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On the First-Order Operators in Bimodules

MICHEL DUBOIS-VIOLETTE and THIERRY MASSON
Laboratoire de Physique Théorique et Hautes Energies, Université Paris XI, Bâtiment 211 91 405, Orsay Cedex, France. e-mail: flad@qcd.th.u-psud.fr & masson@qcd.th.u-psud.fr*

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Abstract. We analyse the structure of the first-order operators in bimodules introduced by A. Connes. We apply this analysis to the theory of connections on bimodules, thereby generalizing several proposals.

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1. Introduction

It is well known that the notion of linear partial differential operator admits the following algebraic formulation. Let $\mathcal{A}$ be a commutative associative algebra over $\mathbb{C}$ and let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{A}$-modules. A linear mapping D of $\mathcal{M}$ into $\mathcal{N}$ is called an operator of order 0 of $\mathcal{M}$ into $\mathcal{N}$ if it is a $\mathcal{A}$-module homomorphism, i.e. if $[D, f] = 0$, $\forall f \in \mathcal{A}$, ($f \in \mathcal{A}$ being identified with the multiplication by f in $\mathcal{M}$ and in $\mathcal{N}$). Then, one defines inductively the operators of order $k \in \mathbb{N}$: A linear mapping D of $\mathcal{M}$ into $\mathcal{N}$ is an operator of order $k+1$ of $\mathcal{M}$ into $\mathcal{N}$ if $[D, f]$ is an operator of order k of $\mathcal{M}$ into $\mathcal{N}$, $\forall f \in \mathcal{A}$. For bad choices of $\mathcal{A}$, it may happen that this notion is not very appealing. However, it makes sense and when $\mathcal{A}$ is the algebra of smooth functions on a smooth manifold X and when $\mathcal{M}$ and $\mathcal{N}$ are the modules of smooth sections of two smooth complex vector bundles E and F over X, then this notion is just the usual notion of linear partial differential operators of order $k \in \mathbb{N}$ of E into F. The troubles start if one tries to replace $\mathcal{A}$ by a noncommutative associative algebra and $\mathcal{M}$ and $\mathcal{N}$ by left (or right) $\mathcal{A}$-module. Then the definition breaks down because multiplications by elements f and g of $\mathcal{A}$ do not commute. However, it was noticed by A. Connes [3] that if one works with bimodules, there is a natural noncommutative generalization of first-order operators, since the multiplications on the right and on the left commute. Moreover, in [3] it was pointed out that the first-order condition for the generalized Dirac operator plays an important role in the connection between the noncommutative Poincaré duality and cyclic cohomology. On the other hand, it was observed at several places, e.g. [6, 7], that the natural generalization of the notion of module over a commutative algebra

* Laboratoire associé au Centre National de la Recherche Scientifique – URA D0063.

is not necessarily the notion of left (or right) $\mathcal{A}$-module over a noncommutative algebra $\mathcal{A}$, but can be a notion of bimodule over $\mathcal{A}$. Furthermore, it was claimed in [7] that the latter point of view is unavoidable if one wants to discuss reality conditions and if one takes the Jordan algebras of Hermitian elements of complex $*$-algebras as the noncommutative analogue of algebras of real functions (e.g. as in quantum theory).

In Section 2, we recall the definition of first order operators in a form which is convenient for our purposes. In Section 3, we give some basic general examples. In Section 4, we establish the general structure of first-order operators and describe the appropriate notion of symbols.

In Section 5, we apply our result to the theory of connections on bimodules and show the relation between the symbols and previously introduced twistings (generalized transpositions) in the case of noncommutative generalizations of linear connections [9].

