Abbas et al. *Fixed Point Theory and Applications* (2016) 2016:17
DOI 10.1186/s13663-016-0502-y

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the fixed point theory of soft metric spaces

Mujahid Abbas[1,2], Ghulam Murtaza[3] and Salvador Romaguera[4*]

*Correspondence:
sromague@mat.upv.es
[4]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Valencia, 46022, Spain
Full list of author information is available at the end of the article

Abstract
The aim of this paper is to show that a soft metric induces a compatible metric on the collection of all soft points of the absolute soft set, when the set of parameters is a finite set. We then show that soft metric extensions of several important fixed point theorems for metric spaces can be directly deduced from comparable existing results. We also present some examples to validate and illustrate our approach.

Keywords: soft mapping; soft metric space; soft contraction; soft Caristi mapping

1 Introduction

It is a usual practice to use mathematical tools to study the behavior of different aspects of a system and its different subsystems. So it is very natural to deal with uncertainties and imprecise data in various situations. Different kind of difficulties arise in dealing with the uncertainties and imprecision either already existing in the data or due to the mathematical tools used to solve the model featuring various situations. Fuzzy set theory, initiated by Zadeh [1], has evolved as an important tool to solve the issues of uncertainties and ambiguities. Theories such as probability theory and rough set theory have been introduced by mathematicians and computer scientists to handle the problems associated with the uncertainties and imprecision of real world models. The contribution made by probability theory, fuzzy set theory, vague sets, rough sets, and interval mathematics to deal with uncertainty is of vital importance but these theories have their own limitations. To overcome these peculiarities, in 1999, Molodtsov introduced in [2] soft sets as a mathematical tool to handle the uncertainty associated with real world data based problems. It provides enough tools to deal with uncertainty in a data and to represent it in a useful way. The problem of inadequacy of parameters has been successfully resolved by this theory. Now it has become a full-fledged research area and has attracted the attention of several mathematicians, economists, and computer scientists [3–21]. The distinguishing attribute of soft set theory is that unlike probability and fuzzy set theory, it does not uphold a precise quantity.

A vast amount of mathematical activity has been carried out to obtain many remarkable results showing the applicability of soft set theory in decision making, demand analysis, forecasting, information science, mathematics, and other disciplines (see for detailed survey [22–31]).

The notion of a soft topology on a soft set was introduced by Cagman *et al.* [32] and some basic properties of soft topological spaces were studied.

Das and Samanta introduced in [33] the notions of soft real set and soft real number, and discussed their properties. Based on these notions, they introduced in [34] the concept of a soft metric. They showed that each soft metric space is also a soft topological space. Abbas *et al.* [35] introduced the notion of soft contraction mapping based on the theory of soft elements of soft metric spaces. They studied fixed points of soft contraction mappings and obtained among others results, a soft Banach contraction principle. Almost simultaneously, Chen and Lin [36] obtained a soft metric version of the celebrated Meir-Keeler fixed point theorem. Here we show that, under some restriction, each (complete) soft metric induces a (complete) usual metric, and we deduce in a direct way soft metric versions of several important fixed point theorems for metric spaces, as the Banach contraction principle, Kannan and Meir-Keeler fixed point theorems, and Caristi-Kirk's theorem. By means of appropriate examples we also show that the aforementioned restriction is essential.

2 Preliminaries

In the sequel, the letters U, E, and $P(U)$ will denote the universal set, the set of parameters, and the power set of U, respectively.

According to [2] if F is a set valued mapping on $A \subset E$ taking values in $P(U)$, then a pair (F,A) is called a soft set over U. We denote the collection of soft sets over a common universe U by $S(U)$.

Basic notions and properties related to soft set theory may be found in [2, 34, 37, 38]. In particular, a soft set (F,A) over U is said to be a soft point if there is exactly one $\lambda \in A$ such that $F(\lambda) = \{x\}$ for some $x \in U$ and $F(e) = \emptyset$, for all $e \in A \setminus \{\lambda\}$. We shall denote such a soft point by (F_λ^x, A) or simply by F_λ^x. A soft point F_λ^x is said to belong to (F,A), denoted by $F_\lambda^x \tilde{\in} (F,A)$, if $F_\lambda^x(\lambda) = \{x\} \subset F(\lambda)$.

The collection of all soft points of (F,A) is denoted by $\mathrm{SP}(F,A)$.

Now let $\mathbb{R}$ be the set of real numbers. We denote the collection of all nonempty bounded subsets of $\mathbb{R}$ by $B(\mathbb{R})$.

Definition 1 [33] A soft real set denoted by $(\widehat{f},A)$, or simply by $\widehat{f}$, is a mapping $\widehat{f}: A \to B(\mathbb{R})$. If $\widehat{f}$ is a single valued mapping on $A \subset E$ taking values in $\mathbb{R}$, then the pair $(\widehat{f},A)$ or simply $\widehat{f}$, is called a soft element of $\mathbb{R}$ or a soft real number. If $\widehat{f}$ is a single valued mapping on $A \subset E$ taking values in the set $\mathbb{R}^+$ of nonnegative real numbers, then a pair $(\widehat{f},A)$, or simply $\widehat{f}$, is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers (corresponding to A) by $\mathbb{R}(A)^*$. A constant soft real number $\overline{c}$ is a soft real number such that for each $e \in A$, we have $\overline{c}(e) = c$, where c is some real number.

Definition 2 [34] For two soft real numbers $\widehat{f}$, $\widehat{g}$, we say that

(i) $\widehat{f} \tilde{\leq} \widehat{g}$ if $\widehat{f}(e) \leq \widehat{g}(e)$, for all $e \in A$,
(ii) $\widehat{f} \tilde{\geq} \widehat{g}$ if $\widehat{f}(e) \geq \widehat{g}(e)$, for all $e \in A$,
(iii) $\widehat{f} \tilde{<} \widehat{g}$ if $\widehat{f}(e) < \widehat{g}(e)$, for all $e \in A$, and
(iv) $\widehat{f} \tilde{>} \widehat{g}$ if $\widehat{f}(e) > \widehat{g}(e)$, for all $e \in A$.

