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On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices

E. Kilic *, A.P. Stakhov

TOBB Economics and Technology University, Mathematics Department, Sogutozu , 06560 Ankara, Turkey
International Club of the Golden Section, 6 McCreary Trail, Bolton, Ont., Canada L7E 2C8

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Abstract

In this paper we consider certain generalizations of the well-known Fibonacci and Lucas numbers, the generalized Fibonacci and Lucas p-numbers. We give relationships between the generalized Fibonacci p-numbers, $F_p(n)$, and their sums, $\sum_{i=1}^{n} F_p(i)$, and the 1-factors of a class of bipartite graphs. Further we determine certain matrices whose permanents generate the Lucas p-numbers and their sums.

1. Introduction

The well-known Fibonacci $\{F_n\}$ and Lucas $\{L_n\}$ sequences are defined by the following equations, for $n > 1$

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

where $F_0 = 0$, $F_1 = 1$, and

$$L_n = L_{n-1} + L_{n-2} \tag{2}$$

where $L_0 = 2$, $L_1 = 1$.

Modern science, particularly physics, [23,24,28–33,57], widely applies these sequences $\{F_n\}$ and $\{L_n\}$ which result from application of the following recurrence relations: for $n > 1$

$$F(n) = F(n-1) + F(n-2) \tag{3}$$

$F(0) = 0, \quad F(1) = 1$

$$L(n) = L(n-1) + L(n-2) \tag{4}$$

$L(0) = 2$, $L(1) = 1$

In [46,48], the authors considered the rules (3) and (4), then gave the generalization of the Fibonacci and Lucas numbers, called the Fibonacci and Lucas p-numbers as, for any given $p (p = 1, 2, 3, \ldots)$ and $n > p + 1$

* Corresponding author.
E-mail address: ekilic@etu.edu.tr (E. Kilic).

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$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{5}$$

with initial conditions $F_p(0) = 0$, $F_p(1) = \ldots = F_p(p) = F_p\ (p+1) = 1$, and for $n > p$

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{6}$$

where $L_p(0) = p+1$, $L_p(1) = \ldots = L_p(p) = 1$, respectively.

The ratio of the successive Fibonacci number is as known *golden ratio*, that is

$$\tau = \frac{F_{n+1}}{F_n}$$

which is approximately equal 1.618. Also the golden ratio satisfies the characteristic equation of the Fibonacci numbers $t^2 - t - 1 = 0$. When the case of the Fibonacci p-numbers, the well-known golden ratio is called *p-proportion* and shown by τ_p which satisfies the following equation:

$$t^{p+1} = t^p + 1$$

There are many applications of the golden ratio and p-proportion in many places of mathematics and physics. For example, we can take the values of the resistors in Fig. 1 as follows:

$$R1 = \tau_p^{-p} R; \quad R2 = \tau_p^{p+1} R; \quad R3 = \tau_p R$$

where τ_p is the golden p-proportion, $p \in \{0, 1, 2, 3, \ldots\}$.

It is clear that the divider in Fig. 1 gives an infinite number of the different resistor dividers because every p originates a new divider. In particular, for the case $p = 0$ the value of the golden 0-proportion $\tau_0 = 2$ and the divider is reduced to the classical *binary* divider. For the case $p = 1$ the resistors $R1$, $R2$, $R3$ take the following values:

$$R1 = \tau^{-1} R; \quad R2 = \tau^2 R; \quad R3 = \tau R$$

where $\tau = (1 + \sqrt{5})/2$ is the classical golden mean. More details can be found in [40].

Furthermore, as examples of physical applications of the golden ratio, in a general theory of high energy particle theory, i.e., the golden mean have been widely used. For other applications of the golden mean, we can refer to the well-known works of El Naschie and Marek-Crnjac [20–22,34–39].

Recently, in [40–53,58], many interesting properties and applications of these recurrences have been studied by several authors. Especially, in [50,52,53], Stakhov gave the generating matrices of the Fibonacci p-numbers called "golden" matrices and their inverses. Then the author gave the interesting applications to the coding theory called "golden" cryptography. Also in [47], one can find the Binet type formulas for these recurrences and many interesting properties. In [16], considering the generating matrix of the sequence of the Fibonacci p-numbers, the author used the matrix methods and then gave the Binet formula, sums and combinatorial representations of the Fibonacci p-numbers. One can find many properties of the Fibonacci and Lucas p-numbers in webpage of the Museum of Harmony and Golden Section "http://www.goldenmuseum.com".

