

ON THE FIRST PASSAGE OF THE INTEGRATED WIENER PROCESS

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The rate of first passage of the integrated Wiener process to $x > 0$ is determined in terms of the " $\frac{1}{2}$ -winding time" distribution of H. P. McKean, Jr. The probability that the integrated Wiener process is currently at its maximum is approximated.

1. Introduction. Let $e(t)$ be the standard representation of the 1-dimensional Wiener process ($e(0) = 0$). In [2] McKean studied the winding around the origin of the two-dimensional Markov process (u, v) where $v(t) = b + e(t)$ and $u(t) = a + \int_0^t v(s) ds$. In addition to many interesting strong limit theorems, he obtained the joint distribution of the " $\frac{1}{2}$ winding time" τ_1 and the hitting place h_1

$$(1.1) \quad P_{0,1}(\tau_1 \in dt, h_1 \in dh) = \frac{3h \exp[-2(1-h+h^2)/t]}{2^{\frac{3}{2}}\pi t^2} \\ \times \left(\int_0^{4h/t} \frac{e^{-3\theta/2}}{(\pi\theta)^{\frac{3}{2}}} d\theta \right) dt dh$$

where $\tau_1 = \min(t; t > 0, u(t) = 0)$, $h_1 = |v(\tau_1)|$ and $a = 0, b = 1$. By means of the scaling $e(t) \rightarrow ce(t/c^2)$ for $c > 0$, the requirement $b = 1$ can be converted to $b \neq 0$. In [3] the integral in (1.1) is shown to be expressible in terms of theta-functions.

Whereas [2] deals with the first return of $u(t)$ to its initial value, this note takes up the first passage of $u(t)$ to $x > 0$ when $u(0) = a = 0$ and $v(0) = b \leq 0$. The assumption $a = 0$ is not restrictive. Let $p(t, \xi, \eta, x, y)$ be the transition density for $(u(t), v(t))$, $\tau_x = \min(t; t > 0, u(t) = x)$ and $\phi_b(x, t) = (d/dt)P(\tau_x \leq t)$. We show that

(a (Proposition 1)) $\phi_b(x, t) = \frac{1}{2}[3/(2\pi t^3)]^{\frac{1}{2}}(3xt^{-1} - b) \exp[-3(x-bt)^2/2t^3] + \int_0^\infty d\xi \int_0^t \int_0^\infty \xi P_{0,\xi}(\tau_1 \in ds, h_1 \in dh)[p(t-s, 0, b, x, \xi) - p(t-s, 0, b, x, -\xi)]$.

(b (Proposition 2)) As $t \rightarrow \infty$, $\phi_0(x, t) \sim \text{const.} \times x^{1/6} t^{-(5/4)}$.

(c) $P(u(t) = \max_{0 \leq s \leq t} u(s))$ is approximately 0.372 when $a = b = 0$.

2. Basics. We let $P_{a,b}$ be the probability measure of the process (u, v) where $(u(0), v(0)) = (a, b)$ and $g_{0,b}(t, x, y) dx dy = P_{a,b}(\max_{0 \leq s \leq t} u(s) = u(t) \in dx, v(t) \in dy)$. As in [2] the transition density for $P_{a,b}(u(t) \in dx, v(t) \in dy)$,

$$p(t, a, b, x, y) = 3^{\frac{3}{2}}/\pi t^2 \exp \left[-\frac{2}{t}(y-b)^2 + \frac{6}{t^2}(y-b)(x-a-bt) \right. \\ \left. - \frac{6}{t^3}(x-a-bt)^2 \right].$$

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It is well known that for $z \geq 0$ and $t > 0$

$$(2.1) \quad P_{0,b}(u(t) \in dx, v(t) \in dy, \max_{0 \leq s \leq t} u(s) \leq z) \\ = p(t, 0, b, x, y) dx dy - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, z, \xi) p(s, z, \xi, x, y) d\xi dx dy.$$

If $x = z = 0, b < 0, y < 0$, (2.1) is equivalent to (3.1) of [2]. If $y > 0$, we may take the limit $z \rightarrow x^+$ and obtain

$$(2.2) \quad g_{0,b}(t, x, y) = p(t, 0, b, x, y) - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi) p(s, x, \xi, x, y) d\xi.$$

If (2.1) is integrated on x and y we obtain

$$(2.3) \quad P_{0,b}(\max_{0 \leq s \leq t} u(s) \leq z) = 1 - \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, z, \xi) d\xi.$$