Remark 1 The notion of a soft mapping may be found in [35, 39]. Recall that if f is a soft mapping from a soft set (F,A) to a soft set (G,B) (denoted by $f:(F,A)\tilde{\to}(G,B)$), then for each soft point $F^x_\lambda \tilde{\in} (F,A)$ there exists only one soft point $G^y_\mu \tilde{\in} (G,B)$ such that $f(F^x_\lambda) = G^y_\mu$.

The definition of a soft metric introduced in [34] is given below.

Definition 3 [34] Let U be a universe, A be a nonempty subset of parameters and $\tilde{U}$ be the absolute soft set, *i.e.*, $F(\lambda) = U$ for all $\lambda \in A$, where $(F,A) = \tilde{U}$. A mapping $d : \mathrm{SP}(\tilde{U}) \times \mathrm{SP}(\tilde{U}) \to \mathbb{R}(A)^*$ is said to be a soft metric on $\tilde{U}$ if for any $U^x_\lambda, U^y_\mu, U^z_\gamma \in \mathrm{SP}(\tilde{U})$ (equivalently, $U^x_\lambda, U^y_\mu, U^z_\gamma \tilde{\in} \tilde{U}$), the following hold:

M1. $d(U^x_\lambda, U^y_\mu) \tilde{\geq} \bar{0}$.
M2. $d(U^x_\lambda, U^y_\mu) = \bar{0}$ if and only if $U^x_\lambda = U^y_\mu$.
M3. $d(U^x_\lambda, U^y_\mu) = d(U^x_\mu, U^y_\lambda)$.
M4. $d(U^x_\lambda, U^z_\gamma) \tilde{\leq} d(U^x_\lambda, U^y_\mu) + d(U^y_\mu, U^z_\gamma)$.

The soft set $\tilde{U}$ endowed with a soft metric d is called a soft metric space and is denoted by $(\tilde{U},d,A)$, or simply by $(\tilde{U},d)$ if no confusion arises.

See [34] for several basic properties of the structure of soft metric spaces. In order to help the reader we recall the following notions, which will be used later on.

Given a soft metric space $(\tilde{U},d)$, a net $\{U^{x_\alpha}_{\lambda_\alpha}\}_{\alpha\in\Lambda}$ of soft points in $\tilde{U}$ will be simply denoted by $\{U^x_{\lambda,\alpha}\}_{\alpha\in\Lambda}$. In particular, a sequence $\{U^{x_n}_{\lambda_n}\}_{n\in\mathbb{N}}$ of soft points in $\tilde{U}$ will be denoted by $\{U^x_{\lambda,n}\}_n$.

Definition 4 [34] Let $(\tilde{U},d)$ be a soft metric space. A sequence $\{U^x_{\lambda,n}\}_n$ of soft points in $\tilde{U}$ is said to be convergent in $(\tilde{U},d)$ if there is a soft point $U^y_\mu \tilde{\in} \tilde{U}$ such that $d(U^x_{\lambda,n}, U^y_\mu) \to \bar{0}$ as $n\to\infty$. This means that, for every $\widehat{\varepsilon} \tilde{>} \bar{0}$, chosen arbitrary, there exists an $m\in\mathbb{N}$ such that $d(U^x_{\lambda,n}, U^y_\mu) \tilde{<} \widehat{\varepsilon}$, whenever $n \geq m$.

Proposition 1 [34] *The limit of a sequence $\{U^x_{\lambda,n}\}_n$ in a soft metric space $(\tilde{U},d)$, if it exists, is unique.*

Definition 5 [34] A sequence $\{U^x_{\lambda,n}\}_n$ of soft points in a soft metric space $(\tilde{U},d)$ is said to be a Cauchy sequence in $(\tilde{U},d)$ if, for each $\widehat{\varepsilon} \tilde{>} \bar{0}$, there exists an $m\in\mathbb{N}$ such that $d(U^x_{\lambda,i}, U^x_{\lambda,j}) \tilde{<} \widehat{\varepsilon}$, for all $i,j \geq m$. That is, $d(U^x_{\lambda,i}, U^x_{\lambda,j}) \to \bar{0}$ as $i,j\to\infty$.

Proposition 2 [34] *Every convergent sequence $\{U^x_{\lambda,n}\}_n$ in a soft metric space $(\tilde{U},d)$ is a Cauchy sequence.*

Definition 6 [34] A soft metric space $(\tilde{U},d)$ is called complete if every Cauchy sequence in $(\tilde{U},d)$ converges to some soft point of $\tilde{U}$. In this case, we say that the soft metric d is complete.

3 Soft metrics inducing compatible metrics

In this short section we show that if $(\tilde{U},d,A)$ is a soft metric space with A a (nonempty) finite set, then d induces in a natural way a compatible metric on $\mathrm{SP}(\tilde{U})$. This fact will be crucial in establishing our main results in Section 4.

Theorem 1 *Let $(\tilde{U}, d, A)$ be a soft metric space with A a finite set. Define a function m_d : $\mathrm{SP}(\tilde{U}) \times \mathrm{SP}(\tilde{U}) \to \mathbb{R}^+$ as*

$$m_d\bigl(U_\lambda^x, U_\mu^y\bigr) = \max_{\eta \in A} d\bigl(U_\lambda^x, U_\mu^y\bigr)(\eta),$$

for all $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$. Then the following hold:

(1) *m_d is a metric on $\mathrm{SP}(\tilde{U})$.*

(2) *For any sequence $\{U_{\lambda,n}^x\}_n$ of soft points and a soft point U_λ^y, we have*

(2a) *$\{U_{\lambda,n}^x\}_n$ is a Cauchy sequence in $(\tilde{U}, d, A)$ if and only if it is a Cauchy sequence in $(\mathrm{SP}(\tilde{U}), m_d)$.*

(2b) *$d(U_\lambda^y, U_{\lambda,n}^x) \to \overline{0}$ if and only if $m_d(U_\lambda^y, U_{\lambda,n}^x) \to 0$.*

(3) *$(\tilde{U}, d, A)$ is complete if and only if $(\mathrm{SP}(\tilde{U}), m_d)$ is complete.*

Proof (1) Let $U_\lambda^x, U_\mu^y, U_\upsilon^z \in \mathrm{SP}(\tilde{U})$. Then we have:

(i) $m_d(U_\lambda^x, U_\mu^y) \geq 0$, by condition M1 of Definition 3.