Further in the earlier works, one can find another generalizations of the Fibonacci and Lucas numbers. For example, in [25], the author defined the k-generalized Fibonacci numbers as

$$f_n = \sum_{j=1}^{k} f_{n-j} \quad \text{for } n > k \geqslant 2 \tag{7}$$

with

$$f_0 = f_1 = \ldots = f_{k-2} = 0, \quad f_{k-1} = f_k = 1$$

where f_n is the nth k-generalized Fibonacci number.

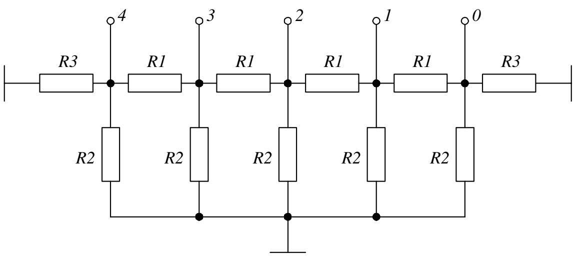

Fig. 1.

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In [4], Er considered the definition of k-generalized Fibonacci number, and then defined k sequences of generalized order-k Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i \quad \text{for } n > 0 \quad \text{and } 1 \leqslant i \leqslant k \tag{8}$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise} \end{cases} \quad 1 - k \leqslant n \leqslant 0 \tag{9}$$

where g_n^i is the nth term of the ith sequence.

In [56], the authors defined k sequences of generalized order-k Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 \leqslant i \leqslant k \tag{10}$$

with initial conditions

$$l_n^i = \begin{cases} -1 & \text{if } n = 1 - i, \\ 2 & \text{if } n = 2 - i, 1 - k \leqslant n \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

where l_n^i is the nth term of the ith sequence. For the relationships between the generalized order-k Fibonacci, Lucas numbers and their Binet formulas and combinatorial representations see [10].

Let A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Define the generalized Fibonacci sequence, $\{u_n\}$, and the generalized Lucas sequence, $\{v_n\}$, by for all $n \geqslant 2$

$$u_n = Au_{n-1} - Bu_{n-2} \tag{11}$$
$$v_n = Av_{n-1} - Bv_{n-2} \tag{12}$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$. If $A = 1$ and $B = -1$, then $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number).

As a special case of the sequence $\{u_n\}$, in [5], the authors consider k-Fibonacci numbers and their some properties.

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$per\ A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. The most important applications of permanents are in the areas of physics and chemistry. One can find more applications of permanents in [27].

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

A matrix is said to be a $(0, 1)$-matrix if each of its entries 0 or 1.

Many connections between permanents or determinants of tridiagonal matrices and the Fibonacci, Lucas numbers can be found in literature. For example, Minc [26] defined an $n \times n$ super-diagonal $(0, 1)$-matrix $F(n,k)$ for $n + 1 \geqslant k$, and showed that the permanent of $F(n,k)$ equals a generalized order-k Fibonacci number. When $k = 2$, the matrix $F(n,2)$ is reduced to the tridiagonal matrix and its permanent equals a usual Fibonacci number. Also in [54,55], the authors defined a family of tridiagonal matrices $M(n)$ and showed that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [19], Lehmer discussed the relationships between permanents of tridiagonal matrices, recurrence relations, and continued fractions. In [9], the authors defined two tridiagonal matrices and then gave the relationships of the permanents and determinants of these matrices and the second order linear recurrences given by (11) and (12). In [11], the authors present a result involving the permanent of an $(-1, 0, 1)$-matrix and the Fibonacci number F_{n+1}. The authors then explore similar directions involving the positive subscripted Fibonacci and Lucas Numbers as well as their uncommon negatively subscripted counterparts. For further similar relationships, we can refer to [2,3,12,13,15,17,18].

