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On the Finsler group and an almost symplectic structure on a tangent bundle

By

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In the preceding paper [5], the present author has found a Lie group F(n) which is called the Finsler group and has investigated a tangent bundle T(M) admitting an F(n)-structure in the sense of the theory of G-structure. Especially it has been shown that the base manifold M is a Finsler manifold if and only if T(M) admits an F(n)-structure satisfying a certain condition. Therefore, an F(n)-structure which is defined on T(M) as a reduction of the standard tangent structure has been called an almost Finsler structure. Moreover, in the case where a non-linear connection is assigned on T(M), the almost Finsler structure has been studied in detail. For example, it has been shown that any G-connection relative to the almost Finsler structure in the present case is nothing but a so-called linear connection of Finsler type whose induced Finsler connection is metrical.

In the present paper, first, we minutely study almost Finsler structures without the assignment of a non-linear connection, and find a necessary and sufficient condition for T(M) to admit an almost Finsler structure, which is expressed in terms of some quantities in the base manifold M.

Since the Finsler group F(n) is a subgroup of the symplectic group, T(M) admits an almost symplectic structure if it admits an almost Finsler structure. So, in this case, we can introduce a special 2-form on T(M). In §2, we are concerned with this 2-form and deal with the case where the 2-form is closed or has an integrating factor. The 2-form, anyway, plays an important role in the development of the theory of almost Finsler structures.

Lastly, §3 is devoted to consideration on almost Hamilton vectors with respect to the almost symplectic structure derived from the almost Finsler structure. In the case of Finsler manifolds, Hamilton vectors and Hamilton systems are treated and Hamilton functions are shown concretely.

Throughout the paper, we use the following indices and notation:

$A, B, C, \dots, P, Q, R, \dots$ run over the range $\{1, 2, 3, \dots, 2n\}$;
$a, b, c, \dots, i, j, k, \dots$ run over the range $\{1, 2, 3, \dots, n\}$;
$\bar{a}, \bar{b}, \dots, \bar{i}, \bar{j}, \dots$ stand for $a+n, b+n, \cdots, i+n, j+n, \cdots$ respectively;
With respect to any canonical coordinate system in a tangent bundle,

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$(x^A)=(x^a, x^{\bar{a}})=(x^a, y^a)$, i. e., $x^{\bar{a}}=y^a$, and the notation ∂_i and $\dot{\partial}_i$ stand for $\partial/\partial x^i$ and $\partial/\partial y^i$ respectively.

§1. The homogeneous almost Finsler structure $\mathscr{F}^*$.

Let M be an n-dimensional differentiable manifold, $\{(U, x^i)\}$ be a system of local coordinate neighbourhoods which covers M. Then, the tangent bundle T(M) over M is covered by the system of canonical local coordinate neighbourhoods $\{(\pi^{-1}(U), (x^i, y^i))\}$ where π is the natural projection $T(M) \longrightarrow M$.

As is well-known, T(M) admits the standard tangent structure $\mathscr{F}_0$ and its structure tensor Q is given by $Q=\begin{pmatrix} 0 & 0 \\ E_n & 0 \end{pmatrix}$ with respect to the canonical local coordinate (x^i, y^i) ([1], [2]).

In the preceding paper [5] the Finsler group is introduced as a linear Lie group such that

$$F(n)=\left\{\begin{pmatrix} A & 0 \\ SA & A \end{pmatrix} \middle| A \in O(n),\ S \in \mathrm{Symm}(n)\right\}.$$

And, if T(M) admits an F(n)-structure as a reduction of $\mathscr{F}_0$, i. e., T(M) admits an F(n)-structure depending on $\mathscr{F}_0$, the structure is called an almost Finsler structure and is denoted by $\mathscr{F}$. If T(M) admits an almost Finsler structure satisfying the homogeneity condition, the structure is called a homogeneous almost Finsler structure and is denoted by $\mathscr{F}^*$. The condition for T(M) to admit the structure $\mathscr{F}^*$ is given by the following:

(1) T(M) admits an F(n)-structure in the sense of the theory of G-structures ([3], [6], [14]), i. e., in any two canonical local coordinate neighbourhoods $(\pi^{-1}(U), (x^i, y^i))$ and $(\pi^{-1}(\bar{U}), (\bar{x}^i, \bar{y}^i))$, there exist adapted 2n-frames $\{Z_A\}$ and $\{\bar{Z}_A\}$ respectively which satisfy the condition $\bar{Z}_A=P_A^B Z_B$ $((P_A^B \in F(n))$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$ if $\pi^{-1}(U) \cap \pi^{-1}(\bar{U}) \neq \phi$.

(2) In each $\pi^{-1}(U)$, the adapted frame $\{Z_A\}=\{\bar{Z}_a, \bar{Z}_{\bar{a}}\}$ has the following components

$$Z_a=\gamma_a^i \partial/\partial x^i+\gamma_a^{\bar{i}} \partial/\partial y^i, \quad Z_{\bar{a}}=\gamma_a^i \partial/\partial y^i,$$

where $\det|\gamma_a^i| \neq 0$ and γ_a^i is positively homogeneous of degree 0 for y^i and $\gamma_a^{\bar{i}}$ is positively homogeneous of degree 1 for y^i.