(ii) $m_d(U_\lambda^x, U_\mu^y) = 0 \Leftrightarrow U_\lambda^x = U_\mu^y$, by condition M2 of Definition 3.

(iii) $m_d(U_\lambda^x, U_\mu^y) = m_d(U_\mu^y, U_\lambda^x)$, by condition M3 of Definition 3.

(iv) $m_d(U_\lambda^x, U_\mu^y) \leq m_d(U_\lambda^x, U_\upsilon^z) + m_d(U_\mu^y, U_\upsilon^z)$, by condition M4 of Definition 3. Indeed, we have

$$\begin{aligned} m_d\bigl(U_\lambda^x, U_\mu^y\bigr) &= \max_{\eta \in A} d\bigl(U_\lambda^x, U_\mu^y\bigr)(\eta) \leq \max_{\eta \in A} d\bigl(U_\lambda^x, U_\upsilon^z\bigr)(\eta) + \max_{\eta \in A} d\bigl(U_\upsilon^z, U_\mu^y\bigr)(\eta) \\ &= m_d\bigl(U_\lambda^x, U_\upsilon^z\bigr) + m_d\bigl(U_\mu^y, U_\upsilon^z\bigr). \end{aligned}$$

(2a) Let $\{U_{\lambda,n}^x\}_n$ be a Cauchy sequence in $(\tilde{U}, d, A)$. Given $\varepsilon > 0$, take the constant soft real number $\overline{\varepsilon} \tilde{>} \bar{0}$. Then there exists an $m \in \mathbb{N}$ such that $d(U_{\lambda,i}^x, U_{\lambda,j}^x) \tilde{<} \overline{\varepsilon}$ for all $i, j \geq m$. Hence $d(U_{\lambda,i}^x, U_{\lambda,j}^x)(\eta) < \varepsilon$ for all $\eta \in A$ and $i, j \geq m$. Thus $m_d(U_{\lambda,i}^x, U_{\lambda,j}^x) < \varepsilon$ for all $i, j \geq m$. We deduce that $\{U_{\lambda,n}^x\}_n$ is a Cauchy sequence in $(m_d, \mathrm{SP}(\tilde{U}))$.

Conversely, let $\{U_{\lambda,n}^x\}_n$ be a Cauchy sequence in $(m_d, \mathrm{SP}(\tilde{U}))$. Given $\widehat{\varepsilon} \tilde{>} \bar{0}$, there exists $\varepsilon = \min_{\eta \in A} \widehat{\varepsilon}(\eta) > 0$, because A is a finite set. Then there exists an $m \in \mathbb{N}$ such that $m_d(U_{\lambda,i}^x, U_{\lambda,j}^x) < \varepsilon$ for all $i, j \geq m$. Hence $d(U_{\lambda,i}^x, U_{\lambda,j}^x)(\eta) < \varepsilon \leq \widehat{\varepsilon}(\eta)$ for all $\eta \in A$ and $i, j \geq m$. We deduce that $\{U_{\lambda,n}^x\}_n$ is a Cauchy sequence in $(\tilde{U}, d, A)$.

Feature (2b) follows from (2a) and (3) is a consequence of (2a) and (2b). □

4 Fixed point theorems for complete soft metric spaces

In [35] we established a Banach contraction principle for those complete soft metric spaces $(\tilde{U}, d, A)$ such that A is a (nonempty) finite set, and showed that the condition that A is finite cannot be omitted. We start this section by applying Theorem 1 to deduce the soft version of Banach's contraction principle cited above.

Theorem 2 [35] *Let $(\tilde{U}, d, A)$ be a complete soft metric space with A a finite set. Suppose that the soft mapping $f : \tilde{U} \tilde{\to} \tilde{U}$ satisfies*

$$d\bigl(f\bigl(U_\lambda^x\bigr), f\bigl(U_\mu^y\bigr)\bigr) \tilde{\leq} \overline{c} d\bigl(U_\lambda^x, U_\mu^y\bigr), \tag{1}$$

for all $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$, where $\bar{0} \tilde{\leq} \overline{c} \tilde{<} \bar{1}$. Then f has a unique fixed point, i.e., there is a unique soft point U_λ^x such that $f(U_\lambda^x) = U_\lambda^x$.

Proof Consider the metric m_d on $\mathrm{SP}(\tilde{U})$ as constructed in Theorem 1. Since $(\tilde{U}, d, A)$ is complete it follows from Theorem 1(3) that $(\mathrm{SP}(\tilde{U}), m_d)$ is a complete metric space.

Since for each $U_\lambda^x \in \mathrm{SP}(\tilde{U})$ (or equivalently, $U_\lambda^x \tilde{\in} \tilde{U}$) there is a unique soft point U_μ^y such that $f(U_\lambda^x) = U_\mu^y$ (see Remark 1), the restriction of f to $\mathrm{SP}(\tilde{U})$ is a self mapping on $\mathrm{SP}(\tilde{U})$, also denoted by f. Note also that the real number c generating the constant soft real number $\bar{c}$ satisfies $0 \leq c < 1$. Finally, we obtain the following contraction condition, for each $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$,

$$\begin{aligned} m_d\big(f\big(U_\lambda^x\big), f\big(U_\mu^y\big)\big) &= \max_{\eta \in A} d\big(f\big(U_\lambda^x\big), f\big(U_\mu^y\big)\big)(\eta) \leq \max_{\eta \in A} c\big(d\big(U_\lambda^x, U_\mu^y\big)(\eta)\big) \\ &= c\Big[\max_{\eta \in A} d\big(U_\lambda^x, U_\mu^y\big)(\eta)\Big] = c m_d\big(U_\lambda^x, U_\mu^y\big). \end{aligned}$$

Hence f has a unique fixed point by the Banach contraction principle. □

Remark 2 Example 3.22 of [35] shows that condition 'A is a finite set' cannot be omitted in the above theorem. In fact, it shows that 'A is a finite set' cannot be replaced with 'A is a countable set'.

