L^\infty(\boldsymbol{R}^n)$. If the resolvent operators $(R_\lambda^{A_1})_{\lambda>0}$ are strong Feller operators, then the resolvent $(R_\lambda^A)_{\lambda>0}$ is also strongly Fellerian.*

PROOF. Note that R_λ^A, $R_\lambda^{A_1}$ are *a priori* defined on $L^2(\boldsymbol{R}^n)$. Since λR_λ^A, $\lambda R_\lambda^{A_1}$ are sub-Markovian operators, they have an extension to $L^\infty(\boldsymbol{R}^n)$; the assumption says that $R_\lambda^{A_1}$ can be defined on $B_b(\boldsymbol{R}^n)$. Using the second resolvent equation $R_\lambda^{A_1} - R_\lambda^A = R_\lambda^{A_1}(A_1 - A)R_\lambda^A$ we get for $u \in B_b(\boldsymbol{R}^n)$ or $L^\infty(\boldsymbol{R}^n)$ that $R_\lambda^A u = R_\lambda^{A_1}u + R_\lambda^{A_1}BR_\lambda^A u$. Since BR_λ^A is bounded on $L^\infty(\boldsymbol{R}^n)$, the assertion follows from Lemma 4.1 and the strong Feller property of $R_\lambda^{A_1}$. □

Since $R_\lambda^A = \int_0^\infty e^{-\lambda t} T_t^A \, dt$, it is clear from Lebesgue's dominated convergence theorem that R_λ^A inherits the (strong) Feller property from T_t^A. Conversely, if $(R_\lambda^A)_{\lambda>0}$ is strongly Fellerian, T_t need not have the strong Feller property. Things are different if we consider the Feller property. The following result is a consequence of the Hille-Yosida theory, cf. sections III.4–6 in Rogers and Williams [15].[2]

PROPOSITION 4.3. *Let $(R_\lambda)_{\lambda>0}$ be a sub-Markovian resolvent and assume that it corresponds to a sub-Markovian semigroup $(T_t)_{t\geqslant 0}$. If the operators R_λ have the Feller property and if* $\lim_{\lambda\to\infty} \lambda R_\lambda u(x) = u(x)$ *for all $u \in C_\infty(\boldsymbol{R}^n)$, then the semigroup operators T_t, $t > 0$ are Fellerian and, as operators on $C_\infty(\boldsymbol{R}^n)$, strongly continuous.*

Now we can apply Propositions 4.2 and 4.3.

COROLLARY 4.4. *Let A and A_1 be the extensions of $-p(x, D)$ and $-p_1(x, D)$ described at the beginning of Section* 4; *In particular we know that* (8), (9) *hold. If the L^2 sub-Markovian contraction resolvent $R_\lambda^{A_1}$ has the strong Feller property, then the L^2-semigroup $(T_t^A)_{t\geqslant 0}$ generated by A is a Feller semigroup, i.e., a strongly continuous sub-Markovian contraction semigroup on $C_\infty(\boldsymbol{R}^n)$.*

PROOF. Note that $B := A - A_1$ is a bounded operator on $L^\infty(\boldsymbol{R}^n)$. Proposition 4.2 therefore shows that R_λ^A inherits the strong Feller property from $R_\lambda^{A_1}$. This means, in particular, that $R_\lambda^A : C_\infty(\boldsymbol{R}^n) \to C_b(\boldsymbol{R}^n)$. To get the Feller property, it remains to show that $\lim_{|x|\to\infty} R_\lambda^A u(x) = 0$ for all $\lambda > 0$.

From Theorem 2.2 we know that $C_0^\infty(\boldsymbol{R}^n) \subset \mathcal{D}(A)$ (note that $\mathcal{D}(A)$ is the L^2 domain of the operator A). From the estimate (10) we find for all $v \in C_0^\infty(\boldsymbol{R}^n)$

$$\|\mu R_\mu^A v - v\|_\infty = \|R_\mu^A A v\|_\infty \leqslant \frac{1}{\mu} \|Av\|_\infty \leqslant \frac{C}{\mu} \sum_{|\alpha|\leqslant 2} \|\partial^\alpha v\|_\infty .$$

Since $C_0^\infty(\boldsymbol{R}^n)$ is dense in $C_\infty(\boldsymbol{R}^n)$ and since $\|\mu R_\mu^A\|_\infty \leqslant 1$, we get with a routine 3ε-argument that $\lim_{\mu\to\infty} \|\mu R_\mu^A u - u\|_\infty = 0$ for all $u \in C_\infty(\boldsymbol{R}^n)$.

