

On the Finite-Time Splash and Splat Singularities for the 3-D Free-Surface Euler Equations

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Abstract: We prove that the 3-D free-surface incompressible Euler equations with regular initial geometries and velocity fields have solutions which can form a finite-time “splash” (or “splat”) singularity first introduced in Castro et al. (Splash singularity for water waves, <http://arxiv.org/abs/1106.2120v2>, 2011), wherein the evolving 2-D hypersurface, the moving boundary of the fluid domain, self-intersects at a point (or on surface). Such singularities can occur when the crest of a breaking wave falls unto its trough, or in the study of drop impact upon liquid surfaces. Our approach is founded upon the Lagrangian description of the free-boundary problem, combined with a novel approximation scheme of a finite collection of local coordinate charts; as such we are able to analyze a rather general set of geometries for the evolving 2-D free-surface of the fluid. We do not assume the fluid is irrotational, and as such, our method can be used for a number of other fluid interface problems, including compressible flows, plasmas, as well as the inclusion of surface tension effects.

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1. Introduction

1.1. The Eulerian description of the free-boundary problem. For $0 \leq t \leq T$, the evolution of a three-dimensional incompressible fluid with a moving free-surface is modeled by the incompressible Euler equations:

$$u_t + u \cdot Du + Dp = 0 \quad \text{in } \Omega(t), \tag{1.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t), \tag{1.1b}$$

$$p = 0 \quad \text{on } \Gamma(t), \tag{1.1c}$$

$$\mathcal{V}(\Gamma(t)) = u \cdot n \tag{1.1d}$$

$$u = u_0 \quad \text{on } \Omega(0), \tag{1.1e}$$

$$\Omega(0) = \Omega_0. \tag{1.1f}$$

The open subset $\Omega(t) \subset \mathbb{R}^3$ denotes the changing volume occupied by the fluid, $\Gamma(t) := \partial\Omega(t)$ denotes the moving free-surface, $\mathcal{V}(\Gamma(t))$ denotes normal velocity of $\Gamma(t)$, and $n(t)$ denotes the exterior unit normal vector to the free-surface $\Gamma(t)$. The vector-field $u = (u_1, u_2, u_3)$ denotes the Eulerian velocity field, and p denotes the pressure function. We use the notation $D = (\partial_1, \partial_2, \partial_3)$ to denote the gradient operator. We have normalized the equations to have all physical constants equal to 1.

This is a free-boundary partial differential equation to determine the velocity and pressure in the fluid, as well as the location and smoothness of the a priori unknown free-surface. In the case that the fluid is irrotational, $\operatorname{curl} u = 0$, the coupled system of Euler equations (1.1) can be reduced to an evolution equation for the free-surface (with potential flow in the interior), in which case (1.1) simplifies to the water waves equation. *We do not make any irrotationality assumptions.*

We will prove that the 3-D Euler equations (1.1) admit classical solutions which evolve regular initial data onto a state, at finite-time $T > 0$, at which the free-surface self-intersects, and the flow map loses injectivity. The self-intersection can occur at a point, causing a “splash,” or on a surface, creating a “splat.”

1.2. Local-in-time well-posedness. We begin with a brief history of the local-in-time existence theory for the free-boundary incompressible Euler equations. For the *irrotational* case of the water waves problem, and for 2-D fluids (and hence 1-D interfaces), the earliest local existence results were obtained by Nalimov [22], Yosihara [33], and Craig [11] for initial data near equilibrium. Beale et al. [6] proved that the linearization of the 2-D water wave problem is well-posed if the Rayleigh–Taylor sign condition

$$\left. \frac{\partial p}{\partial n} \right|_{t=0} < 0 \quad \text{on } \Gamma|_{t=0} \tag{1.2}$$

is satisfied by the initial data (see [24, 27]). Wu [29] established local well-posedness for the 2-D water waves problem and showed that, due to irrotationality, the Taylor sign condition is satisfied. Later Ambrose and Masmoudi [4], proved local well-posedness of the 2-D water waves problem as the limit of zero surface tension. For 3-D fluids (and 2-D interfaces), Wu [30] used Clifford analysis to prove local existence of the water waves

problem with *infinite depth*, again showing that the Rayleigh–Taylor sign condition is always satisfied in the irrotational case by virtue of the maximum principle holding for the potential flow. Lannes [20] provided a proof for the *finite depth case with varying bottom*. Recently, Alazard et al. [2] have established low regularity solutions (below the Sobolev embedding) for the water waves equations.

The first local well-posedness result for the 3-D incompressible Euler equations without the irrotationality assumption was obtained by Lindblad [21] in the case that the domain is diffeomorphic to the unit ball using a Nash–Moser iteration. In Coutand and Shkoller [14], we obtained the local well-posedness result for *arbitrary initial geometries* that have H^3 -class boundaries and without derivative loss (this framework, employing local coordinate charts in the Lagrangian configuration, is ideally suited for the splash and splat singularity problems that we study herein). Shatah and Zeng [26] established a priori estimates for this problem using an infinite-dimensional geometric formulation, and Zhang and Zhang proved well-posedness by extending the complex-analytic method of Wu [30] to allow for vorticity. Again, in the latter case the domain was with infinite depth.

1.3. Long-time existence. It is of great interest to understand if solutions to the Euler equations can be extended for all time when the data is sufficiently smooth and small, or if a finite-time singularity can be predicted for other types of initial conditions.

Because of irrotationality, the water waves problem does not suffer from vorticity concentration; therefore, singularity formation involves only the loss of regularity of the interface. In the case that the irrotational fluid is infinite in the horizontal directions, certain dispersive-type properties can be made use of. For sufficiently smooth and small data, Alvarez-Samaniego and Lannes [3] proved existence of solutions to the water waves problem on large time-intervals (larger than predicted by energy estimates), and provided a rigorous justification for a variety of asymptotic regimes. By constructing a transformation to remove the quadratic nonlinearity, combined with decay estimates for the linearized problem (on the infinite half-space domain), Wu [31] established an almost global existence result (existence on time intervals which are exponential in the size of the data) for the 2-D water waves problem with sufficiently small data. Wu [32] then proved global existence in 3-D for small data. Using the method of spacetime resonances, Germain et al. [18] also established global existence for the 3-D irrotational problem for sufficiently small data.

1.4. Splashing of liquids and the finite-time splash singularity. The study of splashing, and in particular, of drop impact on liquid surfaces has a long history that goes back to the end of the last century when Worthington [28] studied the process by means of single-flash photography. Numerical studies show both fascinating and unexpected fluid behavior during the splashing process (see, for example, Oğuz and Prosperetti [23]), with agreement from matched asymptotic analysis by Howison et al. [19].

The problem of rigorously establishing a finite-time singularity for the fluid interface has recently been explored for the 2-D water waves equations by Castro et al. [9, 10], where it was shown that a smooth initial curve exhibits a finite-time singularity via self-intersection at a point; they refer to this type of singularity as a “splash” singularity, and we will continue to use this terminology. (We will give a precise definition of the splash domain in our 3-D framework in Sect. 3.1.2 and we define the splat domain in Sect. 9.)

Their work follows earlier results by Castro et al. [7, 8] for both the Muskat and water waves equations wherein the authors proved that an initial curve which is a graph, that

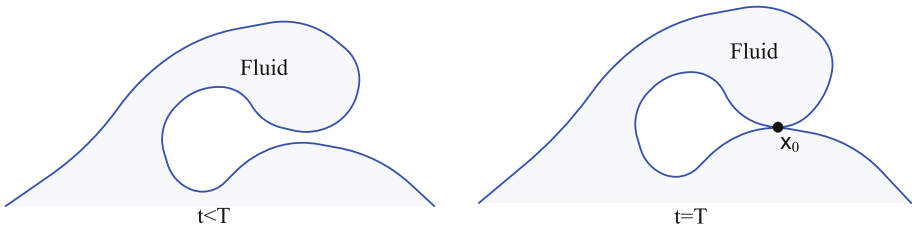


Fig. 1. The splash singularity wherein the top of the crest touches the trough at a point x_0 in finite time T

satisfies the Rayleigh–Taylor sign condition, reaches a regime in finite time in which it is no longer a graph and can become unstable due to a reversal of the sign in the Rayleigh–Taylor condition.

Herein, we develop a new framework for analyzing the finite-time splash and splat singularity for 3-D incompressible fluid flows with vorticity. Our motivation is to produce a general methodology which can also be applied to compressible fluids, as well as to ionized fluids, governed by the equations of magnetohydrodynamics. Our method is founded upon the transformation of (1.1) into Lagrangian variables. We are thus not restricted to potential flows, nor to any special geometries. Furthermore, our method of analysis does not, in any significant way, distinguish between flow in different dimensions. While we present our results for the case of 3-D fluid flow, they are equally valid in the 2-D case (Fig. 1).

1.5. Main result. The main result of this paper states that there exist initial domains Ω_0 of Sobolev class $H^{4.5}$ together with initial velocity vectors $u_0 \in H^4(\Omega_0)$ which satisfy the Rayleigh–Taylor sign condition (1.2), such that after a finite-time $T > 0$ the solution of the Euler equations reaches a “splash” (or “splat”) singularity. At such a time T , particles which were separated at time $t = 0$ collide at a point x_0 (or on a surface Γ_0), the flow map $\eta(T)$ loses injectivity, and $\partial[\Omega^c]$ forms a cusp. In short, T is the time at which the crest of a 3-D wave turns-over and touches the trough. This statement is made precise in Theorems 5.1 and 5.2.

Note that the use of $H^{4.5}$ -regularity for the domain Ω_0 and $H^4(\Omega_0)$ -regularity for velocity field u_0 is due to the functional framework that we employ for the a priori estimates in Theorem A.1. For 3-D incompressible fluid flow, we find that this is the most natural functional setting; of course, we could also employ any H^s -framework for $s \geq 4.5$ or a Hölder space framework as well.

1.6. The Lagrangian description. We transform the system (1.1) into Lagrangian variables. We let $\eta(x, t)$ denote the “position” of the fluid particle x at time t . Thus,

$$\partial_t \eta = u \circ \eta \quad \text{for } t > 0 \text{ and } \eta(x, 0) = x,$$

where \circ denotes composition so that $[u \circ \eta](x, t) := u(\eta(x, t), t)$. We set

- $v = u \circ \eta$ (Lagrangian velocity),
- $q = p \circ \eta$ (Lagrangian pressure),
- $A = [D\eta]^{-1}$ (inverse of the deformation tensor),
- $J = \det[D\eta]$ (Jacobian determinant of the deformation tensor),
- $a = J A$ (cofactor of the deformation tensor).

Whenever $\operatorname{div} u = 0$, it follows that $\det D\eta = 1$, and hence the cofactor matrix of $D\eta$ is equal to $[D\eta]^{-1}$, i.e., $a = A$. Using Einstein's summation convention, and using the notation $F_{,k}$ to denote $\frac{\partial F}{\partial x_k}$, the k^{th} -partial derivative of F for $k = 1, 2, 3$, the Lagrangian version of Eqs. (1.1) is given on the fixed reference domain Ω by

$$\eta(t) = e + \int_0^t v \quad \text{in } \Omega \times [0, T], \tag{1.3a}$$

$$v_t + A^T Dq = 0 \quad \text{in } \Omega \times (0, T], \tag{1.3b}$$

$$\operatorname{div}_\eta v = 0 \quad \text{in } \Omega \times [0, T], \tag{1.3c}$$

$$q = 0 \quad \text{on } \Gamma \times [0, T], \tag{1.3d}$$

$$(\eta, v) = (e, u_0) \quad \text{in } \Omega \times \{t = 0\}, \tag{1.3e}$$

where $e(x) = x$ denotes the identity map on Ω , and where the i^{th} -component of $A^T Dq$ is $A_i^k q_{,k}$. (A^T denotes the transpose of the matrix A .) By definition of the Lagrangian flow $\eta(t)$, the free-surface is given by

$$\Gamma(t) = \eta(t)(\Gamma).$$

We will also use the notation $\eta(t, \Gamma) = \Gamma(t)$, and $\eta(t, \Omega) = \Omega(t)$. The Lagrangian divergence is defined by $\operatorname{div}_\eta v = A_i^j v^i_{,j}$. Solutions to (1.3) which are sufficiently smooth to ensure that $\eta(t)$ are diffeomorphisms, give solutions to (1.1) via the change of variables indicated above.

1.7. The splash singularity for other hyperbolic PDEs. Our methodology can be applied to a host of other time-reversible PDEs that have a local well-posedness theorem.

- (1) *Surface tension.* Our main result also holds if surface tension is added to the Euler equations. In this case Eq. (1.3d) is replaced with

$$qn = -\sigma \Delta_g(\eta),$$

where $\sigma > 0$ denotes the surface tension parameter, n is the outward unit-normal to $\Gamma(t)$, Δ_g denotes the surface Laplacian with respect to the induced metric g where $g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}$. This is the Lagrangian version of the so-called Laplace–Young boundary condition for pressure: $p = \sigma H$, where H is the mean curvature of the free-surface $\Gamma(t)$. We have established well-posedness for this case in [14]. The only modifications required for the case of positive surface tension is to consider initial domains Ω_0 of Sobolev class H^6 with initial velocity fields $u_0 \in H^{4.5}(\Omega_0)$. Our main theorem then provides for a finite-time splash singularity for the case that $\sigma > 0$.

- (2) *Physical vacuum boundary of a compressible gas.* We can also consider the evolution of the free-surface compressible Euler equations which model the expansion of a gas into vacuum. We established the well-posedness of this system of degenerate and characteristic multi-D conservation laws in [16]. In this setting, our methodology shows that there exist initial domains Ω_0 of class H^4 , initial velocity fields $u_0 \in H^{3.5}(\Omega_0)$, and initial density functions $\rho_0 \in H^4(\Omega_0)$ such after time $T > 0$, a splash singularity is formed by the evolving vacuum interface.
- (3) *Other physical models.* In fact, we can establish existence of a finite-time splash singularity for a wide class of hyperbolic systems of PDE which evolve a free-boundary in a sufficiently smooth functional framework, and which are locally well-posedness. Examples of equations (not mentioned above) include nonlinear elasticity and magnetohydrodynamics.

2. Notation, Local Coordinates, and Some Preliminary Results

2.1. *Notation for the gradient vector.* Throughout the paper the symbol D will be used to denote the three-dimensional gradient vector $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$.

2.2. *Notation for partial differentiation and Einstein’s summation convention.* The k^{th} partial derivative of F will be denoted by $F_{,k} = \frac{\partial F}{\partial x_k}$. Repeated Latin indices i, j, k , etc., are summed from 1 to 3, and repeated Greek indices α, β, γ , etc., are summed from 1 to 2. For example, $F_{,ii} = \sum_{i=1}^3 \frac{\partial^2 F}{\partial x_i \partial x_i}$, and $F^i{}_{,\alpha} I^{\alpha\beta} G^i{}_{,\beta} = \sum_{i=1}^3 \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \frac{\partial F^i}{\partial x_\alpha} I^{\alpha\beta} \frac{\partial G^i}{\partial x_\beta}$.

2.3. *The divergence and curl operators.* For a vector field u on Ω , we set

$$\begin{aligned} \operatorname{div} u &= u^1{}_{,1} + u^2{}_{,2} + u^3{}_{,3} , \\ \operatorname{curl} u &= \left(u^3{}_{,2} - u^2{}_{,3} , u^1{}_{,3} - u^3{}_{,1} , u^2{}_{,1} - u^1{}_{,2} \right) . \end{aligned}$$

With the permutation symbol ε_{ijk} given by $\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } \{1, 2, 3\}. \\ -1, & \text{odd permutation of } \{1, 2, 3\}. \\ 0, & \text{otherwise.} \end{cases}$ the i^{th} -component of $\operatorname{curl} u$ is given by

$$(\operatorname{curl} u)_i = \varepsilon_{ijk} u^k{}_{,j} .$$

2.4. *The Lagrangian divergence and curl operators.* We will write $\operatorname{div}_\eta v = \operatorname{div} u \circ \eta$ and $\operatorname{curl}_\eta v = \operatorname{curl} u \circ \eta$. From the chain rule,

$$\operatorname{div}_\eta v = A_r^s v^r{}_{,s} \quad \text{and} \quad (\operatorname{curl}_\eta v)_i = \varepsilon_{ijk} A_j^s v^k{}_{,s} .$$

2.5. *Local coordinates near Γ .* In Appendix A, we establish the a priori estimates for solutions of the 3-D free-surface Euler equations (following our local well-posedness theory in [14, 15]). Such solutions evolve a moving two-dimensional surface which is of Sobolev class H^4 . This boundary regularity implies a three-dimensional domain of class $H^{4.5}$, constructed via a collection of $H^{4.5}$ -class local coordinates.

Let $\Omega \subset \mathbb{R}^3$ denote an open subset of class $H^{4.5}$ and let $\{U_l\}_{l=1}^K$ denote an open covering of $\Gamma = \partial\Omega$, such that for each $l \in \{1, 2, \dots, K\}$, with

$$\begin{aligned} B &= B(0, 1), \text{ denoting the open ball of radius 1 centered at the origin and,} \\ B^+ &= B \cap \{x_3 > 0\}, \\ B^0 &= \overline{B} \cap \{x_3 = 0\}, \end{aligned}$$

there exist $H^{4.5}$ -class charts θ_l which satisfy

$$\theta_l : B \rightarrow U_l \text{ is an } H^{4.5} \text{ diffeomorphism,} \tag{2.1a}$$

$$\theta_l(B^+) = U_l \cap \Omega, \quad \theta_l(B^0) = U_l \cap \Gamma . \tag{2.1b}$$

Next, for $L > K$, we let $\{U_l\}_{l=K+1}^L$ denote a family of open sets contained in Ω such that $\{U_l\}_{l=1}^L$ is an open cover of Ω , and we such that there exist diffeomorphisms $\theta_l : B \rightarrow U_l$ (Fig. 2).