If $z > 0, t > 0, p$ is a genuine density so $P_{0,b}(u(t) = z) = 0$ and

$$(2.4) \quad P(\max_{0 \leq s \leq t} u(s) \leq z) = P(\tau_z > t) = 1 - P(\tau_z \leq t).$$

When $-y < 0$, (2.2) can be interpreted as

$$(2.5) \quad 0 = -p(t, 0, b, x, -y) + \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi) p(s, x, \xi, x, -y) d\xi.$$

Following [1] we let $p^*(t, a, b, x, y) = p(t, a, b, x, y) - p(t, a, b, x, -y)$ and $k^*(t, y, \xi) = p(t, x, \xi, x, -y) - p(t, x, \xi, x, y) = (3^{\frac{1}{2}}/\pi t^2)[\exp(-2(y^2 - y\xi + \xi^2)/t) - \exp(-2(y^2 + y\xi + \xi^2)/t)] = p^*(t, 0, -\xi, 0, y)$.

Upon adding (2.2) and (2.5) for $-y < 0$ we obtain

$$(2.6) \quad g_{0,b}(t, x, y) = p^*(t, 0, b, x, y) + \int_0^t ds \int_0^\infty \xi g_{0,b}(t-s, x, \xi) k^*(s, y, \xi) d\xi.$$

Since $k^*(t, y, \xi) \geq 0$ for positive t, y , and ξ , we have by iteration

$$(2.7) \quad g_{0,b}(t, x, y) \geq p^*(t, 0, b, x, y) + \int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, x, \xi) k^*(r, y, \xi) d\xi \\ + \int_0^t ds \int_0^{-s} dr \int_0^\infty d\xi \int_0^\infty \xi \eta p^*(t-s-r, 0, b, x, \xi) k^*(r, \eta, \xi) \\ \cdot k^*(s, \eta, y) d\eta + \dots.$$

For $b < 0$ and each $x \geq 0$ the expression on the right-hand side of (2.7), when multiplied by y and integrated on y and t from 0 to ∞ , can be seen to be unity. From (2.3) $\int_0^\infty ds \int_0^\infty y g_{0,b}(s, x, y) dy$ cannot exceed unity. Therefore equality must hold in (2.7) almost everywhere (y, t) . We do not carry out the above calculation.

Let $\psi(t, x, \xi) = \int_0^\infty y p^*(t, 0, -\xi, x, y) dy$. Since $k^*(t, y, \xi) = p^*(t, 0, -\xi, 0, y)$ so $\psi(t, 0, \xi) = \int_0^\infty y k^*(t, y, \xi) dy$. Therefore, (2.7) with the aforementioned equality a.e. implies

$$(2.8) \quad \int_0^\infty y g_{0,b}(t, x, y) dy = \psi(t, x, -b) + \int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, x, \xi) \psi(r, 0, \xi) d\xi \\ + \int_0^t ds \int_0^{-s} dr \int_0^\infty d\xi \int_0^\infty \xi \eta p^*(t-s-r, 0, b, x, \xi) k^*(r, \eta, \xi) \\ \cdot \psi(s, 0, \eta) d\eta + \dots$$

and

$$(2.9) \quad \int_0^\infty yg_{0,b}(t, 0, y)dy = \psi(t, 0, -b) + \int_0^t dr \int_0^\infty \xi p^*(t-r, 0, b, 0, \xi) \\ \cdot \psi(r, 0, \xi)d\xi + \int_0^t ds \int_0^{t-s} dr \int_0^\infty d\xi \int_0^\infty \xi \eta p^*(t-s-r, 0, b, 0, \xi) \\ \cdot k^*(r, \eta, \xi)\psi(s, 0, \eta)d\eta + \dots$$

In view of (2.9), (2.8) can be identified as

$$(2.10) \quad \int_0^\infty yg_{0,b}(t, x, y)dy \\ = \psi(t, x, -b) + \int_0^t ds \int_0^\infty (\int_0^\infty yg_{0,-\xi}(s, 0, y)dy)p^*(t-s, 0, b, x, \xi)d\xi$$

in view of the associativity of convolution on the time-like variables. If $b < 0$, it is clear that equations (2.3) and (2.4) apply to $z = 0$ providing τ_z is interpreted as t_1 of Section 1 and [2]. Therefore $\int_0^\infty yg_{0,b}(t, 0, y)dydt = P_{0,b}(t_1 \in dt)$. By means of the scaling $e \rightarrow ce(t/c^2)$

$$P_{0,\xi}(t_1 \in dt, h_1 \in dh) = \frac{3h}{\pi 2^{\frac{1}{2}} t^2} \exp(-2(\xi^2 - \xi h + h^2)/t) \int_0^{4\xi h/t} \frac{e^{-3\theta/2}}{(\pi\theta)^{\frac{1}{2}}} d\theta dt dh.$$

