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On the First Eigenvalue of the p-Laplacian in a Riemannian Manifold

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1. Introduction and results.

Let Ω be a bounded domain in a Riemannian manifold (M, g) of dimension m. We consider the following Dirichlet problem:

$$\begin{aligned} \Delta_p u + \lambda |u|^{p-2}u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|_g^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < \infty$. In local coordinates,

$$\Delta_p u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})}\, g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x^j}\right),$$

where $|\nabla u|^2 = |\nabla u|_g^2 = \sum_{ij} g^{ij}(\partial u/\partial x^i)(\partial u/\partial x^j)$, and $(g^{ij}) = (g_{ij})^{-1}$. The *first eigenvalue* $\lambda_{1,p}(\Omega)$ *of the p-Laplacian* is defined as the least real number λ for which the Dirichlet problem (1) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$. Here the Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|u\|_{1,p} = \{\int_\Omega (|u|^p + |\nabla u|^p) dv_g\}^{1/p}$. It can be also characterized by

$$\lambda_{1,p}(\Omega) = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^p dv_g}{\int_\Omega |u|^p dv_g}, \tag{2}$$

where u runs over $W_0^{1,p}(\Omega)$ and dv_g denotes the volume element of M. We would like to estimate the $\lambda_{1,p}(\Omega)$. For the case $p = 2$, there have been several results, such as the Faber-Krahn inequality [1], the Cheeger inequality [2], and the Cheng inequality [3]. The purpose of this paper is to give inequalities for their p-Laplacian analogue. More precisely we show the following theorems.

THEOREM 1 (the Faber-Krahn type inequality). *Let M_k be a complete simply connected Riemannian manifold of constant sectional curvature κ. Let B be the geodesic*

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ball in M_κ, whose volume is equal to that of the domain Ω in M_κ. Then the following inequality holds:

$$\lambda_{1,p}(\Omega)\geq\lambda_{1,p}(B)\,. \tag{3}$$

The equality holds only for the case the domain Ω is the ball B in M_κ.

Next we define the Cheeger constant $h(\Omega)$ of Ω to be

$$h(\Omega)=\inf_{\Omega'}\frac{\mathrm{Vol}(\partial\Omega')}{\mathrm{Vol}(\Omega')}\,,$$

where Ω' ranges over all open submanifold of Ω with compact closure in Ω and smooth boundary $\partial\Omega'$. $\mathrm{Vol}(\Omega')$ and $\mathrm{Vol}(\partial\Omega')$ denote the volumes of Ω' and $\partial\Omega'$ respectively.

THEOREM 2 (the Cheeger type inequality). *For any bounded domain Ω with piecewise smooth boundary in a complete Riemannian manifold, we have the following inequality*:

$$\lambda_{1,p}(\Omega)\geq\left(\frac{h(\Omega)}{p}\right)^p\,. \tag{4}$$

THEOREM 3 (the Cheng type inequality). *Let M be an m-dimensional complete Riemannian manifold with Ricci curvature satisfying* $\mathrm{Ric}(v)\geq k(m-1)$ *for any unit vector $v\in TM$. Let $B(x_0, r)$ be the geodesic ball in M of radius r with center x_0, and $V(k, r)$ be a ball of radius r with center $\tilde{x}_0$ in the space form of curvature k. Then we have*

$$\lambda_{1,p}(B(x_0, r))\leq\lambda_{1,p}(V(k, r))\,, \tag{5}$$

with equality if and only if $B(x_0, r)$ is isometric to $V(k, r)$.

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2. Proof of Theorem 1.

Let f be a nonnegative eigenfunction of p-Laplacian in Ω associated with $\lambda_{1,p}(\Omega)$. Consider the set $\Omega_t=\{x\in\Omega\,;\,f(x)>t\}$ and $\Gamma_t=\{x\in\Omega\,;\,f(x)=t\}$. Using a symmetrization procedure, we construct the geodesic ball B_t in M_k such that $\mathrm{Vol}(B_t)=\mathrm{Vol}(\Omega_t)$ for each t, and $B_0=B$. We define a function $F: B\to\mathbf{R}^+$ such that F is a radially decreasing function and $\partial B_t=\{x\in B\,;\,F(x)=t\}$.