In the present paper we mainly treat the homogeneous almost Finsler structure $\mathscr{F}^*$.

As to $J=\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$, ${}^tPJP=J$ holds for any $P \in F(n)$. So, the Finsler group F(n) is a subgroup of the symplectic group Sp(n). Hence, if T(M) admits a homogeneous almost Finsler structure $\mathscr{F}^*$, T(M) also admits an almost symplectic structure, i.e., T(M) admits a non-degenerate 2-form ([3], [7], [9]). By $\Omega=\omega_{AB} dx^A \wedge dx^B$ $(\omega_{BA}=-\omega_{AB})$ we denote the 2-form and call it an almost Finsler 2-form associated with $\mathscr{F}^*$.

Now we put

$$\begin{pmatrix}\gamma^i_a & 0\\ \gamma^{\bar{i}}_a & \gamma^i_{\bar{a}}\end{pmatrix}^{-1}=\begin{pmatrix}\beta^a_i & 0\\ \beta^{\bar{a}}_i & \beta^{\bar{a}}_{\bar{i}}\end{pmatrix},\quad J=(J_{PQ})=\begin{pmatrix}0 & -\delta_{ij}\\ \delta_{ij} & 0\end{pmatrix},$$

$$g_{ij}=\sum_{a=1}^{n}\beta^a_i\beta^a_j,\quad \alpha_{ij}=\sum_{a=1}^{n}(\beta^a_i\beta^{\bar{a}}_j-\beta^{\bar{a}}_i\beta^a_j),$$

then g_{ij} and β^a_i are positively homogeneous of degree 0 for y^i, and α_{ij} and $\beta^{\bar{a}}_i$ are positively homogeneous of degree 1 for y^i. Since $\omega_{AB}=J_{PQ}\beta^P_A\beta^Q_B$, we have

$$\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j, \tag{1.1}$$

i. e.,
$$(\omega_{AB})=\begin{pmatrix}\alpha_{ij} & -g_{ij}\\ g_{ij} & 0\end{pmatrix}.$$

In this case, we can define a singular inner product of rank n by

$$\langle Z_a,\ Z_b\rangle=\delta_{ab},\ \langle Z_a,\ Z_{\bar{b}}\rangle=0,\ \langle Z_{\bar{a}},\ Z_{\bar{b}}\rangle=0.$$

Due to the property of the Finsler group F(n), we can easily verify that this is the globally defined one on T(M). Moreover we have

$$\langle\partial/\partial x^i, \partial/\partial x^j\rangle=g_{ij},\ \langle\partial/\partial x^i,\ \partial/\partial y^j\rangle=0,\ \langle\partial/\partial y^i,\ \partial/\partial y^j\rangle=0,$$

namely, $g_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$ is a singular Riemann metric of rank n valid on T(M). And g_{ij} is nothing but a generalized metric defined on M in the sense of Moór [10].

Next, ω_{AB} is a skew-symmetric tensor field on T(M). So, after direct calculation, we can see, in each $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$ where $U\cap\bar{U}\neq\phi$, the following transformation rules of g_{ij} and α_{ij} hold:

$$\begin{cases} g_{ij}=\bar{g}_{pq}\dfrac{\partial\bar{x}^p}{\partial x^i}\dfrac{\partial\bar{x}^q}{\partial x^j},\\ \alpha_{ij}=\bar{\alpha}_{pq}\dfrac{\partial\bar{x}^p}{\partial x^i}\dfrac{\partial\bar{x}^q}{\partial x^j}-\bar{g}_{pq}\dfrac{\partial\bar{x}^p}{\partial x^i}\dfrac{\partial^2\bar{x}^q}{\partial x^j\partial x^m}y^m+\bar{g}_{pq}\dfrac{\partial^2\bar{x}^p}{\partial x^i\partial x^m}y^m\dfrac{\partial\bar{x}^q}{\partial x^j}.\end{cases} \tag{1.2}$$

Thus we obtain

Theorem 1. *If a tangent bundle $T(M)$ admits a homogeneous almost Finsler structure, then $T(M)$ admits an almost symplectic structure whose associated 2-form is given by $\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j$. Here, α_{ij} is a quantity such that $\alpha_{ji}=-\alpha_{ij}$ and is positively homogeneous of degree* 1 *for y^i, g_{ij} is a generalized metric of M, and the transformation rules of α_{ij} and g_{ij} are given by* (1.2).

Now, let $\overset{\circ}{N}$ be a non-linear connection defined on T(M) ([6]. [8]) and $\overset{\circ}{N}{}^i_j$ be the components of $\overset{\circ}{N}$ with respect to the caninical local coordinate (x^i, y^i). Then $\overset{\circ}{N}{}^i_j$ satisfies the transformation rule

$$\frac{\partial\bar{x}^p}{\partial x^m}\overset{\circ}{N}{}^m_j-\overset{\circ}{\bar{N}}{}^p_q\frac{\partial\bar{x}^q}{\partial x^j}=\frac{\partial^2\bar{x}^p}{\partial x^j\partial x^m}y^m.$$

By using this equation, we can show easily that $\beta_{ij}=\alpha_{ij}+g_{im}\overset{\circ}{N}{}^m_j-g_{jm}\overset{\circ}{N}{}^m_i$ is a skew-symmetric quasi tensor on M [6] and is positively homogeneous of degree