2. First-Order Operators in Bimodules

In this Letter, $\mathcal{A}$ and $\mathcal{B}$ are unital associative algebras over $\mathbb{C}$, their units $\mathbb{1}_{\mathcal{A}}$ and $\mathbb{1}_{\mathcal{B}}$ will be simply denoted by $\mathbb{1}$ when no confusion arises. Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A},\mathcal{B})$-bimodules, (i.e. $\mathcal{A}\otimes\mathcal{B}^{op}$-modules), and let $\mathcal{L}(\mathcal{M},\mathcal{N})$ be the vector space of all linear mappings of $\mathcal{M}$ into $\mathcal{N}$. Among the elements $\mathcal{L}(\mathcal{M},\mathcal{N})$, one can distinguish several natural subclasses. The most natural subspace of $\mathcal{L}(\mathcal{M},\mathcal{N})$ is the space $\mathrm{Hom}_{\mathcal{A}}^{\mathcal{B}}(\mathcal{M},\mathcal{N})$ of all $(\mathcal{A},\mathcal{B})$-bimodule homomorphisms of $\mathcal{M}$ into $\mathcal{N}$. Other natural subspaces are the space $\mathrm{Hom}^{\mathcal{B}}(\mathcal{M},\mathcal{N})$ of all right $\mathcal{B}$-module homomorphisms of $\mathcal{M}$ into $\mathcal{N}$ and the space $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ of all left $\mathcal{A}$-module homomorphisms of $\mathcal{M}$ into $\mathcal{N}$. However, from the point of view of the $(\mathcal{A},\mathcal{B})$-bimodule structure, it was pointed out in [3] that there is a more symmetrical subspace of $\mathcal{L}(\mathcal{M},\mathcal{N})$ which contains both $\mathrm{Hom}^{\mathcal{B}}(\mathcal{M},\mathcal{N})$ and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ which we now describe. One has the following lemma.

LEMMA 1. *The following conditions* (a) *and* (b) *are equivalent for an element D of $\mathcal{L}(\mathcal{M},\mathcal{N})$.*

(a) *for any $f\in\mathcal{A}$, $m\mapsto D(fm)-fD(m)$ is a right $\mathcal{B}$-module homomorphism.*
(b) *for any $g\in\mathcal{B}$, $m\mapsto D(mg)-D(m)g$ is a left $\mathcal{A}$-module homomorphism.*

Proof. [3]. Let L_f be the left multiplication by $f\in\mathcal{A}$ and let R_g be the right multiplication by $g\in\mathcal{B}$ in $\mathcal{M}$ and $\mathcal{N}$. Condition (a) reads $[[D,L_f],R_g]=0,\forall f\in\mathcal{A},\forall g\in\mathcal{B}$, and condition (b) reads $[[D,R_g],L_f]=0,\forall f\in\mathcal{A},\forall g\in\mathcal{B}$. On the other hand, one has $[L_f,R_g]=0$ which implies that $[[D,L_f],R_g]=[[D,R_g],L_f]$. □

An element D of $\mathcal{L}(\mathcal{M},\mathcal{N})$ which satisfies the above equivalent conditions (a) and (b) is called [3] *a first-order operator or an operator of order* 1 *of $\mathcal{M}$ into $\mathcal{N}$.*

The set of all first-order operators of $\mathcal{M}$ into $\mathcal{N}$ is a subspace of $\mathcal{L}(\mathcal{M},\mathcal{N})$ which will be denoted by $\mathcal{L}_1(\mathcal{M},\mathcal{N})$:

$$\mathcal{L}_1(\mathcal{M},\mathcal{N}) = \{D \in \mathcal{L}(\mathcal{M},\mathcal{N}) | [[D, L_f], R_g] = 0, \forall f \in \mathcal{A}, \forall g \in \mathcal{B}\}.$$

This terminology is, of course, suggested by the fact that when $\mathcal{A}$ and $\mathcal{B}$ coincide with the algebra $C^\infty(X)$ of smooth functions on a smooth manifold X and when $\mathcal{M}$ and $\mathcal{N}$ are the smooth sections of smooth vector bundles E and F over X, then $\mathcal{L}_1(\mathcal{M},\mathcal{N})$ is just the space of ordinary first-order differential operators of E into F.