From the resolvent equation one easily sees that $\lambda \mapsto \lambda R_\lambda^A v$ is increasing for measurable functions $v \geqslant 0$. Thus, we find for $u \in C_\infty(\boldsymbol{R}^n)$, $x \in \boldsymbol{R}^n$ and all $\mu > \lambda$

$$|\lambda R_\lambda^A u(x)| \leqslant \lambda R_\lambda^A |u|(x) \leqslant \mu R_\mu^A |u|(x) \leqslant \|\mu R_\mu^A |u| - |u|\|_\infty + |u(x)| .$$

Letting first $\mu \to \infty$ and then $|x| \to \infty$ proves $\lim_{|x|\to\infty} |\lambda R_\lambda^A u(x)| = 0$. The assertion follows now from Proposition 4.3. □

5. A Harnack inequality. Let $p(x,\xi)$, $j(x,y)$, A and $p_1(x,\xi)$, $j_1(x,y)$, A_1 be as described at the beginning of Section 4. We write $(X_t)_{t\geqslant 0}$ resp. $(Y_t)_{t\geqslant 0}$ for the Hunt processes generated by A resp. A_1; since the corresponding Dirichlet forms are equivalent, we can assume that the exceptional set N is the same for both processes. Note that this ensures that all assumptions of Corollary 3.8 are satisfied.

[2] We are grateful to the editorial committee of the journal, for pointing out this reference.

In this section we will prove the Harnack inequality for the process generated by the modified operator A_1. Our argument follows closely the methods developed by Bass and co-authors [2, 1], see also Song and Vondraček [19].

Let D be a domain in $\boldsymbol{R}^n$. A function h defined on $\boldsymbol{R}^n \setminus N$ is said to be A_1-*harmonic* in $D \setminus N$ if it is not identically infinite in $D \setminus N$ and if for all bounded open sets $U \subset \bar{U} \subset D$

$$h(x) = \boldsymbol{E}_x[h(Y_{\sigma_U})], \quad x \in U \setminus N\,,$$

where σ_U is the first exit time from U of the process $(Y_t)_{t \geqslant 0}$ belonging to A_1.

We say that the *Harnack inequality* holds for the process if for every domain $D \subset \boldsymbol{R}^n$ and every compact set $K \subset D$ there exists a constant $C = C_{D,K} > 0$ such that for all positive harmonic functions h in $D \setminus N$ the following inequality holds

$$\sup_{x \in K \setminus N} h(x) \leqslant C \inf_{x \in K \setminus N} h(x)\,.$$

Recalling that $j_1(x,y)$ behaves like $j(x,y)$ if $|x-y|$ is small and like $e^{-|x-y|}$ otherwise, the following Lemma follows at once from (11) and (12).

LEMMA 5.1. *For all $x \in \boldsymbol{R}^n$, $r < 1$ and all $v, y, z \in \boldsymbol{R}^n$ satisfying $|v-x| > r$ and $|y-x| < r/2$, $|z-x| < r/2$ we have $j_1(y,v) \leqslant \gamma\, r^{\alpha-\beta}\, j_1(z,v)$ with an absolute constant $0 < \gamma < \infty$.*

PROPOSITION 5.2. *Let $p_1(x,\xi)$ and $j_1(x,y)$ be as above, $x \in \boldsymbol{R}^n \setminus N$ and $0 < r < 1 \wedge \rho^{-1}$. Then there exists a constant c such that for all $H \in B_b^+(\boldsymbol{R}^n)$ with support* supp $H \subset B_r^c(x)$ *the inequality $\boldsymbol{E}_y H(Y_{\sigma^x_{r/2}}) \leqslant c\, r^{2(\alpha-\beta)}\, \boldsymbol{E}_z H(Y_{\sigma^x_{r/2}})$ holds for all $y, z \in B_{r/4}(x) \setminus N$.*