1 for y^i. Hence, if we put $N^i_j = \mathring{N}^i_j - \frac{1}{2} g^{im}\beta_{mj}$, then we can show directly that N^i_j gives T(M) a non-linear connection and satisfies $\alpha_{ij} = -g_{im}N^m_j + g_{jm}N^m_i$. Thus we obtain

Theorem 2. *Let $g = (g_{ij})$ and $\alpha = (\alpha_{ij})$ be the quantities defined in Theorem* 1. *On $T(M)$, there always exists a non-linear connection N satisfying the condition*

$$\alpha = -gN + {}^tNg. \tag{1.3}$$

Next, let N and $\tilde{N}$ be any two non-linear connections satisfying the condition (1. 3). Then $\tilde{N} - N$ is a (1, 1) quasi tensor field on M and is positively homogeneous of degree 1 for y^i. Now, $k = (k_{ij}) = g(\tilde{N} - N)$ is a (0, 2) quasi tensor field on M and is positively homogeneous of degree 1 for y^i and satisfies

$$^tk = ({}^t\tilde{N} - {}^tN)g = (\alpha + g\tilde{N}) - (\alpha + gN) = k.$$

That is, k is a symmetric quasi tensor field.

Conversely, let N be a non-linear connection shown in Theorem 2 and k be any symmetric (0, 2) quasi tensor field and be positively homogeneous of degree 1 for y^i. Then $\tilde{N} = N + g^{-1}k$ satisfies

$$-g\tilde{N} + {}^t\tilde{N}g = -gN + {}^tNg = \alpha.$$

Thus we obtain

Theorem 3. *In a tangent bundle admitting a homogeneous almost Finsler structure, let N be a non-linear connection satisfying the condition* (1. 3). *If $\tilde{N}$ is another non-linear connection satisfying the condition* (1. 3), *then $\tilde{N}$ is written as $\tilde{N} = N + g^{-1}k$ where k is a* (0, 2) *symmetric quasi tensor field on M and is positively homogeneous of degree* 1 *for y^i. And the converse is also true.*

Now let us consider the converse of Theorem 1. That is to say, we assume that a manifold M admits a generalized metric g and a skew-symmetric quantity $\alpha = (\alpha_{ij})$ which is positively homogeneous of degree 1 for y^i and satisfies the transformation rule (1. 2). In this case, $\Omega = \alpha_{ij}dx^i \wedge dx^j - 2g_{ij}dx^i \wedge dy^j$ is a globally defined non-degenerate 2-form on T(M). First, we consider a local coordinate neighbourhood (U, x^i). With respect to the generalized metric g, it is easy to find, in U, n linearly independent local covariant quasi vectors σ^a_i such that $g_{ij} = \sum_{a=1}^{n} \sigma^a_i \sigma^a_j$. That is, $g = {}^t\sigma\sigma$ where $\sigma = (\sigma^a_i)$. Now, we put $\tau = (\tau^i_a) = \sigma^{-1}$. Of course, σ^a_i and τ^i_a are positively homogeneous of degree 0 for y^i. Let N be a non-linear connection shown in Theorem 2, i. e., N satisfies $\alpha = -gN + {}^tNg$. Then we can define, on $\pi^{-1}(U)$, local 2n-frame $\{Z_A\}$ by

$$Z_a = \tau^i_a(\partial/\partial x^i - N^m_i \partial/\partial y^m), \quad Z_{\bar{a}} = \tau^i_a \partial/\partial y^i.$$

The quantities σ, τ, N and $\{Z_A\}$ always exist on $\pi^{-1}(U)$. However, they can not be determined uniquely. Next, let $(\bar{U}, \bar{x}^i)$ be another local coordinate neighbourhood such that $U\cap\bar{U}\neq\phi$. Then, on $(\pi^{-1}(\bar{U}),\ (\bar{x}^i,\ \bar{y}^i))$, we can define similarly σ, τ, N and $\{Z_A\}$, which we denote by $\bar{\lambda}$, $\bar{\mu}$, $\bar{\tilde{N}}$ and $\{\tilde{Z}_A\}$ respectively. Now, on $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, we can consider these quantities in terms of the local canonical coordinate system $(x^i,\ y^i)$, which we denote by λ, μ, $\tilde{N}$ and $\{\tilde{Z}_A\}$ respectively. Then, we see

$$\tilde{Z}_a=\mu^i_a(\partial/\partial x^i-\tilde{N}^m_i\partial/\partial y^m),\ \ \tilde{Z}_{\bar{a}}=\mu^i_a\partial/\partial y^i.$$

Now, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, $\{Z_A\}$ and $\{\tilde{Z}_A\}$ have, of course, the relation

$$\tilde{Z}_A=P^B_AZ_B \text{ where } (P^B_A)=\begin{pmatrix}P^b_a & P^b_{\bar{a}}\\ P^{\bar{b}}_a & P^{\bar{b}}_{\bar{a}}\end{pmatrix}\in GL(2n,\ R).$$