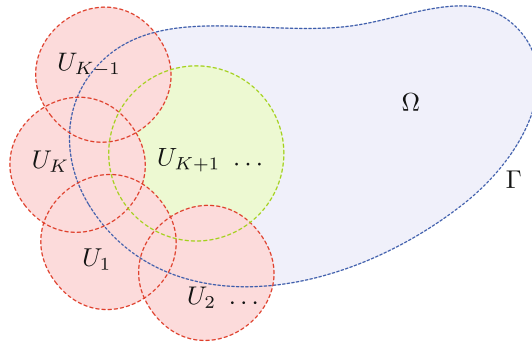


Fig. 2. Indexing convention for the open cover $\{U_l\}_{l=1}^L$ of Ω

2.6. *Tangential (or horizontal) derivatives.* On each boundary chart $U_l \cap \Omega$, for $1 \leq l \leq K$, we let $\bar{\partial}$ denote the *tangential derivative* whose α^{th} -component given by

$$\bar{\partial}_\alpha f = \left(\frac{\partial}{\partial x_\alpha} [f \circ \theta_l] \right) \circ \theta_l^{-1} = \left((Df \circ \theta_l) \frac{\partial \theta_l}{\partial x_\alpha} \right) \circ \theta_l^{-1}.$$

For functions defined directly on B^+ , $\bar{\partial}$ is simply the horizontal derivative $\bar{\partial} = (\partial_{x_1}, \partial_{x_2})$.

2.7. *Sobolev spaces.* For integers $k \geq 0$ and a domain U of \mathbb{R}^3 , we define the Sobolev space $H^k(U)$ ($H^k(U; \mathbb{R}^3)$) to be the completion of $C^\infty(\bar{U})$ ($C^\infty(\bar{U}; \mathbb{R}^3)$) in the norm

$$\|u\|_{k,U}^2 = \sum_{|a| \leq k} \int_U |D^a u(x)|^2,$$

for a multi-index $a \in \mathbb{Z}_+^3$, with the convention that $|a| = a_1 + a_2 + a_3$. When there is no possibility for confusion, we write $\|\cdot\|_k$ for $\|\cdot\|_{k,U}$. For real numbers $s \geq 0$, the Sobolev spaces $H^s(U)$ and the norms $\|\cdot\|_{s,U}$ are defined by interpolation. We will write $H^s(U)$ instead of $H^s(U; \mathbb{R}^3)$ for vector-valued functions.

2.8. *Sobolev spaces on a surface Γ .* For functions $u \in H^k(\Gamma)$, $k \geq 0$, we set

$$\|u\|_{k,\Gamma}^2 = \sum_{|a| \leq k} \int_\Gamma |\bar{\partial}^a u(x)|^2,$$

for a multi-index $a \in \mathbb{Z}_+^2$. For real $s \geq 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $\|\cdot\|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality. That is, for real $s \geq 0$, $H^{-s}(\Gamma) = H^s(\Gamma)'$.

2.9. *The norm of a standard domain Ω .*

Definition 2.1. A domain Ω is of class $H^{4.5}$ if for each $l = 1, \dots, L$, each diffeomorphism θ_l is of class $H^{4.5}$. The $H^{4.5}$ -norm of Ω is defined by

$$\left(\sum_{l=1}^K \|\theta_l\|_{4.5,B^+}^2 + \sum_{l=K+1}^L \|\theta_l\|_{4.5,B}^2 \right)^2. \tag{2.2}$$

In particular if $e : \Omega \rightarrow \Omega$ is the identity map, then $\|e\|_{4.5,\Omega}$ is given by (2.2).

We can, of course, replace $H^{4.5}$ with any H^s , $s > 2.5$ to define domains Ω of class H^s .

2.10. *Local well-posedness for the free-surface Euler problem.*

Theorem 2.1 (Coutand and Shkoller [14]). *With $E(t)$ given by (A.8), suppose that $E(0) \leq M_0$ and that the initial pressure function satisfies the Rayleigh–Taylor sign condition. Then there exists a solution to (1.1) on $[0, T]$ where $T > 0$ depends $E(0)$, and $\sup_{t \in [0, T]} E(t) \leq 2M_0$. Moreover, the solution satisfies*

$$\begin{aligned} \eta &\in C([0, T]; H^{4.5}(\Omega)), \quad v \in C([0, T]; H^4(\Omega)), \\ \operatorname{curl}_\eta v &\in C([0, T]; H^{4.5}(\Omega)), \quad v_t \in C([0, T]; H^{3.5}(\Omega)). \end{aligned}$$

3. The Splash Domain Ω_s and its Approximation by Standard Domains Ω^ϵ

3.1. *The splash domain.*

3.1.1. *The meta-definition.* A splash domain Ω_s is an open and bounded subset of \mathbb{R}^n which is locally on one side of its boundary, except at a point $x_0 \in \partial\Omega_s$, where the domain is locally on each side of the tangent plane at x_0 . The domain Ω_s satisfies the cone property and can be approximated (in a sense to be made precise below) by domains which have a smooth boundary.

We observe that the Sobolev spaces $H^r(\Omega_s)$ are defined for the splash domain Ω_s in the same way as for a domain which is locally on one side of its boundary; moreover, as the bounded splash domain Ω_s satisfies the cone property, interpolation theorems and most of the important Sobolev embedding results hold (see, for examples, Chaps. 4 and 5 of Adams [1]).

The main difference between bounded splash domains with the cone property and domains that have the uniform H^r -regularity property is with regards to trace theorems: For the splash domain Ω_s , a function f in $H^{4.5}(\Omega_s)$ has a trace in $H^4(\Gamma')$ for any smooth subset Γ' of $\partial\Omega_s$ whose closure does not contain x_0 . At x_0 there is not a well-defined (global) trace for f , in the sense of coming from both sides of the tangent plane at x_0 , although it is indeed possible to define *local traces* for f at x_0 with respect to each of the local coordinate charts containing x_0 .

3.1.2. *The definition of the splash domain.*

- (1) We suppose that $x_0 \in \Gamma := \partial\Omega_s$ is the unique boundary self-intersection point, i.e., Ω_s is locally on each side of the tangent plane to $\partial\Omega_s = \Gamma_s$ at x_0 . For all other boundary points, the domain is locally on one side of its boundary. Without loss of generality, we suppose that the tangent plane at x_0 is the horizontal plane $x_3 - (x_0)_3 = 0$.
- (2) We let U_0 denote an open neighborhood of x_0 in \mathbb{R}^3 , and then choose additional L open sets $\{U_l\}_{l=1}^L$ such that the collection $\{U_l\}_{l=0}^L$ is an open cover of Γ_s , and $\{U_l\}_{l=0}^L$ is an open cover of Ω_s and such that there exists a sufficiently small open subset $\omega \subset U_0$ containing x_0 with the property that

$$\overline{\omega} \cap \overline{U_l} = \emptyset \quad \text{for all } l = 1, \dots, L.$$

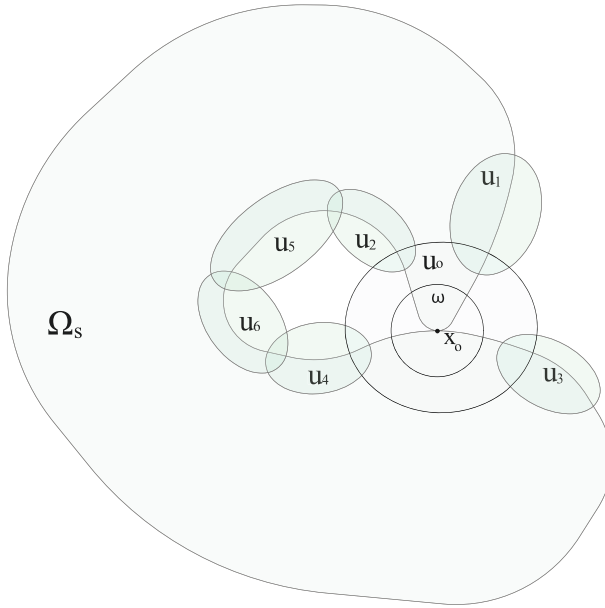


Fig. 3. Splash domain Ω_s , and the collection of open set $\{U_0, U_1, U_2, \dots, U_K\}$ covering Γ

We set

$$U_0^+ = U_0 \cap \Omega_s \cap \{x_3 > (x_0)_3\} \quad \text{and} \quad U_0^- = U_0 \cap \Omega_s \cap \{x_3 < (x_0)_3\}.$$

Additionally, we assume that $\overline{U_0} \cap \overline{\Omega_s} \cap \{x_3 = (x_0)_3\} = \{x_0\}$, which implies in particular that U_0^+ and U_0^- are connected (Fig. 3).

- (3) For each $l \in \{1, \dots, K\}$, there exists an $H^{4.5}$ -class diffeomorphism θ_l satisfying

$$\begin{aligned} \theta_l : B := B(0, 1) &\rightarrow U_l, \\ U_l \cap \Omega_s = \theta_l(B^+) \quad \text{and} \quad \overline{U_l} \cap \Gamma_s &= \theta_l(B^0), \end{aligned}$$

where

$$\begin{aligned} B^+ &= \{(x_1, x_2, x_3) \in B : x_3 > 0\}, \\ B^0 &= \{(x_1, x_2, x_3) \in \overline{B} : x_3 = 0\}. \end{aligned}$$

- (4) For $L > K$, let $\{U_l\}_{l=K+1}^L$ denote a family of open sets contained in Ω_s such that $\{U_l\}_{l=0}^L$ is an open cover of Ω_s , and for $l \in \{K + 1, \dots, L\}$, $\theta_l : B \rightarrow U_l$ is an $H^{4.5}$ diffeomorphism.
- (5) To the open set U_0 we associate two $H^{4.5}$ -class diffeomorphisms θ_+ and θ_- of B onto U_0 with the following properties:

$$\begin{aligned} \theta_+(B^+) &= U_0^+, & \theta_-(B^+) &= U_0^-, \\ \theta_+(B^0) &= \overline{U_0^+} \cap \Gamma_s, & \theta_-(B^0) &= \overline{U_0^-} \cap \Gamma_s, \end{aligned}$$

such that

$$\{x_0\} = \theta_+(B^0) \cap \theta_-(B^0),$$

and

$$\theta_+(0) = \theta_-(0) = x_0.$$

We further assume that

$$\overline{\theta_{\pm}(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B^+)} = \emptyset \quad \text{for } l = 1, \dots, K,$$

and

$$\overline{\theta_{\pm}(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B)} = \emptyset \quad \text{for } l = K + 1, \dots, L.$$

Definition 3.1 (Splash domain Ω_s). *We say that Ω_s is a splash domain, if it is defined by a collection of open covers $\{U_l\}_{l=0}^L$ and associated maps $\{\theta_{\pm}, \theta_1, \theta_2, \dots, \theta_L\}$ satisfying the properties (1)–(5) above. Because each of the maps is an $H^{4.5}$ diffeomorphism, we say that the splash domain Ω_s defines a self-intersecting generalized $\mathbf{H}^{4.5}$ -domain.*

3.2. *A sequence of standard domains approximating the splash domain.* We approximate the two distinguished charts θ_- and θ_+ by charts θ_-^ϵ and θ_+^ϵ in such a way as to ensure that

$$\theta_-^\epsilon(B^0) \cap \theta_+^\epsilon(B^0) = \emptyset \quad \forall \quad \epsilon > 0,$$

and which satisfy

$$\theta_-^\epsilon \rightarrow \theta_- \quad \text{and} \quad \theta_+^\epsilon \rightarrow \theta_+ \quad \text{as } \epsilon \rightarrow 0.$$

We choose $\tau > 0$ sufficiently small so that

$$\theta_-(B^+(0, 2\tau)) \subset \omega \quad \text{and} \quad \theta_+(B^+(0, 2\tau)) \subset \omega,$$

and then we let $\psi \in \mathcal{D}(B(0, \tau))$ denote a smooth bump-function satisfying $0 \leq \psi \leq 1$ and $\psi(0) = 1$. For $\epsilon > 0$ taken small enough, we define

$$\begin{aligned} \theta_-^\epsilon(x) &= \theta_-(x) - \epsilon \psi(x) \mathbf{e}_3, \\ \theta_+^\epsilon(x) &= \theta_+(x) + \epsilon \psi(x) \mathbf{e}_3, \end{aligned}$$

where $\mathbf{e}_3 = (0, 0, 1)$ denotes the vertical basis vector of the standard basis \mathbf{e}_i of \mathbb{R}^3 . By choosing $\psi \in \mathcal{D}(B(0, \tau))$, we ensure that the modification of the domain is localized to a small neighborhood of x_0 and away from the boundary of U_0 and the image of the other maps θ_l (Fig. 4). Then, for $\epsilon > 0$ sufficiently small,

$$\theta_-^\epsilon(\overline{B^+}) \cap \theta_+^\epsilon(\overline{B^+}) = \emptyset.$$

Since the maps θ_{\pm}^ϵ are a modification of the maps θ_{\pm} in a very small neighborhood of $0 \in B$, we have that for $\epsilon > 0$ sufficiently small,

$$\theta_{\pm}^\epsilon(B^+ \cap B(0, 1/2)) \cap \theta_l(B^+) = \emptyset \quad \text{for } l = 1, \dots, K,$$

and

$$\theta_{\pm}^\epsilon(B^+ \cap B(0, 1/2)) \cap \theta_l(B) = \emptyset \quad \text{for } l = K + 1, \dots, L.$$

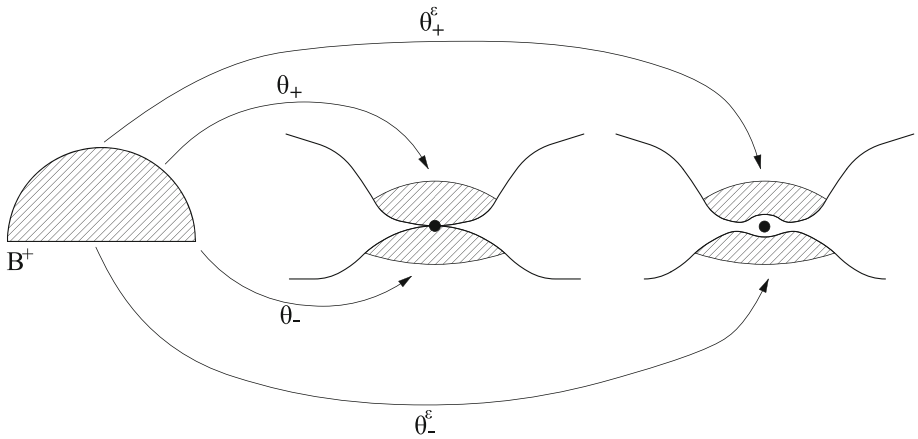


Fig. 4. The black dot denotes the point x_0 where the boundary self-intersects (middle). For $\epsilon > 0$, the approximate domain Ω^ϵ does not intersect itself (right)

For $l \in \{1, \dots, L\}$ we set $\theta_l^\epsilon = \theta_l$. Then $\theta_-^\epsilon : B^+ \rightarrow U_0$, $\theta_+^\epsilon : B^+ \rightarrow U_0$, and $\theta_l^\epsilon : B^+ \rightarrow U_l$, $l \in \{1, \dots, K\}$, $\theta_l^\epsilon : B \rightarrow U_l$, $l \in \{K + 1, \dots, L\}$, is a collection of $H^{4.5}$ coordinate charts as given in Sect. 2.5, and so we have the following

Lemma 3.1 (The approximate domains Ω^ϵ). *For each $\epsilon > 0$ sufficiently small, the set Ω^ϵ , defined by the local charts $\theta_-^\epsilon : B^+ \rightarrow U_0$, $\theta_+^\epsilon : B^+ \rightarrow U_0$, and $\theta_l^\epsilon : B^+ \rightarrow U_l$, $l \in \{1, \dots, K\}$, $\theta_l^\epsilon : B \rightarrow U_l$, $l \in \{K + 1, \dots, L\}$ is a domain of class $H^{4.5}$, which is locally on one side of its H^4 boundary.*

By choosing $0 < r_0 < \frac{1}{2}$ such that $1 \geq \psi \geq \frac{1}{2}$ in $B(0, r_0)$, we see that

$$|(\theta_-^\epsilon(x) - \theta_+^\epsilon(y)) \cdot \mathbf{e}_3| \geq \epsilon \text{ for any } x, y \in B^+ \cap B(0, r_0).$$

With r_0 chosen, due to the fact that by assumption (2) the images of θ_- and θ_+ only intersect the plane $\{x_3 = (x_0)_3\}$ at the point x_0 , there exists $\delta(r_0) > 0$ such that $(\theta_-^\epsilon(x) - x_0) \cdot \mathbf{e}_3 < -\delta(r_0)$ and $(\theta_+^\epsilon(x) - x_0) \cdot \mathbf{e}_3 > \delta(r_0)$ for all $x \in B^+$ with $|x| \geq r_0$. This, in turn, implies that if $x \in B^+$ with $|x| \geq r_0$ and $y \in B^+$, we then have that

$$|(\theta_-^\epsilon(x) - \theta_+^\epsilon(y)) \cdot \mathbf{e}_3| \geq \delta(r_0) - 2\epsilon \geq \epsilon \text{ if } \epsilon \leq \frac{\delta(r_0)}{3}.$$

We have therefore established the following fundamental inequality: for $0 < \epsilon \leq \frac{\delta(r_0)}{3}$,

$$\forall (x, y) \in B^+ \times B^+, \quad |(\theta_-^\epsilon(x) - \theta_+^\epsilon(y)) \cdot \mathbf{e}_3| \geq \epsilon. \tag{3.1}$$

We henceforth assume that $0 < \epsilon \leq \frac{\delta(r_0)}{3}$ (Fig. 4).