Furthermore, evaluation of $\psi(t, x, -b)$ is routine. Therefore, we have

PROPOSITION 1. *If $b \leq 0$ and $x > 0$*

$$(2.11) \quad \phi_b(x, t) = \frac{d}{dt} P(\tau_x \leq t) = \int_0^\infty yg_{0,b}(t, x, y)dy \\ = \psi(t, x, -b) + \int_0^\infty d\xi \int_0^t \int_0^\infty \xi P_{0,\xi}(t_1 \in ds, h_1 \in dh) \\ \cdot [p(t-s, 0, b, x, \xi) - p(t-s, 0, b, x, -\xi)]$$

where $\psi(t, x, -b) = (3/8\pi t^3)^{\frac{1}{2}}(3xt^{-1} - b) \exp(-3(x-bt)^2/2t^3)$,

$$P_{0,\xi}(t_1 \in ds, h_1 \in dh) = \int_0^{4\xi h/s} \frac{3h}{\pi 2^{\frac{1}{2}} s^2 (\pi\theta)^{\frac{1}{2}}} \exp\left[-\frac{2}{s}(\xi^2 - \xi h + h^2) - \frac{3\theta}{2}\right] d\theta ds dh,$$

and

$$p(r, 0, b, x, \xi) = \frac{3^{\frac{1}{2}}}{(\pi r^2)} \exp\left[-\frac{2}{r}(\xi - b)^2 + \frac{6}{r^2}(\xi - b)(x - bt) - \frac{6}{r^3}(x - bt)^2\right].$$

3. Asymptotic behavior of the first passage density. Since the integral in (2.11) is quite complicated we describe its asymptotic behavior as $t \rightarrow \infty$ in the case $x > 0$, $b = 0$. We have

$$(3.1) \quad \int_0^t ds \int_0^\infty d\xi \int_0^\infty dh \int_0^{4h\xi/s} \frac{3h\xi}{\pi 2^{\frac{1}{2}} s^2 (\pi\theta)^{\frac{1}{2}}} \exp\left(-\frac{2}{s}(\xi^2 - \xi h + h^2) - \frac{3\theta}{2}\right) \\ \cdot p^*(t-s, 0, 0, x, \xi)d\theta$$

where $p^*(r, 0, 0, x, \xi) = 2(3)^{\frac{1}{2}}/(\pi r^2) \exp(-2\xi^2/r - 6x^2/r^3) \sinh(6\xi x/r^2)$ by the previous definitions. By means of the successive substitutions $h \rightarrow s^{\frac{1}{2}}h$, $\xi \rightarrow s^{\frac{1}{2}}\xi$, $t-s = w^{-\frac{1}{2}}$, $h \rightarrow h\xi$, $\xi \rightarrow (tw^{\frac{1}{2}} - h + h^2)^{-\frac{1}{2}}\xi$, $h \rightarrow t^{\frac{1}{2}}h$, $\theta \rightarrow t^{-\frac{1}{2}}\theta$, and $h \rightarrow w^{\frac{1}{2}}h$ expression (3.1) can be seen to be

$$(3.2) \quad \frac{3(6)^{\frac{1}{2}}}{\pi^2 t^{5/4}} \int_{t^{-3}}^{\infty} \frac{dw}{w} \int_0^{\infty} d\xi \int_0^{\infty} dh \int_0^{\alpha(w, \xi, h, t)} \frac{h\xi^3}{(1 - w^{-\frac{1}{2}}t^{-\frac{1}{2}}h + h^2)^2(\pi\theta)^{\frac{1}{2}}} \cdot \exp\left(-2\xi^2 - 6wx^2 - \frac{3\theta}{2t^{\frac{1}{2}}}\right) \sinh\left(6x\xi w^{\frac{1}{2}} \left[\frac{1 - t^{-1}w^{-\frac{1}{2}}}{1 - w^{-\frac{1}{2}}t^{-\frac{1}{2}}h + h^2}\right]^{\frac{1}{2}}\right) d\theta$$

where $\alpha(w, \xi, h, t) = 4h\xi^2 w^{-\frac{1}{2}}(1 - w^{-\frac{1}{2}}t^{-\frac{1}{2}}h + h^2)^{-1}$. If $w \geq t^{-3}$, $w^{-\frac{1}{2}}t^{-\frac{1}{2}} \leq 1$ so $1 - w^{-\frac{1}{2}}t^{-\frac{1}{2}}h + h^2 \geq (1 + h^2)/2$. Also, $1 - t^{-1}w^{-\frac{1}{2}} \leq 1 - w^{-\frac{1}{2}}t^{-\frac{1}{2}}h + h^2$. If I_t is the indicator function of the region where $w > t^{-3}$, the integrand in (3.2) is bounded by

$$(3.3) \quad w^{-1}h^3(1 + h^2)^{-2}\theta^{-\frac{1}{2}} \exp(-2\xi^2 - 6wx^2) \sinh(6x\xi w^{\frac{1}{2}}) 4I_t(w, \xi, h, \theta).$$