First, $Z_{\bar{a}}=P^m_{\bar{a}}Z_m+P^{\bar{m}}_{\bar{a}}Z_{\bar{m}}$ can be rewritten as

$$\mu^i_a\partial/\partial y^i=P^m_{\bar{a}}\tau^i_m(\partial/\partial x^i-N^r_i\partial/\partial y^r)+P^{\bar{m}}_{\bar{a}}\tau^i_m\partial/\partial y^i.$$

Hence we have $P^m_{\bar{a}}=0$ and $P^{\bar{m}}_{\bar{a}}=\sigma^m_r\mu^r_a$. Secondly, $\tilde{Z}_a=P^m_aZ_m+P^{\bar{m}}_aZ_{\bar{m}}$ can be rewritten as

$$\mu^i_a(\partial/\partial x^i-\tilde{N}^m_i\partial/\partial y^m)=P^m_a\tau^i_m(\partial/\partial x^i-N^r_i\partial/\partial y^r)+P^{\bar{m}}_a\tau^i_m\partial/\partial y^i.$$

Hence we have $P^m_a=\sigma^m_r\mu^r_a$ and $P^{\bar{m}}_a=\sigma^m_rN^r_t\tau^t_sP^s_a-\sigma^m_r\tilde{N}^r_l\mu^l_a$. Putting $A=(P^m_a)$, we see

$${}^tAA={}^t(\sigma\mu)(\sigma\mu)={}^t\mu g\mu={}^t\mu{}^t\lambda\lambda\mu={}^t(\lambda\mu)(\lambda\mu)=E_n,$$

i. e., $A\in O(n)$. Next, putting $B=(P^{\bar{m}}_a)$, we see, by virtue of Theorem 3,

$$\begin{aligned}B&=\sigma N\tau A-\sigma\tilde{N}\mu=\sigma N\tau A-\sigma N\tau A-\sigma g^{-1}k\tau A\\&=-\sigma\tau{}^t\tau k\tau A=-{}^t\tau k\tau A\end{aligned}$$

where k is a symmetric matrix. So, putting $S=-{}^t\tau k\tau$, we have ${}^tS=S$, i. e., $S\in\mathrm{Symm}(n)$. Thus we get $(P^A_B)=\begin{pmatrix}A & 0\\ SA & A\end{pmatrix}$ where $A\in O(n)$ and $S\in\mathrm{Symm}(n)$. That is to say, $(P^A_B)\in\mathrm{F(n)}$. And, for the relations

$$Z_a=\tau^i_a\partial/\partial x^i-\tau^i_aN^m_i\partial/\partial y^m \quad\text{and}\quad Z_{\bar{a}}=\tau^i_a\partial/\partial y^i,$$

we have seen already that det $|\tau^i_a|\neq0$ and τ^i_a is positively homogeneous of degree 0 for y^i, and $\tau^i_aN^m_i$ is positively homogeneous of degree 1 for y^i. Moreover, ${}^t(\tau^{-1})\tau^{-1}={}^t\sigma\sigma=g$ and $\begin{pmatrix}\tau & 0\\ -N\tau & \tau\end{pmatrix}^{-1}=\begin{pmatrix}\sigma & 0\\ \sigma N & \sigma\end{pmatrix}=\begin{pmatrix}\sigma^a_i & \sigma^a_{\bar{i}}\\ \sigma^{\bar{a}}_i & \sigma^{\bar{a}}_{\bar{i}}\end{pmatrix}$. So, we have

$$\begin{aligned}\sum_{a=1}^n(\sigma^{\bar{a}}_i\sigma^a_j-\sigma^a_i\sigma^{\bar{a}}_j)&=\sum_{a=1}^n(\sigma^a_mN^m_i\sigma^a_j-\sigma^a_i\sigma^a_mN^m_j)\\&=g_{mj}N^m_i-g_{im}N^m_j=\alpha_{ij}.\end{aligned}$$

Thus, as the converse of Theorem 1, we obtain

Theorem 4. *Assume that a manifold M admits a generalized metric* g_{ij} *and a* *skew-symmetric quantity α_{ij} which is positively homogeneous of degree 1 for y^i and satisfies the transformation rule* (1. 2). *Then $T(M)$ admits a homogeneous almost Finsler structure whose associated almost Finsler 2-form is given by*

$$\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j.$$

§2. Finsler structures.

Let T(M) be a tangent bundle admitting a homogeneous almost Finsler structure. Applying the exterior differentiation d to the almost Finsler 2-form $\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j$, we get

$$\mathrm{d}\Omega=\partial_k\alpha_{ij}\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j+(\dot{\partial}_k\alpha_{ij}+2\partial_j g_{ik})\mathrm{d}y^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j$$
$$-2\dot{\partial}_k g_{ij}\mathrm{d}y^k\wedge\mathrm{d}x^i\wedge\mathrm{d}y^j.$$

So, the condition for Ω to be closed can be written as

$$\begin{cases}\dot{\partial}_k g_{ij}-\dot{\partial}_j g_{ik}=0,\\ \dot{\partial}_k\alpha_{ij}+2\partial_j g_{ik}-\dot{\partial}_k\alpha_{ji}-2\partial_i g_{jk}=0,\\ \partial_k\alpha_{ij}+\partial_i\alpha_{jk}+\partial_j\alpha_{ki}=0.\end{cases}$$