Our next result provides a soft metric generalization of the celebrated Kannan fixed point theorem [40].

Theorem 3 *Let $(\tilde{U}, d, A)$ be a complete soft metric space with A a finite set. Suppose that the soft mapping $f : \tilde{U} \tilde{\rightarrow} \tilde{U}$ satisfies*

$$d\big(f\big(U_\lambda^x\big), f\big(U_\mu^y\big)\big) \tilde{\leq} \bar{c}\big\{d\big(U_\lambda^x, f\big(U_\lambda^x\big)\big) + d\big(U_\mu^y, f\big(U_\mu^y\big)\big)\big\}, \tag{2}$$

for all $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$, where $\bar{0} \tilde{\leq} \bar{c} \tilde{<} \overline{1/2}$. Then f has a unique fixed point.

Proof Since $(\tilde{U}, d, A)$ is complete it follows from Theorem 1(3) that $(\mathrm{SP}(\tilde{U}), m_d)$ is a complete metric space.

Moreover, the restriction of f to $\mathrm{SP}(\tilde{U})$ is a self mapping on $\mathrm{SP}(\tilde{U})$, exactly as in the proof of Theorem 2. Note also that the real number c generating the constant soft real number $\bar{c}$ satisfies $0 \leq c < 1/2$. Finally, we obtain the following contraction condition, for each $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$:

$$\begin{aligned} m_d\big(f\big(U_\lambda^x\big), f\big(U_\mu^y\big)\big) &= \max_{\alpha \in A} d\big(f\big(U_\lambda^x\big), f\big(U_\mu^y\big)\big)(\alpha) \\ &\leq \max_{\alpha \in A} c\big(\big(d\big(U_\lambda^x, f\big(U_\lambda^x\big)\big) + d\big(U_\mu^y, f\big(U_\mu^y\big)\big)\big)(\alpha)\big) \\ &\leq c\big\{m_d\big(U_\lambda^x, f\big(U_\lambda^x\big)\big) + m_d\big(U_\mu^y, f\big(U_\mu^y\big)\big)\big\}. \end{aligned}$$

Hence f has a unique fixed point by Kannan's fixed point theorem. □

The following modification of [35], Example 3.22, shows that condition 'A is a finite set' cannot be omitted in the preceding theorem (compare Remark 2).

Example 1 Let $U = A = \{1/n : n \in \mathbb{N}\}$. According to [34], Example 4.3, the mapping $d : \mathrm{SP}(\tilde{U}) \times \mathrm{SP}(\tilde{U}) \to \mathbb{R}(A)^*$ given by

$$d(U_\lambda^x, U_\mu^y) = |\overline{x} - \overline{y}| + |\overline{\lambda} - \overline{\mu}|,$$

for all $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$, where $|\cdot|$ denotes the modulus of soft real numbers, is a soft metric on $\tilde{U}$. Furthermore, the soft metric space $(\tilde{U}, d)$ is complete [35], Example 3.21.

Let $f : \tilde{U} \tilde{\to} \tilde{U}$ such that $f(U_\lambda^x) = U_1^{x/4}$ for all $x \in U$, $\lambda \in A$. We show that f satisfies the contraction condition (2) of Theorem 3 with $\overline{c} = \overline{1/3}$. In fact, given $x, y \in U$ and $\lambda, \mu \in A$, for each $\eta \in A$ we have

$$\begin{aligned} d(f(U_\lambda^x), f(U_\mu^y))(\eta) &= d(U_1^{x/4}, U_1^{y/4})(\eta) = \frac{1}{4}|x - y| \leq \frac{1}{4}(x + y) \\ &= \frac{1}{3}\left(\left(x - \frac{x}{4}\right) + \left(y - \frac{y}{4}\right)\right) \\ &\leq \frac{1}{3}\left(d(U_\lambda^x, f(U_\lambda^x)) + d(U_\mu^y, f(U_\mu^y))\right)(\eta). \end{aligned}$$

Therefore $d(f(U_\lambda^x), f(U_\mu^y)) \tilde{\leq} \overline{1/3}\{d(U_\lambda^x, f(U_\lambda^x)) + d(U_\mu^y, f(U_\mu^y))\}$. However, f has no fixed point.

Now we present an example where we can apply Theorem 3 but not Theorem 2.

Example 2 Let $U = \mathbb{R}^+$ and $A = \{0, 1\}$. Again, according to [34], Example 4.3, the mapping $d : \mathrm{SP}(\tilde{U}) \times \mathrm{SP}(\tilde{U}) \to \mathbb{R}(A)^*$ given by

$$d(U_\lambda^x, U_\mu^y) = |\overline{x} - \overline{y}| + |\overline{\lambda} - \overline{\mu}|,$$

for all $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$, is a soft metric on $\tilde{U}$. Since $\mathbb{R}^+$ is complete for the Euclidean metric, we deduce that $(\tilde{U}, d)$ is a complete soft metric space.

Let $f : \tilde{U} \tilde{\to} \tilde{U}$ such that $f(U_0^x) = f(U_1^x) = U_0^0$ if $x \in [0, 2)$, and $f(U_0^x) = f(U_1^x) = U_0^{1/2}$ if $x \in [2, \infty)$.

Let $\overline{c}$ be a constant soft real number such that $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \overline{1}$. Then there is a real number $c \in [0, 1)$ such that $c = \overline{c}(\eta)$ for all $\eta \in A$. Choose $y \in [0, 2)$ such that $c(2 - y) < 1/2$. Then, for each $\eta \in A$, we have

$$d(f(U_0^2), f(U_0^y))(\eta) = d(U_0^{1/2}, U_0^0)(\eta) = \frac{1}{2} > c(2 - y) = cd(U_0^2, U_0^y)(\eta).$$

Therefore f does not satisfy condition (1) of Theorem 2 for any $\overline{c}$ satisfying $\overline{0} \tilde{\leq} \overline{c} \tilde{<} \overline{1}$.