PROOF. We consider first functions $\phi \in C^2_\infty(\boldsymbol{R}^n) \subset \mathcal{D}(A_1)$, with $\phi \geqslant 0$ and $\operatorname{supp}\phi \subset B_r^c(x)$. To simplify notation we write $\sigma := \sigma^x_{r/2}$. For all $z \in B_{r/4}(x)$ we find using Dynkin's formula

$$\begin{aligned}
\boldsymbol{E}_z\phi(Y_\sigma) &= \boldsymbol{E}_z\phi(Y_\sigma) - \phi(z) = \boldsymbol{E}_z\left(\int_0^\sigma A_1\phi(Y_s)ds\right)\\
&= \boldsymbol{E}_z\left(\int_0^\sigma \text{p.v.}\!\int (\phi(Y_s+h) - \phi(Y_s))\, j_1(Y_s, Y_s+h)\,dh\,ds\right)\\
&= \boldsymbol{E}_z\left(\int_0^\sigma\!\!\int \phi(v)\, j_1(Y_s,v)\,dv\,ds\right) \leqslant (\boldsymbol{E}_z\sigma)\int \phi(v) \sup_{y \in B_{r/2}(x)} j_1(y,v)\,dv\,.
\end{aligned}$$

From Corollary 3.8 we get for $r < 1/\rho$ and $y, z \in B_{r/2}(x)$ that $\boldsymbol{E}_z\sigma \leqslant C\, r^{\alpha-\beta}\, \boldsymbol{E}_y\sigma$, where $C = C_{\kappa,\alpha,\beta,\rho}$; with Lemma 5.1 we conclude

$$\begin{aligned}
\boldsymbol{E}_z\phi(Y_\sigma) &\leqslant C r^{\alpha-\beta}(\boldsymbol{E}_y\sigma)\int \phi(v)\gamma r^{\alpha-\beta} \inf_{y \in B_{r/2}(x)} j_1(y,v)\,dv\\
&\leqslant C' r^{2(\alpha-\beta)} \boldsymbol{E}_y\left(\int_0^\sigma\!\!\int \phi(v)\, j_1(Y_s,v)\,dv\,ds\right)\\
&= C' r^{2(\alpha-\beta)}\, \boldsymbol{E}_y\left(\int_0^\sigma A_1\phi(Y_s)ds\right) = C' r^{2(\alpha-\beta)}\, \boldsymbol{E}_y\phi(Y_\sigma)\,.
\end{aligned}$$

Since we can approximate any positive measurable function H with $C^2_\infty(\boldsymbol{R}^n)$-functions, the claim follows. □

PROPOSITION 5.3. *There exists a constant $c > 0$ such that for all $x \in \boldsymbol{R}^n$, $0 < r < 1/2 \wedge 1/\rho$, $D \subset B_r(x)$ and $y \in B_{r/2}(x)$ the following inequality holds*:

$$\boldsymbol{P}_y(\tau_D < \sigma_r^x) \geqslant c r^{\beta-\alpha} \frac{|D|}{|B_r(x)|},$$

where τ_D is the first entrance time into D and σ_r^x is the first exit time from $B_r(x)$.

PROOF. Fix $y \in B_{r/2}(x) \setminus N$. If $\boldsymbol{P}_y(\tau_D < \sigma_r^x) \geqslant 1/4$, the claim is obviously true. This happens, in particular, if $y \in D \setminus N$. Let us, therefore, assume that $y \notin D$ and that $\boldsymbol{P}_y(\tau_D < \sigma_r^x) < 1/4$. Pick a sequence $\phi_j \in C_0^2(\boldsymbol{R}^n) \subset \mathcal{D}(A_1)$ such that $0 \leqslant \phi_j \uparrow \mathbf{1}_D$ and $\phi_j(y) = 0$. Then we get from Dynkin's formula and the form (26) of the generator A_1