The first condition means that g_{ij} is a Finsler metric [10]. The second condition leads us to $\dot{\partial}_k\alpha_{ij}=\partial_i g_{jk}-\partial_j g_{ik}$. Since α_{ij} is positively homogeneous of degree 1 for y^i, we obtain

$$\alpha_{ij}=y^m(\partial_i g_{jm}-\partial_j g_{im}). \tag{2.1}$$

Conversely, let g_{ij} be a Finsler metric and α_{ij} be the quantity given by (2.1). From the well-known equation $y^m\dot{\partial}_k g_{im}=0$, we get $\dot{\partial}_k\alpha_{ij}=\partial_i g_{jk}-\partial_j g_{ik}$. Hence, the second condition is clearly satisfied. In this case, moreover, we see

$$\begin{aligned}&\partial_k\alpha_{ij}+\partial_i\alpha_{jk}+\partial_j\alpha_{ki}\\ &=y^m(\partial_k\partial_i g_{jm}-\partial_k\partial_j g_{im}+\partial_i\partial_j g_{km}-\partial_i\partial_k g_{jm}+\partial_j\partial_k g_{im}-\partial_j\partial_i g_{km})\\ &=0.\end{aligned}$$

That is, the third condition is also satisfied. Thus we obtain

Theorem 5. *Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure. The almost Finsler 2-form $\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j$ is closed if and only if g_{ij} is a Finsler metric and α_{ij} is given by $\alpha_{ij}=y^m(\partial_i g_{jm}-\partial_j g_{im})$.*

In the case of Theorem 5, we have

$$\begin{aligned}\Omega&=y^m(\partial_i g_{jm}-\partial_j g_{im})\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j\\ &=\mathrm{d}(2y^m g_{mj}\mathrm{d}x^j).\end{aligned}$$

That is, Ω is the well-known exact form [14]. In the paper [5], we have called this Ω the Finsler form associated with a Finsler metric and denote it by Ω^*.

Since Ω^* is determined by a Finsler metric only, it seems to us that Theorem 5 tells us a new definition and a new treatment of a Finsler manifold.

Next, let there be given a scalar field $\sigma(x, y)$ on T(M), which is positively homogeneous of degree 0 for y^i. If T(M) admits a homogeneous almost Finsler structure whose associated 2-form is given by $\Omega=\alpha_{ij}dx^i\wedge dx^j-2g_{ij}dx^i\wedge dy^j$, then T(M) also admits another 2-form $\tilde{\Omega}=e^{\sigma(x,y)}\Omega$. Putting $\tilde{g}_{ij}=e^{\sigma(x,y)}g_{ij}$ and $\tilde{\alpha}_{ij}=e^{\sigma(x,y)}\alpha_{ij}$ we have $\tilde{\Omega}=\tilde{\alpha}_{ij}dx^i\wedge dx^j-2\tilde{g}_{ij}dx^i\wedge dy^j$. Of course, $\tilde{g}_{ij}$ is a generalized metric. With respect to $\tilde{\alpha}_{ij}$, it is easy to verify

$$\tilde{\alpha}_{ij}=\bar{\tilde{\alpha}}_{pq}\frac{\partial\bar{x}^p}{\partial x^i}\frac{\partial\bar{x}^q}{\partial x^j}-\bar{\tilde{g}}_{pq}\frac{\partial\bar{x}^p}{\partial x^i}\frac{\partial^2\bar{x}^q}{\partial x^j\partial x^m}y^m+\bar{\tilde{g}}_{pq}\frac{\partial^2\bar{x}^p}{\partial x^i\partial x^m}y^m\frac{\partial\bar{x}^q}{\partial x^j}.$$

Thus T(M) admits another homogeneous almost Finsler structure whose associated 2-form is $\tilde{\Omega}$ itself. The condition for $\tilde{\Omega}$ to be closed is given by

(1) $\tilde{g}_{ij}$ is a Finsler metric (2) $\tilde{\alpha}_{ij}=y^m(\partial_i\tilde{g}_{jm}-\partial_j\tilde{g}_{im})$.

The condition (1) implies that the generalized metric g_{ij} is conformal to a Finsler metric. From the condition (2), we have

$$\alpha_{ij}=y^m(\partial_i g_{jm}-\partial_j g_{im})+\partial_i\sigma g_{jm}y^m-\partial_j\sigma g_{im}y^m. \tag{2.2}$$

Conversely, let $\Omega=\alpha_{ij}dx^i\wedge dx^j-2g_{ij}dx^i\wedge dy^j$ be a 2-form on T(M). If there exists such a scalar field $\sigma=\sigma(x, y)$ that $\sigma(x, y)$ is positively homogeneous of degree 0 for y^i, $e^\sigma g_{ij}$ is a Finsler metric and the relation (2.2) holds, then $e^\sigma\alpha_{ij}=y^m\{\partial_i(e^\sigma g_{jm})-\partial_j(e^\sigma g_{im})\}$ holds good and $e^\sigma\Omega$ becomes closed. Thus we obtain