Let $A = [a_{ij}]$ be an $m \times n$ real matrix having row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say that A is *contractible on column* (resp. *row*.) k *if column* (resp. *row*.) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of* A *on column* k *relative to rows* i *and* j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}{}^{T}]^{T}$ is called *the contraction of* A *on row* k *relative to* *columns i and j*. Every contraction used in this paper will be on the first column using the first and second rows. One can find the following fact in [1]: let A be a real matrix of order $n>1$ and let B be a contraction of A. Then

$$per\,A = per\,B. \tag{13}$$

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A 1-factor (or *perfect matching*) of a graph with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and sheduling problems. Let $A(G)$ be the adjacency matrix of the bipartite graph G, and let $\mu(G)$ denote the number of 1-factors of G. Then, one can find the following fact in [27]: $\mu(G) \leqslant \sqrt{per\,A(G)}$.

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the *bipartite adjacent matrix* $B(G) = [b_{ij}]$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and 0 otherwise. Then, in [6,27], the number of 1-factors of bipartite graph G equals the permanent of its bipartite adjacency matrix.

In [17], the authors consider the relationship between the k -generalized Fibonacci numbers given in (7) and 1-factors of a class of bipartite graph. Also in [7], the authors determine the class of bipartite graph whose number of 1-factors is the Lucas number, L_n. Also the authors consider the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of certain bipartite graphs. In [8], the authors determine the classes of bipartite digraphs whose number of 1-factors is the generalized order-k Lucas number, l_n^k given by (10) and the sums of the generalized order-k Fibonacci and Lucas numbers, $\sum_{j=1}^{n} g_j^k$ and $\sum_{j=1}^{n} l_j^k$, respectively.

In this paper, we find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrices are the generalized Fibonacci p-numbers and a sum of consecutive generalized Fibonacci p-numbers. Further, we give relationships between permanents of certain matrices and the Lucas p-numbers and their sums.

2. Fibonacci p-numbers

In this section, we determine a class of bipartite graph whose number of 1-factors is the generalized Fibonacci p-number.

Let n and p be positive integers such that $n > p \geqslant 1$.

Definition 1. Let $M(n,p) = [m_{ij}]$ be the $n \times n$ (0,1)-matrix with $m_{i+1,i} = m_{i,i} = m_{i,i+p} = 1$ for a fixed integer p and all i,j, and 0 otherwise.

Clearly,

$$M(n,p) = \begin{array}{c} \begin{array}{cccccccc} & & & & (p+1)\text{th} & & & \\ & & & & \downarrow & & & \end{array} \\ \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Then we have the following Theorem.

Theorem 2. *Let $G(M(n,p))$ be the bipartite graph with bipartite adjacency matrix $M(n,p)$, $n \geqslant 3$. Then the number of 1-factors of $G(M(n,p))$ is the $(n+1)$ th generalized Fibonacci p-number, $F_p(n+1)$.*

Proof. Let $M^{(k)}(n,p) = [m_{ij}^{(k)}]$ be the kth contraction of $M(n,p)$ for $1 \leqslant k \leqslant p-1$. Since the definition of the matrix $M(n,p)$, the matrix can be contracted on column 1 so that

$$M^{(1)}(n,p)=\begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

Since the matrix $M^1(n,p)$ can be contracted on column 1 and $F_p(0)=0, F_p(1)=F_p(2)=F_p(3)=1$,

$$M^{(2)}(n,p)=\begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 1 & 1 \end{pmatrix}$$

where $m^{(2)}_{1,p-1}=F_p(3)$, $m^{(2)}_{1,p}=F_p(2)$, $m^{(2)}_{1,p+1}=F_p(1)$. Continuing this process and since $F_p(p+1)=F_p(p)=\dots=F_p(1)=1$, $F_p(0)=0$, we obtain

$$M^{(p-1)}(n,p)=\begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & 1 & 0 & 0 \\ 0 & \dots & 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

where $m^{(p-1)}_{1,1}=F_p(p+1)$, $m^{(p-1)}_{1,2}=F_p(p),\dots,m^{(p-1)}_{1,p}=F_p(2)$, $m^{(p-1)}_{1,p+1}=F_p(p)$.