$$\begin{aligned}
\boldsymbol{P}_y(\tau_D < \sigma_r^x) \geqslant \boldsymbol{E}_y \mathbf{1}_D(Y_{\tau_D\wedge\sigma_r^x}) - \mathbf{1}_D(y) &= \sup_j \boldsymbol{E}_y\left(\int_0^{\tau_D\wedge\sigma_r^x} A_1\phi_j(Y_s)\,ds\right)\\
&= \sup_j \boldsymbol{E}_y\left(\int_0^{\tau_D\wedge\sigma_r^x}\!\!\int \phi_j(Y_s+h)\, j_1(Y_s, Y_s+h)\,dh\,ds\right)\\
&\overset{(11)}{\geqslant} \sup_j \boldsymbol{E}_y\left(\int_0^{\tau_D\wedge\sigma_r^x}\!\!\int \phi_j(Y_s+h)\frac{c}{|h|^{\alpha+n}}\,dh\,ds\right)\\
&= \boldsymbol{E}_y\left(\int_0^{\tau_D\wedge\sigma_r^x}\!\!\int \mathbf{1}_D(v)\frac{c}{|Y_s-v|^{\alpha+n}}\,dv\,ds\right)\\
&\geqslant \boldsymbol{E}_y\left(\int_0^{\tau_D\wedge\sigma_r^x}\!\!\int \mathbf{1}_D(v)\frac{c}{(2r)^{\alpha+n}}\,dv\,ds\right)\\
&= c'\,\boldsymbol{E}_y(\tau_D\wedge\sigma_r^x)\, r^{-\alpha}\,\frac{|D|}{|B_r(x)|}.
\end{aligned}$$

In the penultimate step we used that $|Y_s - v| \leqslant |Y_s - x| + |x - v| \leqslant 2r$ for all $s < \sigma_r^x$.

From the Markov inequality and Corollary 3.8 we get for any $T > 0$

$$\begin{aligned}
\boldsymbol{E}_y(\tau_D \wedge \sigma_r^x) &\geqslant T(1 - \boldsymbol{P}_y(\tau_D < \sigma_r^x) - \boldsymbol{P}_y(\sigma_r^x < T))\\
&\geqslant T(3/4 - \boldsymbol{P}_y(\sigma_r^x < T)) \geqslant T(3/4 - c_n T r^{-\beta}).
\end{aligned}$$

The last expression reaches its maximum at $T = 3r^\beta/(8c_n)$, and we find $\boldsymbol{E}_y(\tau_D \wedge \sigma_r^x) \geqslant (9/64c_n)r^\beta$. Inserting this into the first estimate we finally arrive at

$$\boldsymbol{P}_x(\tau_D < \sigma_r^x) \geqslant c'' r^{\beta-\alpha} \frac{|D|}{|B_r(x)|}.$$ □

We can now show the Harnack inequality. If we take into account the exceptional set N, the proof is almost literally the same as Bass and Kaßmann's proof of Theorem 4.1 in [1]; we will therefore only state the result.

THEOREM 5.4. *Let $A_1 = -p_1(x, D)$, $p_1(x, \xi)$ and $j_1(x, y)$ be as above. For all $R > 0$ and $x_0 \in \boldsymbol{R}^n$, there exists a constant $C = C(R, x_0) > 0$ such that for all positive* *bounded functions h on $\mathbf{R}^n \setminus N$ which are A_1-harmonic in $B_R(x_0)$, the following Harnack inequality holds*:

$$h(x) \leqslant Ch(y) \quad \text{for all } x, y \in B_{R/2}(x_0) \setminus N\,.$$

6. The Feller property of the semigroup e^{tA}. Let $p(x,\xi)$, $j(x,y)$, A and $p_1(x,\xi)$, $j_1(x,y)$, A_1 be as in the previous section. We write $(X_t)_{t\geqslant 0}$ (resp. $(Y_t)_{t\geqslant 0}$) for the Hunt processes generated by A (resp. A_1); without loss of generality we can assume that the exceptional set N is the same for both processes. *We will from now on assume that $\alpha = \beta$.*