In summary, we have approximated the self-intersecting splash domain Ω_s with a sequence of $H^{4.5}$ -class domains Ω^ϵ , $0 < \epsilon \leq \frac{\delta(r_0)}{3}$ (such that $\partial\Omega^\epsilon$ does not self-intersect). As such, each one of these domains Ω^ϵ , $\epsilon > 0$, will thus be amenable to our local-in-time well-posedness theory for free-boundary incompressible Euler equations with Taylor sign condition satisfied.

We also note that Ω^ϵ and Ω_s are the same domain, except on the two patches $\theta_-^\epsilon(B^+ \cap B(0, \frac{1}{2}))$ and $\theta_+^\epsilon(B^+ \cap B(0, \frac{1}{2}))$. In particular, as θ_\pm differ from θ_\pm^ϵ on a set properly contained in $\omega \subset U_0$, we may use the same covering $\{U_l\}_{l=0}^L$ for Ω^ϵ as for Ω_s .

Lemma 3.2. For $0 < \epsilon \leq \frac{\delta(r_0)}{3}$, the $H^{4.5}$ -norm of Ω^ϵ is bounded independently of ϵ .

Proof. The assertion follows from the following inequality:

$$\|\theta_\pm^\epsilon\|_{4.5, B^+} \leq \|\theta_\pm\|_{4.5, B^+} + \frac{\delta(r_0)}{3} \|\psi\|_{4.5, B^+}.$$

□

3.3. *A uniform cut-off function on the unit-ball B .* Let $B_{1-\alpha} = B(0, 1-\alpha)$ for $0 < \alpha < 1$. For $\alpha > 0$ taken sufficiently small, we have that $\overline{\theta_-(B_{1-\alpha}^+) } \subset U_0$ and $\overline{\theta_+(B_{1-\alpha}^+) } \subset U_0$ and for each $l = 1, \dots, K$, $\overline{\theta_l(B_{1-\alpha}^+) } \subset U_l$, and for each $l = K+1, \dots, L$, $\overline{\theta_l(B_{1-\alpha}^+) } \subset U_l$, and the open sets $\theta_-(B_{1-\alpha}^+)$, $\theta_+(B_{1-\alpha}^+)$, $\theta_l(B_{1-\alpha}^+)$ ($1 \leq l \leq K$), $\theta_l(B_{1-\alpha})$ ($K+1 \leq l \leq L$), are also an open cover of Ω_s . Since the diffeomorphisms θ_\pm^ϵ are modifications for θ_\pm in a very small neighborhood of the origin, it is clear that independently of $\epsilon > 0$, the sets $\theta_-^\epsilon(B_{1-\alpha}^+)$, $\theta_+^\epsilon(B_{1-\alpha}^+)$, $\theta_l(B_{1-\alpha}^+)$ ($1 \leq l \leq K$), $\theta_l(B_{1-\alpha})$ ($K+1 \leq l \leq L$) are also an open cover of each Ω^ϵ .

Definition 3.2 (Uniform cut-off function ζ). Let $\zeta \in \mathcal{D}(B(0, 1))$ such that $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ for $|x| < 1 - \alpha$ and $\zeta = 0$ for $|x| \geq 1 - \frac{\alpha}{2}$.

We set $\varsigma = 1 - \frac{\alpha}{2}$, so that

$$0 \leq \zeta \in \mathcal{D}(B(0, \varsigma)) \leq 1. \quad (3.2)$$

4. Construction of the Splash Velocity Field u_s at the Time of the Splash Singularity

We can now define the so-called *splash velocity* u_s associated with the generalized $\mathbf{H}^{4.5}$ -class splash domain Ω_s , as well as a sequence of approximations u_s^ϵ set on our $H^{4.5}$ -class approximations Ω^ϵ of the splash domain Ω_s .

4.1. The splash velocity u_s .

Definition 4.1 (Splash velocity u_s). A velocity field u_s on an $\mathbf{H}^{4.5}$ -class splash domain Ω_s is called a *splash velocity* if it satisfies the following properties:

- (1) $\zeta u_s \circ \theta_\pm \in H^{4.5}(B^+)$, $\zeta u_s \circ \theta_l \in H^{4.5}(B^+)$ for each $1 \leq l \leq K$ and $u_s \in H^{4.5}(\omega)$ for each $\bar{\omega} \subset \Omega_s$;
- (2) so that under the motion of the fluid, the sets U_0^+ and U_0^- relatively move towards each other, we require that

$$u_s^3 \circ \theta_- > C_- , \quad -u_s^3 \circ \theta_+ > C_+ \quad \text{in } B^+ \text{ and } C_- + C_+ > 0, \quad (4.1)$$

where C_- and C_+ are constants.

Definition 4.2 (Splash pressure p_s). *A pressure function p_s on an $\mathbf{H}^{4.5}$ -class splash domain Ω_s is called a splash pressure associated to the splash velocity u_s if it satisfies the following properties:*

(1) p_s is the unique solution of

$$-\Delta p_s = -\frac{\partial u_s^i}{\partial x_j} \frac{\partial u_s^j}{\partial x_i} \quad \text{in } \Omega_s, \tag{4.2a}$$

$$\zeta p_s \circ \theta_{\pm} = 0 \quad \text{on } B^0, \tag{4.2b}$$

$$\zeta p_s \circ \theta_l = 0 \quad \text{on } B^0 \text{ for } l = 1, \dots, K; \tag{4.2c}$$

(2) the splash pressure $p_s \in H^{4.5}(\Omega_s)$ and satisfies the local version of the Rayleigh–Taylor sign condition:

$$\begin{aligned} \frac{\partial}{\partial x_3} (\zeta p_s \circ \theta_{\pm}) > C_{RT} > 0 \quad \text{and} \quad \frac{\partial}{\partial x_3} (\zeta p_s \circ \theta_l) > C_{RT} > 0 \\ \text{on } B^0 \text{ for } l = 1, \dots, K. \end{aligned} \tag{4.3}$$

Note that the outward unit normal to $\partial B^+ \cap B^0$ points in the direction of $-\mathbf{e}_3$.

Remark 1. As $x_0 = \theta_-(0) = \theta_+(0)$, and as $p \circ \theta_-(0) = p \circ \theta_+(0) = 0$, the conditions (4.2b) and (4.2c) are equivalent to having the usual vanishing trace $p = 0$ on Γ_s . As such, $p \in H_0^1(\Omega_s) \cap H^{4.5}(\Omega_s)$.

For property (1) in Definition 4.2, we note that p_s is the unique $H_0^1(\Omega_s)$ weak solution of (4.2) guaranteed by the Lax–Milgram theorem in Ω_s . The usual methods of elliptic regularity theory show that $\zeta p_s \circ \theta_{\pm}$ and each $\zeta p_s \circ \theta_l \in H^{4.5}(B^+)$ for $l = 1, \dots, L$, and thus that $p_s \in H^{4.5}(\Omega_s)$. (Notice that it is the regularity of our charts θ_{\pm} and θ_l which limits the regularity of the splash pressure p_s .)

As we have defined in property (2) of Definition 4.2, at the point of self-intersection x_0 , the gradient Dp_s has to be defined from each side of the tangent plane at x_0 ; namely, we can define $Dp_s \circ \theta_-$ and $Dp_s \circ \theta_+$ on B^0 , and these two vectors are not equal at the origin 0 which is the pre-image of x_0 under both θ_- and θ_+ .

It is always possible to choose a splash velocity u_s so that (4.3) holds. For example, if we choose u_s to satisfy $\text{curl } u_s = 0$, then (4.3) holds according to the maximum principle [29,30]. On the other hand, it is not necessary to choose an irrotational splash velocity, and we will not impose such a constraint. Essentially, as long as the velocity field induces a positive pressure function, then (4.3) is satisfied.

4.2. *A sequence of approximations u_s^ϵ to the splash velocity.* For $\epsilon > 0$, we proceed to construct a sequence of approximations $u_s^\epsilon : \Omega^\epsilon \rightarrow \mathbb{R}^3$ to the velocity field $u_s : \Omega_s \rightarrow \mathbb{R}^3$ in the following way:

$$u_s^\epsilon \circ \theta_l = u_s \circ \theta_l, \quad \text{in } B^+, \text{ for } l = 1, \dots, K; \tag{4.4a}$$

$$u_s^\epsilon \circ \theta_l = u_s \circ \theta_l, \quad \text{in } B, \text{ for } l = K + 1, \dots, L; \tag{4.4b}$$

$$u_s^\epsilon \circ \theta_-^\epsilon = u_s \circ \theta_-, \text{ and, } u_s^\epsilon \circ \theta_+^\epsilon = u_s \circ \theta_+, \text{ in } B^+. \tag{4.4c}$$

We then have the existence of constants $A > 0$, $B > 0$ such that

$$\|u_s^\epsilon\|_{4.5, \Omega^\epsilon} \leq A \left(\|\zeta u_s^\epsilon \circ \theta_-^\epsilon\|_{4, B^+} + \|\zeta u_s^\epsilon \circ \theta_+^\epsilon\|_{4, B^+} + \sum_{l=1}^K \|\zeta u_s^\epsilon \circ \theta_l^\epsilon\|_{4.5, B^+} + \sum_{l=K+1}^L \|\zeta u_s^\epsilon \circ \theta_l^\epsilon\|_{4.5, B} \right) \leq B \|u_s\|_{4.5, \Omega_s}. \quad (4.5)$$

We next define the approximate pressure function p_s^ϵ in Ω^ϵ as the $H_0^1(\Omega^\epsilon)$ weak solution of

$$-\Delta p_s^\epsilon = \frac{\partial u_s^{\epsilon i}}{\partial x_j} \frac{\partial u_s^{\epsilon j}}{\partial x_i} \quad \text{in } \Omega^\epsilon, \quad (4.6a)$$

$$p_s^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon. \quad (4.6b)$$

Again, standard elliptic regularity theory then shows that $p_s^\epsilon \in H^{4.5}(\Omega^\epsilon)$. Furthermore, since $\theta_\pm^\epsilon \rightarrow \theta_\pm$ and $\theta_l^\epsilon \rightarrow \theta_l$ in $H^{4.5}(B^+)$, we infer from the definition of u_s^ϵ in (4.4) that $\zeta p_s^\epsilon \circ \theta_\pm^\epsilon \rightarrow \zeta p \circ \theta_\pm$ and $\zeta p_s^\epsilon \circ \theta_l^\epsilon \rightarrow \zeta p \circ \theta_l$ in $H^{4.5}(B^+)$. We may thus conclude from the pressure condition (4.3) that we also have, uniformly in $\epsilon > 0$ small enough, that

$$\frac{\partial}{\partial x_3} (\zeta p_s^\epsilon \circ \theta_\pm^\epsilon) > \frac{C_{RT}}{2} > 0 \quad \text{and} \quad \frac{\partial}{\partial x_3} (\zeta p_s^\epsilon \circ \theta_l^\epsilon) > \frac{C_{RT}}{2} > 0, \quad (4.7)$$

on B^0 for each $1 \leq l \leq K$.

4.3. Solving the Euler equations backwards-in-time from the final states Ω^ϵ and u_s^ϵ . Because the Euler equations are time-reversible, we can solve the following system of free-boundary Euler equations backward-in-time:

$$\eta^\epsilon(t) = e + \int_0^t v^\epsilon \quad \text{in } \Omega^\epsilon \times [-T^\epsilon, 0], \quad (4.8a)$$

$$v_i^\epsilon + [A^\epsilon]^T Dq^\epsilon = 0 \quad \text{in } \Omega^\epsilon \times [-T^\epsilon, 0], \quad (4.8b)$$

$$\operatorname{div}_{\eta^\epsilon} v^\epsilon = 0 \quad \text{in } \Omega^\epsilon \times [-T^\epsilon, 0], \quad (4.8c)$$

$$q^\epsilon = 0 \quad \text{on } \Gamma^\epsilon \times [-T^\epsilon, 0], \quad (4.8d)$$

$$(\eta^\epsilon, v^\epsilon) = (e, u_s^\epsilon) \quad \text{in } \Omega^\epsilon \times \{t = 0\}, \quad (4.8e)$$

where $A^\epsilon(x, t) = [D\eta^\epsilon(x, t)]^{-1}$. Thanks to Lemma 3.1, (4.5), and (4.7), we may apply our local well-posedness Theorem 2.1 for (4.8) backward-in-time. This then gives us the existence of $T^\epsilon > 0$, such that there exists a Lagrangian velocity field

$$v^\epsilon \in L^\infty(-T^\epsilon, 0; H^4(\Omega^\epsilon)), \quad (4.9)$$

and a Lagrangian flow map

$$\eta^\epsilon \in L^\infty(-T^\epsilon, 0; H^{4.5}(\Omega^\epsilon)) \quad (4.10)$$

which solve the free-boundary Euler equations (4.8) with *final data* u_s^ϵ and *final domain* Ω^ϵ .

Denoting the corresponding Eulerian velocity field by

$$u^\epsilon = v^\epsilon \circ \eta^{\epsilon^{-1}}, \quad (4.11)$$

it follows that u^ϵ is in $L^\infty((-T^\epsilon, 0); H^4(\Omega^\epsilon(t)))$, where $\Omega^\epsilon(t)$ denotes the image of Ω^ϵ under the flow map $\eta^\epsilon(t)$.

In the remainder of the paper we will prove that the time of existence $T^\epsilon > 0$ (for our sequence of backwards-in-time Euler equations) is, in fact, independent of ϵ ; that is, T^ϵ is equal to a time $T > 0$, and that $\|u^\epsilon(t)\|_{H^4(\Omega^\epsilon(t))}$ and $\|\eta^\epsilon(t)\|_{H^{4.5}(\Omega^\epsilon)}$ are bounded on $[-T, 0]$ independently of ϵ . This will then provide us with the existence of a solution which culminates in the *splash singularity* Ω_s at $t = 0$, from the initial data

$$\begin{aligned} u_0 &= \lim_{\epsilon \rightarrow 0} u^\epsilon(-T), \\ \Omega_0 &= \lim_{\epsilon \rightarrow 0} \Omega^\epsilon(-T). \end{aligned}$$

In particular, when solving the Euler equations forward-in-time from the initial states Ω_0 and u_0 , the smooth $H^{4.5}$ domain Ω_0 is dynamically mapped onto the $\mathbf{H}^{4.5}$ -class splash domain Ω_s after a time T , and the boundary “splashes onto itself” creating the self-intersecting *splash singularity* at the point x_0 .

5. The Main Results

Theorem 5.1 (Finite-time splash singularity). *There exist initial domains Ω_0 of class $H^{4.5}$ and initial velocity fields $u_0 \in H^4(\Omega_0)$, which satisfy the Taylor sign condition (1.2), such that after a finite time $T > 0$, the solution to the Euler equation $\eta(t)$ (with such data) maps Ω_0 onto the splash domain Ω_s , satisfying Definition 3.1, with final velocity u_s . This final velocity u_s satisfies the local Taylor sign condition on the splash domain Ω_s in the sense of (4.3). The splash velocity u_s has a specified relative velocity on the boundary of the splash domain given by (4.1).*

The proof of Theorem 5 is given in Sects. 6–8. In Sects. 9–10 we define the splat domain Ω_s and associated splat velocity \mathbf{u}_s and establish the following

Theorem 5.2 (Finite-time splat singularity). *There exist initial domains Ω_0 of class $H^{4.5}$ and initial velocity fields $u_0 \in H^4(\Omega_0)$, which satisfy the Taylor sign condition (1.2), such that after a finite time $T > 0$, the solution to the Euler equation $\eta(t)$ (with such data) maps Ω_0 onto the splat domain Ω_s , satisfying Definition 9.1, with final velocity \mathbf{u}_s . This final splat velocity \mathbf{u}_s satisfies the local Taylor sign condition on the splat domain Ω_s in the sense of (4.3). The splat velocity \mathbf{u}_s has a specified relative velocity on the boundary of the splat domain as stated in Definition 10.1.*

6. Euler Equations Set on a Finite Number of Local Charts

For each $\epsilon > 0$, the functions v^ϵ , η^ϵ , and u^ϵ , given by (4.9)–(4.11), are solutions to the Euler equations (4.8) on the time interval $[-T^\epsilon, 0]$.

For the purpose of obtaining estimates for this sequence of solutions which do not depend on $\epsilon > 0$, we pull-back the Euler equations (4.8) set on Ω^ϵ by our charts θ_\pm^ϵ and θ_l^ϵ , $l = 1, \dots, L$; in this way we can analyze the equations on the half-ball B^+ .

It is convenient to extend the index l to include both $l = 0$ and $l = -1$; in particular, we set

$$\begin{aligned} \theta_{-1}^\epsilon &= \theta_-^\epsilon & \text{and} & & \theta_{-1} &= \theta_- , \\ \theta_0^\epsilon &= \theta_+^\epsilon & & & \theta_0 &= \theta_+ . \end{aligned}$$

Furthermore, since for $l = -1, 0, 1, 2, \dots, K$, the domain of θ_l is the half-ball B^+ , and for $l = K + 1, \dots, L$, the domain of θ_l is the unit-ball B , it is convenient to write

$$\begin{aligned} \theta_- : \mathcal{B} &\rightarrow U_0^-, \quad \theta_+ : \mathcal{B} \rightarrow U_0^+, \quad \theta_l : \mathcal{B} \rightarrow U_l \cap \Omega \quad \text{for } l = 1, \dots, K, \\ \theta_l : \mathcal{B} &\rightarrow U_l \quad \text{for } l = K + 1, \dots, L, \end{aligned}$$

so that \mathcal{B} denotes B^+ for $l = -1, 0, 1, 2, \dots, K$ and \mathcal{B} denotes B for $l = K + 1, \dots, L$. The Euler equations, set on \mathcal{B} , then take the following form:

$$\eta^\epsilon = e + \int_0^t v^\epsilon \quad \text{in } \Omega^\epsilon \times [-T^\epsilon, 0], \tag{6.1a}$$

$$\partial_t v^\epsilon \circ \theta_l^\epsilon + [b_l^\epsilon]^T D(q^\epsilon \circ \theta_l^\epsilon) = 0 \quad \text{in } \mathcal{B} \times [-T^\epsilon, 0], \tag{6.1b}$$

$$\operatorname{div}_{\eta^\epsilon \circ \theta_l^\epsilon} v^\epsilon \circ \theta_l^\epsilon = 0 \quad \text{in } \mathcal{B} \times [-T^\epsilon, 0], \tag{6.1c}$$

$$q^\epsilon \circ \theta_l^\epsilon = 0 \quad \text{on } B_0 \times [-T^\epsilon, 0], \tag{6.1d}$$

$$(\eta^\epsilon \circ \theta_l^\epsilon, v^\epsilon \circ \theta_l^\epsilon) = (\theta_l^\epsilon, u_s^\epsilon \circ \theta_l^\epsilon) \quad \text{on } \mathcal{B} \times \{t = 0\}, \tag{6.1e}$$

where $[b_l^\epsilon]^T$ denotes the transpose of the matrix b_l^ϵ , and where for any $l = -1, 0, 1, 2, \dots, L$, $b_l^\epsilon(x, t) = [D(\eta^\epsilon(\theta_l^\epsilon(x), t))]^{-1}$. For $l = K + 1, \dots, L$, the boundary condition (6.1d) is not imposed.

