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ON THE FORCE ASSOCIATED WITH ABSORPTION OF SPECTRAL LINE RADIATION

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SUMMARY

The body force on stellar material produced when radiation is absorbed or scattered by atoms, that is, the force derived from the radiative stresses, is studied for the case in which the radiation is produced by line emission of a two-level atom according to the mechanism of non-coherent scattering with complete redistribution. The force is considered for cases with either Doppler or Voigt absorption profiles in static atmospheres with and without the effect of an overlapping continuum, and also in rapidly expanding atmospheres. Approximate analytic results are obtained in several asymptotic regimes. It is suggested that for static atmospheres the force is represented to order-of-magnitude accuracy by the result for LTE with the damping wings and the overlapping continuum omitted. This gives a simple relation, that for optical depths which are greater than unity in the line, but less than unity in the continuum, the force varies as the inverse of the optical depth. The force due to the line alone is very small for optical depths greater than unity in the continuum. In rapidly expanding atmospheres the force is also represented by this formula, except that the optical depth takes the velocity gradient into account. A consequence of these relations is that the force due to an optically thick line is independent of the strength of the line.

1. INTRODUCTION

The most plausible explanation for stellar winds in early-type stars appears to be a flow which is driven by the force of radiation in the spectral lines acting on the material (*cf.* Lucy & Solomon 1970). Simple estimates, based in part on the results of this paper, and to be reported elsewhere (Castor 1974, in preparation) indicate that a very large number of lines may make important contributions to the total force. In such a situation one must use the simplest possible expressions for the radiation force due to individual lines in terms of atmospheric properties such as temperature, velocity, and optical depth. The purpose of this paper is to gather in one place a number of results of this kind, for subsequent use in a statistical treatment of the line contribution to momentum balance in the flow. Some of these results are believed to be new; others have been reported previously. It was thought to be most useful if all the results were brought together in a homogeneous notation.

The force contributed by radiation in a spectral line is computed in this paper for three radiative transfer situations: a line formed by a two-level atom according to the process of non-coherent scattering and for which continuum processes are neglected; the same case with continuum processes included; and the case of a line formed according to non-coherent scattering in a spherical atmosphere with a large velocity gradient, for which continuum processes are again neglected.

These cases are treated in Sections 2, 3 and 4 of the present paper, respectively. The main object in each case is to obtain an asymptotic expression for the radiation force in the relevant regimes of the parameters. The techniques used are drawn heavily from the monograph by Ivanov (1973), except in the third case for which the result has been given by Sobolev (1957).

2. TWO-LEVEL ATOM WITHOUT CONTINUUM

In this model it is assumed that the line is formed by transitions between two bound states of an atom, with all other states and other stages of ionization being neglected. Further, it is supposed that the only processes affecting the population of the two states are the radiative transitions themselves, and inelastic collisions of the atom in either state with another particle such as an electron. It is also assumed that a photon emitted by one of these atoms can be absorbed only by a similar atom, i.e. that there is no competing absorption mechanism. In this case the mass absorption coefficient, that is, that quantity which when multiplied by the mass density gives the probability per unit distance of an absorption, is given by

$$\kappa_L \phi(x).$$

In this expression x is the frequency displacement from the centre frequency of the line, measured in units of the Doppler width. The latter quantity is $\Delta\nu_D = \nu_0 v_{th}/c$ where $v_{th}/\sqrt{2}$ is the rms value of a component of the atom's velocity. In this section it will be assumed that the stellar material is at rest, so $\phi(x)$ will be independent of direction and given by the usual Doppler or Voigt formula. It is assumed to be normalized:

$$\int_{-\infty}^{\infty} \phi(x)dx = 1.$$

The energy emitted in the line near the frequency x for a unit mass and unit intervals of time, solid angle and x is equal to

$$\kappa_L \phi(x) S$$

where S is the line source function, which can be expressed in terms of the populations of the atomic states. On the assumption that the photons are completely redistributed in angle and frequency after each scattering, S is independent of angle and frequency and depends only on depth in the atmosphere. With the additional assumption of plane symmetry, S is given by

$$S = \frac{1-\epsilon}{2} \int_{-1}^{1} d\mu \int_{-\infty}^{\infty} \phi(x)dx I(x, \mu) + \epsilon B \tag{2.1}$$

in terms of the intensity of the radiation at frequency x and angle $\cos^{-1}\mu$ to the outward direction. In this expression B is the Planck function at the line frequency for the local electron temperature, and ϵ is the probability that an excited atom will not decay radiatively. When the equation of transfer is introduced, and the familiar manipulations are performed (see, for example, Hummer & Rybicki 1971), the following is obtained:

$$S(\tau) = (1-\epsilon) \int_0^{\infty} S(t) K_1(t-\tau)dt + \epsilon B(\tau). \tag{2.2}$$

This is the basic equation of the two-level atom transfer problem. The kernel function $K_1(t)$ which appears here is defined as

$$K_1(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} \phi^2(x) E_1(|t|\phi(x))dx$$

and its properties are studied in detail by Ivanov (1973, this work will be referred to henceforth as IB; it should be noted that Ivanov uses different conventions on normalization than those of the present paper). The independent variable in equation (2.2) is line optical depth defined by

$$\tau = \int_r^{\infty} \kappa_L \rho dr'.$$

Equation (2.2) may be solved by a variety of techniques. The solution will be quoted here in the form which arises from an application of the Wiener–Hopf method (Kourganoff 1963; Busbridge 1960). Let equation (2.2) be used to define $S(\tau)$ for all real values of τ. ($B(\tau) = 0$ for $\tau < 0$.) Separate S into positive and negative parts as follows:

$$S(\tau) = S_+(\tau) + S_-(\tau)$$

$$S_+(\tau) = 0,\ \tau < 0;\quad S_-(\tau) = 0,\ \tau \geq 0,$$

so that equation (2.2) may be written

$$S_+(\tau) + S_-(\tau) = (1-\epsilon)\int_{-\infty}^{\infty} S_+(t) K_1(t-\tau)dt + \epsilon B(\tau). \tag{2.3}$$

The Fourier transform, defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ik\tau} f(\tau) d\tau$$

is applied to equation (2.3), with the result

$$\tilde{S}_+(k)[1-(1-\epsilon)\tilde{K}_1(k)] + \tilde{S}_-(k) = \epsilon \tilde{B}(k). \tag{2.4}$$

The function $1-(1-\epsilon)\tilde{K}_1(k)$ is then factored as follows

$$1-(1-\epsilon)\tilde{K}_1(k) = \frac{1}{H(1/ik)H(-1/ik)},$$

where $H(1/ik)$ is analytic and non-vanishing in the complex k plane cut on the positive imaginary axis. The function $H(u)$ is studied in detail in IB (Chapter 5, Section 4). Multiplication of equation (2.4) by $H(-1/ik)$ gives

$$\frac{\tilde{S}_+(k)}{H(1/ik)} + H\left(-\frac{1}{ik}\right)\tilde{S}_-(k) = \epsilon H\left(-\frac{1}{ik}\right)\tilde{B}(k). \tag{2.5}$$

An inspection of equation (2.3), bearing in mind that K_1 is normalizable, indicates that $\tilde{S}_+(k)$ is analytic in the lower half plane, not including the real axis, while $\tilde{S}_-(k)$ is analytic in the upper half plane, including the real axis. A similar decomposition of the right-hand side of equation (2.5) can be effected with Cauchy's integral theorem. This yields

$$\epsilon H\left(-\frac{1}{ik}\right)\tilde{B}(k) = \frac{1}{2\pi i}\int_{C+}\epsilon H\left(-\frac{1}{ik'}\right)\tilde{B}(k')\frac{dk'}{k'-k} + \frac{1}{2\pi i}\int_{C-}\epsilon H\left(-\frac{1}{ik'}\right)\tilde{B}(k')\frac{dk'}{k'-k},$$

where the contours C_+ and C_- are illustrated in Fig. 1. The origin is included within C_+. When the terms which are analytic in each region are identified on each side of equation (2.5), the following result is found

$$\tilde{S}_+(k) = H\left(\frac{1}{ik}\right)\frac{1}{2\pi i}\int_{C+}\epsilon H\left(-\frac{1}{ik'}\right)\tilde{B}(k')\frac{dk'}{k'-k}. \qquad (2.6)$$

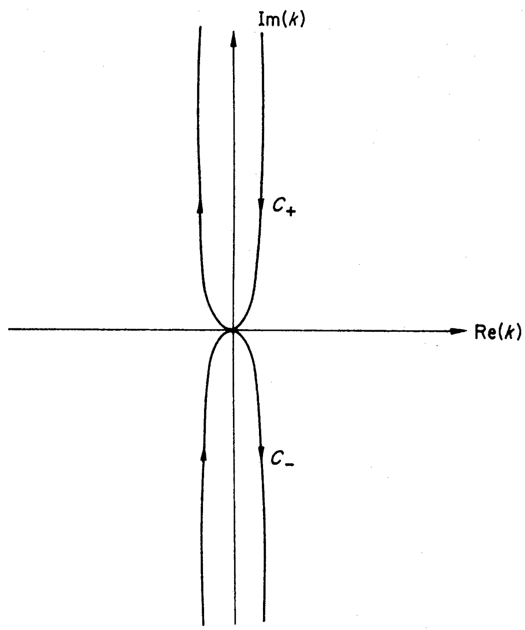

FIG. 1. *Contours in the complex k plane used for the integral in equation* (2.6), *and elsewhere.*

The uniqueness of this separation and the neglect of the integrals over large semicircles is guaranteed by the fact that $\tilde{S}_+$, $\tilde{S}_-$, and $\tilde{B}$, being Laplace transforms of functions which are finite at the origin, are $O(1/k)$ for large k.