Then it suffices to prove

$$\int_\Omega f^p dv_g=\int_B F^p dv_g\,, \tag{6}$$

(7) $$\int_\Omega |\nabla f|^p dv_g \geq \int_B |\nabla F|^p dv_g .$$

Indeed for (6), using coarea formula [4],

$$\int_\Omega f^p dv_g = \int_0^\infty \int_{\Gamma_t} \frac{f^p}{|\nabla f|} dA_t dt = \int_0^\infty t^p \left(\int_{\Gamma_t} \frac{dA_t}{|\nabla f|} \right) dt$$
$$= -\int_0^\infty t^p \frac{d}{dt} \mathrm{Vol}(\Omega_t) dt = -\int_0^\infty t^p \frac{d}{dt} \mathrm{Vol}(B_t) dt = \int_B F^p dv_g ,$$

where dA_t is the $(m-1)$-dimensional volume element on Γ_t. Here we have used the identity

$$\frac{d}{dt} \mathrm{Vol}(\Omega_t) = -\int_{\Gamma_t} |\nabla f|^{-1} dA_t .$$

Next we shall prove (7). Using the Hölder inequality, we have

$$\int_{\Gamma_t} dA_t = \int_{\Gamma_t} |\nabla f|^{1-1/p} \cdot |\nabla f|^{-1+1/p} dA_t$$
$$\leq \left(\int_{\Gamma_t} |\nabla f|^{p-1} dA_t \right)^{1/p} \left(\int_{\Gamma_t} |\nabla f|^{-1} dA_t \right)^{(p-1)/p}$$
$$= \left(\int_{\Gamma_t} |\nabla f|^{p-1} dA_t \right)^{1/p} \left(-\frac{d}{dt} \mathrm{Vol}(\Omega_t) \right)^{(p-1)/p} .$$

Thus we have, using isoperimetric inequality,

$$\int_{\Gamma_t} |\nabla f|^{p-1} dA_t \geq \frac{\mathrm{Vol}(\Gamma_t)^p}{\left(-\frac{d}{dt} \mathrm{Vol}(\Omega_t) \right)^{p-1}}$$
$$\geq \frac{\mathrm{Vol}(\Gamma_t^*)^p}{(\int_{\Gamma_t^*} |\nabla F|^{-1} dA_t^*)^{p-1}} = \int_{\Gamma_t^*} |\nabla F|^{p-1} dA_t^* ,$$

where $\Gamma_t^* = \{x \in B \,;\, F(x) = t\}$, and dA_t^* is the $(m-1)$-dimensional volume element on Γ_t^*. Integrating in t, we get (7).

3. Proof of Theorem 2.

Let u be a nonnegative eigenfunction of the p-Laplacian in Ω associated with $\lambda_{1,p}(\Omega)$. Then we may assume $u(x) > 0$ for $x \in \Omega$. Integrating the identity

$$-u \Delta_p u = \lambda_{1,p}(\Omega) |u|^{p-2} u^2$$

by parts, we have

$$\lambda_{1,p}(\Omega)=\frac{\int_\Omega |\nabla u|^p dv_g}{\int_\Omega |u|^p dv_g}.$$

Hence we have used Green's formula:

$$\int_\Omega -u\Delta_p u dv_g = -\int_\Omega u \operatorname{div}(|\nabla u|^{p-2}\nabla u)dv_g = \int_\Omega |\nabla u|^p dv_g .$$

By $\nabla u^p = pu^{p-1}\nabla u$ and the Hölder inequality,

(8)
$$\lambda_{1,p}(\Omega)\geq\left(\frac{\int_\Omega |\nabla u^p| dv_g}{p\int_\Omega |u|^p dv_g}\right)^p .$$

Now by the coarea formula,

(9)
$$\begin{aligned}\int_\Omega |\nabla u^p| dv_g &= \int_{-\infty}^{\infty} \operatorname{Vol}(A(t))dt \\ &\geq \inf_t\left(\frac{\operatorname{Vol}(A(t))}{\operatorname{Vol}(V(t))}\right)\int_{-\infty}^{\infty} \operatorname{Vol}(V(t))dt \\ &\geq h(\Omega)\int_\Omega |u|^p dv_g ,\end{aligned}$$

where $A(t)=\{x\,;|u(x)|^p=t\}$ and $V(t)=\{x\,;|u(x)|^p>t\}$. Combining (8) and (9), we get (4) in Theorem 2.