Now we consider the case $p\leqslant k\leqslant n-4$. Since the matrix $M^{(p-1)}(n,p)$ can be contracted on column 1 and $F_p(p+2)=2$

$$M^{(p)}(n,p)=\begin{pmatrix} 2 & 1 & \dots & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

where $m^{(p)}_{1,1}=F_p(p+2)$, $m^{(p)}_{1,2}=F_p(p+1),\dots,m^{(p)}_{1,p}=F_p(3)$, $m^{(p)}_{1,p+1}=F_p(p+1)$. Since the matrix $M^{(p)}(n,p)$ can be contracted on column 1

$$M^{(p+1)}(n,p)=\begin{pmatrix} 3 & 1 & \dots & 1 & 2 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\ 0 & \dots & 0 & 0 & 1 & 1 & 0 & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{pmatrix}$$

where $m^{(p+1)}_{1,1}=F_p(p+3)$, $m^{(p+1)}_{1,2}=F_p(p+2),\dots,m^{(p+1)}_{1,p}=F_p(4), m^{(p+1)}_{1,p+1}=F_p(p+2)$.

We only consider the first row because the entries of other rows of $M^{(k)}(n,p)$ are 0 are 1 for all $k>1$. Continuing this process, that is, by repeated contractions, we have

$$\begin{aligned}
m_{1,1}^{(n-p-1)} &= F_p(n-p+1),\\
m_{1,2}^{(n-p-1)} &= F_p(n-2p+1),\\
m_{1,3}^{(n-p-1)} &= F_p(n-2p+2),\\
&\dots\\
m_{1,p+1}^{(n-p-1)} &= F_p(n-p).
\end{aligned}$$

That is, the matrix $M^{(n-p-1)}(n,p)$ is as follows:

$$\begin{bmatrix}
F_p(n-p+1) & F_p(n-2p+1) & \dots & \dots & \dots & F_p(n-p-1) & F_p(n-p)\\
1 & 1 & 0 & \dots & 0 & 0 & 0\\
0 & 1 & 1 & 0 & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & \dots & 0 & 1 & 1 & 0 & 0\\
0 & 0 & \dots & 0 & 1 & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & 1
\end{bmatrix}.$$

By repeating contractions

$$\begin{aligned}
m_{11}^{(n-p)} &= F_p(n-p+1)+F_p(n-2p+1) = F_p(n-p+2)\\
m_{12}^{(n-p)} &= F_p(n-2p+2),\\
m_{13}^{(n-p)} &= F_p(n-2p+3)\\
&\vdots\\
m_{1,p}^{(n-p)} &= F_p(n-p)
\end{aligned}$$

Thus, we have

$$M^{(n-3)}(n,p) = \begin{bmatrix}
Fp(n-1) & F_p(n-p-1) & F_p(n-p-1)\\
1 & 1 & 0\\
0 & 1 & 1
\end{bmatrix}$$

Then we have, by contraction of $M^{(n-3)}(n,p)$ on column 1

$$M^{(n-2)}(n,p) = \begin{bmatrix} F_p(n) & F_p(n-p)\\ 1 & 1 \end{bmatrix}$$

By the Eq. (13), $per\, M^{(n-2)}(n,p) = per\, M(n,p) = F_p(n+1)$.

So the proof is complete. □

For example, if we take $p=2$, then we have, by Theorem 2

$$per\, M(n,2) = per \begin{bmatrix}
1 & 0 & 1 & 0 & \dots & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & \dots & 0 & 0\\
0 & 1 & 1 & 0 & 1 & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & \dots & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 1
\end{bmatrix}_{n\times n} = F_2(n+1)$$

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3. Sums of the consecutive generalized Fibonacci p-numbers

In this section, we determine a class of bipartite graph whose number of 1-factors is the sums of the consecutive generalized Fibonacci p-number, $\sum_{i=1}^{n} F_p(i)$.

Let n and p be positive integers such that $n > p \geqslant 1$.

Definition 3. Let $T(n,p) = [t_{ij}]$ be the $n \times n$ $(0,1)$-matrix with $t_{1,j} = 1$ for all j, $t_{i+1,i} = 1$ for $1 \leqslant i \leqslant n-1$, $t_{i,i} = t_{i,i+p} = 1$ for $2 \leqslant i \leqslant n$ and a fixed integer p, and, 0 otherwise.

Clearly

$$T(n,p) = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 & \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{pmatrix}$$

Then we have the following Theorem.

Theorem 4. Let $G(T(n,p))$ be the bipartite graph with bipartite adjacency matrix $T(n,p), n \geqslant 3$. Then the number of 1-factors of $G(T(n,p))$ is the sums of the consecutive generalized Fibonacci p-number, $\sum_{i=1}^{n} F_p(i)$.