The limit as $t \rightarrow \infty$ of the integrand in (3.2) is a constant multiple of (3.3) so an integration of that limit shows the integrability of (3.3) and shows that limit as $t \rightarrow \infty$ and integration operations may be interchanged. The result of this is

$$\int_0^{\infty} dw \int_0^{\infty} d\xi \int_0^{\infty} dh \int_0^{4h\xi^2 w^{-\frac{1}{2}}(1 + h^2)^{-1}} \frac{h\xi^3}{(1 + h^2)^2(\pi\theta)^{\frac{1}{2}}} \exp(-2\xi^2 - 6wx^2) \cdot \sinh\left(\frac{6x\xi w^{\frac{1}{2}}}{(1 + h^2)^{\frac{1}{2}}}\right) d\theta.$$

The series for sinh and integration on θ give

$$\int_0^{\infty} dw \int_0^{\infty} d\xi \int_0^{\infty} \sum_{n=0}^{\infty} \frac{4h^{\frac{3}{2}}\xi^{2n+5} w^{n+5/12-1} (6x)^{2n+1}}{\pi^{\frac{1}{2}}(1 + h^2)^{n+3} (2n+1)!} \exp(-2\xi^2 - 6wx^2) dh.$$

The ξ and w integrations result in $\Gamma(k)$ for various k ; $\int_0^{\infty} 2h^{\frac{3}{2}}/(1 + h^2)^{n+3} dh = B(n + \frac{7}{4}, \frac{5}{4})$; the first term in (2.11) tends to zero more rapidly than $t^{-\frac{5}{4}}$. We therefore have

PROPOSITION 2. If $x > 0$, as $t \rightarrow \infty$

$$\phi_0(x, t) \sim t^{-1} \left(\frac{x^2}{t^3}\right)^{1/12} \frac{9(6)^{\frac{1}{12}} \Gamma(5/4)\Gamma(7/4)\Gamma(5/12)}{4\pi^2 \Gamma(3/2)} {}_2F_1\left(\frac{5}{12}, \frac{7}{4}; \frac{3}{2}; \frac{3}{4}\right).$$

4. The probability that u is at its maximum. From the definition of $g_{a,b}$

$$(4.1) \quad P_{a,b}(u(t) = \max_{0 \leq s \leq t} u(s)) = \int_0^{\infty} \int_0^{\infty} g_{a,b}(x, y, t) dx dy.$$

If $a = b = 0$ this probability is independent of t and is the limit as $t \rightarrow \infty$ of (4.1) in other cases.

We integrate the first two terms on the right-hand side of (2.7)

$$\int_0^\infty \int_0^\infty p^*(t, 0, 0, x, y) dx dy = \frac{3^{\frac{1}{2}}}{\pi t^2} \int_0^\infty \int_0^\infty \left[\exp\left(-\frac{6x^2}{t^3} + \frac{6xy}{t^2} - \frac{2y^2}{t}\right) - \exp\left(-\frac{6x^2}{t^3} - \frac{6xy}{t^2} - \frac{2y^2}{t}\right) \right] dx dy.$$

The substitutions $x \rightarrow 3^{-\frac{1}{2}}t^{\frac{1}{2}}x, y \rightarrow \frac{1}{2}t^{\frac{1}{2}}y$ yield

$$1/2\pi \int_0^\infty \int_0^\infty [\exp(-\frac{1}{2}(x^2 + (y - 3^{\frac{1}{2}}x)^2)) - \exp(-\frac{1}{2}(x^2 + (y + 3^{\frac{1}{2}}x)^2))] dx dy.$$

A difference of two integrals may be formed, changed to polar coordinates and evaluated to give

$$\int_0^\infty \int_0^\infty p^*(t, 0, 0, x, y) dx dy = \frac{5}{12} - \frac{1}{12} = \frac{1}{3}.$$