Theorem 6. *Let $\Omega=\alpha_{ij}dx^i\wedge dx^j-2g_{ij}dx^i\wedge dy^j$ be the almost Finsler form associated with a homogeneous almost Finsler structure defined on a tangent bundle $T(M)$. Let $\sigma=\sigma(x, y)$ be a scalar field on $T(M)$ which is positively homogeneous of degree 0 for y^i. In order that $e^\sigma\Omega$ be closed, it is necessary and sufficient that $e^\sigma g_{ij}$ is a Finsler metric and the relation (2.2) holds good.*

Let g be a Finsler metric, Ω^* be the Finsler form associated with g, and $\sigma=\sigma(x)$ be a scalar field on M. Then $\tilde{g}=e^{\sigma(x)}g$ is a Finsler metric. So, let $\tilde{\Omega}^*$ be the Finsler form associated with $\tilde{g}$. Then we have $\tilde{\Omega}^*=e^{\sigma(x)}\Omega^*+e^{\sigma(x)}y^m(\partial_i\sigma g_{jm}-\partial_j\sigma g_{im})$. Therefore, the condition $\tilde{\Omega}^*=e^{\sigma(x)}\Omega^*$ is written as $\partial_i\sigma g_{jm}-\partial_j\sigma g_{im}=0$. Applying the differentiation $\dot{\partial}_k$ and multiplying g^{jk} to this equation, we have $\partial_i\sigma=0$, i. e., σ is constant. Conversely, if σ is constant, it is evident that $\tilde{\Omega}^*=e^\sigma\Omega^*$. Thus we obtain

Theorem 7. *Let g and $\tilde{g}$ be Finsler metrics defined on M and be conformal to each other, namely, $\tilde{g}=e^{\sigma(x)}g$. Let Ω^* and $\tilde{\Omega}^*$ be the Finsler forms associated with g and $\tilde{g}$ respectively. Then $\tilde{\Omega}^*=e^{\sigma(x)}\Omega^*$ holds true if and only if $\tilde{g}$ is homothetic to g.*

§3. Hamilton vector fields in T(M).

Let V be a vector field in T(M) and Q be the standard tangent structure tensor. With respect to a local canonical coordinate, V and Q are written as

$V=v^i(x, y)\partial/\partial x^i+v^{\bar{i}}(x, y)\partial/\partial y^i$ and $Q=(Q^A_B)=\begin{pmatrix}0 & 0\\ \delta^i_j & 0\end{pmatrix}$. Now, calculating the Lie derivation $\mathcal{L}_V Q$, we have

$$\mathcal{L}_V Q^i_j=-\dot{\partial}_j v^i,\quad \mathcal{L}_V Q^i_{\bar{j}}=0,$$

$$\mathcal{L}_V Q^{\bar{i}}_j=-\dot{\partial}_j v^{\bar{i}}+\partial_j v^i,\quad \mathcal{L}_V Q^{\bar{i}}_{\bar{j}}=\dot{\partial}_j v^i.$$

Therefore, if $\mathcal{L}_V Q=0$ holds, V must take the form

$$V=v^i(x)\partial/\partial x^i+(y^m\partial_m v^i(x)+u^i(x))\partial/\partial y^i.$$

And the converse is also true. Here, $v^i(x)\partial/\partial x^i+y^m\partial_m v^i(x)\partial/\partial y^i$ is called the complete lift of a vector field $v(x)=v^i(x)\partial/\partial x^i$ to the tangent bundle T(M) and is denoted by $(v(x))^C$, and $u^i(x)\partial/\partial y^i$ is called the vertical lift of a vector field $u(x)=u^i(x)\partial/\partial x^i$ to T(M) and is denoted by $(u(x))^V$ ([6], [14]). Hence we obtain

Theorem 8. *Let V be a vector field in a tangent bundle $T(M)$ and Q be the standard tangent structure tensor of $T(M)$. $\mathcal{L}_V Q=0$ holds good if and only if $V=(v(x))^C+(u(x))^V$ where $(v(x))^C$ is the complete lift of a vector field $v(x)$ in M and $(u(x))^V$ is the vertical lift of a vector field $u(x)$ in M.*

Now, we suppose that the tangent bundle T(M) admits a homogeneous almost Finsler structure $\mathcal{F}^*$. Let V be a vector field in T(M). In what follows, we consider the case where the local 1-parameter group of local transformations generated by V preserves the structure $\mathcal{F}^*$. The condition to be demanded is written as $\mathcal{L}_V Q=0$ and $\mathcal{L}_V \Omega=0$. By virtue of Theorem 8, it is enough to consider the two cases where V is the complete lift or V is the vertical lift of a vector field in the base manifold M.

First, we consider the case where V is the complete lift of a vector field $v(x)$ in M. Now, let us calculate $\mathcal{L}_{v^C}\omega_{AB}=0$ for $(\omega_{AB})=\begin{pmatrix}\omega_{ij} & \omega_{i\bar{j}}\\ \omega_{\bar{i}j} & \omega_{\bar{i}\bar{j}}\end{pmatrix}=\begin{pmatrix}\alpha_{ij} & -g_{ij}\\ g_{ij} & 0\end{pmatrix}$. Using the relations

$$\mathcal{L}_V\omega_{AB}=V^D\frac{\partial\omega_{AB}}{\partial x^D}+\frac{\partial V^D}{\partial x^A}\omega_{DB}+\omega_{AD}\frac{\partial V^D}{\partial x^B}$$