Remark. One has $D \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) \Leftrightarrow [D, L_f] = 0\ \forall f \in \mathcal{A}$ and $D \in \mathrm{Hom}^{\mathcal{B}}(\mathcal{M},\mathcal{N}) \Leftrightarrow [D, R_g] = 0\,\forall g \in \mathcal{B}$. Therefore, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ and $\mathrm{Hom}^{\mathcal{B}}(\mathcal{M},\mathcal{N})$ are subspaces of $\mathcal{L}_1(\mathcal{M},\mathcal{N})$. When f runs over $\mathcal{A}$, $[D, L_f]$ are the obstructions for D to be a left $\mathcal{A}$-module homomorphism and, thus, condition (a) of Lemma 1 means that theses obstructions are right $\mathcal{B}$-module homomorphisms. Condition (b) can be formulated similarily by exchange of left and right.

3. Examples

3.1. THE STRUCTURE OF $\mathcal{L}_1(\mathcal{A},\mathcal{N})$ FOR A $(\mathcal{A},\mathcal{A})$-BIMODULE $\mathcal{N}$

Recall that a *derivation of $\mathcal{A}$ into* a bimodule $\mathcal{N}$ over $\mathcal{A}$ is a linear mapping $\delta\colon \mathcal{A} \to \mathcal{N}$ satisfying

$$\delta(fg) = \delta(f)g + f\delta(g), \quad \forall f, g \in \mathcal{A}.$$

The space of all derivations of $\mathcal{A}$ into $\mathcal{N}$ is denoted by $\mathrm{Der}(\mathcal{A},\mathcal{N})$. For each element $n \in \mathcal{N}$, one defines a derivation $\mathrm{ad}(n) \in \mathrm{Der}(\mathcal{A},\mathcal{N})$ by $\mathrm{ad}(n)(f) = nf - fn, \forall f \in \mathcal{A}$. The subspace $\mathrm{ad}(\mathcal{N})$ of $\mathrm{Der}(\mathcal{A},\mathcal{N})$ is denoted by $\mathrm{Int}(\mathcal{A},\mathcal{N})$ and its elements are called *inner (or interior) derivations of $\mathcal{A}$ into $\mathcal{N}$*. By the very definition, a $D \in \mathcal{L}(\mathcal{A},\mathcal{N})$ is a first-order operator, i.e. $D \in \mathcal{L}_1(\mathcal{A},\mathcal{N})$, if and only if one has

$$D(fg) = D(f)g + fD(g) - fD(\mathbb{1})g, \forall f, g \in \mathcal{A}.$$

It follows that the derivations of $\mathcal{A}$ into $\mathcal{N}$ are exactly the first-order operators of $\mathcal{A}$ into $\mathcal{N}$ which vanish on the unit $\mathbb{1}$ of $\mathcal{A}$, i.e. one has

$$\mathrm{Der}(\mathcal{A},\mathcal{N}) = \{D \in \mathcal{L}_1(\mathcal{A},\mathcal{N}) | D(\mathbb{1}) = 0\}.$$

One defines two projections p_R and p_L of $\mathcal{L}_1(\mathcal{A},\mathcal{N})$ onto $\mathrm{Der}(\mathcal{A},\mathcal{N})$ by setting

$$p_R(D)(f) = D(f) - D(\mathbb{1})f \quad \text{and} \quad p_L(D)(f) = D(f) - fD(\mathbb{1}).$$

Notice that $p_L(D) - p_R(D) = \mathrm{ad}(D(\mathbb{1}))$ so $\mathrm{Im}(p_L - p_R) = \mathrm{Int}(\mathcal{A},\mathcal{N})$. It is clear that $D \mapsto (p_L(D), D(\mathbb{1}))$ and $D \mapsto (p_R(D), D(\mathbb{1}))$ are both isomorphisms of $\mathcal{L}_1(\mathcal{A},\mathcal{N})$ onto $\mathrm{Der}(\mathcal{A},\mathcal{N}) \oplus \mathcal{N}$. Thus one has

$$\mathcal{L}_1(\mathcal{A},\mathcal{N}) \simeq \mathrm{Der}(\mathcal{A},\mathcal{N}) \oplus \mathcal{N}.$$