The integral of the second term on the right-hand side of (2.7) is

$$\left(\frac{3}{\pi}\right)^2 \int_0^\infty dx \int_0^\infty dy \int_0^t ds \int_0^\infty \xi s^{-2}(t-s)^{-2} \exp\left[-\frac{2}{s}y^2 - \frac{2}{s}\xi^2 - \frac{6x^2}{(t-s)^3} - \frac{2\xi^2}{t-s}\right] \cdot \left[2 \sinh\left(\frac{2y\xi}{s}\right)\right] \left[2 \sinh\left(\frac{6x\xi}{(t-s)^2}\right)\right] d\xi.$$

With the substitutions $y \rightarrow \frac{1}{2}s^{\frac{1}{2}}y, \xi \rightarrow (t-s)^{\frac{1}{2}}\xi, x \rightarrow ((t-s)/12)^{\frac{1}{2}}x, s \rightarrow ts,$ and $(1-s)/s = w,$ this is just

$$(4.2) \quad \frac{3^{\frac{1}{2}}}{4\pi^2} \int_0^\infty dx \int_0^\infty dy \int_0^\infty dw \int_0^\infty \frac{\xi w^{\frac{1}{2}}}{w+1} \exp\left(-\frac{y^2}{2} - \frac{x^2}{2} - 2(w+1)\xi^2\right) \cdot [2 \sinh(y\xi w^{\frac{1}{2}})][2 \sinh(3^{\frac{1}{2}}x\xi)] d\xi.$$

If $c > 0, \int_0^\infty e^{-z^2/2} \sinh cz dz = e^{c^2/2} \int_0^c e^{-z^2/2} dz$ so (4.2) is

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} dw}{w+1} \int_0^\infty \xi \exp[-(3w+1)\xi^2/2] \left[\int_0^{\xi w^{\frac{1}{2}}} e^{-y^2/2} dy \int_0^{3^{\frac{1}{2}}\xi} e^{-x^2/2} dx \right] d\xi.$$

We may integrate by parts so that $\xi \exp[-(3w+1)\xi^2/2]$ is integrated and the product of integrals is differentiated. This gives

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} w^{\frac{1}{2}} dw}{(w+1)(3w+1)} \int_0^\infty d\xi \int_0^{3^{\frac{1}{2}}\xi} \exp[-\frac{1}{2}(4w+1)\xi^2 - \frac{1}{2}x^2] dx + \frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} 3^{\frac{1}{2}} dw}{(w+1)(3w+1)} \int_0^\infty d\xi \int_0^{\xi w^{\frac{1}{2}}} \exp[-\frac{1}{2}(3w+4)\xi^2 - \frac{1}{2}y^2] dy.$$

By means of the usual change to polar coordinates this is

$$\frac{3^{\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{w \tan^{-1}[3/(4w+1)]^{\frac{1}{2}}}{(w+1)(3w+1)(4w+1)^{\frac{1}{2}}} dw + \frac{3}{\pi^2} \int_0^\infty \frac{w^{\frac{1}{2}} \tan^{-1}[w/(3w+4)]^{\frac{1}{2}}}{(w+1)(3w+1)(3w+4)^{\frac{1}{2}}} dw.$$

The rationalizing substitutions $\theta^2 = 3/(4w+1)$ and $\theta^2 = w/(3w+4)$ give

$$\frac{2}{\pi^2} \int_0^{3^{1/2}} \frac{(3-\theta^2) \tan^{-1} \theta}{(1+\theta^2)(9+\theta^2)} d\theta + \frac{24}{\pi^2} \int_0^{1/3^{1/2}} \frac{\theta^2 \tan^{-1} \theta}{(1+\theta^2)(1+9\theta^2)} d\theta.$$

After integration by parts we have, finally,

$$(4.3) \quad \frac{24}{\pi^2} \int_0^{3^{1/2}} \frac{\theta(\tan^{-1} \theta)^2}{(9+\theta^2)^2} d\theta + \frac{12}{\pi^2} \left[\frac{1}{12} \left(\frac{\pi}{6} \right)^2 - \int_0^{1/3^{1/2}} \frac{2\theta(\tan^{-1} \theta)^2}{(1+9\theta^2)^2} d\theta \right].$$

The summands in (4.3) are approximately 0.021 and 0.018, respectively. Since (4.1) is less than

$$P(u(t) > 0, v(t) > 0) = \int_0^\infty \int_0^\infty p(t, 0, 0, x, y) dx dy = \frac{5}{12},$$

and $0.333 + 0.021 + 0.018 = 0.372$, we see that

$$P(u(t) = \max_{0 \leq s \leq t} u(s) \mid u(t) > 0, v(t) > 0)$$

is quite close to unity.

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