LEMMA 6.1. *Let $0 < r < 1/2 \wedge 1/\rho$. Then we have for all $x \in \mathbf{R}^n \setminus N$ and $R > 2r$*

$$\mathbf{P}_x(Y_{\sigma_r^x} \notin B_R(x)) \leqslant c''\,\frac{r^\alpha}{R^\alpha}\,.$$

PROOF. Pick a sequence $\phi_j \in C_0^2(\mathbf{R}^n) \subset \mathcal{D}(A_1)$ with $0 \leqslant \phi_j \uparrow \mathbf{1}_{B_R^c(x)}$. From Dynkin's formula and the structure (26) of $A_1 = -p_1(x,D)$ we get

$$\begin{aligned}\mathbf{P}_x(Y_{\sigma_r^x} \notin B_R(x)) &= \sup_j \mathbf{E}_x(\phi_j(Y_{\sigma_r^x})) = \sup_j \mathbf{E}_x\left(\int_0^{\sigma_r^x} A_1\phi_j(Y_s)ds\right)\\ &= \sup_j \mathbf{E}_x\left(\int_0^{\sigma_r^x}\!\!\int \phi_j(Y_s+h)j_1(Y_s,Y_s+h)dhds\right)\\ &\leqslant C\,\mathbf{E}_x\left(\int_0^{\sigma_r^x}\!\!\int \mathbf{1}_{B_R^c(x)}(v)\,\frac{dvds}{|Y_s-v|^{\alpha+n}}\right).\end{aligned}$$

The last estimate follows from (11) if $|Y_s - v| \leqslant 1$, and from the trivial estimate $e^{-|Y_s-v|} \leqslant |Y_s-v|^{-n-\alpha}$, if $|Y_s - v| > 1$. For $s < \sigma_r^x$ we have $|Y_s - x| < r$ which means that for $v \in B_R^c(x)$ the inequality $|Y_s - v| \geqslant |x-v| - |Y_s - x| \geqslant R - r \geqslant R/2$ obtains; in particular, $B_R^c(x) \subset B_{R/2}^c(Y_s)$. Thus,

$$\mathbf{P}_x(Y_{\sigma_r^x} \notin B_R(x)) \leqslant C\mathbf{E}_x\left(\int_0^{\sigma_r^x}\!\!\int_{B_{R/2}^c(Y_s)} \frac{dvds}{|Y_s-v|^{\alpha+n}}\right) = C_n'\,(\mathbf{E}_x\sigma_r^x)\int_{R/2}^\infty \frac{du}{|u|^{\alpha+1}} \leqslant C_n''\,\frac{r^\alpha}{R^\alpha}$$

where we used Corollary 3.8 for the last estimate. □

THEOREM 6.2. *Let $0 < r < 1/2 \wedge 1/\rho$ and $x_0 \in \mathbf{R}^n$. There exist constants $c < \infty$ and $\kappa > 0$ such that for all bounded functions $h \in \mathbf{R}^n \setminus N$ which are A_1-harmonic in $B_r(x_0)$*

$$|h(x) - h(y)| \leqslant c\,\|h\|_\infty\,|x-y|^\kappa, \quad x, y \in B_{r/2}(x_0)\,.$$

The constants c, κ are independent of $0 < r < 1/2 \wedge 1/\rho$, x_0 and h.

PROOF. From Proposition 5.3 we know that for all $D \subset B_r(x)$ with $|D| \geqslant |B_r(x)|/3$ there exists a constant $c > 0$ such that $\mathbf{P}_y(\tau_D < \sigma_r^x) \geqslant c/3$. Take some h as in the statement of the theorem; by adding a suitable constant we can achieve that $h \geqslant 0$. Set

$$M := \|h\|_\infty\,, \quad \eta^2 := 1 - \frac{c}{12} \quad \text{and} \quad \theta^\alpha := \frac{c}{24c''}\sqrt{1-\frac{c}{12}}$$

where c, c'' are the constants from Proposition 5.3 and Lemma 6.1, respectively. Without loss of generality we can assume that $c'' > \rho$, where ρ is as in Corollary 3.8. Obviously, $\eta < 1$ and we can choose c'' so large that $\theta^\alpha/\eta < 1$ and $\theta < 1/\rho \wedge 1/2$.