If a specific form is taken for $B(\tau)$, the contour integral in equation (2.6) can be evaluated. An application of the inversion formula for the Laplace transform (another integral over the contour C_+) then gives $S(\tau)$. This last step will not be required. Since the neglect of continuous absorption can be justified only if the continuous optical depth is small, the most realistic assumption regarding $B(\tau)$ is that it is constant. (Chromospheres are ruled out by this assumption.) In that case $\tilde{B}(k) = B/ik$ so that the integrand in equation (2.6) is analytic within the contour, except for a simple pole at the origin. Equation (2.6) becomes ($H(\infty) = 1/\sqrt{\epsilon}$)

$$\tilde{S}_+(k) = \sqrt{\epsilon}\,\frac{1}{ik}\,H\left(\frac{1}{ik}\right)B, \qquad (2.7)$$

which is the form that will be used in the succeeding development.

Once the source function has been obtained, the next problem is to evaluate the radiation force. This is computed by first finding the rate at which a unit mass of material absorbs energy from a parallel beam of monochromatic radiation. This quantity is then divided by c to give the rate at which momentum is absorbed, and multiplied by μ to give the rate at which the radial component of momentum is absorbed. Finally, an integration over all frequencies and angles gives the force per unit mass of material. There is no force associated with the emission process since the emission occurs isotropically. The following expression is found

$$f_{\text{rad}} = \frac{2\pi\kappa_{\text{L}}\Delta\nu_{\text{D}}}{c}\int_{-1}^{1}\mu d\mu\int_{-\infty}^{\infty}\phi(x)dxI(x,\mu). \tag{2.8}$$

(See also Milne 1930, p. 100.) From the same analysis as that which led to equation (2.2) one finds

$$f_{\text{rad}} = \frac{4\pi\kappa_{\text{L}}\Delta\nu_{\text{D}}}{c}\int_{0}^{\infty}S(t)K_{22}(t-\tau)dt, \tag{2.9}$$

where the kernel function $K_{22}(t)$, following the notation of IB, is

$$K_{22}(t) = \text{sgn}(t)\,\tfrac{1}{2}\int_{-\infty}^{\infty}\phi^2(x)E_2(|t|\phi(x))dx.$$

It will be convenient to write this as the derivative of another kernel:

$$K_{22}(t) = -\frac{d}{dt}K_{31}(t),$$

$$K_{31}(t) = \tfrac{1}{2}\int_{-\infty}^{\infty}\phi(x)E_3(|t|\phi(x))dx. \tag{2.10}$$

Since equation (2.9) expresses f_{rad} as the convolution of K_{22} with S_+, it suggests that the Fourier transform be applied to equation (2.9). The result of this operation, and use of the relation between K_{22} and K_{31}, is

$$\tilde{f}_{\text{rad}}(k) = \frac{4\pi\kappa_{\text{L}}\Delta\nu_{\text{D}}}{c}ik\tilde{K}_{31}(k)\tilde{S}_+(k). \tag{2.11}$$

Finally, equations (2.7) and (2.11) can be combined to give

$$\tilde{f}_{\text{rad}}(k) = \frac{4\pi\kappa_{\text{L}}\Delta\nu_{\text{D}}}{c}\sqrt{\epsilon}\,B\tilde{K}_{31}(k)H\left(\frac{1}{ik}\right). \tag{2.12}$$

Two results of interest can be obtained directly from equation (2.12). First, by setting ϵ to unity and noting that in that case the H function is identically unity, the LTE result is found:

$$f_{\text{rad}}(\tau) = \frac{4\pi\kappa_{\text{L}}\Delta\nu_{\text{D}}}{c}BK_{31}(\tau). \tag{2.13}$$

Second, suppose it is desired to find f_{rad} for optical depths large compared with the thermalization depth (where $1-\tilde{K}_1(1/\tau) \approx \epsilon$). The main contributions to f_{rad} in the Fourier inversion will come from values of k for which $|1-\tilde{K}_1(k)| \ll \epsilon$ and therefore $H(1/ik) = 1/\sqrt{\epsilon}$. It follows that the result (2.13) obtains in that case also.

No simple analytic result has been found for the value of $f_{\rm rad}$ at $\tau = 0$. Equation (2.12) suggests that it should be of the order $4\pi\kappa_{\rm L}\Delta\nu_{\rm D}\sqrt{\epsilon}\, B/c$ since $H(1/ik)$ and K_{31} are both of order unity for k of order unity. The main result of interest to be found from equation (2.12) is the behaviour of $f_{\rm rad}$ for optical depths which are large compared with unity, but small compared with the thermalization depth. That is, τ is made to become infinite, and simultaneously ϵ tends to zero, so that $\epsilon/(1-\tilde{K}_1(1/\tau))$ tends to zero. As ϵ tends to zero, the H function tends to a limit, the H function for conservative scattering, designated $\tilde{H}(z)$ in IB. (This notation will be avoided.) This limit is non-uniform in z, but for any value of ϵ, $H(z)$ agrees with the limiting value to moderate accuracy if z is smaller than the thermalization depth for that value of ϵ. The accuracy is better the smaller is z with respect to the thermalization depth. The assumption that τ is small compared with the thermalization depth guarantees that the main contributions to $f_{\rm rad}$ come from values of k for which the H function is well approximated by the conservative function. Therefore H in equation (2.12) will be considered to be that function. The next step is to seek limiting expressions for $\tilde{K}_{31}(k)$ at small k, and $H(z)$ at large z, and introduce these into equation (2.12). The Fourier inversion then gives $f_{\rm rad}$ for large τ. Since these limiting forms depend on the line profile function, the discussion will be carried out separately for the two profiles of interest: Doppler and Voigt.

2.1 *Doppler profile*

The inversion formula for the Fourier transform is applied to equation (2.12), following which the path of integration is deformed in the k plane so that it follows C_+ in reverse. This gives the result

$$f_{\rm rad}(\tau) = \frac{4\pi\kappa_{\rm L}\Delta\nu_{\rm D}}{c}\sqrt{\epsilon}\, B\,\frac{1}{2\pi i}\int_0^\infty e^{-\tau y}dy\left[H\left(-\frac{1}{y}+i\delta\right)\tilde{K}_{31}(iy-\delta) - H\left(-\frac{1}{y}-i\delta\right)\tilde{K}_{31}(iy+\delta)\right].$$

δ is a positive infinitesimal. With the substitution $t = y\tau$, this becomes

$$f_{\rm rad}(\tau) = \frac{4\pi\kappa_{\rm L}\Delta\nu_{\rm D}}{c}\frac{\sqrt{\epsilon B}}{\tau}\frac{1}{2\pi i}\int_0^\infty e^{-t}dt\left[H\left(-\frac{\tau}{t}+i\delta\right)\tilde{K}_{31}\left(\frac{it}{\tau}-\delta\right) - H\left(-\frac{\tau}{t}-i\delta\right)\tilde{K}_{31}\left(\frac{it}{\tau}+\delta\right)\right]. \quad (2.14)$$

As τ tends to ∞ with t fixed, τ/t tends to ∞ and t/τ tends to 0. Therefore, expansions of H and $\tilde{K}_{31}$ for large and small argument, respectively, are required.

The asymptotic form of $H(z)$ can be found using the technique of IB, Section 5.4, but generalized to complex values of z not lying on the negative real axis. The method proceeds from the integral equation (IB 5.4.18′)

$$\frac{1}{2}H(z)\int_0^\infty \frac{z'H(z')}{z+z'}G(z')dz' = 1 \quad (2.15)$$

with

$$G(z) = 2\int_{x(z)}^\infty \phi^2(x)dx$$

where $x(z)$ is the solution of $\phi(x) = 1/z$. For the Doppler profile, $G(z)$ is related to the complementary error function, so that its asymptotic form is known. The asymptotic form for $H(z)$ is then found by inserting a hypothetical form for $H(z)$, z complex, with the known form for $G(z)$ into equation (2.15) and requiring that the resultant equation be satisfied, in the limit of z large. In this way one finds

$$H(z) \sim \frac{2}{\sqrt{\pi}} z^{1/2} \left(\ln \frac{|z|}{\sqrt{\pi}}\right)^{1/4}, \qquad (2.16)$$

$$-\pi < \arg z < \pi.$$

The kernel K_{31} is defined by equation (2.10). Applying the Fourier transform gives

$$\tilde{K}_{31}(k) = 2 \int_0^\infty \frac{\phi^3(x)}{k^3} \left[\frac{k}{\phi(x)} - \tan^{-1} \frac{k}{\phi(x)}\right] dx. \qquad (2.17)$$

The variable of integration in equation (2.17) is now changed to y, defined by

$$x = yx(1/|k|)$$

with the result

$$\tilde{K}_{31}(k) = 2x(1/|k|) \int_0^\infty \frac{\phi^3(yx(1/|k|))}{k^3} \left[\frac{k}{\phi(yx(1/|k|))} - \tan^{-1}\left(\frac{k}{\phi(yx(1/|k|))}\right)\right] dy.$$

It is possible at this point to use a special property of the Doppler profile:

$$\lim_{k\to 0} \frac{\phi(yx(1/|k|))}{|k|} = \lim_{k\to 0} \frac{\phi(yx(1/|k|))}{\phi(x(1/|k|))} = \begin{cases} \infty & y < 1 \\ 0 & y > 1 \end{cases}$$

and the following two limits

$$\lim_{z\to 0} \frac{1}{z^3}(z - \tan^{-1} z) = \frac{1}{3}$$

$$\lim_{z\to \infty} \frac{1}{z^3}(z - \tan^{-1} z) = 0 \qquad (\mathrm{Re}\, z \neq 0)$$

to find

$$\tilde{K}_{31}(k) \sim \frac{2}{3} x(1/|k|) = \frac{2}{3} \sqrt{\ln \frac{1}{\sqrt{\pi}|k|}}. \qquad (2.18)$$