Proof. We will use the induction method to prove that $per\,T(n,p) = \sum_{i=1}^{n} F_p(i)$. If $n = 3$, then we have, for $p = 2$

$$per\,T(3,2) = per \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \sum_{i=1}^{3} F_2(i) = 3$$

If $n = 4$, then we have, for fixed $p = 2$

$$per\,T(4,2) = per \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \sum_{i=1}^{4} F_2(i) = 5$$

Now we suppose that the equation holds for $n, n > p \geqslant 1$. If we compute the $per\,T(n,p)$ by the Laplace expansion of permanent with respect to the first column, then we have that

$$per\,T(n,p) = per \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} + per \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \\ 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}$$

Considering the definitions of the matrices $T(n,p)$ and $M(n,p)$, we write that

$$per\,T(n,p) = per\,T(n-1,p) + per\,M(n-1,p)$$

By our assumption and the Theorem 2, we obtain that

$$per\,T(n,p)=\sum_{i=1}^{n}F_p(i)+F_p(n+1)=\sum_{i=1}^{n+1}F_p(i)$$

So the proof is complete. $\square$

For example, when $p=1$, then the sequence $\{F_p(n)\}$ is reduced to the well-known usual Fibonacci sequence $\{F_n\}$, then by Theorem 4

$$per\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & 1 & 1 & \ddots & \vdots \\ \dots & \dots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix}_{n\times n}=\sum_{i=1}^{n}F_i$$

which is the well-known result from [14].

4. On the Lucas p-numbers and permanent of certain matrices

In the above results we also determine relationships between the permanents of certain square matrices and the Fibonacci p-numbers. Here we determine the similar directions for the Lucas p-numbers. For these purposes, we define a new $(n\times n)$ matrix $H(n,p)$.

Definition 5. For $n>p$ and $p\geqslant 1$, let $H(n,p)=[h_{ij}]$ be the $n\times n$ matrix with $h_{i+1,i}=h_{i,i+p}=1$ for a fixed integer p and all $i,j,h_{ii}=1$ for all i except from $i=p+1,h_{n-p,n-p}=p+1$ and 0 otherwise.

Clearly,

$$H(n,p)=\begin{array}{c} \begin{array}{c}(\dot{p}+1)\text{th}\\ \downarrow\end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & p+1 & 0 & & & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \rightarrow (n-p)th$$

Then we have the following Theorem.

Theorem 6. *For* $n>p\geqslant 1$,

$$per\,H(n,p)=L_p(n).$$

Proof. (Induction on n) First, we consider the case $n=p+1$. Then the matrix $H(p+1,p)$ takes the following form:

$$H(p+1,p)=\begin{bmatrix} p+1 & 0 & \dots & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 1 \end{bmatrix}.$$

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Thus expanding by the Laplace expansion of permanent with respect to the first row gives us

$$per\,H(p+1,p) = per\begin{bmatrix} p+1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \dots \\ \dots & 0 & 1 & 1 & 0 \\ \dots & \dots & 0 & 1 & 1 \end{bmatrix} + per\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & \dots & \vdots \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix} = p+2 \tag{14}$$

By the definition of the Lucas p-numbers, we have for $n = p+1$,

$$L_p(p+1) = L_p(p) + L_p(0) = 1 + p + 1 = p + 2. \tag{15}$$

Considering (14) and (15), we have the conclusion for the first case $n = p+1$.

Now we consider the case $n > p+1$. Suppose that the claim is true for $n > p+1$. Then we show that the claim is true for $n+1$. Thus if we expand the $per\,H(n,p)$ by the Laplace expansion of permanent with respect to first row, then we obtain by the definition of matrix $H(n,p)$

$$per\,H(n+1,p) = per\,H(n,p) + per\,H(n-p,p)$$

By our assumption and the definition of Lucas p-numbers, we may write

$$per\,H(n+1,p) = per\,H(n,p) + per\,H(n-p,p) = L_p(n) + L_p(n-p) = L_p(n+1)$$

Thus the proof is complete. □

For example, when $n = 6$ and $p = 2$, the matrix $H(6,2)$ takes the following form:

$$H(6,2) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then by Theorem 6, we have

$$per\,H(6,2) = 10$$

Indeed by the definition of the Lucas 2-numbers, we see that $L_2(6) = per\,H(6,2) = 10$.