Following the idea of Bass and Levin [2] we are going to show that

$$\sup h(B_{\theta^k}(x) \setminus N) - \inf h(B_{\theta^k}(x) \setminus N) \leqslant M\eta^k \quad \text{for all } k \in \boldsymbol{Z}. \tag{30}$$

We write $B_j := B_{\theta^j}(x) \setminus N$, $\sigma_j := \sigma^x_{\theta^j}$, $a_j := \inf h(B_j)$ and $b_j := \sup h(B_j)$ for $j \in \boldsymbol{Z}$. Since $\eta < 1$, (30) clearly holds for all negative $k \in \boldsymbol{Z}$. For $k \in \boldsymbol{N}$ we use induction. Assume that (30) is true for $0, 1, 2, \ldots, k$. Fix $\varepsilon > 0$; from the definition of a_{k+1} and b_{k+1} we find some $y, z \in B_{k+1}$ such that

$$b_{k+1} - a_{k+1} \leqslant h(y) - h(z) + \varepsilon .$$

Define $D' := \{z \in B_k : h(z) \leqslant (a_k + b_k)/2\}$. We may assume that $|D'| \geqslant |B_k|/2$—otherwise we consider $M - h$ instead of h. Since Lebesgue measure is inner regular, we can find a compact set $D \subset D'$ such that $|D| \geqslant |B_k|/3$.

Since h is harmonic, we use the strong Markov property to deduce

$$\begin{aligned} h(y) - h(z) &= \boldsymbol{E}_y(h(Y_{\sigma_k}) - h(z)) \\ &= \boldsymbol{E}_y(h(Y_{\sigma_k}) - h(z);\ \sigma_k > \tau_D) + \boldsymbol{E}_y(h(Y_{\sigma_k}) - h(z);\ \sigma_k \leqslant \tau_D) \\ &= \boldsymbol{E}_y(h(Y_{\tau_D}) - h(z);\ \sigma_k > \tau_D) + \boldsymbol{E}_y(h(Y_{\sigma_k}) - h(z);\ \sigma_k \leqslant \tau_D,\ Y_{\sigma_k} \in B_{k-1}) \\ &\quad + \sum_{j=1}^{\infty} \boldsymbol{E}_y(h(Y_{\sigma_k}) - h(z);\ \sigma_k \leqslant \tau_D,\ Y_{\sigma_k} \in B_{k-j-1} \setminus B_{k-j}) . \end{aligned}$$

By the very definition of the set D, the first term on the right is bounded by

$$\left(\frac{1}{2}(a_k + b_k) - a_k\right) \boldsymbol{P}_y(\tau_D < \sigma_k) = \frac{1}{2}(b_k - a_k)\, \boldsymbol{P}_y(\tau_D < \sigma_k) .$$

The second term is less than or equal to

$$(b_{k-1} - a_k)\boldsymbol{P}_y(\sigma_k \leqslant \tau_D) \leqslant (b_{k-1} - a_{k-1})(1 - \boldsymbol{P}_y(\sigma_k > \tau_D)) .$$

By the induction assumption and Lemma 6.1 we find that the third term is dominated by

$$\sum_{j=1}^{\infty} (b_{k-j-1} - a_{k-j-1})\, \boldsymbol{P}_y(Y_{\sigma_k} \notin B_{k-j}) \leqslant \sum_{j=1}^{\infty} M\eta^{k-j-1}\, c''\, \frac{\theta^{\alpha k}}{\theta^{(k-j)\alpha}} = c'' M\, \frac{\theta^\alpha}{\eta}\, \frac{1}{1 - \theta^\alpha/\eta} .$$