Inserting equations (2.16) and (2.18) into equation (2.14), one finds

$$f_{\mathrm{rad}}(\tau) \sim \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c} \frac{\sqrt{\epsilon B}}{\sqrt{\tau}} \frac{4}{3\pi^{3/2}} \int_0^\infty t^{-1/2} e^{-t} \left(\ln \frac{\tau}{\sqrt{\pi t}}\right)^{3/4} dt,$$

but, since for large τ with t fixed,

$$\ln \frac{\tau}{\sqrt{\pi t}} \sim \ln \frac{\tau}{\sqrt{\pi}},$$

the logarithmic factor can be taken outside the integral to give the final result

$$f_{\mathrm{rad}}(\tau) \sim \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c} \frac{\sqrt{\epsilon B}}{\sqrt{\tau}} \frac{4}{3\pi} \left(\ln \frac{\tau}{\sqrt{\pi}}\right)^{3/4}. \qquad (2.19)$$

This can be compared with the asymptotic form of the source function itself for the same regime (IB 5.5.42b)

$$S(\tau) \sim \sqrt{\epsilon} B \frac{4\sqrt{\tau}}{\pi} \left(\ln \frac{\tau}{\sqrt{\pi}}\right)^{1/4}. \tag{2.20}$$

2.2 *Voigt profile*

An analysis analogous to that which led to equation (2.16) gives in the case of a Voigt profile

$$H(z) \sim \left(\frac{9z}{2\pi a}\right)^{1/4},$$

$$-\pi < \arg z < \pi. \tag{2.21}$$

A different technique must be used to evaluate $\tilde{K}_{31}$ for the Voigt profile than that used in the Doppler case. The method of Abramov, Dykhne and Napartovich, described in IB, Section 2.4, is suitable for this case. Let

$$k = |k| e^{i\alpha}.$$

The substitution

$$y = \frac{|k|}{\phi(x)}$$

is made in equation (2.17), giving

$$\tilde{K}_{31}(k) = 2 \int_0^\infty \frac{e^{-3i\alpha}}{y^3} (ye^{i\alpha} - \tan^{-1}(ye^{i\alpha})) \frac{x'(y/|k|)dy}{|k|}, \tag{2.22}$$

where the derivative of the function $x(z)$ appears. Since for fixed y and small k, x becomes large, the asymptotic form of $x(z)$ for the Voigt profile may be used:

$$x(z) \sim \left(\frac{az}{\pi}\right)^{1/2}$$

$$x'(z) \sim \frac{1}{2}\left(\frac{a}{\pi z}\right)^{1/2}.$$

When these results are used in equation (2.22) one gets

$$\tilde{K}_{31}(k) \sim \left(\frac{a}{\pi|k|}\right)^{1/2} \int_0^\infty \frac{e^{-3i\alpha}}{y^{7/2}} (ye^{i\alpha} - \tan^{-1}(ye^{i\alpha}))dy. \tag{2.23}$$

The further substitution

$$u = ye^{i\alpha}$$

then gives

$$\tilde{K}_{31}(k) \sim \left(\frac{a}{\pi|k|e^{i\alpha}}\right)^{1/2} \int_0^\infty u^{-7/2}(u - \tan^{-1} u)du,$$

where the path of integration is the line arg $(u) = \alpha$. This path may be deformed to become either the positive or negative real axis, depending on α. The result when the integral is evaluated is

$$\int_0^\infty u^{-7/2}(u-\tan^{-1}u)du = \frac{2}{5}\int_0^\infty \frac{u^{-1/2}du}{1+u^2} = \begin{cases} -i\dfrac{\pi\sqrt{2}}{5} & -\pi < \alpha < -\dfrac{\pi}{2} \\ \dfrac{\pi\sqrt{2}}{5} & -\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}, \\ i\dfrac{\pi\sqrt{2}}{5} & \dfrac{\pi}{2} < \alpha < \pi \end{cases}$$

which gives the asymptotic form of $\tilde{K}_{31}$:

$$\tilde{K}_{31}(k) \sim \begin{cases} \dfrac{1}{5}\sqrt{\dfrac{2\pi a}{-k}} & \text{Re}(k) < 0 \\ \dfrac{1}{5}\sqrt{\dfrac{2\pi a}{k}} & \text{Re}(k) > 0. \end{cases} \tag{2.24}$$

The asymptotic expression for f_{rad} is obtained by combining equations (2.14), (2.21) and (2.24):

$$f_{\text{rad}} \sim \frac{4\pi\kappa_L\Delta\nu_D}{c}\sqrt{\epsilon}B\,\frac{\Gamma(1/4)}{5\pi}\left(\frac{18\pi a}{\tau}\right)^{1/4}. \tag{2.25}$$

The corresponding expression for the source function (IB 5.5.42c) is

$$S(\tau) \sim \sqrt{\epsilon}B\,\frac{4}{\Gamma(1/4)}\left(\frac{9\tau}{2\pi a}\right)^{1/4}. \tag{2.26}$$

It is also useful, in conjunction with equation (2.13), to have asymptotic expressions for $K_{31}(\tau)$. These follow directly from IB equation (2.6.38). For the Doppler profile K_{31} becomes

$$K_{31}(\tau) \sim \frac{1}{6\tau\sqrt{\ln \tau/\sqrt{\pi}}}, \tag{2.27}$$

and for the Voigt profile it is

$$K_{31}(\tau) \sim \frac{1}{5}\sqrt{\frac{a}{\tau}}. \tag{2.28}$$

3. TWO-LEVEL ATOM WITH CONTINUUM

The assumptions in this case are identical with those in the preceding case with the exceptions that a background continuum is assumed to exist, so that photons are emitted and absorbed by continuum processes as well as by line processes, and that the Planck function is not assumed to be constant with depth. It is assumed that the continuum is in LTE, so that the continuum source function is equal to the Planck function. The relative magnitudes of line and continuum absorption coefficients are given by the parameter

$$\beta \equiv \frac{\kappa_C}{\kappa_L}$$

which, with ϵ, is assumed to be constant with depth.

Equation (2.1) for the line source function remains valid, but, since the equation of transfer is complicated by absorption and emission at the line frequency in the continuum, equation (2.2) assumes a modified form

$$S(\tau) = (1-\epsilon)\int_0^\infty K_1(\tau-t,\beta)S(t)dt+(1-\epsilon)\beta\int_0^\infty K_{11}(\tau-t,\beta)B(t)dt+\epsilon B(\tau) \quad (3.1)$$

in which new kernels $K_1(t, \beta)$ and $K_{11}(t, \beta)$ appear. These functions are defined and discussed in IB, Section 7.1. (The functions as they will be used in this paper are one half times those defined in IB.) The normalization of K_1 is given by

$$\int_{-\infty}^{\infty} K_1(\tau,\beta)d\tau = 1-\beta\delta(\beta) \quad (3.2)$$

where

$$\delta(\beta) = \int_{-\infty}^{\infty}\frac{\phi(x)}{\phi(x)+\beta}dx.$$

The function $\delta(\beta)$ is essentially the ordinate of the curve of growth, just as $1/\beta$ is the abscissa. Equation (3.1) is solved by the same procedure as in the absence of a continuum. The Fourier transform of the equation is taken, with the result

$$\tilde{S}_+(1-(1-\epsilon)\tilde{K}_1)+\tilde{S}_- = (\epsilon+(1-\epsilon)\beta\tilde{K}_{11})\tilde{B}. \quad (3.3)$$

($B(\tau)$ is taken to vanish for negative τ.) Introducing the H function exactly as before, we see that equation (3.3) becomes

$$\frac{\tilde{S}_+(k)}{H(1/ik)}+H\left(-\frac{1}{ik}\right)\tilde{S}_-(k) = H\left(-\frac{1}{ik}\right)(\epsilon+(1-\epsilon)\beta\tilde{K}_{11}(k))\tilde{B}(k). \quad (3.4)$$

Let the right-hand side of equation (3.4) be used to define a function $\Omega(k)$:

$$\Omega(k) = H\left(-\frac{1}{ik}\right)[\epsilon+(1-\epsilon)\beta\tilde{K}_{11}(k)]\tilde{B}(k). \quad (3.5)$$

This function is separated into parts which are analytic in the upper and lower half planes by means of the Cauchy integral,

$$\Omega(k) = \frac{1}{2\pi i}\int_{C_+}\frac{\Omega(k')dk'}{k'-k}+\frac{1}{2\pi i}\int_{C_-}\frac{\Omega(k')dk'}{k'-k} = \Omega_+(k)+\Omega_-(k).$$

Therefore $\tilde{S}_+(k)$ is given by

$$\tilde{S}_+(k) = H\left(\frac{1}{ik}\right)\Omega_+(k) = H\left(\frac{1}{ik}\right)\frac{1}{2\pi i}\int_{C_+}\frac{\Omega(k')dk'}{k'-k}. \quad (3.6)$$

The intermediate steps are easily justified, as before, since for large k the H function is unity, $\tilde{K}_{11}$ vanishes, and $\tilde{S}$ functions and $\tilde{B}$ are $O(1/k)$. The quantity $\beta\delta(\beta)$ represents the destruction probability of photons by continuous absorption, just as ϵ is the destruction probability by collisional de-excitation of the upper state. If $\beta\delta(\beta)$ is much less than ϵ, then since

$$\tilde{K}_{11}(0) = \int_{-\infty}^{\infty} K_{11}(t,\beta)dt = \delta(\beta), \quad (3.7)$$

it follows that the $\tilde{K}_{11}$ term in equation (3.5) can be neglected, with the result that equation (3.6) reduces to equation (2.6). In that case β also has no effect on the H function, since β affects $\tilde{K}_1(k)$ only for values of k such that

$$|1-\tilde{K}_1(k)| < \epsilon,$$

and therefore where $H(1/ik) \approx 1/\sqrt{\epsilon}$. The line source function for the case that $\beta\delta(\beta)$ is much smaller than ϵ is therefore the same as in the absence of a continuum, discussed in the previous section.

