ON THE FINITISTIC GLOBAL DIMENSION CONJECTURE FOR ARTIN ALGEBRAS

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Abstract. We find a simple condition which implies finiteness of finitistic global dimension for artin algebras. As a consequence we obtain a short proof of the finitistic global dimension conjecture for radical cubed zero algebras. The same condition also holds for algebras of representation dimension less then or equal to three. Hence the finitistic dimension conjecture holds in that case as well.

Let Λ be an Artin algebra (an algebra of finite length over a commutative Artinian ring). Then the *finitistic global dimension conjecture* states that there exists a uniform bound called $findim\Lambda$ for the finite projective dimensions (pd) of all f.g. (left) Λ-modules of finite pd. This conjecture would imply the Nakayama conjecture. Some of the known cases in which the finitistic global dimension conjecture holds are the radical cubed zero case [GZ] and the monomial relation case [GKK] (see also [IZ], [BFGZ]). The conjecture is also true in the case when the category of modules of finite pd is contravariantly finite in the category of all f.g. modules [AR]. However, the converse is not true [IST]. In this paper we give a short proof of the finitistic gl dim conjecture for all modules of radical square zero over any Artin algebra. This is a generalization of the radical cubed zero case since all syzygies have radical square zero in that case. A thorough overview of the state of the finitistic global dimension conjecture can be found in [Z-H].

As another consequence of the main theorem we prove the finitistic dimension conjecture for algebras with weak representation dimension at most 3, and consequently for algebras with representation dimension $repdim\Lambda \leq 3$. The notion of representation dimension was introduced by M. Auslander in his Queen Mary Notes [A1], and he and many others expect this dimension to be bounded by 3. O. Iyama showed that it is always finite [I], many classes of algebras are known to have $repdim\Lambda = 3$, the most recent class being subalgebras of algebras of finite representation type with the same radical [EHIS].

The proof of the main theorem is based on the following well-known elementary observation.

Lemma 1 (Fitting's Lemma)**.** *a) Let M be a module over a Noetherian ring R and let $f : M \to M$ be an endomorphism of M. Then for any*

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f.g. submodule X of M there is an integer $\eta_f(X)$ so that f sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$ for all $m \geq \eta_f(X)$.

b) If Y is a submodule of X then $\eta_f(Y) \leq \eta_f(X)$.

c) If R is an Artin algebra and $X = M$ there is a direct sum decomposition $X = Y \oplus Z$ so that $Z = ker f^m$ and $Y = im f^m$ for all $m \geq \eta_f(X)$.

Let K_0 be the abelian group generated by all symbols $[M]$ where M is a f.g. Λ-module modulo the relations

a) $[C] = [A] + [B]$ if $C \approx A \oplus B$.

b) $[P] = 0$ if P is projective.

Then K_0 is the free abelian group generated by the isomorphism classes of indecomposable f.g. nonprojective Λ-modules. For any f.g. Λ-module M let $L[M] = [\Omega M]$ where ΩM is the first syzygy of M. Since Ω commutes with direct sum and takes projective modules to zero this gives a homomorphism $L : K_0 \to K_0$. For every f.g. Λ-module M let $\langle add M\rangle$ denote the subgroup of K_0 generated by all the indecomposable summands of M. Let

$$\phi(M) := \eta_L\langle add M\rangle.$$

Lemma 2. *a) If M has finite projective dimension then $\phi(M) = pdM$.*

b) If M is indecomposable with $pdM = \infty$ then $\phi(M) = 0$.

c) $\phi(A) \leq \phi(A \oplus B)$.

d) $\phi(kM) = \phi(M)$ if $k \geq 1$.

Proof. (c) follows from Lemma 1(b) since $\langle add A\rangle$ is a subgroup of $\langle add(A \oplus B)\rangle$. (d) follows from the fact that $add\ kM = add M$. (a) and (b) are easy exercises for the reader. □

We need one more definition. For any f.g. Λ-modules M let

$$\psi(M) := \phi(M) + \sup\{pdX \mid pdX < \infty, X \text{ direct summand of } \Omega^{\phi(M)}M\}.$$

Lemma 3. *a) $\psi(M) = \phi(M) = pdM$ whenever $pdM < \infty$.*

b) $\psi(kM) = \psi(M)$ if $k \geq 1$.

c) $\psi(A) \leq \psi(A \oplus B)$.

d) If Z is a summand of $\Omega^n M$ where $n \leq \phi(M)$ and $pdZ < \infty$ then $pdZ + n \leq \psi(M)$.

Proof. (a) is easy and (b) follows from Lemma 2(d). (c) follows from (d) in the case when $M = A \oplus B, n = \phi(A)$ and from Lemma 2(c).