However, taking, without loss of generality, $x \in [0, 2)$ and $y \in [2, \infty)$, we obtain, for $\lambda, \mu, \eta \in A$,

$$\begin{aligned} d(f(U_\lambda^x), f(U_\mu^y))(\eta) &= d(U_0^0, U_0^{1/2})(\eta) = \frac{1}{2} = \frac{1}{3}\left(2 - \frac{1}{2}\right) \leq \frac{1}{3}\left(x + y - \frac{1}{2}\right) \\ &= \frac{1}{3}\left(d(U_\lambda^x, U_0^0) + d(U_\mu^y, U_\mu^{1/2})\right)(\eta). \end{aligned}$$

Therefore f satisfies condition (2) of Theorem 3 for $\overline{c} = \overline{1/3}$. In fact, U_0^0 is the unique fixed point of f.

Meir and Keeler proved in [41] their well-known fixed point theorem: every Meir-Keeler contractive self mapping on a complete metric space has a unique fixed point, where a self mapping T on a metric space (X,d) is said to be a Meir-Keeler contractive mapping if it satisfies the following condition:

for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in X$,

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \quad \Rightarrow \quad d\bigl(T(x), T(y)\bigr) < \varepsilon.$$

In a recent paper [36], Chen and Lin discussed the extension of the Meir and Keeler fixed point theorem to soft metric spaces. To this end, they introduced the following notion [36], Definition 15:

Let $(\tilde{U}, d, A)$ be a soft metric space and let $\varphi : A \to A$. A soft mapping $(f,\varphi) : \tilde{U} \tilde{\to} \tilde{U}$ is called a soft Keir-Meeler contractive mapping if it satisfies the following condition:

for each soft real number $\widehat{\varepsilon} \tilde{>} \bar{0}$ there exists $\widehat{\delta} \tilde{>} \bar{0}$ such that for each $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$,

$$\widehat{\varepsilon} \tilde{\leq} d\bigl(U_\lambda^x, U_\mu^y\bigr) \tilde{<} \widehat{\varepsilon} + \widehat{\delta} \quad \Rightarrow \quad d\bigl((f,\varphi)\bigl(U_\lambda^x\bigr), (f,\varphi)\bigl(U_\mu^y\bigr)\bigr) \tilde{<} \widehat{\varepsilon}.$$

Then Chen and Lin [36], Theorem 1, established that every soft Keir-Meeler contractive mapping on a complete soft metric space has a unique fixed point.

The following examples show that this result is not correct-even for the case that A is a finite set (the error in the proof seems to occur on page 4, lines 18-19: compare Definition 2 above).

Example 3 Let $U = \{2\}$, $A = \{0,1\}$, and d a soft metric on $\tilde{U}$ defined as:

$$d\bigl(U_\lambda^2, U_\lambda^2\bigr) = \overline{0} \quad \text{for all } \lambda \in A, \quad \text{and}$$

$$d\bigl(U_0^2, U_1^2\bigr)(0) = d\bigl(U_1^2, U_0^2\bigr)(0) = 0, \qquad d\bigl(U_0^2, U_1^2\bigr)(1) = d\bigl(U_1^2, U_0^2\bigr)(1) = 1.$$

Since m_d is the discrete metric on $\mathrm{SP}(\tilde{U})$ it follows from Theorem 1(3) that $(\tilde{U}, d)$ is a complete soft metric space.

For $f : U \to U$, it necessarily follows that $f(2) = 2$. Let $\varphi : A \to A$ given by $\varphi(0) = 1$ and $\varphi(1) = 0$. Then $(f,\varphi)(U_0^2) = U_1^2$ and $(f,\varphi)(U_1^2) = U_0^2$. From the fact that for each $\widehat{\varepsilon} \tilde{>} \bar{0}$ we have $d(U_0^2, U_1^2)(0) = 0 < \widehat{\varepsilon}(0)$, it follows that condition $\widehat{\varepsilon} \tilde{\leq} d(U_\lambda^2, U_\mu^2)$ is not satisfied for any $\lambda, \mu \in A$, and thus (f,φ) is trivially a soft Keir-Meeler contractive mapping on $(\tilde{U}, d)$. However, (f,φ) has no fixed point.

The above example suggests the following modification of [36], Definition 15.

Definition 7 Let $(\tilde{U}, d, A)$ be a soft metric space and let $\varphi : A \to A$. A soft mapping $(f,\varphi) : \tilde{U} \tilde{\to} \tilde{U}$ is called a soft contraction of Meir-Keeler type if it satisfies the following condition:

for each soft real number $\widehat{\varepsilon} \tilde{>} \bar{0}$ there exists $\widehat{\delta} \tilde{>} \bar{0}$ such that for each $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$,

$$d\bigl(U_\lambda^x, U_\mu^y\bigr) \tilde{<} \widehat{\varepsilon} + \widehat{\delta}, \quad \text{and} \quad \widehat{\varepsilon}(\eta) \leq d\bigl(U_\lambda^x, U_\mu^y\bigr)(\eta) \quad \text{for some } \eta \in A$$
$$\Rightarrow \quad d\bigl((f,\varphi)\bigl(U_\lambda^x\bigr), (f,\varphi)\bigl(U_\mu^y\bigr)\bigr) \tilde{<} \widehat{\varepsilon}.$$

Remark 3 Let $(\tilde{U}, d)$ be the complete soft metric space of Example 1. Define $f(x) = x/4$ for all $x \in U$, and $\varphi(\lambda) = 1$ for all $\lambda \in A$. Then $(f, \varphi)(U_\lambda^x) = U_1^{x/4}$ for all $x \in U$ and $\lambda \in A$. Although (f, φ) has no fixed point, it is easy to check that the conditions of Definition 7 hold. However, we can state the following positive result.