The overlapping continuum has a dominant effect if $\beta\delta(\beta)$ is large compared with ϵ. In this case it is ϵ which has a negligible effect on the source function. In many cases of interest the Planck function is essentially constant for line optical depths which are small compared with $1/\beta$, that is, for small continuum optical depths. In that case the major contributions to the integral in equation (3.6) come from values of k' of order β. Therefore if k is large compared with β, we can write

$$\tilde{S}_+(k) \approx \frac{1}{ik} H\left(\frac{1}{ik}\right) . \frac{-1}{2\pi} \int_{C+} \Omega(k')dk'. \tag{3.8}$$

The integral in this equation can be related to the surface value of the source function:

$$S(0) = \lim_{k\to\infty} ik\tilde{S}_+(k) = -H(0)\frac{1}{2\pi}\int_{C+}\Omega(k')dk' = -\frac{1}{2\pi}\int_{C+}\Omega(k')dk' \tag{3.9}$$

and so the transform of the source function for k large compared with β becomes

$$\tilde{S}_+(k) = S(0)\frac{1}{ik} H\left(\frac{1}{ik}\right). \tag{3.10}$$

This is the same as equation (2.7) for the case $\beta = 0$, except that $S(0)$ is no longer equal to $\sqrt{\epsilon}B$. The H function is also different, in principle, but for $k \gg \beta$ the H function is well approximated by that for conservative scattering, considered above. The value of $S(0)$ can be found from equations (3.5) and (3.9) if a form is taken for $B(\tau)$. In particular, if B is constant, then $S(0)$ becomes

$$S(0) = \frac{-1}{2\pi i}\int_{C+} H\left(-\frac{1}{ik}\right)[\epsilon+(1-\epsilon)\beta\tilde{K}_{11}(k',\beta)]B\frac{dk'}{k'}. \tag{3.11}$$

The singularities of the integrand within the contour consist of a simple pole at the origin, and the branch cut associated with $\tilde{K}_{11}$. The residue at the pole is

$$H(\infty)[\epsilon+(1-\epsilon)\beta\tilde{K}_{11}(0,\beta)]B.$$

Using equation (3.2) and the definition of the H function gives

$$H(\infty) = (\epsilon+(1-\epsilon)\beta\delta(\beta))^{-1/2}.$$

In the notation of IB,

$$\tilde{\lambda} = (1-\epsilon)(1-\beta\delta(\beta))$$

and

$$1-\tilde{\lambda} = \epsilon+(1-\epsilon)\beta\delta(\beta), \tag{3.12}$$

so that the residue at the pole is

$$(1-\tilde{\lambda})^{1/2}B.$$

To evaluate the contribution to $S(0)$ of the cut, we require the representation of $\tilde{K}_{11}$:

$$\tilde{K}_{11}(k, \beta) = \int_\beta^\infty \frac{1}{s^2+k^2} G_0\left(\frac{1}{s-\beta}\right) ds, \tag{3.13}$$

where

$$G_0(z) = 2\int_{x(z)}^\infty \phi(x)dx.$$

From equation (3.13) we see that $\tilde{K}_{11}$ has a branch cut on the imaginary axis from $i\beta$ to $i\infty$ and from $-i\beta$ to $-i\infty$. Along the cut we have the relations

$$\tilde{K}_{11}(is\pm 0, \beta) = \mathscr{P}\int_\beta^\infty G_0\left(\frac{1}{s'-\beta}\right)\frac{ds'}{(s')^2-s^2} \mp \frac{i\pi}{2s} G_0\left(\frac{1}{|s|-\beta}\right), \tag{3.14}$$

where s is real and of magnitude greater than β. An application of equation (3.14) gives for the contribution of the cut in equation (3.11) the following (with $k' = is$):

$$-\frac{1}{2}(1-\epsilon)\beta B\int_\beta^\infty H\left(\frac{1}{s}\right) G_0\left(\frac{1}{s-\beta}\right)\frac{ds}{s^2}, \tag{3.15}$$

giving the final expression for $S(0)$ in the isothermal case

$$S(0) = B\left[(1-\tilde{\lambda})^{1/2} - \frac{1}{2}(1-\epsilon)\beta\int_0^{1/\beta} H(z) G_0\left(\frac{z}{1-\beta z}\right) dz\right]. \tag{3.16}$$

The integral in equations (3.15) and (3.16) is called α_{00} in IB (Section 7.5) and is tabulated (Tables 34 and 35) for Voigt profiles in the case $\epsilon = 0$. From the relationships

$$\int_0^{1/\beta} G_0\left(\frac{z}{1-\beta z}\right) dz = \delta(\beta)$$

$$H(z) \le H(\infty) = (1-\tilde{\lambda})^{-1/2}$$

we obtain the limits

$$\frac{1}{2}(1-\tilde{\lambda})^{1/2}\left(1+\frac{\epsilon}{1-\tilde{\lambda}}\right) B \le S(0) \le (1-\tilde{\lambda})^{1/2} B \tag{3.17}$$

given previously by Hummer (1968, for $\rho = 1$). For more precise results, it is necessary to use numerical values of α_{00}.

While equations (3.10) and (3.17) may give a satisfactory representation of $S(\tau)$ for $\tau \ll 1/\beta$, results are still desired for the other limit, $\tau \gg 1/\beta$. In this case it is possible to take advantage of the exponential decay of K_1 and K_{11} at large τ, and treat the atmosphere as if it were infinite. For an infinite atmosphere the equation analogous to equation (3.1) can be solved directly by Fourier transforms giving

$$\tilde{S}_+(k) \approx \tilde{S}_a(k) = \frac{\epsilon + (1-\epsilon)\beta\tilde{K}_{11}(k, \beta)}{1-(1-\epsilon)\tilde{K}_1(k, \beta)} \tilde{B}(k). \tag{3.18}$$

By introducing the H function and using equation (3.5), equation (3.18) can be written

$$\tilde{S}_a(k) = H\left(\frac{1}{ik}\right)\Omega(k),$$

so that the error committed in treating the atmosphere as infinite is given by

$$\tilde{S}_+(k) - \tilde{S}_a(k) = -H\left(\frac{1}{ik}\right)\frac{1}{2\pi i}\int_{C_-}\frac{\Omega(k')dk'}{k'-k}. \tag{3.19}$$

Since $\tilde{B}(k)$ is analytic in the lower half of the k-plane, while $\tilde{K}_{11}$ and the H function are analytic in the strip $-\beta < \text{Im}(k) < \beta$, it is necessary only for the contour C_- to enclose the portion of the imaginary axis from $-i\beta$ to $-i\infty$. Therefore the integral in equation (3.19) is analytic in the region $\text{Im}(k) > -\beta$. The H function in equation (3.19) is analytic in the region $\text{Im}(k) < \beta$, so $\tilde{S}_+(k) - \tilde{S}_a(k)$ is analytic in the strip $-\beta < \text{Im}(k) < \beta$. This suggests that $|S(\tau) - S_a(\tau)|$ is exponentially bounded for large τ:

$$|S(\tau) - S_a(\tau)| < Me^{-s\tau}, \tag{3.20}$$

where s is any positive real number less than β. Therefore if $S(\tau)$ is replaced by $S_a(\tau)$ in the calculation of the radiation force, the error which is committed is of order $\exp(-\beta\tau)$.

The expression for the radiation force given in equation (2.9) must be generalized to account for the continuum processes. The total radiation force, continuum and line together, is given by

$$f_{\text{rad}}(\tau) = \frac{4\pi\kappa_L\Delta\nu_D}{c}\int_{-\infty}^{\infty}(\phi(x)+\beta)dx\,\frac{1}{2}\int_0^{\infty}[\phi(x)S(\tau')+\beta B(\tau')] \times \text{sgn}(\tau'-\tau)E_2((\phi(x)+\beta)|\tau'-\tau|)d\tau'. \tag{3.21}$$

In order to obtain a finite result, the integration over x in equation (3.21) must be taken over some very large finite interval containing the line. Equation (3.21) can be re-written in terms of new kernel functions as follows:

$$f_{\text{rad}}(\tau) = \frac{4\pi\kappa_L\Delta\nu_D}{c}\left[\int_0^{\infty}S(\tau')K_{2a}(\tau'-\tau)d\tau' + \beta\int_0^{\infty}B(\tau')K_{2b}(\tau'-\tau)d\tau'\right] \tag{3.22}$$

where

$$K_{2a}(\tau) = \tfrac{1}{2}\,\text{sgn}(\tau)\int_{-\infty}^{\infty}\phi(x)(\phi(x)+\beta)E_2((\phi(x)+\beta)|\tau|)dx \tag{3.23}$$

and

$$K_{2b}(\tau) = \tfrac{1}{2}\,\text{sgn}(\tau)\int_{-\infty}^{\infty}(\phi(x)+\beta)E_2((\phi(x)+\beta)|\tau|)dx. \tag{3.24}$$

The integral in equation (3.24) is cut off as described following equation (3.21). In order to find the line contribution to the force, the continuum part must be subtracted. The continuum force is found by simply ignoring the presence of the line within the band; this is accomplished by replacing $\phi(x)$ with zero in equation (3.21) giving

$$f_{\text{rad, c}}(\tau) = \frac{4\pi\kappa_L\Delta\nu_D}{c}\int_{-\infty}^{\infty}\beta dx\,\frac{1}{2}\int_0^{\infty}\beta B(\tau')\,\text{sgn}(\tau'-\tau)E_2(\beta|\tau'-\tau|)d\tau'. \tag{3.25}$$