and $v^C=v^i(x)\partial/\partial x^i+y^m\partial_m v^i(x)\partial/\partial y^i$, after some calculation, we get

$$\mathcal{L}_{v^C}\omega_{\bar{i}\bar{j}}=0,$$

$$\mathcal{L}_{v^C}\omega_{\bar{i}j}=v^h\partial_h g_{ij}+y^m\partial_m v^h\dot{\partial}_h g_{ij}+\partial_i v^h g_{hj}+g_{ih}\partial_j v^h=\mathcal{L}_v g_{ij},$$

$$\mathcal{L}_{v^C}\omega_{ij}=v^h\partial_h\alpha_{ij}+y^m\partial_m v^h\dot{\partial}_h\alpha_{ij}+\partial_i v^h\alpha_{hj}+\alpha_{ih}\partial_j v^h$$
$$+g_{jh}\frac{\partial^2 v^h}{\partial x^i\partial x^m}y^m-g_{ih}\frac{\partial^2 v^h}{\partial x^j\partial x^m}y^m,$$

where $\mathcal{L}_v g_{ij}$ is the well-known formula of the Lie derivative of the generalized metric g_{ij} [13].

As is well-known ([4], [11], [12]), in a manifold admitting a symplectic structure whose associated 2-form is Ω, a vector field V satisfying $\mathcal{L}_V\Omega=0$ is called a Hamilton vector. And similarly, in a manifold admitting an almost symplectic structure whose associated 2-form is Ω, a vector field V satisfying $\mathcal{L}_V\Omega=0$ is said to be an almost Hamilton vector. Now we obtain

Theorem 9. *Let $T(M)$ be a tangent bundle admitting a homogeneous almost Finsler structure $\mathcal{F}^*$, let $\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j$ be the almost Finsler form associated with $\mathcal{F}^*$, and let $v=v^i(x)\partial/\partial x^i$ be a vector field in the base manifold M. Then, the complete lift of v is an almost Hamilton vector of $\mathcal{F}^*$ if and only if*
(1) *v is a Killing vector field of the generalized metric g_{ij},*

$$(2)\quad v^h\partial_h\alpha_{ij}+y^m\partial_m v^h\dot{\partial}_h\alpha_{ij}+\partial_i v^h\alpha_{hj}+\alpha_{ih}\partial_j v^h+g_{jh}\frac{\partial^2 v^h}{\partial x^i\partial x^m}y^m-g_{ih}\frac{\partial^2 v^h}{\partial x^j\partial x^m}y^m=0$$

hold good.

In the case where $d\Omega=0$, i. e., g_{ij} is a Finsler metric and $\alpha_{ij}=y^m(\partial_i g_{jm}-\partial_j g_{im})$, the left hand side of the condition (2) of Theorem 9 can be rewritten, after some calculation due to $y^m\dot{\partial}_h g_{im}=0$, as

$$v^h y^m\frac{\partial^2 g_{jm}}{\partial x^h\partial x^i}-v^h y^m\frac{\partial^2 g_{im}}{\partial x^h\partial x^j}+y^m\partial_m v^h\partial_i g_{jh}-y^m\partial_m v^h\partial_j g_{ih}$$
$$+y^m\partial_i v^h\partial_h g_{jm}-y^m\partial_i v^h\partial_j g_{hm}+y^m\partial_j v^h\partial_i g_{hm}-y^m\partial_j v^h\partial_h g_{im}$$
$$+y^m g_{jh}\frac{\partial^2 v^h}{\partial x^i\partial x^m}-y^m g_{ih}\frac{\partial^2 v^h}{\partial x^j\partial x^m}.$$

Thus we can rewrite the condition (2) as

$$\partial_i(y^m\mathcal{L}_v g_{jm})-\partial_j(y^m\mathcal{L}_v g_{im})=0.$$

Therefore we obtain

Theorem 10. *Let g be a Finsler metric of a manifold M, $v=v^i(x)\partial/\partial x^i$ be a vector field in M and $\mathcal{F}^*$ be the symplectic structure on $T(M)$ derived from $\Omega^*=\mathrm{d}(2y^m g_{mj}\mathrm{d}x^j)$. Then v^C is a Hamilton vector of $\mathcal{F}^*$ if and only if v is a Killing vector of the Finsler metric g.*

It is well-known ([4], [11]) that, for any p-form, the relation $\mathcal{L}_V=\mathrm{i}_V\mathrm{d}+\mathrm{d}\mathrm{i}_V$ holds good where i_V is the interior product by V and d is the exterior differential operator. If v is a Killing vector field of a Finsler metric g, then we have $\mathcal{L}_{v^C}\Omega^*=0$. Of course, $\mathrm{d}\Omega^*=0$ holds. So, we have $\mathrm{di}_{v^C}\Omega^*=0$. That is, the so-called Hamilton system $\mu=\omega_{BA}(v^C)^B dx^A$ is closed. Putting $H_A=\omega_{AB}(v^C)^B$, we have

$$H_i=y^r\{(\partial_m g_{ir}-\partial_i g_{mr})v^m+g_{mi}\partial_r v^m\},\ H_{\bar{i}}=-g_{mi}v^m.$$