Define also the $n \times n$ matrix $E(n,p)$ with $e_{i+1,\,i} = e_{i,\,i+p} = 1$ for a fixed integer p and all $i,j, e_{ii} = 1$ for all i except from $i = p+1, e_{p+1,\,p+1} = p+1$ and 0 otherwise.

Clearly the matrix $E(n,p)$ have the form

$$E(n,p) = \begin{bmatrix} 1 & 0 & \dots 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & \dots & 0 \\ & & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 1 & 0 & \dots & 0 & 1 & 0 \\ \dots & 0 & 1 & p+1 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix}$$

For example when $n = 5, p = 2$, then

$$E(5,2) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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Considering the definitions of matrices $H(n,p)$ and $E(n,p)$, we have the following corollary without proof by the result of Theorem 6.

Corollary 7. *For* $n > p \geqslant 1$

$$perE(n,p) = L_p(n)$$

Now we derive relationship between the sums of the Lucas $p-$ numbers subscripted from 0 to n and permanent of a certain matrix. For this purpose, we give the following definition.

For compactness, we define the $n\times n$ matrix $G(n,p)$ as in the following form:

$$G(n,p) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ 0 & & E(n-1,p) & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

where $E(n,p)$ be as before.

Then we have the following Theorem.

Theorem 8. *For* $n > p \geqslant 1$

$$per\,G(n,p) = \sum_{k=0}^{n-1} L_p(k)$$

Proof. We will use the induction method to prove Theorem 8. Let $p = 1$ and so $n = 2$. Then

$$per\,G(2,1) = per\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 3$$

Since $L_1(0) = 2, L_1(1) = 1, \sum_{k=0}^{1} L_1(k) = 3$. Thus the proof is complete for $n = 2$ and $p = 1$. Consider the case $n = 3$ and $p = 1$. Then

$$per\,G(3,1) = per\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 6$$

Since also $L_1(2) = 3, \sum_{k=0}^{2} L_1(k) = 6$. So the proof is complete for $n = 3$ and $p = 1$. Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. If we extend the $per\,G(n+1,p)$ according to the first column, then we obtain by the definitions of the matrices $G(n,p)$ and $E(n,p)$

$$per\,G(n+1,p) = per\,G(n,p) + E(n,p)$$

By our assumption and the result of Corollary 7, we can write

$$per\,G(n+1,p) = per\,G(n,p) + E(n,p) = \sum_{k=0}^{n-1} L_p(k) + L_p(n) = \sum_{k=0}^{n} L_p(k)$$

Thus the proof is complete. □

In the above results, we give relationships between the Fibonacci, Lucas p-numbers and the permanents of certain matrices. Here we give relationships between determinants of certain matrices and the Fibonacci and Lucas p-numbers and their sums.

A matrix A is called *convertible* if there is an $n\times n(1,-1)$-matrix H such that $per\,A = \det(A\circ H)$, where $A\circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be a $(1,-1)$-matrix of order n, defined by

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & -1 & 1 \end{bmatrix}$$

Denote $M(n,p)\circ S, T(n,p)\circ S, H(n,p)\circ S$ and $G(n,p)\circ S$ by $\widehat{M}(n,p), \widehat{T}(n,p), \widehat{H}(n,p)$ and $\widehat{G}(n,p)$, respectively.

Then we have the following Corollaries without proof.

Corollary 9. *For* $n > p \geqslant 1$

$$\det \widehat{M}(n,p) = F_p(n+1)$$

Corollary 10. *For* $n > p \geqslant 1$

$$\det \widehat{T}(n,p) = \sum_{i=1}^{n+1} F_p(i)$$

Corollary 11. *For* $n > p \geqslant 1$

$$\det \widehat{H}(n,p) = L_p(n)$$

Corollary 12. *For* $n > p \geqslant 1$

$$\det \widehat{G}(n,p) = \sum_{k=0}^{n-1} L_p(k).$$

For example, when $n = 5$ and $p = 2$

$$\det \widehat{G}(5,2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} = 14.$$

Since the definition of the Lucas 2-numbers, we have $\sum_{i=0}^{4} L_2(i) = 14$.

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