The line contribution to the force is then the difference of equation (3.22) and (3.25)

$$f_{\text{rad, L}}(\tau) = \frac{4\pi\kappa_L\Delta\nu_D}{c}\left[\int_0^{\infty}S(\tau')K_{2a}(\tau'-\tau)d\tau' - \beta\int_0^{\infty}B(\tau')K_{2c}(\tau'-\tau)d\tau'\right], \tag{3.26}$$

where

$$K_{2c}(\tau) = \tfrac{1}{2}\,\text{sgn}(\tau)\int_{-\infty}^{\infty}[\beta E_2(\beta|\tau|) - (\phi(x)+\beta)E_2((\phi(x)+\beta)|\tau|)]dx. \tag{3.27}$$

No cut-off is necessary in equation (3.27) to assure convergence. The two kernel functions in equation (3.26) can also be represented as follows

$$K_{2a}(\tau) = -\frac{d}{d\tau} K_{3a}(\tau) \tag{3.28}$$

and

$$K_{2c}(\tau) = -\frac{d}{d\tau} K_{3c}(\tau), \tag{3.29}$$

where

$$K_{3a}(\tau) = \tfrac{1}{2}\int_{-\infty}^{\infty} \phi(x)E_3((\phi(x)+\beta)|\tau|)dx, \tag{3.30}$$

and

$$K_{3c}(\tau) = \tfrac{1}{2}\int_{-\infty}^{\infty} [E_3(\beta|\tau|)-E_3((\phi(x)+\beta)|\tau|)]dx. \tag{3.31}$$

By integrating by parts in equation (3.26) and making use of equations (3.28) and (3.29), the line force can be cast into the following form

$$f_{\mathrm{rad,\,L}}(\tau) = \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c}\left\{S(0)K_{3a}(\tau)+\int_0^{\infty}\frac{dS}{d\tau'}K_{3a}(\tau'-\tau)d\tau' - B(0)\beta K_{3c}(\tau)-\int_0^{\infty}\frac{dB}{d\tau'}\beta K_{3c}(\tau'-\tau)d\tau'\right\}. \tag{3.32}$$

From equations (3.30) and (3.31) we see that the normalization of K_{3a} and K_{3c} is given by

$$\int_{-\infty}^{\infty} K_{3a}(\tau)d\tau = \int_{-\infty}^{\infty} \beta K_{3c}(\tau)d\tau = \tfrac{1}{3}\delta(\beta). \tag{3.33}$$

If $B(\tau)$ is essentially constant over intervals in τ which are small compared with $1/\beta$, then $dB/d\tau$ is of order βB. From this assumption and equation (3.33) we see that the terms in B in equation (3.32) are no larger than order $\beta\delta(\beta)B$. At small values of the continuum optical depth, i.e. for $\tau \ll 1/\beta$, the S terms in equation (3.32) are dominated by the contribution from optical depths smaller than the thermalization depth. (In the region interior to the thermalization depth $dS/d\tau' \approx dB/d\tau' = O(\beta B)$, so the contribution from this region is no larger than order $\beta\delta(\beta)B$.) However, in the region exterior to the thermalization depth $S(\tau')$ is given, according to the arguments above, by equation (2.20) or equation (2.26), in which, from equation (3.17), ϵ is replaced by $1-\tilde{\lambda}$, or approximately by the larger of ϵ and $\beta\delta(\beta)$. Furthermore, exterior to the thermalization depth τ' is necessarily smaller than $1/\beta$, so the function K_{3a} can be approximated by K_{31}. In short, the line force reduces to that calculated in the previous section, except for the replacement of ϵ with $1-\tilde{\lambda}$, and the understanding that B is an average over the range $0 \leq \tau \leq$ thermalization depth. It can be verified that the force is larger than $\beta\delta(\beta)B$.

For continuum optical depths larger than unity, evidently the line force altogether is of order $\beta\delta(\beta)B$. In order to evaluate it more precisely we use equation (3.18), which is valid in this region. Equations (3.26), (3.28) and (3.29) give for the Fourier transform of the line force the expression

$$\tilde{f}_{\mathrm{rad,\,L}}(k) = \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c}ik[\tilde{K}_{3a}(k)\tilde{S}_+(k)-\beta\tilde{K}_{3c}(k)\tilde{B}(k)]. \tag{3.34}$$

With equation (3.18), equation (3.34) becomes

$$f_{\mathrm{rad,\,L}} \approx \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c}\, ik\left[\frac{\epsilon+(1-\epsilon)\beta\tilde{K}_{11}}{1-(1-\epsilon)\tilde{K}_{1}}\,\tilde{K}_{3a}-\beta\tilde{K}_{3c}\right]\tilde{B}. \tag{3.35}$$

We seek to obtain a diffusion-type expression from equation (3.35). To this end we need expansions in powers of k of the transforms of the kernels appearing in equation (3.35). These are given by

$$\begin{aligned}\tilde{K}_1 &= \frac{1}{k}\int_{-\infty}^{\infty}\phi^2(x)\tan^{-1}\frac{k}{\phi(x)+\beta}\,dx\\ &= 1-\beta\delta(\beta)-\frac{k^2}{3}\int_{-\infty}^{\infty}\frac{\phi^2}{(\phi+\beta)^3}+\frac{k^4}{5}\int_{-\infty}^{\infty}\frac{\phi^2}{(\phi+\beta)^5}\,dx-+\ldots\end{aligned} \tag{3.36}$$

$$\begin{aligned}\tilde{K}_{11} &= \frac{1}{k}\int_{-\infty}^{\infty}\phi(x)\tan^{-1}\frac{k}{\phi(x)+\beta}\,dx\\ &= \delta(\beta)-\frac{k^2}{3}\int_{-\infty}^{\infty}\frac{\phi}{(\phi+\beta)^3}\,dx+\frac{k^4}{5}\int_{-\infty}^{\infty}\frac{\phi}{(\phi+\beta)^5}\,dx-+\ldots\end{aligned} \tag{3.37}$$

$$\begin{aligned}\tilde{K}_{3a} &= \frac{1}{k^3}\int_{-\infty}^{\infty}\phi(x)(\phi(x)+\beta)^2\left(\frac{k}{\phi(x)+\beta}-\tan^{-1}\frac{k}{\phi(x)+\beta}\right)dx\\ &= \frac{1}{3}\,\delta(\beta)-\frac{k^2}{5}\int_{-\infty}^{-}\frac{\phi\,dx}{(\phi+\beta)^3}+\frac{k^4}{7}\int_{-\infty}^{\infty}\frac{\phi}{(\phi+\beta)^5}\,dx-+\ldots\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\tilde{K}_{3c} &= \frac{1}{k^3}\int_{-\infty}^{\infty}\left[(\phi(x)+\beta)^2\tan^{-1}\frac{k}{\phi(x)+\beta}-\beta^2\tan^{-1}\frac{k}{\beta}-k\phi(x)\right]dx\\ &= \frac{1}{3}\,\frac{\delta(\beta)}{\beta}-\frac{k^2}{5\beta^3}\int_{-\infty}^{\infty}\frac{\phi^3+3\phi^2\beta+3\phi\beta^2}{(\phi+\beta)^3}\,dx\\ &\quad+\frac{k^4}{7\beta^5}\int_{-\infty}^{\infty}\frac{\phi^5+5\phi^4\beta+10\phi^3\beta^2+10\phi^2\beta+5\phi\beta^4}{(\phi+\beta)^5}\,dx-+\ldots\end{aligned} \tag{3.39}$$

If expansions (3.36)–(3.39) are inserted in equation (3.35), and the resulting expansion for the expression in brackets is truncated at the k^2 term, the following result is found

$$f_{\mathrm{rad,\,L}} = \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c}\left[\left(\frac{1}{5}-\frac{1}{9}\,\frac{1-\epsilon}{1-\bar{\lambda}}\,\beta\delta(\beta)\right)\frac{1}{\beta}\int_{-\infty}^{\infty}\frac{\phi}{(\phi+\beta)^2}\,dx+\frac{1}{5}\,\frac{\delta(\beta)}{\beta^2}\right]ik^3\tilde{B}, \tag{3.40}$$

which indicates that in the region $\tau > 1/\beta$ the line force is given by

$$\begin{aligned}f_{\mathrm{rad,\,L}}(\tau) = -\frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c}&\left[\left(\frac{1}{5}-\frac{1}{9}\,\frac{1-\epsilon}{1-\bar{\lambda}}\,\beta\delta(\beta)\right)\frac{1}{\beta}\int_{-\infty}^{\infty}\frac{\phi}{(\phi+\beta)^2}\,dx\right.\\ &\left.+\frac{1}{5}\,\frac{\delta(\beta)}{\beta^2}\right]\frac{d^3B}{d\tau^3}+O\left(\frac{d^5B}{d\tau^5}\right)+O(e^{-\beta\tau}).\end{aligned} \tag{3.41}$$

The first part of the error is due to neglected higher terms in k in equation (3.35); the second part is due to the error in equation (3.18).