In fact, one has

$$\ker(p_R) = \mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{N}), \qquad \ker(p_L) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{N})$$

and the mapping $D \mapsto D(\mathbb{1})$ of $\mathcal{L}(\mathcal{A},\mathcal{N})$ into $\mathcal{N}$ induces isomorphisms

$$\alpha_R\colon \mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{N}) \xrightarrow{\simeq} \mathcal{N} \quad \text{and} \quad \alpha_L\colon \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{N}) \xrightarrow{\simeq} \mathcal{N},$$

so the two above isomorphisms of $\mathcal{L}_1(\mathcal{A},\mathcal{N})$ onto $\mathrm{Der}(\mathcal{A},\mathcal{N}) \oplus \mathcal{N}$ correspond to the canonical splitting of the exact sequences of vector spaces

$$0 \to \mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{N}) \xrightarrow{\subset} \mathcal{L}_1(\mathcal{A},\mathcal{N}) \xrightarrow{p_R} \mathrm{Der}(\mathcal{A},\mathcal{N}) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{N}) \xrightarrow{\subset} \mathcal{L}_1(\mathcal{A},\mathcal{N}) \xrightarrow{p_L} \mathrm{Der}(\mathcal{A},\mathcal{N}) \to 0$$

associated to the inclusion $\mathrm{Der}(\mathcal{A},\mathcal{N}) \subset \mathcal{L}_1(\mathcal{A},\mathcal{N})$.

Notice, finally, that one has in $\mathcal{L}_1(\mathcal{A},\mathcal{N})$

$$\mathrm{Int}(\mathcal{A},\mathcal{N}) = \mathrm{Der}(\mathcal{A},\mathcal{N}) \cap (\mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{N}) + \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{N})).$$

3.2. FIRST-ORDER OPERATORS OF $\mathcal{A}$ INTO ITSELF

In the case where $\mathcal{N} = \mathcal{A}$, the space $\mathcal{L}_1(\mathcal{A},\mathcal{A})$, $\mathrm{Der}(\mathcal{A},\mathcal{A})$ and $\mathrm{Int}(\mathcal{A},\mathcal{A})$ will be simply denoted by $\mathcal{L}_1(\mathcal{A})$, $\mathrm{Der}(\mathcal{A})$ and $\mathrm{Int}(\mathcal{A})$. If D_1 and D_2 are elements of $\mathcal{L}_1(\mathcal{A})$, it is easy to see that $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is again an element of $\mathcal{L}_1(\mathcal{A})$. Thus, $\mathcal{L}_1(\mathcal{A})$ is a Lie algebra; it is also, in an obvious way, a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$. The subspace $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra and a $Z(\mathcal{A})$-submodule of $\mathcal{L}_1(\mathcal{A})$ and $\mathrm{Int}(\mathcal{A})$ is a Lie ideal of $\mathrm{Der}(\mathcal{A})$ and also a $Z(\mathcal{A})$-submodule. The quotient $\mathrm{Out}(\mathcal{A}) = \mathrm{Der}(\mathcal{A})/\mathrm{Int}(\mathcal{A})$ is a Lie algebra and a $Z(\mathcal{A})$-module. The spaces $\mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{A})$ are both Lie ideals and $Z(\mathcal{A})$-submodules of $\mathcal{L}_1(\mathcal{A})$ and the exact sequences