Theorem 4 *Let $(\tilde{U}, d, A)$ be a complete soft metric space with A a finite set. Then every soft contraction of Meir-Keeler type on $(\tilde{U}, d, A)$ has a unique fixed point.*

Proof We first note that, by Theorem 1(3), the metric space $(\mathrm{SP}(\tilde{U}), m_d)$ is complete.

Now let (f, φ) be a soft contraction of Meir-Keeler type on $(\tilde{U}, d, A)$. As in the proof of Theorem 2, the restriction of (f, φ) to $\mathrm{SP}(\tilde{U})$ is a self mapping on $\mathrm{SP}(\tilde{U})$, which is also denoted by (f, φ).

We want to show that (f, φ) is a Meir-Keeler contractive mapping on $(\mathrm{SP}(\tilde{U}), m_d)$. Indeed, given $\varepsilon > 0$ consider the constant soft real number $\overline{\varepsilon}$. Since $\overline{\varepsilon} \tilde{>} \bar{0}$, there exists $\widehat{\delta} \tilde{>} \bar{0}$ for which the conditions of Definition 7 are satisfied. Also, $\delta = \min_{\eta \in A} \widehat{\delta}(\eta) > 0$ because A is finite.

Take $U_\lambda^x, U_\mu^y \in \mathrm{SP}(\tilde{U})$ satisfying $\varepsilon \leq m_d(U_\lambda^x, U_\mu^y) < \varepsilon + \delta$. Then

$$d\big(U_\lambda^x, U_\mu^y\big)(\eta) < \varepsilon + \delta \leq (\overline{\varepsilon} + \widehat{\delta})(\eta),$$

for all $\eta \in A$, so $d(U_\lambda^x, U_\mu^y) \tilde{<} \overline{\varepsilon} + \widehat{\delta}$. Furthermore, from $\varepsilon \leq m_d(U_\lambda^x, U_\mu^y)$ we deduce that

$$\overline{\varepsilon}(\eta_0) \leq d\big(U_\lambda^x, U_\mu^y\big)(\eta_0),$$

where $m_d(U_\lambda^x, U_\mu^y) = d(U_\lambda^x, U_\mu^y)(\eta_0)$, $\eta_0 \in A$.

Since (f, φ) is a soft mapping of Meir-Keeler type, we deduce that

$$d\big((f, \varphi)\big(U_\lambda^x\big), (f, \varphi)\big(U_\mu^y\big)\big) \tilde{<} \overline{\varepsilon}.$$

From this relation it follows that $m_d((f, \varphi)(U_\lambda^x), (f, \varphi)(U_\mu^y)) < \varepsilon$. We deduce that (f, φ) is a Meir-Keeler contractive mapping on $(\mathrm{SP}(\tilde{U}), m_d)$. Hence (f, φ) has a unique fixed point. □

We conclude the paper by obtaining a soft metric extension of the celebrated Caristi-Kirk's [42, 43] theorem that a metric space (X, d) is complete if and only if every Caristi mapping on (X, d) has a fixed point.

Let us recall that a self mapping T on a metric space (X, d) is a Caristi mapping provided that there exists a lower semicontinuous function $\phi : X \to \mathbb{R}^+$ such that $d(x, T(x)) + \phi(T(x)) \leq \phi(x)$ for all $x \in X$.

Caristi proved that every Caristi mapping on a complete metric space has a fixed point, while Kirk proved that actually Caristi's fixed point theorem characterizes metric completeness.

In Definition 9 below we propose a notion of a soft Caristi mapping. To this end, we first generalize, in a natural way, Definition 4 to the case of a net. Thus, given a soft metric space $(\tilde{U}, d)$, we say that a net $\{U_{\lambda,\alpha}^x\}_{\alpha \in \Lambda}$ of soft points in $\tilde{U}$ is convergent in $(\tilde{U}, d)$ if there is a soft point U_μ^y such that for each $\widehat{\varepsilon} \tilde{>} \bar{0}$, there exists $\alpha_0 \in \Lambda$ satisfying $d(U_{\lambda,\alpha}^x, U_\mu^y) \tilde{<} \widehat{\varepsilon}$, whenever $\alpha \geq \alpha_0$.

Definition 8 Let $(\tilde{U},d,A)$ be a soft metric space. A mapping $\phi:\mathrm{SP}(\tilde{U})\to\mathbb{R}(A)^*$ is called lower semicontinuous on $(\tilde{U},d,A)$ if whenever $\{U^x_{\lambda,\alpha}\}_{\alpha\in\Lambda}$ is a net of soft points in $\tilde{U}$ that converges in $(\tilde{U},d,A)$ to a soft point U^y_μ, the following holds: for each $\widehat{\varepsilon}\tilde{>}\bar{0}$ there exists $\widehat{\delta}\tilde{>}\bar{0}$ such that $\phi(U^y_\mu)\tilde{\leq}\phi(U^x_{\lambda,\alpha})+\widehat{\varepsilon}$ whenever $d(U^x_{\lambda,\alpha},U^y_\mu)\tilde{<}\widehat{\delta}$.

Definition 9 Let $(\tilde{U},d,A)$ be a soft metric space. A soft mapping $f:\tilde{U}\tilde{\to}\tilde{U}$ is called a soft Caristi mapping if there exists a lower semicontinuous mapping $\phi:\mathrm{SP}(\tilde{U})\to\mathbb{R}(A)^*$ such that $d(U^x_\lambda,f(U^x_\lambda))+\phi(f(U^x_\lambda))\tilde{\leq}\phi(U^x_\lambda)$ for all $U^x_\lambda\in\mathrm{SP}(\tilde{U})$.

Theorem 5 *Let $(\tilde{U},d,A)$ be a soft metric space with A a finite set. Then $(\tilde{U},d,A)$ is complete if and only if every soft Caristi mapping on $(\tilde{U},d,A)$ has a fixed point.*

Proof Suppose that $(\tilde{U},d,A)$ is complete and let f be a soft Caristi mapping on $(\tilde{U},d,A)$. Then there exists a lower semicontinuous mapping $\phi:\mathrm{SP}(\tilde{U})\to\mathbb{R}(A)^*$ such that $d(U^x_\lambda,f(U^x_\lambda))+\phi(f(U^x_\lambda))\tilde{\leq}\phi(U^x_\lambda)$ for all $U^x_\lambda\in\mathrm{SP}(\tilde{U})$.