Equation (3.41) can be simplified by assuming a specific profile function, Doppler or Voigt. In either case there is the relation

$$\frac{1}{\beta}\int_{-\infty}^{\infty}\frac{\phi(x)}{(\phi(x)+\beta)^2}\,dx = -\frac{1}{\beta}\frac{d\delta(\beta)}{d\beta}. \tag{3.42}$$

In the Doppler case, if β is small compared with unity, we have

$$\delta(\beta) \approx 2\sqrt{\ln\frac{1}{\sqrt{\pi}\beta}} \tag{3.43}$$

and

$$\frac{d\delta(\beta)}{d\beta} \approx -\frac{1}{\beta\sqrt{\ln 1/\sqrt{\pi}\beta}}. \tag{3.44}$$

If a Voigt profile is applicable, and the line is on the damping part of the curve of growth (i.e. $a > \pi\beta \ln(1/\sqrt{\pi}\beta)$), then the functions are

$$\delta(\beta) \approx \pi\sqrt{\frac{a}{\beta}} \tag{3.45}$$

and

$$\frac{d\delta(\beta)}{d\beta} \approx -\frac{\pi}{2}\sqrt{\frac{a}{\beta^3}}. \tag{3.46}$$

Furthermore, we can write the derivative of the Planck function in terms of continuum optical depth $\tau_c = \beta\tau$. The result is that the line force in the Doppler case can be written

$$f_{\mathrm{rad,\,L}}(\tau) = -\frac{4\pi\kappa_L\Delta\nu_D}{c}\cdot\frac{2}{5}\,\beta\sqrt{\ln\frac{1}{\sqrt{\pi}\beta}}\,\frac{d^3B}{d\tau_c^3} \tag{3.47}$$

and in the Voigt case it is

$$f_{\mathrm{rad,\,L}}(\tau) = -\frac{4\pi\kappa_L\Delta\nu_D}{c}\left[\frac{3}{10}-\frac{1}{18}\frac{\pi(1-\epsilon)\sqrt{a\beta}}{\epsilon+\pi(1-\epsilon)\sqrt{a\beta}}\right]\pi\sqrt{a\beta}\,\frac{d^3B}{d\tau_c^3}. \tag{3.48}$$

The force exerted in the continuum, in this diffusion regime, in a band of width $\Delta x = \delta(\beta)$ is equal to $(4\pi\kappa_L\Delta\nu_D/3c)\beta\delta(\beta).dB/d\tau_c$. For realistic Planck functions it is also true that $d^3B/d\tau_c^3$ is small compared with $dB/d\tau_c$ if $\tau_c > 1$. Then we see from equations (3.47) and (3.48) that the line force is a small correction to the force exerted in the continuum in the same frequency band. That is, in the part of the atmosphere which is optically thick in the continuum, the inclusion of a line does not alter the radiation force in the lowest order. This statement needs further qualification, since in the present discussion the effect of a line has been considered with the temperature distribution held fixed. Another, indirect effect of inclusion of a line is the alteration of the temperature distribution, i.e. the 'backwarming' effect, which raises the temperature gradient. In the region of the star which is opaque in the continuum this effect is fully taken into account by including the lines in the Rosseland mean opacity used to establish the temperature distribution.

4. LARGE VELOCITY GRADIENT

The third and final radiation transfer situation for which the force of line radiation on the stellar material is desired is that of a line formed in an atmosphere which is expanding (or contracting) rapidly. The continuum contributions to line formation are neglected, except that at the inner boundary radius, r_c, of the line formation region an incident intensity with a continuous spectral distribution is assumed. This situation was considered by Castor (1970), and, except for the incident intensity, by Sobolev (1957), Rublev (1961, 1963), and Lyong (1967) in a number of papers. Rybicki (1970) has given an excellent review of this theory. The radiation field at line frequencies is separated into two parts, one which is simply the attenuated incident continuous radiation, and another, the diffuse component, which is the line radiation emitted within the line forming region. In this section the line forming region will be called the envelope of the star, to distinguish it from the part of the star inside the inner boundary radius, r_c, which will be called the core. In the event that the velocity is large compared with the thermal velocity, and if the velocity increases in magnitude outward, both of which are true for stellar winds, it is very unlikely that a photon emitted in the line in one part of the envelope will be absorbed in a remote part of the envelope. Therefore the transfer problem of the diffuse component of the line radiation field becomes local to an excellent approximation. This simplification makes the problem tractable.

The notation of Castor (1970) will be used. The intensity is given as a function $I(x, p, z)$, where x is the displacement from line centre, in Doppler units, of the inertial-frame frequency, p is the impact parameter of the light ray with respect to the centre of the star, and z is the distance of a point from the mid-plane of the star as measured along the ray, increasing away from the observer. The dominant effect of the velocity is due simply to the Doppler-shift of the line absorption coefficient; other, smaller, effects due to time dilation and the abberation of light will be neglected in this paper. The line source function S is assumed to be independent of frequency and angle. Variations in the thermal Doppler width are neglected. The formal solution of the transfer equation for $I(x, p, z)$ is given by equations (1a) and (1b) of Castor (1970), which are repeated here for convenience:

$$I(x, p, z) = \begin{cases} \displaystyle\int_{\tau(x,p,z)}^{\tau(x,p,\infty)} S(r') \exp\,[\tau(x, p, z) - \tau(x, p, z')]\,d\tau(x, p, z'), & p > r_c \text{ or } z > 0 \\ \displaystyle\int_{\tau(x,p,z)}^{\tau(x,p,z_{\max})} S(r') \exp\,[\tau(x, p, z) - \tau(x, p, z')]\,d\tau(x, p, z') + I_c(p) \exp\,[\tau(x, p, z) - \tau(x, p, z_{\max})] & p < r_c \text{ and } z < 0 \end{cases} \quad (4.1)$$

$$[r = (p^2+z^2)^{1/2} > r_c].$$

The definitions of r' and $z_{\max}$ are

$$r' = (p^2+(z')^2)^{1/2}$$

and

$$z_{\max} = -(r_c{}^2-p^2)^{1/2}.$$

In the second part of equation (4.1) $I_c(p)$ is the intensity of the radiation incident from the core, as a function of the impact parameter to allow for limb darkening. The quantity $\tau(x, p, z)$ is the optical depth at frequency x along the ray with impact parameter p, and is an increasing function of z. It is taken to be zero at the observer ($z = -\infty$), so that $\tau(x, p, \infty)$ is the total optical depth along the ray unless the ray intersects the core. For rays which do intersect the core, the parts with positive and negative z are disconnected, and the zero point of $\tau(x, p, z)$ can be chosen separately in each part. The expression for $\tau(x, p, z)$ is

$$\tau(x, p, z) = \int_{-\infty}^{z} \kappa_L(r')\rho(r')\phi\left(x + \frac{z'}{r'}\frac{v(r')}{v_{th}}\right)dz'. \tag{4.2}$$

In the case that $p < r_c$, then $\tau(x, p, z)$ is undefined for $|z| < (r_c^2 - p^2)^{1/2}$, and if also $z > (r_c^2 - p^2)^{1/2}$ the region $|z'| < (r_c^2 - p^2)^{1/2}$ is omitted from the integral.

Equation (2.8) for the line force, modified for the present case, becomes

$$f_{rad} = \frac{4\pi\kappa_L\Delta\nu_D}{c} \cdot \frac{1}{2} \int_{-1}^{1} \mu d\mu \int_{-\infty}^{\infty} \phi\left(x - \mu\frac{v(r)}{v_{th}}\right) I(x, r(1-\mu^2)^{1/2}, -\mu r)dx. \tag{4.3}$$

When equations (4.1) and (4.3) are combined, the line force becomes the sum of two parts,

$$f_{rad} = f_{rad,\,core} + f_{rad,\,diff}, \tag{4.4}$$

where

$$f_{rad,\,core} = \frac{2\pi\kappa_L\Delta\nu_D}{c} \int_{\left(1-\frac{r_c^2}{r^2}\right)^{1/2}}^{1} \mu d\mu \int_{-\infty}^{\infty} \phi\left(x - \mu\frac{v(r)}{v_{th}}\right) \times I_c(p) \exp\left[\tau(x, p, z) - \tau(x, p, z_{max})\right]dx \tag{4.5}$$

$$(p = (1-\mu^2)^{1/2} r,\ z = -\mu r)$$

and

$$f_{rad,\,diff} = \frac{2\pi\kappa_L\Delta\nu_D}{c} \int_{-1}^{1} \mu d\mu \int_{-\infty}^{\infty} \phi\left(x - \mu\frac{v(r)}{v_{th}}\right)dx \times \int_{\tau(x,p,z)}^{\tau(x,p,z\,max)} S(r') \exp\left[\tau(x, p, z) - \tau(x, p, z')\right]d\tau(x, p, z') \tag{4.6}$$

($p = (1-\mu^2)^{1/2} r$, $z = -\mu r$, $r' = (p^2 + (z')^2)^{1/2}$, $z_{max} = \infty$ if $p > r_c$ or $z > 0$.) The two parts of the line force will be treated separately in the discussion to follow.

The integrand in equation (4.2) is sharply peaked near the point in the envelope where the frequency x is in resonance with the appropriate component of the local fluid velocity. With this in mind, equation (4.2) is simplified by changing to the fluid-frame frequency as the variable of integration then evaluating all slowly varying factors in the integrand at the resonant point on the ray. This gives the expression

$$\tau(x, p, z) = \left[\frac{\tau_0}{1 + \sigma\mu^2}\right]_{res} y\left(x + \frac{z}{r}\frac{v(r)}{v_{th}}\right) \tag{4.7}$$

in terms of the new quantities defined by

$$\tau_0(r) = \frac{\kappa_L \rho v_{th} r}{v(r)} \tag{4.8}$$

$$\sigma = \frac{d\ln v(r)}{d\ln r} - 1 \tag{4.9}$$

$$y(x) = \int_{-\infty}^{x} \phi(x')dx', \tag{4.10}$$

and where the subscript 'res' signifies evaluation of the expression in brackets at the resonant point along the ray. When equation (4.7) is inserted in equation (4.5), a further simplification is possible. This comes about because of the presence of the factor ϕ in the integrand. The result is that the main contributions to the force come from those frequencies for which the point of evaluation of the force is almost the same as the resonant point for each ray. Therefore the expression in brackets in equation (4.7) can be evaluated locally. In this case it is also true that the core is well away from the resonant point and $\tau(x, p, z_{\max})$ can be replaced by $\tau_0/(1+\sigma\mu^2)$. With these substitutions, equation (4.5) becomes

$$f_{\text{rad, core}} = \frac{2\pi\kappa_L\Delta\nu_D}{c}\int_{(1-\frac{r_c^2}{r^2})^{1/2}}^{1} \mu d\mu I_c(p)\int_0^1 \exp\left[-\frac{\tau_0(1-y)}{1+\sigma\mu^2}\right]dy, \tag{4.11}$$

or

$$f_{\text{rad, core}} = \frac{2\pi\kappa_L\Delta\nu_D}{c}\int_{(1-\frac{r_c^2}{r^2})^{1/2}}^{1} \mu d\mu I_c(p)\,\frac{1+\sigma\mu^2}{\tau_0}\left[1-\exp\left(-\frac{\tau_0}{1+\sigma\mu^2}\right)\right]. \tag{4.12}$$