$$0 \to \mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{A}) \to \mathcal{L}_1(\mathcal{A}) \xrightarrow{p_R} \mathrm{Der}(\mathcal{A}) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{A}) \to \mathcal{L}_1(\mathcal{A}) \xrightarrow{p_L} \mathrm{Der}(\mathcal{A}) \to 0$$

are now exact sequences of Lie algebras and of $Z(\mathcal{A})$-modules, whereas the corresponding isomorphisms $\mathcal{L}_1(\mathcal{A}) \simeq \mathrm{Der}(\mathcal{A}) \oplus \mathcal{A}$ are isomorphisms of Lie algebras and of $Z(\mathcal{A})$-modules. One has the isomorphism of Lie algebra and of $Z(\mathcal{A})$-module

$$\mathrm{Out}(\mathcal{A}) \simeq \mathcal{L}_1(\mathcal{A})/(\mathrm{Hom}^{\mathcal{A}}(\mathcal{A},\mathcal{A}) + \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{A})).$$

3.3. FIRST-ORDER OPERATORS ASSOCIATED WITH DERIVATIONS

Let Ω^1_L be a bimodule over $\mathcal{A}$, let Ω^1_R be a bimodule over $\mathcal{B}$ and let d_L be a derivation of $\mathcal{A}$ into Ω^1_L and d_R be a derivation of $\mathcal{B}$ into Ω^1_R. Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A},\mathcal{B})$-bimodules and let D be a linear mapping of $\mathcal{M}$ into $\mathcal{N}$.

(1) Assume that there is a $(\mathcal{A},\mathcal{B})$-bimodule homomorphism σ_L of $\Omega^1_L \otimes_{\mathcal{A}} \mathcal{M}$ into $\mathcal{N}$ such that

$$D(fm) = fD(m) + \sigma_L(d_L(f) \otimes m), \quad \forall m \in \mathcal{M} \text{ and } \forall f \in \mathcal{A},$$

then D is a first-order operator.

(2) Assume that there is a $(\mathcal{A},\mathcal{B})$-bimodule homomorphism σ_R of $\mathcal{M} \otimes_{\mathcal{B}} \Omega^1_R$ into $\mathcal{N}$ such that

$$D(mg) = D(m)g + \sigma_R(m \otimes d_R(g)), \quad \forall m \in \mathcal{M} \text{ and } \forall g \in \mathcal{B},$$

then D is a first-order operator.

Thus, any of the two above conditions implies that D is of first order. We shall now show that, conversely, if D is a first-order operator of $\mathcal{M}$ into $\mathcal{N}$, these conditions are satisfied with an appropriate choice of the (d_L, Ω_L) and (d_R, Ω_R).

4. General Structure of First-Order Operators

Recall that in the category of derivations of $\mathcal{A}$ into the bimodules over $\mathcal{A}$ there is an initial object, $d_u\colon \mathcal{A} \to \Omega^1_u(\mathcal{A})$, which is obtained by the following standard construction [2, 1]. The bimodule $\Omega^1_u(\mathcal{A})$ is the kernel of the multiplication $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $(m(f \otimes g) = fg)$, and the derivation $d_u\colon \mathcal{A} \to \Omega^1_u(\mathcal{A})$ is defined by $d_u(f) = \mathbb{1} \otimes f - f \otimes \mathbb{1}$. This derivation has the following universal property: *For any derivation $\delta\colon \mathcal{A} \to \mathcal{M}$ of $\mathcal{A}$ into a bimodule $\mathcal{M}$ over $\mathcal{A}$, there is a unique bimodule homomorphism i_δ of $\Omega^1_u(\mathcal{A})$ into $\mathcal{M}$ such that $\delta = i_\delta \circ d_u$.* The left (resp. right) $\mathcal{A}$-module $\Omega^1_u(\mathcal{A})$ is isomorphic to $\mathcal{A} \otimes d_u\mathcal{A}$ (resp. $d_u\mathcal{A} \otimes \mathcal{A}$) whereas the kernel of d_u is $\mathbb{C}\mathbb{1}$, i.e. $d_u\mathcal{A} \simeq \mathcal{A}/\mathbb{C}\mathbb{1}$ as vector space. One has the following structure theorem for first-order operators.