Since $\theta^\alpha/\eta \leqslant 1/2$, the last fraction is less than 2. Bearing in mind that $\eta \leqslant 1$ and that $\boldsymbol{P}_y(\tau_D < \sigma_k) > c/3$, we find altogether

$$\begin{aligned} h(y) - h(z) &\leqslant \frac{1}{2}\, M\eta^k\, \boldsymbol{P}_y(\tau_D < \sigma_k) + M\eta^{k-1}\, (1 - \boldsymbol{P}_y(\tau_D < \sigma_k)) + 2\, c'' M\eta^{k-1}\, \frac{\theta^\alpha}{\eta} \\ &\leqslant M\eta^{k-1} \left(1 - \frac{c}{6}\right) + 2\, c'' M\eta^{k-1}\, \frac{\theta^\alpha}{\eta} , \end{aligned}$$

and inserting the definition of θ^α/η we get

$$h(y) - h(z) \leqslant M\eta^{k-1}(1 - c/6 + c/12) = M\eta^{k-1}\, \eta^2 .$$

This finishes the induction step.

The rest is now routine: if $x, y \in B_r(x_0) \setminus N$, let $k \in \boldsymbol{Z}$ be the smallest integer such that $\theta^{k+1} \leqslant |x-y| < \theta^k$. Then $\log|x-y| \geqslant (k+1)\log\theta$, $y \in B_{\theta^k}(x)$, and

$$h(x) - h(y) \leqslant M\eta^k = Me^{k\log\eta} \leqslant Me^{(\log|x-y|/\log\theta - 1)\log\eta} = (M/\eta)|x-y|^{\log\eta/\log\rho}. \quad \square$$

Recall that for all $\lambda > 0$ the resolvent of A_1 is given by

$$R_\lambda^{A_1} f(x) = \int_0^\infty e^{-\lambda t} T_t^{A_1} f(x) dt = \boldsymbol{E}_x\left(\int_0^\infty e^{-\lambda t} f(Y_t) dt\right), \quad f \in L^\infty(\boldsymbol{R}^n).$$

THEOREM 6.3. *For every compact set K there exist constants $C < \infty$ and $\kappa > 0$ such that for every $\lambda > 0$ the resolvent $R_\lambda^{A_1} f$, $f \in L^\infty(\boldsymbol{R}^n)$, is Hölder continuous,*

$$|R_\lambda^{A_1} f(x) - R_\lambda^{A_1} f(y)| \leqslant C\,(1 + 1/\lambda)\, \|f\|_\infty |x-y|^\kappa, \qquad x, y \in K \setminus N.$$

PROOF. Without loss of generality we can assume that $x, y \in K \setminus N$ are so close together that $r := 3|x-y| \leqslant 1/2 \wedge 1/\rho$. (If x and y are further apart, we can link them with a finite chain of neighbouring intermediate points.) Fix $\lambda > 0$ and $f \in L^\infty(\boldsymbol{R}^n)$. An application of the strong Markov property shows for $z = x, y$, $x \notin N$, $y \in B_{r/2}(x) \setminus N$

$$R_\lambda^{A_1} f(z) = \boldsymbol{E}_z\left(\int_0^{\sigma_r^x} e^{-\lambda s} f(Y_s) ds\right) + \boldsymbol{E}_z((e^{-\lambda\sigma_r^x} - 1) R_\lambda^{A_1} f(Y_{\sigma_r^x})) + \boldsymbol{E}_z(R_\lambda^{A_1} f(Y_{\sigma_r^x})).$$

A further application of the strong Markov property reveals that the last term, $z \mapsto \boldsymbol{E}_z(R_\lambda^{A_1} f(Y_{\sigma_r^x}))$ is A_1-harmonic in $B_r(x) \setminus N$. Using the elementary estimates $|e^{-\lambda s} - 1| \leqslant \lambda s$ and $|e^{-\lambda s}| \leqslant 1$, we get from Theorem 6.2 and Corollary 3.8 that

$$\begin{aligned}
&\left|R_\lambda^{A_1} f(x) - R_\lambda^{A_1} f(y)\right| \\
&\quad\leqslant 2\|f\|_\infty \max_{z=x,y}(\boldsymbol{E}_z\sigma_r^x) + 2\lambda\|R_\lambda^{A_1} f\|_\infty \max_{z=x,y}(\boldsymbol{E}_z\sigma_r^x) + c\,\|R_\lambda^{A_1} f\|_\infty |x-y|^\kappa \\
&\quad\leqslant 4c\|f\|_\infty r^\alpha + \frac{c}{\lambda}\,\|f\|_\infty |x-y|^\kappa \\
&\quad\leqslant C(1 + 1/\lambda)(r^\alpha + |x-y|^\kappa)\|f\|_\infty.
\end{aligned}$$