4. Proof of Theorem 3.

Let $\tilde{f}$ be a nonnegative first eigenfunction of p-Laplacian on $\overline{V(k,r)}$. Let $d_{\tilde{x}_0}$ be the distance function with respect to the center $\tilde{x}_0$ of $\overline{V(k,r)}$. Since $\tilde{f}$ depends only on the distance $d_{\tilde{x}_0}$, we may write $\tilde{f}=\varphi\circ d_{\tilde{x}_0}$, where φ is a positive function on $(0,r)$. We define a C^∞ map $\Theta:(0,r)\times S^{m-1}\to M$ by

$$\Theta(t,v)=\exp_x tv ,$$

where S^{m-1} is the unit sphere in T_xM and $\exp_x$ is a local diffeomorphism from a neighbourhood of 0 in T_xM onto a neighbourhood of x in M. We set $\theta(t,v)=t^{m-1}\sqrt{\det g_{ij}(\Theta(t,v))}$, which is a C^∞ function on $(0,r)\times S^{m-1}$. Then we have

$$\Theta^* dv_g=\theta(t,v)dtdv ,$$

where $dtdv$ denotes the canonical product measure on $(0,r)\times S^{m-1}$. When we define $\theta(t,v)$ on $\overline{V(k,r)}$ in the same manner, $\theta(t,v)$ does not depend on $v\in S^{m-1}$. We denote it simply by $\tilde{\theta}(s)$. We have for $0\leq s\leq r$,

$$(10)\quad \begin{aligned}&(p-1)|\varphi'(s)|^{p-2}\varphi''(s)+\frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)}|\varphi'(s)|^{p-2}\varphi'(s)+\lambda_{1,p}(V(k,r))|\varphi(s)|^{p-2}\varphi(s)=0\,,\\ &\varphi(r)=0\,,\qquad \varphi'(0)=0\,.\end{aligned}$$

We take $f(x)=\varphi\circ d_{x_0}(x)$ as a test function on a ball $B(x_0, r)$, which satisfies the boundary condition $f|_{\partial B(x_0,r)}=\varphi(r)=0$. Then we get

$$(11)\qquad \lambda_{1,p}(B(x_0, r))\le\frac{\int_{B(x_0,r)}|\nabla f|^p dv_g}{\int_{B(x_0,r)}|f|^p dv_g}\,.$$

From $|\nabla f|^p=|\varphi'|^p$ we have

$$(12)\qquad \int_{B(x_0,r)}|\nabla f|^p dv_g=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}|\varphi'(s)|^p\theta(s, v)ds\,,$$

$$(13)\qquad \int_{B(x_0,r)}|f|^p dv_g=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}|\varphi(s)|^p\theta(s, v)ds\,,$$

where $a(v)\le r$ such that $\exp_{x_0}(a(v)\cdot v)$ is the cut point of x_0 along the geodesic $t\to\exp_{x_0}(tv)$. By

$$\{\tilde{\theta}(s)|\varphi'(s)|^{p-2}\varphi'(s)\}'=-\lambda_{1,p}(V(k,r))|\varphi(s)|^{p-2}\varphi(s)\tilde{\theta}(s)\le 0$$

and $\varphi'(0)=0$, we can see that $\varphi'(s)\le 0$. Integrating the above equation (12) by parts, we have

$$\begin{aligned}\int_{B(x_0,r)}|\nabla f|^p dv_g&=-\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\Big[\{\varphi|\varphi'|^{p-1}\theta(s, v)\}'-\varphi(|\varphi'|^{p-1}\theta)'\Big]ds\\ &=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\varphi(s)|\varphi'(s)|^{p-2}\left\{-(p-2)\varphi''(s)-\frac{\theta'(s, v)}{\theta(s, v)}\cdot\varphi'(s)\right\}\theta(s, v)ds\,,\end{aligned}$$

where $\theta'(s, v)$ denotes the partial derivative with respect to s. By the Bishop comparison theorem we have $\{\theta(s, v)/\tilde{\theta}(s)\}'\le 0$. Recalling $\varphi'\le 0$, we get

$$\varphi'(s)\cdot\theta'(s, v)/\theta(s, v)\ge\varphi'(s)\cdot\tilde{\theta}'(s)/\tilde{\theta}(s)\,.$$

Thus we have

$$\begin{aligned}&\int_{B(x_0,r)}|\nabla f|^p dv_g\\ &\le\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\varphi(s)|\varphi'(s)|^{p-1}\left\{-(p-1)\varphi''(s)-\tilde{\theta}'(s)/\tilde{\theta}(s)\cdot\varphi'(s)\right\}\theta(s, v)ds\\ &=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\lambda_{1,p}(V(k, r))\varphi^p(s)\theta(s, v)ds=\lambda_{1,p}(V(k, r))\int_{B(x_0,r))}\varphi^p dv_g\,.\end{aligned}$$

This implies that $\lambda_{1,p}(B(x_0, r))\le\lambda_{1,p}(V(k, r))$. If the equality holds, then $\{\theta(s, v)/\tilde{\theta}\}'=0$.

Since the equality holds in the Bishop comparison theorem, $B(x_0, r)$ is of constant curvature k. It follows that $B(x_0, r)$ is isometric to $V(k, r)$.

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