THEOREM 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A},\mathcal{B})$-bimodules and let D be a first-order operator of $\mathcal{M}$ into $\mathcal{N}$. Then there is a unique $(\mathcal{A},\mathcal{B})$-bimodule homomorphism $\sigma_L(D)$ of $\Omega^1_u(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$ into $\mathcal{N}$ and there is a unique $(\mathcal{A},\mathcal{B})$-bimodule homomorphism $\sigma_R(D)$ of $\mathcal{M} \otimes_{\mathcal{B}} \Omega^1_u(\mathcal{B})$ into $\mathcal{N}$ such that one has*

$$D(fmg) = fD(m)g + \sigma_L(D)(d_u f \otimes m)g + f\sigma_R(D)(m \otimes d_u g)$$

for any $m \in \mathcal{M}$, $f \in \mathcal{A}$ and $g \in \mathcal{B}$.

Proof. By definition, one has

$$D(fmg) - D(fm)g - fD(mg) + fD(m)g = 0$$

which can be rewritten as

$$D(fmg) = fD(m)g + (D(fm) - fD(m))g + f(D(mg) - D(m)g).$$

Now we know (Lemma 1(a), etc.), that $m \mapsto D(fm) - fD(m)$ is a right $\mathcal{B}$-module homomorphism. Furthermore, it vanishes whenever $f \in \mathbb{C}\mathbb{1}$, so one defines a $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\tilde{\sigma}_L(D)$ of $\Omega^1_u(\mathcal{A}) \otimes \mathcal{M}$ into $\mathcal{N}$ by setting

$$\tilde{\sigma}_L(D)(f_0 d_u f_1 \otimes m) = f_0(D(f_1 m) - f_1 D(m)).$$

Moreover, one has

$$\begin{aligned}
&\tilde{\sigma}_L(D)(f_0 d_u(f_1)h \otimes m)\\
&\quad = \tilde{\sigma}_L(D)(f_0 d_u(f_1 h) \otimes m) - \tilde{\sigma}_L(D)(f_0 f_1 d_u h \otimes m)\\
&\quad = f_0(D(f_1 hm) - f_1 hD(m)) - f_0 f_1(D(hm) - hD(m))\\
&\quad = f_0(D(f_1 hm) - f_1 D(hm))\\
&\quad = \tilde{\sigma}_L(D)(f_0 d_u f_1 \otimes hm).
\end{aligned}$$

This means that

$$\tilde{\sigma}_L(D)(\alpha h \otimes m) - \tilde{\sigma}_L(D)(\alpha \otimes hm) = 0,$$

for any $\alpha \in \Omega^1_u(\mathcal{A})$, $m \in \mathcal{M}$ and $h \in \mathcal{A}$, i.e. that $\tilde{\sigma}_L(D)$ passes to the quotient and defines a $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\sigma_L(D)$ of $\Omega^1_u(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$ into $\mathcal{N}$. The uniqueness of $\sigma_L(D)$ is obvious by setting $g = \mathbb{1}$ in the statement. One proceeds similarily on the other side for $\sigma_R(D)$. □

In view of 3.3, the converse is also true: any element D of $\mathcal{L}(\mathcal{M}, \mathcal{N})$ for which there are $\sigma_L(D)$ and $\sigma_R(D)$ as above is a first-order operator.

It is clear that $\sigma_L(D)$ and $\sigma_R(D)$ are the appropriate generalization of the notion of symbol in this setting. We shall refer to them as the *left* and *right universal symbols of* D, respectively.