Exactly as in Theorem 2, the restriction of f to $\mathrm{SP}(\tilde{U})$ is a self mapping on $\mathrm{SP}(\tilde{U})$, also denoted by f. Now define $\Phi:\mathrm{SP}(\tilde{U})\to\mathbb{R}^+$ as

$$\Phi\bigl(U^x_\lambda\bigr)=\sum_{\eta\in A}\bigl(\phi\bigl(U^x_\lambda\bigr)(\eta)\bigr),$$

for all $U^x_\lambda\in\mathrm{SP}(\tilde{U})$. It is not difficult to check Φ is lower semicontinuous on the complete metric space $(\mathrm{SP}(\tilde{U}),m_d)$. Furthermore, given $U^x_\lambda\in\mathrm{SP}(\tilde{U})$ let $\eta_{x,\lambda}\in A$ such that $m_d(U^x_\lambda,f(U^x_\lambda))=d(U^x_\lambda,f(U^x_\lambda))(\eta_{x,\lambda})$. Then we obtain

$$\begin{aligned}m_d\bigl(U^x_\lambda,f\bigl(U^x_\lambda\bigr)\bigr)+\Phi\bigl(f\bigl(U^x_\lambda\bigr)\bigr)&=d\bigl(U^x_\lambda,f\bigl(U^x_\lambda\bigr)\bigr)(\eta_{x,\lambda})+\sum_{\eta\in A}\phi\bigl(f\bigl(U^x_\lambda\bigr)(\eta)\bigr)\\&\leq\phi\bigl(U^x_\lambda\bigr)(\eta_{x,\lambda})+\sum_{\eta\in A\setminus\{\eta_{x,\lambda}\}}\phi\bigl(f\bigl(U^x_\lambda\bigr)(\eta)\bigr)\\&=\Phi\bigl(U^x_\lambda\bigr).\end{aligned}$$

We deduce that f is a Caristi mapping on $(\mathrm{SP}(\tilde{U}),m_d)$, and hence it has a fixed point.

Conversely, suppose that every soft Caristi mapping on $(\tilde{U},d,A)$ has a fixed point, and let T be a Caristi mapping on the complete metric space $(\mathrm{SP}(\tilde{U}),m_d)$. Then there exists a lower semicontinuous function $\phi:\mathrm{SP}(\tilde{U})\to\mathbb{R}^+$ such that $m_d(U^x_\lambda,T(U^x_\lambda))+\phi(T(U^x_\lambda))\leq\phi(U^x_\lambda)$ for all $U^x_\lambda\in\mathrm{SP}(\tilde{U})$.

Let $f_T:\tilde{U}\to\tilde{U}$ such that $f_T(U^x_\lambda)=T(U^x_\lambda)$, and define $\varphi:\mathrm{SP}(\tilde{U})\to\mathbb{R}(A)^*$ such that $\varphi(U^x_\lambda)=\overline{\phi(U^x_\lambda)}$ for all $U^x_\lambda\tilde{\in}\tilde{U}$. Clearly φ is lower semicontinuous on $(\tilde{U},d,A)$. Then, for each $U^x_\lambda\tilde{\in}\tilde{U}$ and $\eta\in A$ we obtain

$$\begin{aligned}\bigl(d\bigl(U^x_\lambda,f_T\bigl(U^x_\lambda\bigr)\bigr)+\varphi\bigl(f_T\bigl(U^x_\lambda\bigr)\bigr)\bigr)(\eta)&\leq m_d\bigl(U^x_\lambda,T\bigl(U^x_\lambda\bigr)\bigr)+\phi\bigl(T\bigl(U^x_\lambda\bigr)\bigr)\\&\leq\phi\bigl(U^x_\lambda\bigr)=\varphi\bigl(U^x_\lambda\bigr)(\eta).\end{aligned}$$

We deduce that $d(U^x_\lambda,f(U^x_\lambda))+\varphi(f_T(U^x_\lambda))\tilde{\leq}\varphi(U^x_\lambda)$, and, consequently, f_T is a soft Caristi mapping on $(\tilde{U},d,A)$. Therefore f_T, and hence T has a fixed point. So, $(\mathrm{SP}(\tilde{U}),m_d)$ is complete by Caristi-Kirk's theorem. Completeness of $(\tilde{U},d,A)$ is now a consequence of Theorem 1(3). □

5 Conclusion

In an attempt to reverse the trend of obtaining soft metric extensions of existing fixed point results in the framework of ordinary metric spaces, we showed that a soft metric space, under the restriction that a set of parameters is finite, gives rise to a compatible metric on the collection of all soft points of absolute soft set. We also studied some essential properties of the induced metric thus obtained. We then proved that some recently obtained soft fixed point results can be directly deduced from existing comparable results in ordinary metric spaces. We presented examples to show that the restriction of a finite parameter set is unavoidable. It will be an interesting problem to study the limits to which soft fixed point theory may be extended.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this article. They read and approved the final manuscript.

Author details
[1]Department of Mathematics and Applied Mathematics, University of Pretoria, Hatfield, Pretoria, South Africa. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [3]Department of Mathematics, GC University, Faisalabad, Pakistan. [4]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Valencia, 46022, Spain.