The factor in front of the integral can be combined with the τ_0 in the denominator by means of equations (4.8) and (4.9) to give the form

$$f_{\text{rad, core}} = \frac{2\pi\nu_0}{\rho c^2}\int_{(1-\frac{r_c^2}{r^2})^{1/2}}^{1} \mu d\mu I_c(p)\left[(1-\mu^2)\frac{v}{r}+\mu^2\frac{dv}{dr}\right] \times\left[1-\exp\left(-\frac{\tau_0}{1+\sigma\mu^2}\right)\right]. \tag{4.13}$$

The line will be called optically thin or optically thick according to whether the quantity $\tau_0/(1+\sigma\mu^2)$ is smaller than unity or larger than unity over most of the range in μ. If the line is optically thin, then we see from equation (4.12) that the force reduces to $(\kappa_L\Delta\nu_D/c)F_c$, where F_c is the local continuum flux. This is just what is expected for the optically thin case. If the line is optically thick, then the second expression in square brackets in equation (4.13) can be replaced by unity, so the force becomes

$$f_{\text{rad, core}} = \frac{\nu_0 F_c}{\rho c^2}\left(\frac{dv_l}{dl}\right)_{\text{av}}, \tag{4.14}$$

where the quantity $(dv_l/dl)_{\text{av}}$ is the angle average of the directional derivative of the projected velocity, the quantity in the first set of brackets in equation (4.13). The interpretation of this result is very simple. The radiation in an element of solid angle $d\Omega$ at direction cosine μ, in a frequency band $\Delta\nu$, and crossing an area dA perpendicular to the beam carries with it an amount of radial momentum per unit time equal to

$$\frac{\text{momentum}}{\text{time}} = \frac{\mu I_c}{c}\,d\Omega\Delta\nu dA.$$

If the width of the intrinsic line profile is neglected, each frequency within the band is absorbed at the particular point along the direction of propagation where it becomes resonant with the projected fluid velocity. Therefore the band width $\Delta\nu$ corresponds with a distance Δl along the light path equal to $(c/\nu_0)\Delta\nu/(dv_l/dl)$. If the radiation is completely absorbed, its momentum is given to the material contained in the cylinder with dimensions dA by Δl, of which the mass is $\rho dA\Delta l$.

The corresponding force per unit mass of material is then

$$\frac{\text{force}}{\text{mass}} = \frac{\mu I_c d\Omega}{\rho c^2} \nu_0 \left(\frac{dv_l}{dl}\right).$$

If this result is integrated over solid angle to obtain the total force, equation (4.14) is recovered. An explicit expression can be found for $(dv_l/dl)_{av}$ by transforming, in equation (4.13), to the direction cosine μ' at the core radius r_c in place of the direction cosine μ at radius r. The relation between them is

$$(1-\mu^2) = \frac{r_c^2}{r^2}(1-(\mu')^2),$$

with the result that $(dv_l/dl)_{av}$ can be written

$$\left(\frac{dv_l}{dl}\right)_{av} = \frac{dv}{dr} - \frac{r_c^2}{r^2}\left(1 - \frac{N_c}{H_c}\right)\left(\frac{dv}{dr} - \frac{v}{r}\right), \tag{4.15}$$

where H_c and N_c are the first and third Eddington moments of the continuum radiation emitted by the core, evaluated at the core radius. On the reasonable assumption that I_c increases with μ', but I_c/μ' decreases, the ratio N_c/H_c must lie between 0·5 and 0·6.

The reduction of $f_{\text{rad, diff}}$ follows the analysis of Sobolev (1957), and is attended by several difficulties which do not arise for $f_{\text{rad, core}}$. The angle integration which gives the core component of the force is a sum of positive quantities owing to the fact that the core radiation is confined to the outward direction. In contrast, the diffuse part of the force involves the difference between contributions of the out-going and the in-going radiation, so that approximations which are valid for the core force are not for the diffuse force. The diffuse force owes its existence to whatever anisotropy exists in the diffuse radiation field. Such anisotropy can be due to radial gradients of the line source function and the absorbing atom density, and to curvature in the velocity law. The effects of the absorbing atom density and the velocity can be neglected if the line is optically thick, with the main anisotropy in that case deriving from the source function gradient. This is not the case if the line is optically thin. However, if the line is optically thin, we expect that the diffuse field will be weak compared with the direct radiation from the core, so the diffuse component of the force would be unimportant. In the following discussion only the part of the diffuse force due to the source function gradient will be considered. This is also the approach taken by Sobolev (1957).

Equation (4.6) is simplified by using equation (4.7) to evaluate $\tau(x, p, z)$ and $\tau(x, p, z_{max})$, and also by neglecting the spatial dependence of the quantity $\tau_0/(1+\sigma\mu^2)$, which is assigned its value at the point where the force is to be found. This approximation is equivalent to the neglect of the density and velocity effects discussed above. When this is done, equation (4.6) becomes

$$f_{\text{rad, diff}} = \frac{2\pi\kappa_L \Delta\nu_D}{c} \int_{-1}^{1} \mu d\mu \kappa_L \rho \int_z^{z_{max}} S(r')dz'$$
$$\times \int_{-\infty}^{\infty} \phi\left(x - \frac{\mu v}{v_{th}}\right) \phi\left(x - \frac{\mu' v'}{v_{th}}\right) \exp\left[-\frac{\tau_0}{1+\sigma\mu^2} \int_{x-\frac{\mu v}{v_{th}}}^{x-\frac{\mu' v'}{v_{th}}} \phi(x')dx'\right] dx. \tag{4.16}$$

Equation (4.16) can be written more simply by introducing a new function which is related to the line profile,

$$F(t, \tau) = \int_{-\infty}^{\infty} \left[1 - \exp\left(-\tau \int_{x}^{x+t} \phi(x')dx'\right)\right] dx, \tag{4.17}$$

for which

$$F_t(t, \tau) \equiv \frac{\partial F(t, \tau)}{\partial t} = \tau \int_{-\infty}^{\infty} \phi(x+t) \exp\left(-\tau \int_{x}^{x+t} \phi(x')dx'\right) dx$$
$$= \tau \int_{-\infty}^{\infty} \phi(x) \exp\left[-\tau \int_{x-t}^{x} \phi(x')dx'\right] dx \tag{4.18}$$

and

$$F_{tt}(t, \tau) \equiv \frac{\partial^2 F(t, \tau)}{\partial t^2} = -\tau^2 \int_{-\infty}^{\infty} \phi(x)\phi(x-t) \exp\left[-\tau \int_{x-t}^{x} \phi(x')dx'\right] dx$$
$$= -\tau^2 \int_{-\infty}^{\infty} \phi(x)\phi(x+t) \exp\left[-\tau \int_{x}^{x+t} \phi(x')dx'\right] dx. \tag{4.19}$$

The new form of equation (4.16) is therefore

$$f_{\text{rad, diff}} = \frac{2\pi\kappa_L \Delta\nu_D}{c} \int_{-1}^{1} \mu d\mu \int_{z}^{z_{\max}} S(r')\kappa_L \rho \left(\frac{1+\sigma\mu^2}{\tau_0}\right)^2 \left(-F_{tt}\left(t, \frac{\tau_0}{1+\sigma\mu^2}\right)\right) dz' \tag{4.20}$$

where t is given by

$$t = \frac{\mu v}{v_{\text{th}}} - \frac{\mu' v'}{v_{\text{th}}}. \tag{4.21}$$

The primed variables here and in equation (4.16) refer to evaluation at the point p, z', where p is $r(1-\mu^2)^{1/2}$. The integration variable z' in equation (4.20) is next changed to t, using the approximation

$$z' - z = \frac{-t}{\partial/\partial z'(\mu' v'/v_{\text{th}})} = \frac{v_{\text{th}} r/v(r)}{1+\sigma\mu^2} t. \tag{4.22}$$

The source function is expanded in a MacLaurin series in t,

$$S(r') = S(r) - t \frac{dS}{dr} \frac{\partial r'}{\partial z'} \bigg/ \frac{\partial}{\partial z'} \left(\frac{\mu' v'}{v_{\text{th}}}\right) + \dots \tag{4.23}$$

of which only the second term is kept. (The first term gives zero contribution to equation (4.20).) When the derivatives are evaluated, equation (4.23) becomes

$$S(r') = S(r) - t\mu \frac{v_{\text{th}} r/v(r)}{1+\sigma\mu^2} \frac{dS}{dr} + \dots. \tag{4.24}$$

Equations (4.22) and (4.24) are substituted into equation (4.20), and the factors τ_0 are combined with the line opacity using the definition (4.8), to give the result

$$f_{\text{rad, diff}} = -\frac{4\pi\nu_0 v_{\text{th}}}{\rho c^2} \cdot \frac{dS}{dr} \int_{0}^{1} \mu^2 d\mu \int_{0}^{\infty} t\left(-F_{tt}\left(t, \frac{\tau_0}{1+\sigma\mu^2}\right)\right) dt. \tag{4.25}$$