The system (6.1) will allow us to analyze the behavior of η^ϵ , v^ϵ , and q^ϵ in an ϵ -independent fashion. Fundamental to this analysis is the following

Lemma 6.1 (Equivalence-of-norms lemma). *With the smooth cut-off function ζ given in Definition 3.2, there exist constants $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that for any $\epsilon > 0$ and $f \in H^s(\Omega)$ with $0 \leq s \leq 4.5$,*

$$\tilde{C}_1 \sum_{l=-1}^L \|\zeta f \circ \theta_l^\epsilon\|_{s, \mathcal{B}}^2 \leq \|f\|_{s, \Omega^\epsilon}^2 \leq \tilde{C}_2 \sum_{l=-1}^L \|\zeta f \circ \theta_l^\epsilon\|_{s, \mathcal{B}}^2. \tag{6.2}$$

Proof. Since by construction $\|\theta_l^\epsilon\|_{4.5, \mathcal{B}} \leq C_l$, the first inequality is obvious. For the second inequality, we simply notice that with $E = \{x \in \mathcal{B} | \zeta(x) = 1\}$, $\Omega^\epsilon = \cup_{l=-1}^L \theta_l^\epsilon(E)$, so that

$$\begin{aligned} \|f\|_{s, \Omega^\epsilon} &\leq C \sum_{l=-1}^L \|f\|_{s, \theta_l^\epsilon(E)} \leq C \sum_{l=-1}^L \|\zeta((\theta_l^\epsilon)^{-1})f\|_{s, \theta_l^\epsilon(E)} \\ &\leq C \sum_{l=-1}^L \|\zeta f(\theta_l^\epsilon)\|_{s, E} \|(\theta_l^\epsilon)^{-1}\|_{4.5, \theta_l^\epsilon(\mathcal{B})} \leq C \sum_{l=-1}^L \|\zeta f(\theta_l^\epsilon)\|_{s, \mathcal{B}}, \end{aligned}$$

where we used the fact that $\det D\theta_l^\epsilon > c_l > 0$ for the last inequality. \square

7. Time of Existence $-T$ of Solutions to (4.8) is Independent of ϵ

Recall that for $\epsilon > 0$, the functions v^ϵ , η^ϵ , and u^ϵ , given by (4.9)–(4.11), are solutions to the Euler equations (4.8) on the time interval $[-T^\epsilon, 0]$. We now prove that the time of existence $-T^\epsilon$ is, in fact, independent of ϵ .

We begin by using the fundamental theorem of calculus to express the difference between the flow of two particles x and y as

$$\eta^\epsilon(x, t) - \eta^\epsilon(y, t) = x - y + \int_0^t [v^\epsilon(x, s) - v^\epsilon(y, s)] ds.$$

Next, for any x and y in Ω^ϵ for which we do not have at the same time $x \in \theta_-^\epsilon(B^+)$ and $y \in \theta_+^\epsilon(B^+)$, we see that independently of $\epsilon > 0$ small enough,

$$|\eta^\epsilon(x, t) - \eta^\epsilon(y, t) - (x - y)| \leq C_1 |t| \sup_{[-T^\epsilon, 0]} E^\epsilon(t) |x - y|, \tag{7.1}$$

where we have used the Sobolev embedding theorem and where

$$E^\epsilon(t) = \|\eta^\epsilon(t)\|_{4.5, \Omega^\epsilon}^2 + \|v^\epsilon(t)\|_{4, \Omega^\epsilon}^2 + \|\text{curl } v^\epsilon(t)\|_{3.5, \Omega^\epsilon}^2 + \|v_t^\epsilon(t)\|_{3.5, \Omega^\epsilon}^2.$$

The inequality (7.1) cannot be independent of $\epsilon > 0$ if both $x \in \theta_-^\epsilon(B^+)$ and $y \in \theta_+^\epsilon(B^+)$, for in this case, according to (3.1), $|x - y| = O(\epsilon)$ as $\epsilon \rightarrow 0$, whereas $|v^\epsilon(x, t) - v^\epsilon(y, t)| = O(1)$ as $\epsilon \rightarrow 0$, and this, in turn, yields a global Lipschitz constant for v^ϵ of $O(\frac{1}{\epsilon})$ as $\epsilon \rightarrow 0$.

When $x \in \theta_-^\epsilon(B^+)$ and $y \in \theta_+^\epsilon(B^+)$, there exist constants C_- , C_+ , and a polynomial function P_1 which are each independent of ϵ , such that

$$\begin{aligned} |\eta^\epsilon(x, t) - \eta^\epsilon(y, t)| &\geq |(\eta^\epsilon(x, t) - \eta^\epsilon(y, t)) \cdot \mathbf{e}_3| \\ &\geq |(x - y) \cdot \mathbf{e}_3 + t[u_s^\epsilon(x) - u_s^\epsilon(y)] \cdot \mathbf{e}_3| \\ &\quad - \underbrace{\left| \mathbf{e}_3 \cdot \int_0^t v^\epsilon(x, t') - u_s^\epsilon(x) dt' \right|}_{\mathcal{I}_1} - \underbrace{\left| \mathbf{e}_3 \cdot \int_0^t v^\epsilon(y, t') - u_s^\epsilon(y) dt' \right|}_{\mathcal{I}_2} \\ &\geq \epsilon + (C_- + C_+) |t| - t^2 P_1 \left(\sup_{[-T^\epsilon, 0]} E^\epsilon \right), \end{aligned} \tag{7.2}$$

where the triangle inequality has been employed together with (4.1) and (3.1). In order to obtain the lower bound on the terms \mathcal{I}_1 and \mathcal{I}_2 , we again use the fundamental theorem of calculus, and write

$$v^\epsilon(x, t') = u_s^\epsilon(x) + \int_0^{t'} v_t^\epsilon(x, \tau) d\tau;$$

using the definition of E^ϵ , it follows that

$$\|v^\epsilon(\cdot, t') - u_s^\epsilon(\cdot)\|_{L^\infty(\Omega^\epsilon)} \leq C |t'| P_1 \left(\sup_{[-T^\epsilon, 0]} E^\epsilon \right).$$

We proceed to show how the two inequalities (7.1) and (7.2) (together with the fact that $C_- > 0$ and $C_+ > 0$) are used to prove that the time $-T$ is independent of ϵ , the flow map η^ϵ is injective on $[-T, 0]$, and the a priori estimates for solutions of (4.8) are independent of ϵ on $[-T, 0]$.

We first record our basic polynomial-type a priori estimate, given in Theorem A.1 in the appendix (see also [14, 15]); we find that on $[-T^\epsilon, 0]$,

$$\sup_{t \in [-T^\epsilon, 0]} E^\epsilon(t) \leq M_0^\epsilon + |t| P_2 \left(\sup_{[-T^\epsilon, 0]} E^\epsilon(t) \right), \tag{7.3}$$

where the constant $M_0^\epsilon = P(E^\epsilon(0))$, i.e. the constant M_0^ϵ only depends on initial data (6.1e). By Lemma 3.2 and (4.5), we see that M_0^ϵ is bounded by a constant M_0 which is independent of ϵ , so that $\sup_{t \in [-T^\epsilon, 0]} E^\epsilon(t) \leq 2M_0$.

We therefore see that if we set

$$T = \min \left(\frac{1}{4C_1M_0}, \frac{C_- + C_+}{2P_1(2M_0)}, \frac{M_0}{2P_2(2M_0)} \right), \tag{7.4}$$

Eq. (7.1) implies that on $[-T, 0]$,

$$|\eta^\epsilon(x, t) - \eta^\epsilon(y, t)| \geq \frac{1}{2}|x - y| \quad \text{for } (x, y) \in \theta_l^\epsilon(\mathcal{B}) \times \theta_k^\epsilon(\mathcal{B}), \\ (l, k) \notin \{(-1, 0), (0, -1)\}, \tag{7.5}$$

while equation (7.2) shows that on $[-T, 0]$,

$$|\eta^\epsilon(x, t) - \eta^\epsilon(y, t)| \geq \epsilon + (C_- + C_+) \frac{|t|}{2} \quad \text{for all } (x, y) \in \theta_-^\epsilon(\mathcal{B}) \times \theta_+^\epsilon(\mathcal{B}). \tag{7.6}$$

We then have from (7.5) and (7.6) that the domain $\eta^\epsilon(t, \Omega^\epsilon)$ does not self-intersect for each $t \in [-T, 0]$ and from (7.4) we also have the estimate

$$\sup_{t \in [-T, 0]} E^\epsilon(t) \leq 2M_0. \tag{7.7}$$

Since $T > 0$ is independent of ϵ by (7.4), the estimates we have just obtained will permit the use of weak convergence to find the initial domain Ω_0 at $t = -T$ and the initial velocity field u_0 at $t = -T$, from which the free surface Euler equations, when run forward in time from $t = 0$, will produce the self-intersecting splash domain Ω_s and velocity field u_s at the final time $T > 0$.

8. Asymptotics as $\epsilon \rightarrow 0$ on the Time-Interval $[-T, 0]$

8.1. *Construction of the initial domain Ω_0 : the asymptotic domain at $t = -T$.* Theorem A.2 provides continuity-in-time, and Lemma 6.1 together with the estimate (7.7) shows that

$$\sum_{l=-1}^L \|\zeta \eta^\epsilon(\theta_l^\epsilon, -T)\|_{4.5, \mathcal{B}}^2 \leq \frac{2}{C} M_0.$$

Weak compactness and Rellich's theorem provide the existence of a subsequence (which by abuse of notation we continue to denote by η^ϵ) such that

$$\eta^\epsilon(\cdot, -T) \circ \theta_l^\epsilon \rightharpoonup \Theta_l, \quad \text{as } \epsilon \rightarrow 0, \quad \text{in } H^{4.5}(\mathcal{B}_\zeta), \tag{8.1a}$$

$$\eta^\epsilon(\cdot, -T) \circ \theta_l^\epsilon \rightarrow \Theta_l, \quad \text{as } \epsilon \rightarrow 0, \quad \text{in } H^{3.5}(\mathcal{B}_\zeta), \tag{8.1b}$$

where $\mathcal{B}_\zeta = \mathcal{B} \cap B(0, \zeta)$ and ζ is given in Definition 3.2.

We now define Ω_0 as the union of the sets $\Theta_l(\mathcal{B}_\zeta)$ ($-1 \leq l \leq L$). Due to (7.5), (7.6) and (8.1b), we have that

$$(x, y) \in \mathcal{B}_\zeta \times \mathcal{B}_\zeta, ((l, k) \notin \{(-1, 0), (0, -1)\}) \quad |\Theta_l(x) - \Theta_k(y)| \geq \frac{1}{2} |\theta_l(x) - \theta_k(y)|, \tag{8.2}$$

and from (7.6) on $[-T, 0]$,

$$\forall (x, y) \in \mathcal{B}_\zeta \times \mathcal{B}_\zeta, |\Theta_-(x) - \Theta_+(y)| \geq (C_- + C_+) \frac{T}{2}, \quad (8.3)$$

where $\Theta_- = \Theta_{-1}$ and $\Theta_+ = \Theta_0$. These inequalities show that the boundary of Ω_0 does not self-intersect and that Ω_0 is locally on one side of its boundary. Furthermore, setting $k = l$ in (8.2), we see that each smooth map Θ_l is injective, and thus each $\Theta_l(\mathcal{B}_\zeta)$ is a domain, which implies that Ω_0 is an open set of \mathbb{R}^3 .

Lemma 8.1. Ω_0 is a connected, $H^{4.5}$ -class domain, which is locally on one side of its boundary.

Proof. Step 1. We begin by proving that Ω_0 is connected. To this end, fix X and Y in Ω_0 so that $X \in \Theta_l(\mathcal{B}_\zeta)$ and $Y \in \Theta_j(\mathcal{B}_\zeta)$ ($-1 \leq l, j \leq L$). We let $(x, y) \in \mathcal{B}_\zeta \times \mathcal{B}_\zeta$ be such that $X = \Theta_l(x)$ and $Y = \Theta_j(y)$, and we define

$$\begin{aligned} X^\epsilon &= \eta^\epsilon(\theta_l^\epsilon(x), -T) \in \eta^\epsilon(\Omega^\epsilon, -T) \\ Y^\epsilon &= \eta^\epsilon(\theta_j^\epsilon(y), -T) \in \eta^\epsilon(\Omega^\epsilon, -T). \end{aligned}$$

For $\beta > 0$, we set

$$\Omega_\beta^\epsilon = \{Z \in \Omega^\epsilon \mid \text{dist}(Z, \partial\Omega^\epsilon) > \beta\}.$$

Then for $\beta > 0$ small enough, we have that Ω_β^ϵ is connected and X^ϵ and Y^ϵ are in $\eta^\epsilon(\Omega_\beta^\epsilon, -T)$.

From (8.1b) we infer that each $\eta^\epsilon(\theta_l^\epsilon, -T)$ uniformly converges to Θ_l in \mathcal{B}_ζ ; thus, for $\epsilon > 0$ small enough, we find that

$$\eta^\epsilon(\Omega_\beta^\epsilon, -T) \subset \Omega_0, \quad (8.4a)$$

$$X^\epsilon \in \Theta_l(\mathcal{B}_\zeta), \quad Y^\epsilon \in \Theta_j(\mathcal{B}_\zeta). \quad (8.4b)$$

Now, as Ω_β^ϵ is a connected set, so is $\eta^\epsilon(\Omega_\beta^\epsilon, -T)$. Since X^ϵ and Y^ϵ are in this connected set, we let $C_{X^\epsilon, Y^\epsilon}$ denote a continuous path included in $\eta^\epsilon(\Omega_\beta^\epsilon, -T)$, and having X^ϵ and Y^ϵ as its end-points. From (8.4a), $C_{X^\epsilon, Y^\epsilon} \subset \Omega_0$.

Next since both X and X^ϵ belong to the connected set $\Theta_l(\mathcal{B}_\zeta)$, let C_{X, X^ϵ} denote a continuous path included in $\Theta_l(\mathcal{B}_\zeta) \subset \Omega_0$ and having X and X^ϵ as end-points. Similarly, we let $C_{Y^\epsilon, Y}$ denote a continuous path included in $\Theta_j(\mathcal{B}_\zeta) \subset \Omega_0$ and having Y^ϵ and Y as its end-points. We then see that the union of these three paths joins X to Y and is contained in Ω_0 , which shows that Ω_0 is connected.

Step 2. The fact that Ω_0 is an $H^{4.5}$ -class domain follows immediately from the convergence given in (8.1a).

Step 3. We conclude by showing that Ω_0 is locally on one side of its boundary, and that with $B_\zeta^0 = B^0 \cap B(0, \zeta)$,

$$\partial\Omega_0 = \cup_{l=-1}^K \Theta_l(B_\zeta^0), \quad (8.5)$$

which will indeed complete the proof that Ω_0 is a standard $H^{4.5}$ -class domain.

To this end we first notice from (8.1b) and the fact that η^ϵ is volume preserving, that for each l ,

$$\det D\Theta_l = \det D\theta_l \geq c_l > 0. \quad (8.6)$$

Also, from (8.2) used when $k = l$, we notice that each Θ_l is an injective map, which with (8.6) provides

$$\partial[\Theta_l(\mathcal{B}_\zeta)] = \Theta_l(\partial\mathcal{B}_\zeta).$$

Therefore,

$$\partial\Omega_0 \subset \cup_{l=-1}^L \Theta_l(\partial\mathcal{B}_\zeta). \tag{8.7}$$

Now, let us fix $x \in \partial\mathcal{B}_\zeta \cap \{x_3 > 0\}$. We then have (since the only modified charts are modified close to the origin) that for any $-1 \leq l \leq K$,

$$\theta_l^\epsilon(x) = \theta_l(x).$$

We also notice that there exists $-1 \leq k \leq L$ and $y \in \mathcal{B}_\zeta$ such that $\theta_l(x) = \theta_k(y)$ (since $\Omega = \cup_{k=-1}^L \theta_k(\mathcal{B}_\zeta)$).

We also have that $\theta_k^\epsilon(y) = \theta_k(y)$, for otherwise k would be equal to either -1 or 0 , in which case $\theta_k(y)$ would be in a very small neighborhood of x_0 , which, in turn, would imply that l must be equal to k (since the charts θ_- or θ_+ do not intersect the other charts in a small neighborhood of x_0), but then we would not be able to have x at a distance ζ from the origin.