Acknowledgements
Salvador Romaguera thanks the support of Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

Received: 25 August 2015 Accepted: 14 January 2016 Published online: 28 January 2016

References

1. Zadeh, LA: Fuzzy sets. Inf. Control **8**, 103-112 (1965)
2. Molodtsov, D: Soft set theory - first results. Comput. Math. Appl. **37**, 19-31 (1999)
3. Aktaş, H, Çağman, N: Soft sets and soft groups. Inf. Sci. **177**, 2726-2735 (2007)
4. Ali, MI, Feng, F, Liu, X, Min, WK, Shabir, M: On some new operations in soft set theory. Comput. Math. Appl. **57**, 1547-1553 (2009)
5. Feng, F, Liu, X, Leoreanu-Fotea, V, Jun, YB: Soft sets and soft rough sets. Inf. Sci. **181**, 1125-1137 (2011)
6. Jiang, Y, Tang, Y, Chen, Q, Wang, J, Tang, S: Extending soft sets with description logics. Comput. Math. Appl. **59**, 2087-2096 (2009)
7. Jun, YB: Soft BCK/BCI-algebras. Comput. Math. Appl. **56**, 1408-1413 (2008)
8. Jun, YB, Lee, KJ, Khan, A: Soft ordered semigroups. Math. Log. Q. **56**, 42-50 (2010)
9. Jun, YB, Lee, KJ, Park, CH: Soft set theory applied to ideals in d-algebras. Comput. Math. Appl. **57**, 367-378 (2009)
10. Jun, YB, Park, CH: Applications of soft sets in ideal theory of BCK/BCI-algebras. Inf. Sci. **178**, 2466-2475 (2008)
11. Kong, Z, Gao, L, Wang, L, Li, S: The normal parameter reduction of soft sets and its algorithm. Comput. Math. Appl. **56**, 3029-3037 (2008)
12. Majumdar, P, Samanta, SK: Generalized fuzzy soft sets. Comput. Math. Appl. **59**, 1425-1432 (2010)
13. Li, F: Notes on the soft operations. ARPN J. Syst. Softw. **1**, 205-208 (2011)
14. Maji, PK, Roy, AR, Biswas, R: An application of soft sets in a decision making problem. Comput. Math. Appl. **44**, 1077-1083 (2002)
15. Qin, K, Hong, Z: On soft equality. J. Comput. Appl. Math. **234**, 1347-1355 (2010)
16. Xiao, Z, Gong, K, Xia, S, Zou, Y: Exclusive disjunctive soft sets. Comput. Math. Appl. **59**, 2128-2137 (2009)
17. Xiao, Z, Gong, K, Zou, Y: A combined forecasting approach based on fuzzy soft sets. J. Comput. Appl. Math. **228**, 326-333 (2009)
18. Xu, W, Ma, J, Wang, S, Hao, G: Vague soft sets and their properties. Comput. Math. Appl. **59**, 787-794 (2010)
19. Yang, CF: A note on soft set theory. Comput. Math. Appl. **56**, 1899-1900 (2008)
20. Yang, X, Lin, TY, Yang, J, Li, Y, Yu, D: Combination of interval-valued fuzzy set and soft set. Comput. Math. Appl. **58**, 521-527 (2009)
21. Zhu, P, Wen, Q: Operations on soft sets revisited (2012). arXiv:1205.2857v1
22. Feng, F, Jun, YB, Liu, XY, Li, LF: An adjustable approach to fuzzy soft set based decision making. J. Comput. Appl. Math. **234**, 10-20 (2009)
23. Feng, F, Jun, YB, Zhao, X: Soft semirings. Comput. Math. Appl. **56**, 2621-2628 (2008)
24. Feng, F, Liu, X: Soft rough sets with applications to demand analysis. In: Int. Workshop Intell. Syst. Appl. (ISA 2009), pp. 1-4. (2009)
25. Herawan, T, Deris, MM: On multi-soft sets construction in information systems. In: Emerging Intelligent Computing Technology and Applications with Aspects of Artificial Intelligence, pp. 101-110. Springer, Berlin (2009)
26. Herawan, T, Rose, ANM, Deris, MM: Soft set theoretic approach for dimensionality reduction. In: Database Theory and Application, pp. 171-178. Springer, Berlin (2009)

27. Kim, YK, Min, WK: Full soft sets and full soft decision systems. J. Intell. Fuzzy Syst. **26**, 925-933 (2014). doi:10.3233/IFS-130783
28. Mushrif, MM, Sengupta, S, Ray, AK: Texture classification using a novel, soft-set theory based classification algorithm. Lect. Notes Comput. Sci. **3851**, 246-254 (2006)
29. Roy, AR, Maji, PK: A fuzzy soft set theoretic approach to decision making problems. J. Comput. Appl. Math. **203**, 412-418 (2007)
30. Zhu, P, Wen, Q: Probabilistic soft sets. In: IEEE Conference on Granular Computing (GrC 2010), pp. 635-638 (2010)
31. Zou, Y, Xiao, Z: Data analysis approaches of soft sets under incomplete information. Knowl.-Based Syst. **21**, 941-945 (2008)
32. Cagman, N, Karatas, S, Enginoglu, S: Soft topology. Comput. Math. Appl. **62**, 351-358 (2011)
33. Das, S, Samanta, SK: Soft real sets, soft real numbers and their properties. J. Fuzzy Math. **20**, 551-576 (2012)
34. Das, S, Samanta, SK: Soft metric. Ann. Fuzzy Math. Inform. **6**, 77-94 (2013)
35. Abbas, M, Murtaza, G, Romaguera, S: Soft contraction theorem. J. Nonlinear Convex Anal. **16**, 423-435 (2015)
36. Chen, CM, Lin, IJ: Fixed point theory of the soft Meir-Keeler type contractive mappings on a complete soft metric space. Fixed Point Theory Appl. **2015**, 184 (2015)
37. Feng, F, Li, CX, Davvaz, B, Ali, MI: Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Comput. **14**, 8999-9911 (2010)
38. Maji, PK, Biswas, R, Roy, AR: Soft set theory. Comput. Math. Appl. **45**, 555-562 (2003)
39. Wardowski, D: On a soft mapping and its fixed points. Fixed Point Theory Appl. **2013**, 182 (2013)
40. Kannan, R: Some results on fixed points II. Am. Math. Mon. **76**, 405-408 (1969)
41. Meir, A, Keeler, E: A theorem on contraction mappings. J. Math. Anal. Appl. **28**, 326-329 (1969)
42. Caristi, J: Fixed point theorems for mappings satisfying inwardness conditions. Trans. Am. Math. Soc. **215**, 241-251 (1976)
43. Kirk, WA: Caristi's fixed-point theorem and metric convexity. Colloq. Math. **36**, 81-86 (1976)