In order to make contact with this notion, let us investigate the examples of the last section. So let D be a first-order operator of $\mathcal{A}$ into a bimodule $\mathcal{N}$ and let $p_R(D)$ and $p_L(D)$ be the corresponding derivations of $\mathcal{A}$ into $\mathcal{N}$ as in 3.1, then $\sigma_D(D) = i_{p_R(D)}$ and $\sigma_L(D) = i_{p_L(D)}$ are the canonical homomorphisms of bimodule over $\mathcal{A}$ of $\Omega^1_u(\mathcal{A})$ into $\mathcal{N}$ associated with the derivations $p_R(D)$ and $p_L(D)$, (i.e. such that $p_R(D) = i_{p_R(D)} \circ d_u$ and $p_L(D) = i_{p_L(D)} \circ d_u$); in particular, if δ is a derivation of $\mathcal{A}$ into $\mathcal{N}$, one has $\sigma_R(\delta) = \sigma_L(\delta) = i_\delta$. In the cases of the examples of 3.3, for case (1), one has $\sigma_L(D) = \sigma_L \circ (i_{d_L} \otimes \mathrm{id}_\mathcal{M})$ and for case (2), one has $\sigma_R(D) = \sigma_R \circ (\mathrm{id}_\mathcal{M} \otimes i_{d_R})$, where i_{d_L} is the canonical homomorphism of bimodule over $\mathcal{A}$ of $\Omega^1_u(\mathcal{A})$ into Ω^1_L associated with the derivation d_L, i_{d_R} is the canonical homomorphism of bimodule over $\mathcal{B}$ of $\Omega^1_u(\mathcal{B})$ into Ω^1_R associated with the derivation d_R and where $\mathrm{id}_\mathcal{M}$ is the identity mapping of $\mathcal{M}$ onto itself.

Remark. In the case where the right and left module structures are related, the $\sigma_R(D)$ and the $\sigma_L(D)$ are also related. For instance, if $\mathcal{A} = \mathcal{B}$ is a commutative algebra and if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{A}$-modules, then $\sigma_R(D) = \sigma_L(D) \circ T$, where T is the transposition of $\mathcal{M} \otimes_\mathcal{A} \Omega^1_u(\mathcal{A})$ onto $\Omega^1_u(\mathcal{A}) \otimes_\mathcal{A} \mathcal{M}$ defined by

$$T(m \otimes \alpha) = \alpha \otimes m, \quad \forall m \in \mathcal{M} \text{ and } \forall \alpha \in \Omega^1_u(\mathcal{A}).$$

In particular, this applies if $\mathcal{A} = \mathcal{B} = C^\infty(X)$ and if $\mathcal{M}$ and $\mathcal{N}$ are the smooth sections of smooth vector bundles E and F over the smooth manifold X. In this case, one has $\sigma_R(D) = \sigma(D) \circ (i_d \otimes \mathrm{id}_\mathcal{M})$, where $\sigma(D)$ is the usual symbol of the first-order linear partial differential operator D of E into F and i_d is the canonical homomorphism of $\Omega^1_u(C^\infty(X))$ into $\Omega^1(X)$ (= the ordinary 1-forms on X) associated with the ordinary differential d: $C^\infty(X) \to \Omega^1(X)$.

5. Application to the Theory of Connections

Let Ω be a graded differential algebra, with differential d, such that $\Omega^0 = \mathcal{A}$. The restriction of the differential d to $\mathcal{A}$ is then a derivation of $\mathcal{A}$ into the bimodule Ω^1 over $\mathcal{A}$ (the Ω^n, $n \in \mathbb{N}$, are bimodules over $\mathcal{A}$). Recall that a Ω-connection on a left $\mathcal{A}$-module $\mathcal{M}$ [3, 4] is a linear mapping ∇ of $\mathcal{M}$ into $\Omega^1 \otimes_\mathcal{A} \mathcal{M}$ satisfying

$$\nabla(fm) = f\nabla(m) + \mathrm{d}f \otimes m, \quad \forall m \in \mathcal{M} \text{ and } \forall f \in \mathcal{A}.$$