We then have $\eta^\epsilon(\theta_l^\epsilon(x), -T) = \eta^\epsilon(\theta_k^\epsilon(y), -T)$ which with (8.1b) implies that

$$\Theta_l(x) = \Theta_k(y) \in \Theta_k(\mathcal{B}_\zeta) \subset \Omega_0. \tag{8.8}$$

We can prove the same inclusion in a similar way if $x \in \partial\mathcal{B}_\zeta$ and $K + 1 \leq l \leq L$. With (8.7), this yields

$$\partial\Omega_0 \subset \cup_{l=-1}^K \Theta_l(\mathcal{B}_\zeta^0). \tag{8.9}$$

Now, for $X \in \Theta_l(\mathcal{B}_\zeta^0)$, we have $X = \Theta_l(x)$, with $x \in \mathcal{B}_\zeta^0$. Now for any $y \in \overline{\mathcal{B}_\zeta}$ such that there exists $-1 \leq k \leq L$ satisfying $X = \Theta_k(y)$, we see from (8.3) that $(l, k) \notin \{(-1, 0), (0, -1)\}$. Therefore, from (8.2) we have $\theta_l(x) = \theta_k(y)$. Then, as $\partial\Omega = \cup_{k=-1}^K \theta_k(\mathcal{B}_\zeta^0)$, we see that $y \in \mathcal{B}_\zeta^0$. Therefore, we have proved that X does not belong to $\cup_{k=-1}^L \theta_k(\mathcal{B}_\zeta) = \Omega_0$. Thus $X \in \partial\Omega_0$, which establishes (8.5).

Together with (8.6) and (8.3), this establishes that Ω_0 is a smooth domain locally on one side of its boundary, and concludes the proof. \square

8.2. *Asymptotic velocity at $-T$ in the limit $\epsilon \rightarrow 0$.* From our equivalence Lemma 6.1 and (7.7),

$$\sum_{l=-1}^L \|\zeta v^\epsilon(\theta_l^\epsilon, -T)\|_{4,B}^2 \leq \frac{2M_0}{C},$$

which shows the existence of a subsequence (which we continue to denote by the index ϵ) such that

$$v^\epsilon(\cdot, -T) \circ \theta_l^\epsilon \rightharpoonup V_l \text{ as } \epsilon \rightarrow 0, \text{ in } H^4(\mathcal{B}_\zeta), \tag{8.10a}$$

$$v^\epsilon(\cdot, -T) \circ \theta_l^\epsilon \rightharpoonup V_l \text{ as } \epsilon \rightarrow 0, \text{ in } H^3(\mathcal{B}_\zeta). \tag{8.10b}$$

We now define u_0 on Ω_0 as follows:

$$\forall l \in \{-1, 0, 1, 2, \dots, L\}, \quad u_0(\Theta_l) = V_l \quad \text{on } \mathcal{B}_\zeta. \quad (8.11)$$

In order to justify the definition in (8.11), we have to check that if $\Theta_l(x) = \Theta_j(y)$, for x and y in \mathcal{B}_ζ , then $V_l(x) = V_j(y)$. We first notice that if $\Theta_l(x) = \Theta_j(y)$, then by (8.3) we have $(l, k) \notin \{(-1, 0), (0, -1)\}$. From (8.2), we then infer that $\theta_l(x) = \theta_j(y)$ and thus $|\theta_l^\epsilon(x) - \theta_j^\epsilon(y)| \leq c_\epsilon$, with $\lim_{\epsilon \rightarrow 0} c_\epsilon = 0$.

This then, in turn, shows that

$$\begin{aligned} |v^\epsilon(\theta_l^\epsilon(x), -T) - v^\epsilon(\theta_j^\epsilon(y), -T)| &\leq |Dv^\epsilon(\cdot, -T)|_{L^\infty(\Omega^\epsilon(-T))} |\theta_l^\epsilon(x) - \theta_j^\epsilon(y)| \\ &\leq c_\epsilon |Dv^\epsilon(\cdot, -T)|_{L^\infty(\Omega^\epsilon(-T))}, \end{aligned}$$

which, thanks to (7.7), implies that

$$|v^\epsilon(\theta_l^\epsilon(x), -T) - v^\epsilon(\theta_j^\epsilon(y), -T)| \leq c_\epsilon \sqrt{M_0}.$$

By using (8.10b), this then implies at the limit $\epsilon \rightarrow 0$:

$$|V_l(x) - V_j(y)| \leq 0,$$

which concludes the proof. Also from (8.11), we have that $u_0 \in H^4(\Omega_0)$, with $\|u_0\|_{4, \Omega_0}^2 \leq 2M_0$.

8.3. Asymptotic domain and velocity on $(-T, 0]$ in the limit $\epsilon \rightarrow 0$. From our estimate (7.7) we then infer the existence of a subsequence (of the subsequence constructed in Sect. 8.1 and still denoted by a superscript ϵ) such that for all $l = -1, 0, 1, \dots, L$,

$$\partial_t v^\epsilon \circ \theta_l^\epsilon \rightharpoonup \partial_t v \circ \theta_l, \quad \text{in } L^2(-T, 0; H^{3.5}(\mathcal{B}_\zeta)), \quad (8.12a)$$

$$v^\epsilon \circ \theta_l^\epsilon \rightharpoonup v \circ \theta_l, \quad \text{in } L^2(-T, 0; H^4(\mathcal{B}_\zeta)), \quad (8.12b)$$

$$\eta^\epsilon \circ \theta_l^\epsilon \rightharpoonup \eta \circ \theta_l, \quad \text{in } L^2(-T, 0; H^{4.5}(\mathcal{B}_\zeta)), \quad (8.12c)$$

Next, let $\{\phi_n\}_{n=1}^\infty$ denote a countable dense set in $H^4(\mathcal{B}_\zeta)$. We next define the sequence $f_n^\epsilon : [-T, 0] \rightarrow \mathbb{R}$ by

$$f_n^\epsilon(t) = ([\eta^\epsilon \circ \theta_l^\epsilon](\cdot, t), \phi_n)_4,$$

where $(\cdot, \cdot)_4$ denotes the standard inner-product on $H^4(\mathcal{B}_\zeta)$. Now, for fixed n , the uniform bound (7.7) together with the fundamental theorem of calculus shows that for a positive constant $\mathfrak{M} < \infty$, $\|f_n^\epsilon\|_{C^0([-T, 0])} \leq \mathfrak{M}$ and that f_n^ϵ is equicontinuous (as a sequence of functions indexed by the sequence ϵ). By the Arzela–Ascoli theorem, there exists a subsequence (which we continue to denote by ϵ) such that $f_n^\epsilon \rightarrow f_n$ uniformly on $[-T, 0]$. This uniform convergence then implies for all $t \in [-T, 0]$ that

$$\int_0^t f_n^\epsilon(s) ds \rightarrow \int_0^t f_n(s) ds.$$

Due to (8.12c) we also have (with test function $1_{[0, t]} \phi_n$) that

$$\int_0^t f_n^\epsilon(s) ds \rightarrow \int_0^t ([\eta \circ \theta_l](\cdot, t), \phi_n)_4 ds,$$

which by comparison with the previous relation, then shows that

$$\int_0^t f_n(s) ds = \int_0^t ([\eta \circ \theta_t](\cdot, t), \phi_n)_4 ds.$$

Since both integrands are continuous with respect to time, this provides us by differentiation that for all $t \in [-T, 0]$,

$$f_n(t) = ([\eta \circ \theta_t](\cdot, t), \phi_n)_4.$$

Next, since $\{\phi_n\}$ is countable, we may employ the standard diagonal argument to extract a further subsequence (still denoted by ϵ) such that for all $t \in [-T, 0]$,

$$([\eta^\epsilon \circ \theta_t^\epsilon](\cdot, t), \phi)_4 \rightarrow ([\eta \circ \theta_t](\cdot, t), \phi)_4$$

for any $\phi \in H^4(\mathcal{B}_\zeta)$. This then establishes the existence of a *single* subsequence, such that for all $t \in [-T, 0]$,

$$\eta^\epsilon \circ \theta_t^\epsilon(\cdot, t) \rightarrow \eta \circ \theta_t(\cdot, t), \quad \text{in } H^4(\mathcal{B}_\zeta). \tag{8.13}$$

A similar argument shows that for the same subsequence (refined if necessary) and for all $t \in [-T, 0]$,

$$v^\epsilon \circ \theta_t^\epsilon(\cdot, t) \rightarrow v \circ \theta_t(\cdot, t) \quad \text{in } H^{3.5}(\mathcal{B}_\zeta). \tag{8.14}$$

Theorem A.2 providing continuity-in-time, together with the estimate (7.7), we have that for all $t \in [0, T]$,

$$\|v^\epsilon \circ \theta_t^\epsilon(\cdot, t)\|_{H^4(\mathcal{B}_\zeta)}^2 \leq CM_0, \tag{8.15a}$$

$$\|\eta^\epsilon \circ \theta_t^\epsilon(\cdot, t)\|_{H^{4.5}(\mathcal{B}_\zeta)}^2 \leq CM_0. \tag{8.15b}$$

Together with (8.13), this shows that for all $t \in [-T, 0]$, for the *same sequences* η^ϵ , v^ϵ , and $\tilde{\theta}_t^\epsilon$ as in (8.13) and (8.14), we have the following convergence (by an argument of uniqueness of the weak limit):

$$v^\epsilon \circ \theta_t^\epsilon(\cdot, t) \rightarrow v \circ \theta_t(\cdot, t) \quad \text{in } H^4(\mathcal{B}_\zeta), \tag{8.16a}$$

$$\eta^\epsilon \circ \theta_t^\epsilon(\cdot, t) \rightarrow \eta \circ \theta_t(\cdot, t) \quad \text{in } H^{4.5}(\mathcal{B}_\zeta). \tag{8.16b}$$

Having established the asymptotic limit as $\epsilon \rightarrow 0$ when $t = -T$, we next consider the time interval $(-T, 0)$. We employ the identical argument for taking the limit as $\epsilon \rightarrow 0$ for the case that $-T < t < 0$ as for the case that $t = -T$, leading to an asymptotic domain $\Omega(t)$ of class $H^{4.5}$ and an Eulerian velocity field $u(\cdot, t) \in H^4(\Omega(t))$ with $\|u(t, \cdot)\|_{4, \Omega(t)} \leq M_0$.

At time $t = 0$, there is a slight difference in the asymptotic limit $\epsilon \rightarrow 0$, in the sense that the limit domain is the *splash domain* Ω_s , which is a self-intersecting generalized $\mathbf{H}^{4.5}$ -domain, with the corresponding limit velocity field is $u_s \in H^{4.5}(\Omega_s)$. This limit simply comes from the fact that $\|\theta_t^\epsilon - \theta_t\|_{4.5, B^+} \rightarrow 0$ and $\|u_s^\epsilon \circ \theta_t^\epsilon - u_s \circ \theta_t\|_{4.5, B^+} \rightarrow 0$ as $\epsilon \rightarrow 0$.

8.4. *Asymptotic Euler equations.* It remains for us to prove that

$$u_f(x, t) = u(x, t - T), \quad 0 \leq t \leq T$$

is indeed a solution of the free-surface Euler equations on the moving domain

$$\Omega_f(t) = \Omega(t - T),$$

which evolves the initial velocity u_0 and initial domain Ω_0 onto the final data at time $t = T$ given by u_s and Ω_s . This will, in turn, establish the fact that after a finite time T , the free-surface of the 3-D Euler equations develops a splash singularity.

We again consider the asymptotic limit as $\epsilon \rightarrow 0$. For each $\epsilon > 0$ fixed, we solve the Euler equations forward-in-time using as initial data, $\Omega^\epsilon(-T)$ for the domain, and $u^\epsilon(\cdot, -T)$ for the initial velocity.

To this end, we first define the forward in time quantities for $0 \leq t \leq T$ by

$$\Omega_f^\epsilon(t) = \Omega^\epsilon(t - T),$$

and

$$\begin{aligned} u_f^\epsilon(\cdot, t) &= u^\epsilon(\cdot, t - T) && \text{in } \Omega_f^\epsilon(t), \\ \eta_f^\epsilon(\cdot, t) &= \eta^\epsilon(\cdot, t - T) \circ \eta^\epsilon(\cdot, -T)^{-1} && \text{in } \Omega_f^\epsilon(0), \\ v_f^\epsilon(\cdot, t) &= v^\epsilon(\cdot, t - T) \circ \eta^\epsilon(\cdot, -T)^{-1} && \text{in } \Omega_f^\epsilon(0), \\ p_f^\epsilon(\cdot, t) &= p^\epsilon(\cdot, t - T) && \text{in } \Omega_f^\epsilon(t), \\ q_f^\epsilon(\cdot, t) &= q^\epsilon(\cdot, t - T) \circ \eta^\epsilon(\cdot, -T)^{-1} && \text{in } \Omega_f^\epsilon(0). \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{div} u_f^\epsilon &= 0 && \text{in } \Omega_f^\epsilon(t), \\ v_f^\epsilon &= u_f^\epsilon \circ \eta_f^\epsilon = \partial_t \eta_f^\epsilon && \text{in } \Omega_f^\epsilon(0), \\ \eta_f^\epsilon(\cdot, 0) &= e && \text{in } \Omega_f^\epsilon(0). \end{aligned}$$

From the definitions of v^ϵ , η^ϵ , and u^ϵ in (4.9)–(4.11) and by uniqueness of solutions to (4.8), we see that $(u_f^\epsilon, p_f^\epsilon)$ is a solution of (1.1) on $[0, T]$ with initial domain $\Omega_f^\epsilon(0)$ and initial velocity $u_f^\epsilon(0)$, with the domain and velocity at time $t = T$ equal to Ω^ϵ and u_s^ϵ , respectively.

In order to analyze the limiting behavior of these solutions as $\epsilon \rightarrow 0$, we write the Euler equations in Lagrangian form on the fixed domain \mathcal{B}_ζ by pulling back the equations from the reference domain $\Omega_f^\epsilon(0)$ using the following local coordinate charts:

$$\tilde{\theta}_l^\epsilon = \eta^\epsilon(\theta_l^\epsilon, -T) \quad \text{for } l = -1, 0, 1, 2, \dots, L.$$

Denoting the local inverse-deformation tensor by

$$\tilde{b}_l^\epsilon = [D(\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon)]^{-1},$$

for $-1 \leq l \leq K$, solutions of the Euler equations satisfy

$$\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon = \tilde{\theta}_l^\epsilon + \int_0^t v_f^\epsilon \circ \tilde{\theta}_l^\epsilon \quad \text{in } \mathcal{B}_\zeta \times (0, T], \quad (8.19a)$$

$$\partial_t v_f^\epsilon \circ \tilde{\theta}_l^\epsilon + [\tilde{b}_l^\epsilon]^\top D(q_f^\epsilon \circ \tilde{\theta}_l^\epsilon) = 0 \quad \text{in } \mathcal{B}_\zeta \times (0, T), \quad (8.19b)$$

$$\operatorname{div}_{\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon} v_f^\epsilon \circ \tilde{\theta}_l^\epsilon = 0 \quad \text{in } \mathcal{B}_\zeta \times (0, T), \quad (8.19c)$$

$$q_f^\epsilon \circ \tilde{\theta}_l^\epsilon = 0 \quad \text{on } B_0 \times (0, T), \quad (8.19d)$$

$$(\eta_f^\epsilon, v_f^\epsilon) \circ \tilde{\theta}_l^\epsilon = (e, u_f^\epsilon(0)) \circ \tilde{\theta}_l^\epsilon \quad \text{on } \mathcal{B}_\zeta \times \{t = 0\}, \quad (8.19e)$$

together with

$$\eta_f^\epsilon(\Omega_f^\epsilon(0), T) = \Omega^\epsilon. \quad (8.19f)$$

For $l = K+1, \dots, L$ the same equations are satisfied with the exception of the boundary condition (8.19d).

Our a priori estimate Theorem A.1 shows that for each $l = -1, 0, 1, 2, \dots, L$,

$$\sup_{t \in [0, T]} \left(\|\eta_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|_{4.5, \mathcal{B}_\zeta}^2 + \|v_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|_{4, \mathcal{B}_\zeta}^2 + \|q_f^\epsilon(t) \circ \tilde{\theta}_l^\epsilon\|_{4.5, \mathcal{B}_\zeta}^2 \right) \leq 2\tilde{M}_0^\epsilon,$$

where \tilde{M}_0^ϵ is a constant that depends on the $H^{4.5}$ -norms of $\tilde{\theta}_l^\epsilon$ and the H^4 -norm of $u_f^\epsilon(0)$.

Thanks to Lemma 8.1 and the convergence in (8.10), we see that \tilde{M}_0^ϵ is bounded by a constant which is independent of ϵ . As such, we have the following convergence in two weak topologies and one strong topology:

$$v_f^\epsilon \circ \tilde{\theta}_l^\epsilon \rightharpoonup v_f \circ \Theta_l, \quad \text{in } L^2(0, T; H^4(\mathcal{B}_\zeta)), \quad (8.20a)$$

$$\eta_f^\epsilon \circ \tilde{\theta}_l^\epsilon \rightarrow \eta_f \circ \Theta_l, \quad \text{in } L^2(0, T; H^3(\mathcal{B}_\zeta)), \quad (8.20b)$$

$$q_f^\epsilon \circ \tilde{\theta}_l^\epsilon \rightharpoonup q_f \circ \Theta_l, \quad \text{in } L^2(0, T; H^{4.5}(\mathcal{B}_\zeta)), \quad (8.20c)$$

which together with the convergence in (8.1b) shows, in a manner similar as in Sect. 8.3, that for $l = -1, 0, 1, 2, \dots, K$, the limit as $\epsilon \rightarrow 0$ of the sequence of solutions to (8.19) is indeed a solution of

$$\eta_f \circ \Theta_l = \Theta_l + \int_0^t v_f \circ \Theta_l \quad \text{in } \mathcal{B}_\zeta \times (0, T], \quad (8.21a)$$

$$\partial_t v_f \circ \Theta_l + [\mathfrak{b}_l]^\top D(q_f \circ \Theta_l) = 0 \quad \text{in } \mathcal{B}_\zeta \times (0, T), \quad (8.21b)$$

$$\operatorname{div}_{\eta_f \circ \Theta_l} v_f \circ \Theta_l = 0 \quad \text{in } \mathcal{B}_\zeta \times (0, T), \quad (8.21c)$$

$$q_f \circ \Theta_l = 0 \quad \text{on } B_0 \times (0, T), \quad (8.21d)$$

$$(\eta_f, v_f) \circ \Theta_l = (e, u_0) \circ \Theta_l \quad \text{on } \mathcal{B}_\zeta \times \{t = 0\}, \quad (8.21e)$$

$$\eta_f(T, \Omega_0) = \Omega_s, \quad (8.21f)$$

where $\mathfrak{b}_l = [D(\eta_f \circ \Theta_l)]^{-1}$, and where v_f , q_f and η_f are the forward in time velocity, pressure and displacement fields.