Since $r = 3|x-y|$ the claim follows from

$$|R_\lambda^{A_1} f(x) - R_\lambda^{A_1} f(y)| \leqslant C'\,(1 + 1/\lambda)\|f\|_\infty |x-y|^{\alpha\wedge\kappa}. \qquad \square$$

The following result has been obtained by Komatsu [13] for non-degenerate Lévy-kernels of the form $k(x, y)|x-y|^{-\alpha-n}$ where $0 < c_1 \leqslant k(x, y) \leqslant c_2 < \infty$ using pseudo differential operator methods and a smoothing technique for non-smooth kernels k.

COROLLARY 6.4. *The semigroups $T_t^{A_1}$ and T_t^A have modifications $\tilde{T}_t^{A_1}$ and $\tilde{T}_t^A$ which are Feller semigroups. In particular, we can take the exceptional set $N = \emptyset$ and both $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ are everywhere defined processes.*

PROOF. Theorem 6.3 shows that $R_\lambda^{A_1}$ has a modification $\tilde{R}_\lambda^{A_1}$ which has the strong Feller property, i.e., which maps $B_b(\boldsymbol{R}^n)$ to $C_b(\boldsymbol{R}^n)$. Thus, the Feller property of (a modification of) T_t^A follows from Corollary 4.4. The claim for $T_t^{A_1}$ follows if we take $A = A_1$ in the first place. □

COROLLARY 6.5. *If $h \in L^\infty(\boldsymbol{R}^n) \cap \mathcal{D}(A)$ is such that $Ah \in L^\infty(\boldsymbol{R}^n)$—e.g., if h is A-harmonic on $\boldsymbol{R}^n$—, then $h \in C_b(\boldsymbol{R}^n)$ and even Hölder continuous.*

PROOF. Set $g = Ah$. Since $B = A - A_1$ is a bounded operator in $L^\infty(\boldsymbol{R}^n)$, cf. (29), we see that $g = Ah = A_1 h + Bh$ and this implies $h - A_1 h = h - g + Bh$. Applying $R_1^{A_1}$ on both sides of this equality we get $h = R_1^{A_1}(h - g + Bh)$; since $h - g + Bh \in L^\infty(\boldsymbol{R}^n)$, the claim follows from the strong Feller property of the resolvent operator $R_1^{A_1}$. In view of Theorem 6.3 we have even Hölder continuity. □

REMARK 6.6. There are a few obvious extensions of this paper: one possibility is to use the full power of the estimate (21) rather than an asymptotic version of it. Another possibility is to use the full strength of our perturbation technique. In the present paper we have always assumed that $A - A_1$ is a bounded operator in $L^\infty(\boldsymbol{R}^n)$; this is equivalent to consider only perturbations of the large jump part of $j(x, y)$ where $|x - y| > 1$. In fact, all arguments only require that $(A - A_1)R_\lambda^A$ is a bounded operator on $L^\infty(\boldsymbol{R}^n)$. Using some standard functional analysis this amounts to saying that $\mathcal{D}(A) \subset \mathcal{D}(A_1)$. A typical example would be a perturbation of $j(x, y)$ which satisfies conditions (11) with $\alpha = \beta$ by some $\tilde{j}(x, y)\mathbf{1}_{\{|x-y|<1\}}(x - y) \sim |x - y|^{-n-\tilde{\alpha}}$, where $0 < \tilde{\alpha} < \alpha$. This covers, e.g., jump kernels considered by Song and Vondraček [19].

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