The equation $y^r\mathcal{L}_v g_{ir}=0$ leads us to $H_i=-y^r v^m\partial_i g_{mr}-y^r\partial_i v^m g_{mr}$. Then we have $\mu=-\mathrm{d}(g_{mr}y^r v^m)$. That is, μ is an exact form and $H=g_{mr}y^r v^m$ is a Hamilton function of $\mathcal{F}^*$ ([4], [11], [12]). Thus we obtain

Theorem 11. *Suppose that a manifold M admits a Finsler metric g and a Killing* *vector field* $v=v^i(x)\partial/\partial x^i$ *of* g. *Concerning the symplectic structure* $\mathscr{F}^*$ *derived from* $\Omega^*=\mathrm{d}(2y^m g_{mj}\mathrm{d}x^j)$, $H=g_{mr}y^r v^m$ *is the Hamilton function with respect to the Hamilton vector* v^C *in* $T(M)$.

In the case of Theorem 11, the so-called Hamilton equation is written as

$$\begin{cases} \dfrac{dx^i}{dt}(=v^i)=g^{im}\dfrac{\partial H}{\partial y^m}, \\ \dfrac{dy^i}{dt}(=y^m\partial_m v^i)=-g^{im}\dfrac{\partial H}{\partial x^m}+y^p(\partial_h g_{mp}-\partial_m g_{hp})g^{ih}g^{mr}\dfrac{\partial H}{\partial y^r}. \end{cases}$$

It is a matter of course that the Hamilton function is constant along the integral curve of the Hamilton vector v^C.

Next, we consider the case where $V=u^V$, u being a vector field on M. Calculating $\mathscr{L}_{u^V}\,\omega_{AB}$, we have

$$\mathscr{L}_{u^V}\omega_{ij}=u^m\dot{\partial}_m\alpha_{ij}+g_{mj}\partial_i u^m-g_{im}\partial_j u^m,$$

$$\mathscr{L}_{u^V}\omega_{i\bar{j}}=u^m\dot{\partial}_m g_{ij},$$

$$\mathscr{L}_{u^V}\omega_{\bar{i}\bar{j}}=0.$$

Thus we obtain

Theorem 12. *Let* $T(M)$ *be a tangent bundle admitting a homogeneous almost Finsler structure* $\mathscr{F}^*$, *let* $\Omega=\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j-2g_{ij}\mathrm{d}x^i\wedge\mathrm{d}y^j$ *be the almost Finsler form associated with* $\mathscr{F}^*$, *and let* $u=u^i(x)\partial/\partial x^i$ *be a vector field in the base manifold* M. *Then, the vertical lift of* u *is an almost Hamilton vector of* $\mathscr{F}^*$ *if and only if*

(1) $$u^m\dot{\partial}_m g_{ij}=0,$$

(2) $$u^m\dot{\partial}_m\alpha_{ij}+g_{mj}\partial_i u^m-g_{im}\partial_j u^m=0$$

hold good.

Here we consider the case where $\mathrm{d}\Omega=0$, i. e., g is a Finsler metric and $\Omega=\Omega^*$. By virtue of (2.1) we have

$$\begin{aligned} &u^m\dot{\partial}_m\alpha_{ij}+g_{mj}\partial_i u^m-g_{im}\partial_j u^m \\ &\quad=u^m(\partial_i g_{jm}-\partial_j g_{im})+g_{jm}\partial_i u^m-g_{im}\partial_j u^m. \end{aligned}$$

Let $\overset{*}{\nabla}$ be the covariant differentiation with respect to the Cartan's Finsler connection $\overset{*}{\Gamma}{}^i_{jk}$ ([8], [13]). Using the condition $u^m\dot{\partial}_m g_{ij}=0$ and the well-known relation $\overset{*}{\nabla}_k g_{ij}=0$, we have

$$u^m(\partial_i g_{jm}-\partial_j g_{im})+g_{ir}\overset{*}{\Gamma}{}^r_{mj}u^m-g_{jr}\overset{*}{\Gamma}{}^r_{mi}u^m=0.$$

Hence, we can rewrite the condition (2) as $\overset{*}{\nabla}_i(g_{jm}u^m)-\overset{*}{\nabla}_j(g_{im}u^m)=0$. Therefore we obtain

Theorem 13. *Let* g *be a Finsler metric of a manifold* M, *let* $u=u^i(x)\partial/\partial x^i$ *be a vector field in* M *and let* $\mathscr{F}^*$ *be the homogeneous almost Finsler structure on* $T(M)$ *derived* *from* $\Omega^*=\mathrm{d}(2y^m g_{mj}\mathrm{d}x^j)$. *Then the vertical lift of u is a Hamilton vector of the symplectic structure* $\mathscr{F}^*$ *if and only if*

$$(1)\quad u^m\dot{\partial}_m g_{ij}=0, \qquad (2)\quad \overset{*}{\nabla}_i(g_{jm}u^m)=\overset{*}{\nabla}_j(g_{im}u^m)$$

hold good where $\overset{*}{\nabla}$ *means the covariant differentiation with respect to the Cartan's Finsler connection* $\overset{*}{\Gamma}{}^i_{jk}$.

In the case of Theorem 13, the Hamilton system μ is written as $\mu=g_{im}u^m dx^i$. This μ is, naturally, a closed 1-form, however, is not always an exact form.

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