A similar system holds for the interior charts Θ_l , with $K+1 \leq l \leq L$, with the exception of the boundary condition (8.21d). Therefore, since the charts Θ_l define Ω_0 , we have established that

$$\eta_f = e + \int_0^t v_f \quad \text{in } \Omega_0 \times (0, T], \quad (8.22a)$$

$$\partial_t v_f + A_f^\top Dq_f = 0 \quad \text{in } \Omega_0 \times (0, T), \quad (8.22b)$$

$$\operatorname{div}_{\eta_f} v_f = 0 \quad \text{in } \Omega_0 \times (0, T), \quad (8.22c)$$

$$q_f = 0 \quad \text{on } \partial\Omega_0 \times (0, T), \quad (8.22d)$$

$$(\eta_f, v_f) = (e, u_0) \quad \text{on } \Omega_0 \times \{t = 0\}, \quad (8.22e)$$

$$\eta_f(T, \Omega_0) = \Omega_s, \quad (8.22f)$$

where the matrix $A_f = [D\eta_f]^{-1}$. By a return to Eulerian variables this means that (u_f, p_f) is solution of (1.1) with initial domain and velocity Ω_0 and u_0 , respectively, and final domain and velocity at time $t = T$ equal to the splash domain Ω_s and u_s .

9. The Splat Domain Ω_s and its Approximation by Standard Domains Ω^ϵ

9.1. *The splat domain.* Whereas our splash domain has a boundary which self-intersects a point x_0 , an obvious generalization allows to define the so-called *splat domain* Ω_s , with boundary $\partial\Omega_s$ which self-intersects on an open subset Γ_0 of $\partial\Omega_s$.

9.1.1. *The definition of the splat domain.*

- (1) We suppose that $\Gamma_0 \subset \mathbf{\Gamma}_s := \partial\Omega_s$ is the unique boundary self-intersection surface, i.e., Ω_s is locally on each side of Γ_0 for each $x_0 \in \Gamma_0$. For all other boundary points, the domain is locally on one side of its boundary. We assume the existence of a smooth level set function $\phi \in H^{4.5}(\mathbb{R}^3)$ such that $\Gamma_0 \subset \{\phi = 0\}$.
- (2) We let U_0 denote an open neighborhood of x_0 in \mathbb{R}^3 , and then choose an additional L open sets $\{U_l\}_{l=1}^L$ such that the collection $\{U_l\}_{l=0}^K$ is an open cover of $\mathbf{\Gamma}_s$, and $\{U_l\}_{l=0}^L$ is an open cover of Ω_s and such that there exists a sufficiently small open subset $\omega \subset U_0$ containing Γ_0 with the property that

$$\overline{\omega} \cap \overline{U_l} = \emptyset \quad \text{for all } l = 1, \dots, L.$$

We set

$$U_0^+ = U_0 \cap \Omega_s \cap \{\phi > 0\} \quad \text{and} \quad U_0^- = U_0 \cap \Omega_s \cap \{\phi < 0\}.$$

Additionally, we assume that $\overline{U_0} \cap \overline{\Omega_s} \cap \{\phi = 0\} = \Gamma_0$, which implies in particular that U_0^+ and U_0^- are connected (Fig. 5).

- (3) We furthermore assume that our level set function is such that $\|D\phi\| \geq C_0 > 0$ on $\overline{U_0}$.
- (4) For each $l \in \{1, \dots, K\}$, there exists an $H^{4.5}$ -class diffeomorphism θ_l satisfying

$$\begin{aligned} \theta_l : B := B(0, 1) &\rightarrow U_l, \\ U_l \cap \Omega_s &= \theta_l(B^+) \quad \text{and} \quad \overline{U_l} \cap \mathbf{\Gamma}_s = \theta_l(B^0), \end{aligned}$$

where

$$\begin{aligned} B^+ &= \{(x_1, x_2, x_3) \in B : x_3 > 0\}, \\ B^0 &= \{(x_1, x_2, x_3) \in \overline{B} : x_3 = 0\}. \end{aligned}$$

- (5) For $L > K$, let $\{U_l\}_{l=K+1}^L$ denote a family of open sets contained in Ω_s such that $\{U_l\}_{l=0}^L$ is an open cover of Ω_s , and for $l \in \{K+1, \dots, L\}$, $\theta_l : B \rightarrow U_l$ is an $H^{4.5}$ diffeomorphism.

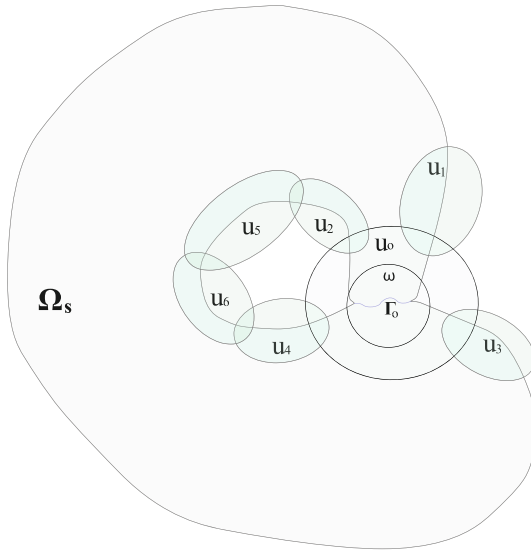


Fig. 5. Splat domain Ω_s , and the collection of open set $\{U_0, U_1, U_2, \dots, U_K\}$ covering Γ_s

- (6) To the open set U_0 we associate two $H^{4.5}$ -class diffeomorphisms θ_+ and θ_- of B onto U_0 with the following properties:

$$\begin{aligned} \theta_+(B^+) &= U_0^+, & \theta_-(B^+) &= U_0^-, \\ \theta_+(B^0) &= \overline{U_0^+} \cap \Gamma_s, & \theta_-(B^0) &= \overline{U_0^-} \cap \Gamma_s, \end{aligned}$$

such that

$$\Gamma_0 = \theta_+(B^0) \cap \theta_-(B^0),$$

and

$$\theta_+ = \theta_- \quad \text{on } \overline{\omega_0} \subset B_0,$$

where ω_0 is a smooth connected domain of B_0 in \mathbb{R}^2 .

We further assume that

$$\overline{\theta_{\pm}(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B^+)} = \emptyset \quad \text{for } l = 1, \dots, K,$$

and

$$\overline{\theta_{\pm}(B^+ \cap B(0, 1/2))} \cap \overline{\theta_l(B)} = \emptyset \quad \text{for } l = K + 1, \dots, L.$$

Definition 9.1 (Splat domain Ω_s). We say that Ω_s is a splat domain, if it is defined by a collection of open covers $\{U_l\}_{l=0}^L$ and associated maps $\{\theta_{\pm}, \theta_1, \theta_2, \dots, \theta_L\}$ satisfying the properties (1)–(6) above. Because each of the maps is an $H^{4.5}$ diffeomorphism, we say that the splat domain Ω_s defines a self-intersecting generalized $\mathbf{H}^{4.5}$ -domain.

9.2. *A sequence of standard domains approximating the splat domain.* We approximate the two distinguished charts θ_- and θ_+ by charts θ_-^ϵ and θ_+^ϵ in such a way as to ensure that

$$\theta_-^\epsilon(B^0) \cap \theta_+^\epsilon(B^0) = \emptyset \quad \forall \epsilon > 0,$$

and which satisfy

$$\theta_-^\epsilon \rightarrow \theta_- \quad \text{and} \quad \theta_+^\epsilon \rightarrow \theta_+ \quad \text{as} \quad \epsilon \rightarrow 0.$$

We let $\psi \in \mathcal{D}(\omega)$ denote a smooth bump-function satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ on Γ_0 . For $\epsilon > 0$ taken small enough, we define the following diffeomorphisms:

$$\begin{aligned} \theta_-^\epsilon(x) &= \theta_-(x) - \epsilon \psi(\theta_-(x)) D\phi(\theta_-(x)), \\ \theta_+^\epsilon(x) &= \theta_+(x) + \epsilon \psi(\theta_+(x)) D\phi(\theta_+(x)). \end{aligned}$$

By choosing $\psi \in \mathcal{D}(\omega)$, we ensure that the modification of the domain is localized to a small neighborhood of Γ_0 and away from the boundary of U_0 and the image of the other maps θ_l . Then, for $\epsilon > 0$ sufficiently small, thanks to item (3) in the definition of the splat domain,

$$\begin{aligned} \phi(\theta_-^\epsilon(x)) &\leq \phi(\theta_-(x)) - \frac{\epsilon}{2} \psi(\theta_-(x)) |D\phi(\theta_-(x))|^2 < 0, \\ \phi(\theta_+^\epsilon(x)) &\geq \phi(\theta_+(x)) + \frac{\epsilon}{2} \psi(\theta_+(x)) |D\phi(\theta_+(x))|^2 > 0, \end{aligned}$$

which shows that

$$\theta_-^\epsilon(\overline{B^+}) \cap \theta_+^\epsilon(\overline{B^+}) = \emptyset.$$

Since the maps θ_\pm^ϵ are a modification of the maps θ_\pm in a very small neighborhood of $0 \in B$, we have that for $\epsilon > 0$ sufficiently small,

$$\theta_\pm^\epsilon(B^+ \cap B(0, 1/2)) \cap \theta_l(B^+) = \emptyset \quad \text{for} \quad l = 1, \dots, K,$$

and

$$\theta_\pm^\epsilon(B^+ \cap B(0, 1/2)) \cap \theta_l(B) = \emptyset \quad \text{for} \quad l = K + 1, \dots, L.$$

For $l \in \{1, \dots, L\}$ we set $\theta_l^\epsilon = \theta_l$. Then $\theta_-^\epsilon : B^+ \rightarrow U_0$, $\theta_+^\epsilon : B^+ \rightarrow U_0$, and $\theta_l^\epsilon : B^+ \rightarrow U_l$, $l \in \{1, \dots, K\}$, $\theta_l^\epsilon : B \rightarrow U_l$, $l \in \{K + 1, \dots, L\}$, is a collection of $H^{4.5}$ coordinate charts as given in Sect. 2.5, and so we have the following

Lemma 9.1 (The approximate domains Ω^ϵ). *For each $\epsilon > 0$ sufficiently small, the set Ω^ϵ , defined by the local charts $\theta_-^\epsilon : B^+ \rightarrow U_0$, $\theta_+^\epsilon : B^+ \rightarrow U_0$, and $\theta_l^\epsilon : B^+ \rightarrow U_l$, $l \in \{1, \dots, K\}$, $\theta_l^\epsilon : B \rightarrow U_l$, $l \in \{K + 1, \dots, L\}$ (given in Definition 9.1) is a domain of class $H^{4.5}$, which is locally on one side of its H^4 boundary.*

Just as for the splash domain, we have approximated the self-intersecting *splat domain* Ω_s with a sequence of $H^{4.5}$ -class standard domains Ω^ϵ locally on one side of its boundary for each $\epsilon > 0$. Also, just as for the splash domain, our approximate domains Ω^ϵ differ from our splat domain Ω_s only on the two patches $\theta_-^\epsilon(B^+)$ and $\theta_+^\epsilon(B^+)$. In particular, as θ_\pm^ϵ differ from θ_\pm on a set properly contained in $\omega \subset U_0$, we continue to use the same covering $\{U_l\}_{l=0}^L$ for Ω^ϵ as for Ω_s .

10. Construction of the Splat Velocity Field u_s at the Time of the Splat Singularity

We can now define the *splat velocity* u_s associated with the generalized $\mathbf{H}^{4.5}$ -class splat domain Ω_s , as well as a sequence of approximations u_s^ϵ set on our $H^{4.5}$ -class approximations Ω^ϵ of the splat domain Ω_s .

10.1. The splat velocity u_s .

Definition 10.1 (Splat velocity u_s). *A velocity field u_s on an $\mathbf{H}^{4.5}$ -class splat domain Ω_s is called a splat velocity if it satisfies the following properties:*

- (1) $\zeta u_s \circ \theta_\pm \in H^{4.5}(B^+)$, $\zeta u_s \circ \theta_l \in H^{4.5}(B^+)$ for each $1 \leq l \leq K$ and $u_s \in H^{4.5}(\omega)$ for each $\bar{\omega} \subset \Omega_s$;
- (2) $u_s \cdot D\phi \circ \theta_-|_{\theta_-(B^+)} > C_-$ and $-u_s \cdot D\phi \circ \theta_+|_{\theta_+(B^+)} > C_+$ with $C_- + C_+ > 0$, so that under the motion of the fluid, the sets U_0^+ and U_0^- are moving relatively towards each other.

We can then define the approximate splat velocity fields $u_s^\epsilon : \Omega_s^\epsilon \rightarrow \mathbb{R}^3$ in the same way as we did for the case of the splash velocity. The results of Sects. 7 and 8 can then proceed in the same fashion as for the splash case, leading to Theorem 5.2.

We note only that the inequality (7.2) must be replaced with

$$-\phi(\eta^\epsilon(x, t)) + \phi(\eta^\epsilon(y, t)) \geq (C_+ + C_-) |t| - t^2 P(\sup_{[0,t]} E^\epsilon), \quad (10.1)$$

for x, y as in (7.2). The estimate (10.1) together with

$$|\phi(\eta^\epsilon(x, t)) - \phi(\eta^\epsilon(y, t))| \leq |D\phi| |\eta^\epsilon(x, t) - \eta^\epsilon(y, t)|,$$

and item (3) of the definition of our splat domain Ω_s then provides

$$|\eta^\epsilon(x, t) - \eta^\epsilon(y, t)| \geq \frac{(C_+ + C_-)}{C_0} |t| - t^2 P_1(\sup_{[0,t]} E^\epsilon). \quad (10.2)$$

This relation is the analogous of (7.2) obtained for the approximated splash domain. Since our splat domain is also bounded, we can derive in the same way as for the splash domain a relation similar to (7.6) for our approximated splat domain, which shows that η^ϵ is also injective for $\epsilon > 0$ small enough. In turn, this allows us to establish ϵ -independent estimates and arrive to the analogous conclusions as those obtained in Sects. 7 and 8.

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Appendix A. A Priori Estimates for the Free-Surface Euler Equations

In this appendix, we establish a priori estimates for the free-surface Euler equations with reference (or initial) domain Ω which is a standard $H^{4.5}$ -class domain, open, bounded, and locally on one side of its boundary.

A.1. Properties of the cofactor matrix a , and a polynomial-type inequality.

A.1.1. Geometry of the moving surface $\Gamma(t)$. With respect to local coordinate charts, the vectors $\eta_{,\alpha}$ for $\alpha = 1, 2$ span the tangent space to the moving surface $\Gamma(t) = \eta(\Gamma)$ in \mathbb{R}^3 . The (induced) surface metric g on $\Gamma(t)$ has components $g_{\alpha\beta} = \eta_{,\alpha} \cdot \eta_{,\beta}$. We let g_0 denote the surface metric of the initial surface Γ . The components of the inverse metric $[g]^{-1}$ are denoted by $[g]^{\alpha\beta}$. We use \sqrt{g} to denote $\sqrt{\det g}$; we note that $\sqrt{g} = |\eta_{,1} \times \eta_{,2}|$, so that $n(\eta) = [\eta_{,1} \times \eta_{,2}]/\sqrt{g}$.

A.1.2. Differentiating the inverse matrix A . Using that $D\eta A = \text{Id}$, we have the following identities:

$$\bar{\partial} A_i^k = -A_i^s \bar{\partial} \eta^r_{,s} A_r^k, \tag{A.1}$$

$$D A_i^k = -A_i^s D \eta^r_{,s} A_r^k, \tag{A.2}$$

$$\partial_t A_i^k = -A_i^s v^r_{,s} A_r^k. \tag{A.3}$$

A.1.3. Relating the cofactor matrix and the unit normal $n(t)$. With N denoting the outward unit normal to Γ , we have the identity

$$n_i(\eta) = a_i^k N_k / |a^T N|,$$

so that

$$A_i^k N_k = J^{-1} \sqrt{g} n_i(\eta) \quad \text{on } \Gamma. \tag{A.4}$$

A.1.4. A polynomial-type inequality. For a constant $M_0 \geq 0$, suppose that $f(t) \geq 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \leq M_0 + t P(f(t)), \tag{A.5}$$

where P denotes a polynomial function. Then for t taken sufficiently small, we have the bound

$$f(t) \leq 2M_0.$$

We use this type of inequality (see [14]) in place of nonlinear Gronwall-type of inequalities.

A.2. Trace and elliptic estimates for vector fields. The normal trace theorem states that the existence of the normal trace $w \cdot N|_\Gamma$ of a velocity field $w \in L^2(\Omega)$ relies on the regularity of $\text{div} w \in L^2(\Omega)$ (see, for example, [25]). If $\text{div} w \in L^2(\Omega)$, then $w \cdot N$ exists in $H^{-0.5}(\Gamma)$. We will use the following variant:

$$|\bar{\partial} w \cdot N|_{-0.5,\Gamma}^2 \leq C \left[\|\bar{\partial} w\|_{0,\Omega}^2 + \|\text{div} w\|_{0,\Omega}^2 \right] \tag{A.6}$$

for some constant C independent of w .