Suppose now that $\mathcal{M}$ is not only a left $\mathcal{A}$-module but is also a bimodule over $\mathcal{A}$. It follows then from 3.3 that a left $\mathcal{A}$-module Ω-connection ∇ on $\mathcal{M}$ as above is a first-order operator of the bimodule $\mathcal{M}$ into the bimodule $\Omega^1 \otimes_\mathcal{A} \mathcal{M}$ over $\mathcal{A}$. Consequently, in view of the structure theorem of Section 4, there is a homomorphism of bimodule over $\mathcal{A}$, $\sigma_R(\nabla)$, of $\mathcal{M} \otimes_\mathcal{A} \Omega^1_u(\mathcal{A})$ into $\Omega^1 \otimes_\mathcal{A} \mathcal{M}$ such that

$$\nabla(mg) = \nabla(m)g + \sigma_R(\nabla)(m \otimes \mathrm{d}_u g), \quad \forall m \in \mathcal{M} \text{ and } \forall g \in \mathcal{A}.$$

The homomorphism $\sigma_R(\nabla)$ is the right universal symbol of ∇, whereas its left universal symbol is simply $\sigma_L(\nabla) = i_d \otimes \mathrm{id}_\mathcal{M}$ where i_d is the canonical bimodule homomorphism of $\Omega^1_u(\mathcal{A})$ into Ω^1 induced by d, (i.e. such that $\mathrm{d} = i_d \circ \mathrm{d}_u$), and $\mathrm{id}_\mathcal{M}$ is the identity mapping of $\mathcal{M}$ onto itself.

In the case where $\sigma_R(\nabla)$ factorizes through a bimodule homomorphism σ of $\mathcal{M} \otimes_\mathcal{A} \Omega^1$ into $\Omega^1 \otimes_\mathcal{A} \mathcal{M}$ as $\sigma_R(\nabla) = \sigma \circ (\mathrm{id}_\mathcal{M} \otimes i_d)$, we call ∇ *a (left) bimodule Ω-connection on the bimodule $\mathcal{M}$*. (One defines similarily a (right) bimodule Ω-connection by starting with a right $\mathcal{A}$-module Ω-connection on $\mathcal{M}$.)

In the case where $\mathcal{M}$ is the bimodule Ω^1 itself, a (left) bimodule Ω-connection on Ω^1 in the above sense is just what is the first part of the proposal of J. Mourad [9] for the definition of linear connections in noncommutative geometry (the second part which relates σ and the product $\Omega^1 \otimes_\mathcal{A} \Omega^1 \to \Omega^2$ makes sense only in this case). This proposal was applied in some simple examples, e.g. [5, 8].

On the other hand, in [7] connections on central bimodules for the derivation-based differential calculus, i.e. for $\Omega = \Omega_{\mathrm{Der}}(\mathcal{A})$, were defined and it was pointed out in [9] that when the central bimodule is $\Omega^1_{\mathrm{Der}}(\mathcal{A})$ itself the connections of [7] are linear connections in the sense of [9], (the choice of σ being fixed). By a similar argument, one sees that the derivation-based connections on central bimodules of [7] are bimodules $\Omega_{\mathrm{Der}}(\mathcal{A})$-connections in the above sense. To be precise, this holds strictly in the finite-dimensional cases, otherwise, one has to introduce completions of the $\Omega^1_{\mathrm{Der}}(\mathcal{A}) \otimes_\mathcal{A} \mathcal{M}$, but this is merely a technicality.

Thus, the above definition of (left) bimodule Ω^1-connections seems very general. However, it is worth noticing here that for the derivation-based connections on central bimodules, the curvatures of these connections are bimodule homomorphisms whereas this is generally not the case for the curvature of a (left) bimodule Ω-connection (it is, of course, always a left $\mathcal{A}$-module homomorphism), although differences of such connections for a fixed σ are bimodule homomorphisms, (see, e.g., [5]).

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