To prove (d) let $k = \phi(M) - n$. Then $\Omega^k Z$ is a direct summand of $\Omega^{\phi(M)}M$ so $pd(\Omega^k Z) + \phi(M) \leq \psi(M)$ by definition of $\psi(M)$. But $p + n = (p - k) + \phi(M) \leq pd(\Omega^k Z) + \phi(M)$. □

Theorem 4. *Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of f.g. Λ-modules so that C has finite projective dimension. Then $pdC \leq \psi(A \oplus B) + 1$.*

Remark 5. This is a generalization of the well known formula $pdC \leq pd(A \oplus B) + 1$. Y. Wang [W] pointed out that $pdB \leq \psi(A \oplus \Omega C) + 1$.

Proof. Since C has finite pd we must have $\Omega^n A \approx \Omega^n B$ for some $n \geq 0$. Take n to be minimal so that $[\Omega^n A] = L^n[A] = L^n[B] = [\Omega^n B]$. Then $n \leq pdC$ and since $[A], [B] \in \langle add(A \oplus B)\rangle$ we also have $n \leq \phi(A \oplus B)$. The n-th syzygies of our short exact sequence then give an exact sequence of the form

$$0 \to X \oplus P \to X \oplus Q \to M \to 0$$

where P, Q are projective and $M = \Omega^n C$. Let $f : X \to X$ be the $X - X$ component of the map $X \oplus P \to X \oplus Q$, i.e. the composition $X \to X \oplus P \to X \oplus Q \to X$. Then by Lemma 1(c) $X = Y \oplus Z$ so that any sufficiently large iterate f^m of f has kernel Z and image Y. This means there is another exact sequence:

$$0 \to Z \oplus P \to Z \oplus Q \to M \to 0 \qquad (1)$$

where the $Z - Z$ component of the map $Z \oplus P \to Z \oplus Q$ is a nilpotent endomorphism g of Z.

We claim that Z has finite projective dimension. To see this suppose that $Ext^k(Z, S) \neq 0$ for some simple module S. Then the endomorphism of $Ext^k(Z, S)$ induced by g must be nilpotent and therefore cannot be surjective. By the exactness of the sequence:

$$Ext^k(Z \oplus Q, S) \to Ext^k(Z \oplus P, S) \to Ext^{k+1}(M, S)$$

we see that $Ext^{k+1}(M, S) \neq 0$.

Since Z is a direct summand of $\Omega^n(A), pdZ + n \leq \psi(A) \leq \psi(A \oplus B)$ by Lemma 3(c) and (d) and $pdM \leq pdZ + 1$ by the exactness of (1). Therefore $pdC = pdM + n \leq pdZ + n + 1 \leq \psi(A \oplus B) + 1$. □

Corollary 6. *If M is a f.g. Λ-module with Loewy length 2 and finite projective dimension then*

$$pdM \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2) + 1.$$

Proof. There is a short exact sequence $0 \to A \to P/rad^2 P \to M \to 0$ where P is the projective cover of M and A is semisimple and $\psi(A \oplus P/rad^2 P) \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2)$ by Lemma 3. □

Corollary 7. *Suppose that Λ is an artin algebra with $(rad\Lambda)^3 = 0$ then*

$$findim\Lambda \leq \psi(\Lambda/rad\Lambda \oplus \Lambda/(rad\Lambda)^2) + 2$$

where $findim\Lambda = sup\{pdM | pdM < \infty\}$.

Proof. ΩM has Loewy length 2 for any Λ-module M. □

The following Corollary gives another class of algebras for which the finitistic dimension conjecture holds.

Corollary 8. *Let $\Lambda = End_\Gamma(P)^{op}$, where P is a projective module over an artin algebra Γ with $gldim\Gamma \leq 3$. Then $findim\Lambda \leq \psi((P, \Gamma)) + 3$ where $(P, \Gamma) = Hom_\Gamma(P, \Gamma)$ is considered as a Λ-module.*

Proof. We use a construction due to Auslander [A2]. Any Λ-module X has a Λ-projective presentation $(P, P_1) \to (P, P_0) \to X \to 0$ with P_0, P_1 in $add(P)$. The associated sequence of projective Γ-modules $0 \to P_3 \to P_2 \to P_1 \to P_0$ induces an exact sequence of Λ-modules $0 \to (P, P_3) \to (P, P_2) \to (P, P_1) \to (P, P_0) \to X \to 0$. Hence $pdX \leq pd(\Omega^2 X) + 2 = pd(coker((P, P_3) \to (P, P_2))) + 2 \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$. □

Corollary 9. *If* $repdim\Lambda \leq 3$ *then* $findim\Lambda < \infty$.

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