The construction of our higher-order energy function is based on the following Hodge-type elliptic estimate:

Proposition A.1. *For an H^r domain Ω with $\Gamma = \partial\Omega$, $r \geq 3$, if $F \in L^2(\Omega; \mathbb{R}^3)$ with $\text{curl } F \in H^{s-1}(\Omega; \mathbb{R}^3)$, $\text{div } F \in H^{s-1}(\Omega)$, and $\bar{\partial} F \cdot N|_{\Gamma} \in H^{s-\frac{3}{2}}(\Gamma)$ for $1 \leq s \leq r$, then there exists a constant $\bar{C} > 0$ depending only on Ω such that*

$$\|F\|_{s,\Omega} \leq \bar{C} \left(\|F\|_{0,\Omega} + \|\text{curl } F\|_{s-1,\Omega} + \|\text{div } F\|_{s-1,\Omega} + |\bar{\partial} F \cdot N|_{s-\frac{3}{2},\Gamma} \right), \quad (\text{A.7})$$

where N denotes the outward unit-normal to Γ .

This well-known inequality follows from the identity $-\Delta F = \text{curl } \text{curl } F - D \text{div } F$.

A.3. The higher-order energy function $E(t)$.

Definition A.1. *We set on $[0, T]$*

$$E(t) = 1 + \|\eta(t)\|_{4.5,\Omega}^2 + \|v(t)\|_{4,\Omega}^2 + \|\text{curl}_\eta v(t)\|_{3.5,\Omega}^2 + \|v_t(t)\|_{3.5,\Omega}^2. \quad (\text{A.8})$$

The function $E(t)$ is the higher-order energy function which we will prove remains bounded on $[0, T]$.

Definition A.2. *We set the constant M_0 to be a particular polynomial function P of $E(0)$ so that $M_0 = P(E(0))$.*

A.3.1. Conventions about constants. We take $T > 0$ sufficiently small so that, using the fundamental theorem of calculus, for constants c_1, c_2 and $t \in [0, T]$,

$$\begin{aligned} c_1 \det g(0) &\leq \det g(t) \leq c_2 \det g(0) \text{ on } \Gamma, \\ \|\eta(t)\|_4 &\leq \|e\|_4 + 1, \quad \|q(t)\|_4 \leq \|q(0)\|_4 + 1, \\ \|v(t)\|_{3.5} &\leq \|u_0\|_{3.5} + 1, \quad \|v_t(t)\|_3 \leq \|v_t(0)\|_3 + 1. \end{aligned}$$

The right-hand sides appearing in the last three inequalities shall be denoted by a generic constant C in the estimates that we will perform. The norms are over Ω .

A.4. Curl and divergence estimates for η , v , and v_t .

Proposition A.2. *For all $t \in (0, T)$,*

$$\|\text{curl } \eta(t)\|_{3.5,\Omega}^2 + \|\text{curl}_\eta v(t)\|_{3.5,\Omega}^2 \leq M_0 + T P\left(\sup_{t \in [0, T]} E(t)\right). \quad (\text{A.9})$$

Proof. By taking the curl of (1.3b), we have that

$$\text{curl}_\eta v_t = 0.$$

It follows that $\partial_t(\text{curl}_\eta v) = B(A, Dv)$, where the k th-component of B is given by

$$[B(A, Dv)]_k = \varepsilon_{kji} A_t^s v^i{}_{,s} = \varepsilon_{kij} v^i{}_{,s} A_p^s v^p{}_{,l} A_j^l;$$

hence,

$$\text{curl}_\eta v(t) = \text{curl } u_0 + \int_0^t B(A(t'), Dv(t')) dt'. \quad (\text{A.10})$$

Step 1. Estimate for curl η . Computing the gradient of (A.10) yields

$$\operatorname{curl}_\eta Dv(t) = D \operatorname{curl} u_0 - \varepsilon_{.ji} DA_j^s v^i_{,s} + \int_0^t DB(A(t'), Dv(t')) dt'. \quad (\text{A.11})$$

(In components, $[\operatorname{curl}_\eta \partial_{x_l} v]_i = \varepsilon_{ijk} v^k_{,l} A_j^r$.) Applying the fundamental theorem of calculus once again, shows that

$$\begin{aligned} \operatorname{curl}_\eta D\eta(t) &= t D \operatorname{curl} u_0 + \varepsilon_{.ji} \int_0^t [A_{rj}^s D\eta^i_{,s} - DA_j^s v^i_{,s}] dt' \\ &\quad + \int_0^t \int_0^{t'} DB(A(t''), Dv(t'')) dt'' dt', \end{aligned} \quad (\text{A.12})$$

and finally that

$$\begin{aligned} D \operatorname{curl} \eta(t) &= t D \operatorname{curl} u_0 - \varepsilon_{.ji} \int_0^t A_{rj}^s(t') dt' D\eta^i_{,s} \\ &\quad + \varepsilon_{.ji} \int_0^t [A_{rj}^s D\eta^i_{,s} - DA_j^s v^i_{,s}] dt' + \int_0^t \int_0^{t'} DB(A(t''), Dv(t'')) dt'' dt'. \end{aligned} \quad (\text{A.13})$$

Using the fact that $\partial_t A_j^s = -A_l^s v^l_{,p} A_j^p$ and $DA_j^s = -A_l^s D\eta^l_{,p} A_j^p$, we see that

$$\begin{aligned} DB(A, Dv) &= -\varepsilon_{kji} [Dv^i_{,s} A_l^s v^l_{,p} A_j^p + v^i_{,s} A_l^s Dv^l_{,p} A_j^p \\ &\quad + v^i_{,s} v^l_{,p} D(A_l^s A_j^p)]. \end{aligned} \quad (\text{A.14})$$

The precise structure of the right-hand side is not very important; rather, the derivative count is the focus, and as such we write

$$DB(A, Dv) \sim D^2 v Dv A A + D^2 \eta Dv Dv A A.$$

Integrating by parts in time in the last term of the right-hand side of (A.13), we see that

$$\begin{aligned} \int_0^t \int_0^{t'} DB(A, Dv) dt'' dt' &\sim - \int_0^t \int_0^{t'} D^2 \eta (Dv A A)_t dt'' dt' \\ &\quad + \int_0^t \int_0^{t'} D^2 \eta Dv Dv A A dt'' dt' + \int_0^t D^2 \eta Dv A A dt'. \end{aligned} \quad (\text{A.15})$$

Thus, we can write

$$\begin{aligned} D \operatorname{curl} \eta(t) &\sim t D \operatorname{curl} u_0 + D^2 \eta \int_0^t Dv A A dt' + \int_0^t D^2 \eta Dv A A dt' \\ &\quad + \int_0^t \int_0^{t'} D^2 \eta Dv Dv A A dt'' dt' + \int_0^t \int_0^{t'} D^2 \eta (Dv A A)_t dt'' dt'. \end{aligned}$$

Our goal is to estimate $\|D \operatorname{curl} \eta\|_{2.5, \Omega}^2$. Thanks to the Sobolev embedding theorem, we have that

$$\|D \operatorname{curl} \eta\|_{2.5, \Omega}^2 \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right),$$

and hence with $\text{curl}_\eta v_t = 0$, that

$$\| \text{curl } \eta \|_{3.5, \Omega}^2 \leq M_0 + T P(\sup_{t \in [0, T]} E(t)).$$

Step 2. Estimate for $\text{curl}_\eta v$. Integrating-by-parts with respect to ∂_t in the time integral in Eq. (A.11), we see that the highest order term in $\text{curl}_\eta Dv$ is given by $\int_0^t D^2 \eta Dv_t A \, dt'$. As $H^{2.5}(\Omega)$ is a multiplicative algebra, it follows that on $[0, T]$,

$$\| \text{curl}_\eta v(t) \|_{3.5, \Omega}^2 \leq M_0 + T P(\sup_{t \in [0, T]} E(t)).$$

□

Proposition A.3. *For all $t \in (0, T)$,*

$$\| \text{div } \eta(t) \|_{3.5, \Omega}^2 + \| \text{div } v(t) \|_{3, \Omega}^2 \leq M_0 + T P(\sup_{t \in [0, T]} E(t)). \tag{A.16}$$

Proof. Since $A_i^j v^i_{,j} = 0$, we see that

$$A_i^j Dv^i_{,j} = -DA_i^j v^i_{,j} . \tag{A.17}$$

Step 1. Estimate for $\text{div } \eta$. It follows that

$$[A_i^j D\eta^i_{,j}]_t = \partial_t A_i^j D\eta^i_{,j} - DA_i^j v^i_{,j} .$$

Using the fact that $\eta(x, 0) = x$,

$$[A_i^j D\eta^i_{,j}](t) = \int_0^t \left(\partial_t A_i^j D\eta^i_{,j} - DA_i^j v^i_{,j} \right) dt' , \tag{A.18}$$

and hence

$$D \text{div } \eta(t) = \int_0^t \partial_t A_i^j D\eta^i_{,j} dt' - \int_0^t DA_i^j v^i_{,j} dt' - \int_0^t \partial_t A_i^j dt' D\eta^i_{,j} .$$

Again, the Sobolev embedding theorem provides us with the estimate

$$\| \text{div } \eta(t) \|_{3.5, \Omega}^2 \leq T P(\sup_{t \in [0, T]} E(t)).$$

Step 2. Estimate for $\text{div } v$. From $A_i^j v^i_{,j} = 0$, we see that

$$\text{div } v(t) = - \int_0^t \partial_t A_i^j dt' v^i_{,j} . \tag{A.19}$$

Hence,

$$\| \text{div } v(t) \|_{3, \Omega}^2 \leq T P(\sup_{t \in [0, T]} E(t)).$$

□

A.5. Pressure estimates. Letting $A_i^j \frac{\partial}{\partial x_j}$ act on (1.3b), for $t \in [0, T]$, the Lagrangian pressure function $q(x, t)$ satisfies the elliptic equation

$$-\left[A_i^j A_i^k q, \right]_{,j} = v^i{}_{,j} A_r^j v^r{}_{,s} A_i^s \quad \text{in } \Omega, \tag{A.20a}$$

$$q = 0 \quad \text{on } \Gamma. \tag{A.20b}$$

Suppose that there exists a weak solution $u \in H_0^1(\Omega)$ to $-\operatorname{div}[\mathcal{A} Du] = f$ in Ω with $u = 0$ on Γ , and where \mathcal{A} is positive-definite and symmetric. Suppose further that $f \in H^k(\Omega)$, $\mathcal{A} \in H^{k+1}(\Omega)$ for integers $1 \geq 2$. Then $u \in H^{k+2}(\Omega) \cap H_0^1(\Omega)$ and satisfies

$$\|u\|_{k+2} \leq C (\|f\|_k + \mathcal{P}(\|A\|_{k+1}) \|f\|_0), \tag{A.21}$$

where \mathcal{P} denotes a polynomial function of its argument. By invoking the Sobolev embedding theorem, the elliptic estimate (A.21) shows that

$$\begin{aligned} \|q\|_4 &\leq C(\|A\|_2, \|v\|_3) \|A\|_3, \\ \|q\|_5 &\leq C(\|A\|_2, \|v\|_3) \|A\|_4, \end{aligned}$$

where the constant has polynomial dependence on $\|A\|_2$ and $\|v\|_3$. Linear interpolation then yields

$$\|q\|_{4.5} \leq C(\|A\|_2, \|v\|_3) \|\eta\|_{4.5}.$$

By time-differentiating (A.20), and using our conventions of Sect. A.3.1 concerning the generic constant C , we have the elliptic estimate on $[0, T]$

$$\|q(t)\|_{4.5} + \|q_t(t)\|_4 \leq C \|\eta(t)\|_{4.5}. \tag{A.22}$$

Remark 2. When the elliptic problem (A.20) is set on the approximate splash domain Ω^ϵ , the elliptic constant a priori depends on $\epsilon > 0$, via the charts θ_\pm^ϵ ; however, thanks to Lemma 3.2, the elliptic constant is independent of ϵ since the charts θ_\pm are bounded in $H^{4.5}$.

A.6. Rayleigh–Taylor condition at time $t > 0$. For each $l = 1, \dots, K$, the fundamental theorem of calculus allows us to write

$$[q(\theta_l^\epsilon(x), t)]_{,3} = [q(\theta_l^\epsilon(x), 0)]_{,3} + \int_0^t [q_t(\theta_l^\epsilon(x), t')]_{,3} dt'.$$

From the assumed Rayleigh–Taylor condition (4.7) on the initial data, it follows that for all $x \in B_0$,

$$[q(\theta_l^\epsilon(x), t)]_{,3} \geq \frac{C_{\text{RT}}}{2} - C \int_0^t \|q_t(\theta_l^\epsilon, t')\|_3 dt'.$$

Thanks to our previously established bound (A.22), we then see that on B_0 ,

$$[q(\theta_l^\epsilon(x), t)]_{,3} \geq \frac{C_{\text{RT}}}{2} - t P\left(\sup_{s \in [0, t]} E(s)\right), \tag{A.23}$$

so that by choosing T sufficiently small, $[q(\theta_l^\epsilon(x), t)]_{,3} \geq \frac{C_{\text{RT}}}{4}$ for all $t \in [0, T]$. In what follows, we will drop the ϵ for notational convenience.

A.7. *Technical lemma.* Our energy estimates require the use of the following

Lemma A.1. *Let $H^{\frac{1}{2}}(\Omega)'$ denote the dual space of $H^{\frac{1}{2}}(\Omega)$. There exists a positive constant C such that*

$$\|\bar{\partial}F\|_{H^{\frac{1}{2}}(\Omega)'} \leq C \|F\|_{\frac{1}{2},\Omega} \quad \forall F \in H^{\frac{1}{2}}(\Omega).$$

Proof. Integrating by parts with respect to the tangential derivative yields for all $G \in H^1(\Omega)$,

$$\begin{aligned} \int_{\Omega} \bar{\partial}F G \, dx &= \sum_{l=1}^L \int_{U_l \cap \Omega} \zeta [(F \circ \theta_l)_{,\alpha}] \circ \theta_l^{-1} G \, dx \\ &= \sum_{l=1}^L \int_{B^+} \zeta \circ \theta_l (F \circ \theta_l)_{,\alpha} G \circ \theta_l \det D\theta_l \, dx \\ &= - \sum_{l=1}^L \int_{B^+} \zeta \circ \theta_l F \circ \theta_l (G \circ \theta_l)_{,\alpha} \det D\theta_l \, dx \\ &\quad - \sum_{l=1}^L \int_{B^+} F \circ \theta_l G \circ \theta_l (\zeta \circ \theta_l \det D\theta_l)_{,\alpha} \, dx \\ &\leq C \|F\|_{0,\Omega} \|G\|_{1,\Omega}, \end{aligned}$$

which shows that there exists $C > 0$ such that

$$\forall F \in L^2(\Omega), \quad \|\bar{\partial}F\|_{H^1(\Omega)'} \leq C \|F\|_{0,\Omega}. \quad (\text{A.24})$$

Interpolating with the obvious inequality

$$\forall F \in H^1(\Omega), \quad \|\bar{\partial}F\|_{L^2(\Omega)} \leq C \|F\|_{1,\Omega}$$

proves the lemma. \square

A.8. *Energy estimates for the normal trace of η and v .* By denoting $\eta_l = \eta \circ \theta_l$ we see that

$$\eta_l(t) : B^+ \rightarrow \Omega(t) \quad \text{for } l = 1, \dots, K.$$

We set $v_l = u \circ \eta_l$, $q_l = p \circ \eta_l$ and $A_l = [D\eta_l]^{-1}$, $J_l = \det D\eta_l$, and $a_l = J_l A_l$. It follows that for $l = 1, \dots, K$,

$$\eta_l(t) = \theta_l + \int_0^t v_l \quad \text{in } B^+ \times [0, T], \quad (\text{A.25a})$$

$$\partial_t v_l + A_l Dq_l = 0 \quad \text{in } B^+ \times (0, T], \quad (\text{A.25b})$$

$$\operatorname{div}_{\eta_l} v_l = 0 \quad \text{in } B^+ \times [0, T], \quad (\text{A.25c})$$

$$q_l = 0 \quad \text{on } B^0 \times [0, T], \quad (\text{A.25d})$$

$$(\eta_l, v_l) = (\theta_l, u_0 \circ \theta_l) \quad \text{in } B^+ \times \{t = 0\}. \quad (\text{A.25e})$$

Proposition A.4. For $t \in [0, T]$,

$$|\partial\eta(t) \cdot N|_{3,\Gamma}^2 + |\bar{\partial}v(t) \cdot N|_{2,5,\Gamma}^2 \leq M_0 + T P(\sup_{t \in [0, T]} E(t)). \quad (\text{A.26})$$

Proof. We compute the following $L^2(B^+)$ inner-product:

$$0 = \left(\zeta \bar{\partial}^4 [\partial_t v_l + A_l Dq_l], \zeta \bar{\partial}^4 v_l \right)_{L^2(B^+)}. \quad (\text{A.27})$$

To simplify the notation, we fix $l \in \{1, \dots, K\}$ and drop the subscript. We have that

$$0 = \underbrace{\frac{1}{2} \frac{d}{dt} \|\zeta \bar{\partial}^4 v(t)\|_{0, B^+}^2}_{\mathcal{I}_1} + \underbrace{\int_{B^+} \zeta^2 \bar{\partial}^4 A_i^k q_{,k} \bar{\partial}^4 v^i dx}_{\mathcal{I}_2} + \underbrace{\int_{B^+} \zeta^2 A_i^k \bar{\partial}^4 q_{,k} \bar{\partial}^4 v^i dx}_{\mathcal{I}_3} + \mathcal{R}, \quad (\text{A.28})$$

where \mathcal{R} denotes integrals over B^+ consisting of lower-order terms (or remainders) which can easily be shown, via the Cauchy–Schwarz inequality, to satisfy

$$\int_0^T |\mathcal{R}(t)| dt \leq M_0 + T P(\sup_{t \in [0, T]} E(t)).$$

Using the identity (A.1), we see that

$$\begin{aligned} \mathcal{I}_2 &= - \int_{B^+} \zeta^2 A_r^k \bar{\partial}^4 \eta^r_{,s} A_i^s q_{,k} \bar{\partial}^4 v^i dx + \mathcal{R} \\ &= - \underbrace{\int_{B^0} \zeta^2 A_r^k \bar{\partial}^4 \eta^r q_{,k} \bar{\partial}^4 v^i A_i^s N_s^0 dx_h}_{\mathcal{I}_{2a}} + \underbrace{\int_{B^+} \zeta^2 A_r^k \bar{\partial}^4 \eta^r A_i^s q_{,k} \bar{\partial}^4 v^i_{,s} dx}_{\mathcal{I}_{2b}} + \mathcal{R}, \end{aligned}$$

where $dx_h = dx_1 dx_2$ denotes the surface measure on B^0 . As $q = 0$ on B^0 , $q_{,1} = 0$ and $q_{,2} = 0$ on Γ , and since the exterior normal on B_0 is $N^0 = -e_3$, we have $A_r^3 = -A_r^k N_k^0$, which then implies

$$\mathcal{I}_{2a} = \int_{B^0} q_{,3} \zeta^2 \bar{\partial}^4 \eta^r A_r^k N_k^0 \bar{\partial}^4 v^i A_i^s N_s^0 dx_h.$$

We define n_l to be the outward unit normal to the moving surface $\eta_l(t, B^0)$, so that from (A.4),

$$A_i^k N_k^0 = J_l^{-1} \sqrt{g_l} n_l(\eta) \quad \text{on } B^0.$$

Dropping the subscript l again and writing n for $n(\eta)$, it follows that

$$\begin{aligned} \mathcal{I}_{2a}(t) &= \int_{B^0} q_{,3} \zeta^2 \bar{\partial}^4 \eta \cdot n \bar{\partial}^4 v \cdot n |\det g| J^{-2} dx_h \\ &= \frac{1}{2} \frac{d}{dt} \underbrace{\int_{B^0} q_{,3} \zeta^2 |\bar{\partial}^4 \eta \cdot n|^2 |\det g| J^{-2} dx_h}_{\mathcal{K}_a} \\ &\quad - \underbrace{\int_{B^0} \frac{1}{2} \zeta^2 \bar{\partial}^4 \eta^i \bar{\partial}^4 \eta^j \partial_t [(n_i n_j |\det g| J^{-2})] dx_h}_{\mathcal{K}_b}. \end{aligned}$$

By the assumption of Sect. A.3.1,

$$\left| \partial_t [n_i n_j | \det g | J^{-2}] \right|_{L^\infty(\Gamma)} \leq C,$$

from which it follows that

$$\int_0^T \mathcal{K}_b(t) dt \leq C T P(\sup_{t \in [0, T]} E(t)).$$

Using our Rayleigh–Taylor condition (A.23) for $q_{,3}(t)$, and bounds for $\det g(t)$, J which can be established similarly on $[0, T]$, we see that

$$\bar{c} |\zeta \bar{\partial}^4 \eta_l(t) \cdot n_l(t)|_{0, B^0}^2 - T P(\sup_{t \in [0, T]} E(t)) \leq \int_0^T \mathcal{I}_{2a}(t) dt,$$

for a constant \bar{c} which depends on C_{RT} , $g(0)$, and $J(0) = \det D\theta_l$. We set

$$N_l = \frac{\theta_{l,1} \times \theta_{l,2}}{|\theta_{l,1} \times \theta_{l,2}|}.$$

By the fundamental theorem of calculus $n_l(t) = N_l + \int_0^t \partial_t n_l(t') dt'$, and by our assumptions in Sect. A.3.1, $\sup_{[0, T]} |\partial_t n_l(t)|_{L^\infty(\Gamma)} \leq C$; hence,

$$\bar{c} |\zeta \bar{\partial}^4 \eta_l(t) \cdot N_l|_{0, B^0}^2 \leq \int_0^T \mathcal{I}_{2a}(t) dt + T P(\sup_{t \in [0, T]} E(t)),$$

and hence

$$\bar{c} |\zeta \bar{\partial} \eta_l(t) \cdot N_l|_{3, B^0}^2 \leq \int_0^T \mathcal{I}_{2a}(t) dt + T P(\sup_{t \in [0, T]} E(t)).$$

It remains to show that the integrals $\int_0^T \mathcal{I}_{2b}(t) dt$ and $\int_0^T \mathcal{I}_3(t) dt$ are both bounded by $T P(\sup_{t \in [0, T]} E(t))$. Using (A.25c),

$$\begin{aligned} \mathcal{I}_{2b}(t) &= - \int_{B^+} \zeta^2 A_r^k \bar{\partial}^4 \eta^r q_{,k} v^i{}_{,s} \bar{\partial}^4 A_i^s dx + \mathcal{R} \\ &\leq C \|\zeta \bar{\partial}^4 \eta(t)\|_{\frac{1}{2}, B^+} \|\zeta \bar{\partial}^4 A(t)\|_{H^{\frac{1}{2}}(B^+)} + \mathcal{R} \\ &\leq C \|\bar{\partial}^4 \eta(t)\|_{\frac{1}{2}} \|\bar{\partial}^3 A(t)\|_{H^{\frac{1}{2}}(\Omega)} + \mathcal{R} \\ &\leq C \sup_{t \in [0, T]} E(t) + \mathcal{R}, \end{aligned}$$

where we have used Lemma A.1 for the second inequality.

Finally,

$$\begin{aligned} \mathcal{I}_3(t) &= - \int_{B^+} \zeta^2 \bar{\partial}^4 q \bar{\partial}^4 v^i{}_{,k} A_i^k dx = \int_{B^+} \zeta^2 \bar{\partial}^4 q v^i{}_{,k} \bar{\partial}^4 A_i^k dx + \mathcal{R} \\ &\leq C \|\zeta \bar{\partial}^3 q(t)\|_{\frac{1}{2}, B^+} \|\zeta \bar{\partial}^4 A(t)\|_{H^{\frac{1}{2}}(B^+)} + \mathcal{R} \\ &\leq C \sup_{t \in [0, T]} E(t) + \mathcal{R}, \end{aligned}$$

where we have used the pressure estimate (A.22) and Lemma A.1 for the last inequality.

Summing the estimates for $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and integrating (A.28) from 0 to T , we obtain the inequality,

$$\sup_{t \in [0, T]} \left(|\zeta \bar{\partial} \eta(t) \cdot N_t|_{3, B^0}^2 + \|\zeta \bar{\partial}^4 v(t)\|_{0, B^+}^2 \right) \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right).$$

According to Proposition A.3,

$$\sup_{t \in [0, T]} \|\operatorname{div} v(t)\|_3^2 \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right),$$

from which it follows that

$$\sup_{t \in [0, T]} \|\zeta \bar{\partial}^3 \operatorname{div} v(t)\|_{0, B^+}^2 \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right).$$

Hence, the normal trace estimate (A.6) shows that

$$\sup_{t \in [0, T]} \left(|\zeta \bar{\partial}^4 v(t) \cdot N_t|_{-\frac{1}{2}, B^0}^2 \right) \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right),$$

from which it follows that

$$\sup_{t \in [0, T]} \left(|\zeta \bar{\partial} \eta(t) \cdot N_t|_{3, B^0}^2 + |\zeta \bar{\partial} v(t) \cdot N_t|_{2.5, B^0}^2 \right) \leq M_0 + T P \left(\sup_{t \in [0, T]} E(t) \right).$$

□

Combining Proposition A.4 with the curl estimates in Proposition A.2 and the divergence estimates in Proposition A.3 for $\eta(t)$ and $v(t)$ and using (A.7) together with the fact that $v_t = -A^T Dq$ provides us with the following

Theorem A.1. *Suppose that the initial pressure p_0 satisfies $\frac{\partial p}{\partial N} < 0$ on Γ and that $E(0) < \infty$. For T taken sufficiently small and for a polynomial function P_2 ,*

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|\eta(t)\|_{4.5, \Omega}^2 + \|v(t)\|_{4, \Omega}^2 + \|\operatorname{curl}_\eta v(t)\|_{3.5, \Omega}^2 + \|v_t(t)\|_{3.5, \Omega}^2 \right) \\ & \leq M_0 + T P_2 \left(\sup_{t \in [0, T]} E(t) \right). \end{aligned}$$

Moreover $\frac{\partial p}{\partial n} < 0$ on $\Gamma(t)$ for $t \in [0, T]$.

(The rigorous construction of solutions to this problem was established in [14] using an approximation scheme founded on the idea of horizontal convolution-by-layers.) We next show that our solutions are continuous in time.

Theorem A.2 (Continuity in time). *The solution satisfies*

$$\begin{aligned} \eta & \in C([0, T]; H^{4.5}(\Omega)), \quad v \in C([0, T]; H^4(\Omega)), \\ \operatorname{curl}_\eta v & \in C([0, T]; H^{3.5}(\Omega)), \quad v_t \in C([0, T]; H^{3.5}(\Omega)). \end{aligned}$$

Proof. It follows immediately from Theorem A.1 that

$$\begin{aligned} \eta & \in C([0, T]; H^4(\Omega)), \quad v \in C([0, T]; H^{3.5}(\Omega)), \\ \operatorname{curl}_\eta v & \in C([0, T]; H^3(\Omega)), \quad v_t \in C([0, T]; H^3(\Omega)). \end{aligned} \tag{A.29}$$

Furthermore, by the same argument used to establish (8.13) and (8.14), it follows that

$$\begin{aligned} \eta & \in C([0, T]; H^{4.5}(\Omega) \text{-w}), \quad v \in C([0, T]; H^4(\Omega) \text{-w}), \\ \operatorname{curl}_\eta v & \in C([0, T]; H^{3.5}(\Omega) \text{-w}), \quad v_t \in C([0, T]; H^{3.5}(\Omega) \text{-w}), \end{aligned} \tag{A.30}$$

the notation $H^s(\Omega)$ -w denoting the weak topology. Thus, it suffices to prove continuity of the norms

$$\|\eta(t)\|_{4,5}, \|v(t)\|_4, \|v_t(t)\|_{3,5}, \text{ and } \|\operatorname{curl}_\eta v(t)\|_{3,5}.$$

For $h > 0$ we define the horizontal difference quotient

$$\bar{\partial}^h u := \frac{1}{h} (u(\cdot + h\mathbf{e}_i) - u(\cdot)), \quad (i = 1, 2),$$

and we proceed as in (A.27), using $\bar{\partial}^h \bar{\partial}^3$ in place of $\bar{\partial}^4$. The same energy estimate then yields

$$\frac{d}{dt} \left(\|\zeta \bar{\partial}^h \bar{\partial}^3 v_l(t)\|_{0,B^+}^2 + |\zeta \sqrt{-q, N} \bar{\partial}^h \bar{\partial}^3 \eta_l(t) \cdot n_l(\eta_l)|_{0,B^0}^2 \right) \leq C \left(\|\eta(t)\|_{4,5}^2 + \|v(t)\|_4^2 \right).$$

With $\mathcal{F}_h(t) := \|\zeta \bar{\partial}^h \bar{\partial}^3 v_l(t)\|_{0,B^+}^2$ and $\mathcal{G}_h(t) := |\zeta \sqrt{-q, N} \bar{\partial}^h \bar{\partial}^3 \eta_l(t) \cdot n_l(\eta_l)|_{0,B^0}^2$, we have that

$$\frac{d}{dt} [\mathcal{F}_h(t) + \mathcal{G}_h(t)] \leq C \left(\|\eta(t)\|_{4,5}^2 + \|v(t)\|_4^2 \right).$$

Integrating from t to $t + \delta$, $0 < \delta \ll 1$, and setting $\mathcal{H}_h := \mathcal{F}_h + \mathcal{G}_h$, we see that

$$|\mathcal{H}_h(t + \delta) - \mathcal{H}_h(t)| \leq \delta C M_0.$$

Since the bounds are independent of $h > 0$, we see that

$$|\mathcal{H}(t + \delta) - \mathcal{H}(t)| \leq \delta C M_0, \tag{A.31}$$

where $\mathcal{H} = \mathcal{F} + \mathcal{G}$, and $\mathcal{F}(t) := \|\zeta \bar{\partial}^4 v_l(t)\|_{0,B^+}^2$ and $\mathcal{G}(t) := |\zeta \sqrt{-q, N} \bar{\partial}^4 \eta_l(t) \cdot n_l(\eta_l)|_{0,B^0}^2$

Hence, $t \mapsto \mathcal{H}(t)$ is uniformly Lipschitz continuous for $t \in [0, T]$. Consider the product topology on the Hilbert space $\mathcal{X} := L^2(B^+) \times L^2(B^0)$, with norm $\|(f, g)\|_{\mathcal{X}}^2 = \|f\|_{L^2(B^+)}^2 + \|g\|_{L^2(B^0)}^2$. The convergence in the norm given by (A.31) together with the continuity into the weak topology, given by (A.30), show that $(\zeta \bar{\partial}^4 v, \zeta \sqrt{-q, N} \bar{\partial}^4 \eta \cdot n(\eta))$ are continuous into \mathcal{X} . We sum over all boundary charts; thanks to (A.29) and the elliptic estimate (A.21), $q \in C([0, T]; H^4(\Omega))$, from which it follows that

$$\bar{\partial}^4 v \in C^0([0, T]; L^2(\Omega)) \text{ and } \bar{\partial}^2 \eta \cdot n(\eta) \in C^0([0, T]; H^2(\Gamma)).$$

In order to prove that $\|\eta(t)\|_{4,5}$ is continuous for each $t \in [0, T]$, we will rely on the Lagrangian divergence and curl identities which we established earlier. From Eqs. (A.12) and (A.15), we see that

$$\|\operatorname{curl}_\eta D\eta(t+h) - \operatorname{curl}_\eta D\eta(t)\|_{2,5} \leq ChM_0,$$

so that $\operatorname{curl}_\eta D\eta(t) \in C^0([0, T]; H^{2.5}(\Omega))$. Similarly, from (A.18),

$$\|\operatorname{div}_\eta D\eta(t+h) - \operatorname{div}_\eta D\eta(t)\|_{2,5} \leq ChM_0,$$

so that $\operatorname{div}_\eta D\eta(t) \in C^0([0, T]; H^{2.5}(\Omega))$.

It follows that for each $l = 1, \dots, K$,

$$\begin{aligned} \operatorname{curl}_{\eta_l}(\zeta_l \bar{\partial}^2 \eta_l) &\in C^0([0, T]; H^{1.5}(B^+)), \\ \operatorname{div}_{\eta_l}(\zeta_l \bar{\partial}^2 \eta_l) &\in C^0([0, T]; H^{1.5}(B^+)), \\ \zeta_l \bar{\partial}^2 \eta_l \cdot n_l(\eta_l) &\in C^0([0, T]; H^2(B^0)). \end{aligned}$$

We let $w_l = \bar{\partial}^2 \eta_l \circ \eta_l^{-1}$ denote the Eulerian counterpart to $\bar{\partial}^2 \eta_l$, so that $w_l(\cdot, t) : \eta_l(B^+, t) \rightarrow \mathbb{R}^3$. Then, by the chain-rule, we see that, due to the continuity provided by (A.29),

$$\begin{aligned} \operatorname{curl} w_l &\in C^0([0, T]; H^{1.5}(\eta_l(B^+, t))), \\ \operatorname{div} w_l &\in C^0([0, T]; H^{1.5}(\eta_l(B^+, t))), \\ w_l \cdot n_l &\in C^0([0, T]; H^2(\eta_l(B^0, t))). \end{aligned}$$

We may then infer from Proposition A.1, that

$$w_l \in C^0([0, T]; H^{2.5}(\eta_l(B^+, t))),$$

with bound depending only on $\eta_l \in C^0([0, T]; H^{2.5}(B^+))$. It follows that for each $l = 1, \dots, K$, $\bar{\partial}^2 \eta_l \in C^0([0, T]; H^{2.5}(B^+))$. It follows that $\bar{\partial}^2 D\eta_l \in C^0([0, T]; H^{1.5}(B^+))$, and hence the trace satisfies $D\eta_l \in C^0([0, T]; H^3(B^0))$. Summing over $l = 1, \dots, K$, we see that

$$D\eta \in C^0([0, T]; H^3(\Gamma)).$$

Therefore, we have the following elliptic system:

$$\begin{aligned} \operatorname{curl}_{\eta}(D\eta) &\in C^0([0, T]; H^{2.5}(\Omega)), \\ \operatorname{div}_{\eta}(D\eta) &\in C^0([0, T]; H^{2.5}(\Omega)), \\ D\eta &\in C^0([0, T]; H^3(\Gamma)). \end{aligned}$$

Setting $W = D\eta \circ \eta^{-1}$, and using the fact that $\eta \in C^0([0, T]; H^4(\Omega))$ we see that

$$\begin{aligned} \operatorname{curl} W &\in C^0([0, T]; H^{2.5}(\Omega(t))), \\ \operatorname{div} W &\in C^0([0, T]; H^{2.5}(\Omega(t))), \\ W &\in C^0([0, T]; H^3(\Gamma(t))). \end{aligned}$$

Elliptic estimates then show that

$$W \in C^0([0, T]; H^{3.5}(\Omega(t)))$$

with a bound that depends on $\eta \in C^0([0, T]; H^4(\Omega))$ (but not on $\|\eta(t)\|_{4.5}$). In turn, $D\eta \in C^0([0, T]; H^{3.5}(\Omega))$, and hence

$$\eta \in C^0([0, T]; H^{4.5}(\Omega)).$$

Analogously, we find that $v \in C^0([0, T]; H^4(\Omega))$, which by elliptic estimates shows that $q \in C^0([0, T]; H^{4.5}(\Omega))$. The momentum equation then shows that $v_t \in C^0([0, T]; H^{3.5}(\Omega))$. \square

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