According to equation (4.25), the dependence of the diffuse force upon the profile and the optical depth is contained in the factor

$$\int_0^\infty t\left(-F_{tt}\left(t, \frac{\tau_0}{1+\sigma\mu^2}\right)\right)dt, \tag{4.26}$$

which can also be written, by means of an integration by parts, as

$$\lim_{t\to\infty}\left[F\left(t, \frac{\tau_0}{1+\sigma\mu^2}\right) - tF_t\left(\infty, \frac{\tau_0}{1+\sigma\mu^2}\right)\right]. \tag{4.27}$$

The following properties of F are easily established from equations (4.17) and (4.18):

$$F(0, \tau) = 0 \tag{4.28}$$

$$F_t(0, \tau) = \tau \tag{4.29}$$

and

$$F_t(\infty, \tau) = 1 - e^{-\tau}. \tag{4.30}$$

If the line is optically thin, expression (4.26) can easily be evaluated using equation (4.19), with the result

$$\int_0^\infty t\left(-F_{tt}\left(t, \frac{\tau_0}{1+\sigma\mu^2}\right)\right)dt = \left(\frac{\tau_0}{1+\sigma\mu^2}\right)^2 \int_0^\infty tA(t)dt, \tag{4.31}$$

where $A(t)$ is the autocorrelation function of the profile:

$$A(t) = \int_{-\infty}^\infty \phi(x)\phi(x+t)dx. \tag{4.32}$$

The diffuse force in the optically thin case becomes

$$f_{\text{rad, diff}} = -\frac{4\pi\nu_0 v_{\text{th}}}{\rho c^2}\frac{dS}{dr}\tau_0^2 \int_0^1 \frac{\mu^2}{(1+\sigma\mu^2)^2}d\mu \int_0^\infty tA(t)dt. \tag{4.33}$$

Comparison of this result with equation (4.13) for the core force shows that the diffuse force is smaller by roughly a factor τ_0, which is the factor by which the diffuse radiation field is weaker than the core radiation. Since equation (4.33) is also not expected to be very accurate, owing to the neglected effects of density gradient, etc., no more will be said about the optically thin case.

To treat the optically thick case, equation (4.27) is used, and so the asymptotic form of F is required, when t tends to infinity and τ is large. First it is noted that the quantity in brackets in equation (4.17) is symmetric about $x = -t/2$, so that equation (4.17) can be rewritten

$$F(t, \tau) = 2\int_{-t/2}^\infty dx\left[1 - \exp\left(-\tau\int_x^{x+t}\phi(x')dx'\right)\right]. \tag{4.34}$$

Now the upper limit of the integral over x' can be replaced by ∞, since it always exceeds $t/2$. Furthermore, if x is negative the exponent in equation (4.34) is always more negative then $-\tau/2$, so that the exponential can be neglected. This gives the following result for F in the limit of large t, if τ is also large:

$$F(t, \tau) \sim t + 2\int_0^\infty\left[1 - \exp\left(-\tau\int_x^\infty \phi(x')dx'\right)\right]dx. \tag{4.35}$$

When equations (4.27), (4.30) and (4.35) are combined, it is found that

$$\int_0^\infty t(-F_{tt}(t,\tau))dt = 2\int_0^\infty \left[1-\exp\left(-\tau\int_x^\infty \phi(x')dx'\right)\right]dx. \qquad (4.36)$$

If the profile function is Gaussian, as for Doppler broadening, the technique which led to equation (2.18) can be used, with the result

$$\int_0^\infty t(-F_{tt}(t,\tau))dt \sim 2\sqrt{\ln\frac{\tau}{2\sqrt{\pi}}}, \qquad (4.37)$$

which is between 2 and 6 for the expected range of τ. Equation (4.25) becomes

$$f_{\text{rad, diff}} \sim -\frac{8\pi\nu_0 v_{\text{th}}}{3\rho c^2}\frac{dS}{dr}\int_0^1 3\mu^2\sqrt{\ln\frac{\tau_0}{2\sqrt{\pi}(1+\sigma\mu^2)}}\,d\mu, \qquad (4.38)$$

which is the result given by Sobolev (1957).

If the profile function is actually Voigt, and if τ is so large that the line is on the damping part of the curve of growth, then the following approximation should be used for the integral in the exponent of equation (4.36):

$$\int_x^\infty \phi(x')dx' = \frac{a}{\pi x}, \qquad (4.39)$$

where a is the damping constant. Unfortunately, if this expression is substituted in equation (4.36), the integration over x diverges in $x \to \infty$. This is also the result if equation (4.39) is used directly in equations (4.27) and (4.34) instead of equation (4.36). To assess the seriousness of this state of affairs, a finite upper limit $x_{\max}$ can be taken in the integral, after which the sensitivity to $x_{\max}$ is explored. A reasonable choice for $x_{\max}$ would be V/v_{th}, where V is the maximum velocity in the envelope. The cut-off integral gives

$$\int_0^{x_{\max}} t(-F_{tt}(t,\tau))dt = 2x_{\max}\left(1-E_2\left(\frac{a\tau}{\pi x_{\max}}\right)\right), \qquad (4.40)$$

which takes the following two limiting forms

$$\frac{2a\tau}{\pi}\ln\frac{\pi x_{\max}}{a\tau} \quad \text{for} \quad \frac{a\tau}{\pi x_{\max}} \ll 1, \qquad (4.41)$$

and

$$2x_{\max} \quad \text{for} \quad \frac{a\tau}{\pi x_{\max}} \gg 1. \qquad (4.42)$$

Clearly, if the case (4.42) applies, the dependence on $x_{\max}$ is serious. The significance of the condition $a\tau/\pi = x_{\max}$ can be seen in this picture: For a particular frequency of a photon, and a certain ray in the envelope, there is a region of finite size along the ray within which the photon stands an appreciable change of being absorbed. For a Doppler profile, that is the region in which the velocity is within v_{th} of the resonant velocity for the photon. If the profile is Voigt, and the line is on the damping part of the curve of growth, the region is extended to become that in which the velocity lies within $(a\tau/\pi)v_{\text{th}}$ of the resonant velocity. If this velocity exceeds the maximum available velocity, V, then the region becomes the entire envelope and clearly the assumption of localness of the radiative transfer breaks down completely. That is, the case (4.42) is disastrous for this model. Even in case (4.41) the local approximation gets worse and worse as the limit $a\tau/\pi = x_{max}$ is approached. Reasonable values for a and x_{max} are 10^{-3} and 10^2, respectively (with Wolf–Rayet and Of stars in mind), so that the limit on τ is about 10^5. This limit is likely to be violated by the resonance lines, but probably not by very many others. It should also be pointed out that when the local assumption breaks down, it does so for both the diffuse force term *and* the core force term, although the results in the latter case tend to hide that fact.

It is of considerable interest to compare the core and diffuse forces in the optically thick case, i.e. to compare equation (4.14) with equation (4.25) into which one of equations (4.37), (4.41) and (4.42) has been inserted. One finds

$$\frac{f_{\mathrm{rad,\,diff}}}{f_{\mathrm{rad,\,core}}} \approx \frac{8\pi}{3}\frac{v_{\mathrm{th}}}{v}\frac{(-1/F_c)dS/dr}{(1/v)(dv_l/dl)_{\mathrm{av}}}\begin{cases}\sqrt{\ln\dfrac{\tau_0}{2\sqrt{\pi}}} & \dfrac{a\tau_0}{\pi} < 1\\[2ex] \dfrac{a\tau_0}{\pi}\ln\left(\dfrac{\pi V}{a\tau_0 v_{\mathrm{th}}}\right) & 1 < \dfrac{a\tau_0}{\pi} < \dfrac{V}{v_{\mathrm{th}}}.\\[2ex] \dfrac{V}{v_{\mathrm{th}}} & \dfrac{a\tau_0}{\pi} > \dfrac{V}{v_{\mathrm{th}}}.\end{cases} \tag{4.43}$$

It seems reasonable to assume that the relative gradient in the source function is never larger than the relative gradient of the velocity, so

$$-\frac{1}{F_c}\frac{dS}{dr} \lesssim \frac{1}{v}\left(\frac{dv_l}{dl}\right)_{\mathrm{av}}, \tag{4.44}$$

and consequently the diffuse force is never larger than the core force if $v > v_{\mathrm{th}}$, and only equals it in the disastrous case (4.42). With a Doppler profile, the diffuse force is smaller than the core force by a factor v_{th}/v, and so is negligible in most of the envelope. Equation (4.12) for the core force can therefore be considered to give with adequate accuracy the total force due to the line.

5. CONCLUSION

By piecing together the results of the preceding sections, we can form a fairly complete if not too accurate picture of the dependence of the line contribution to the body force acting on the stellar material upon the parameters τ, ϵ, β, a, and the velocity field if any. We simply assume that in any regime some one of the asymptotic results above is applicable, and the details of the manner in which the transition from one regime to another occurs are ignored. This probably gives results which are accurate to about an order of magnitude, judging from the comparison with exact results for a particular case. For that reason, we will also discard the numerical constants in the formulae, and slowly varying factors like $(\ln \tau)^{1/2}$.

For static atmospheres, or the slowly moving part of expanding atmospheres, we have the results of Sections 2 and 3. In considering the force as a function of line optical depth, we distinguish six cases corresponding to various relative values of the parameters ϵ, β, and a, the Voigt parameter. The force in the six cases is sketched in Fig. 2.

Case I. $a < \beta < \epsilon < 1$. Here the line source function is thermalized at the depth $1/\epsilon$, since the continuum is relatively less important than collisions and the damping wings are unimportant. The surface value of the source function is $\sqrt{\epsilon}B$, so the force for τ less than unity is about $(4\pi\kappa_L\Delta\nu_D/c)\sqrt{\epsilon}B$. For optical depths between unity and the thermalization depth we use equation (2.19). For optical depths between the thermalization depth and continuum depth unity we use equations (2.13) and (2.27). Finally, for depths large in the continuum the force due to the line is negligible. The composite curve is shown in the first panel of Fig. 2.

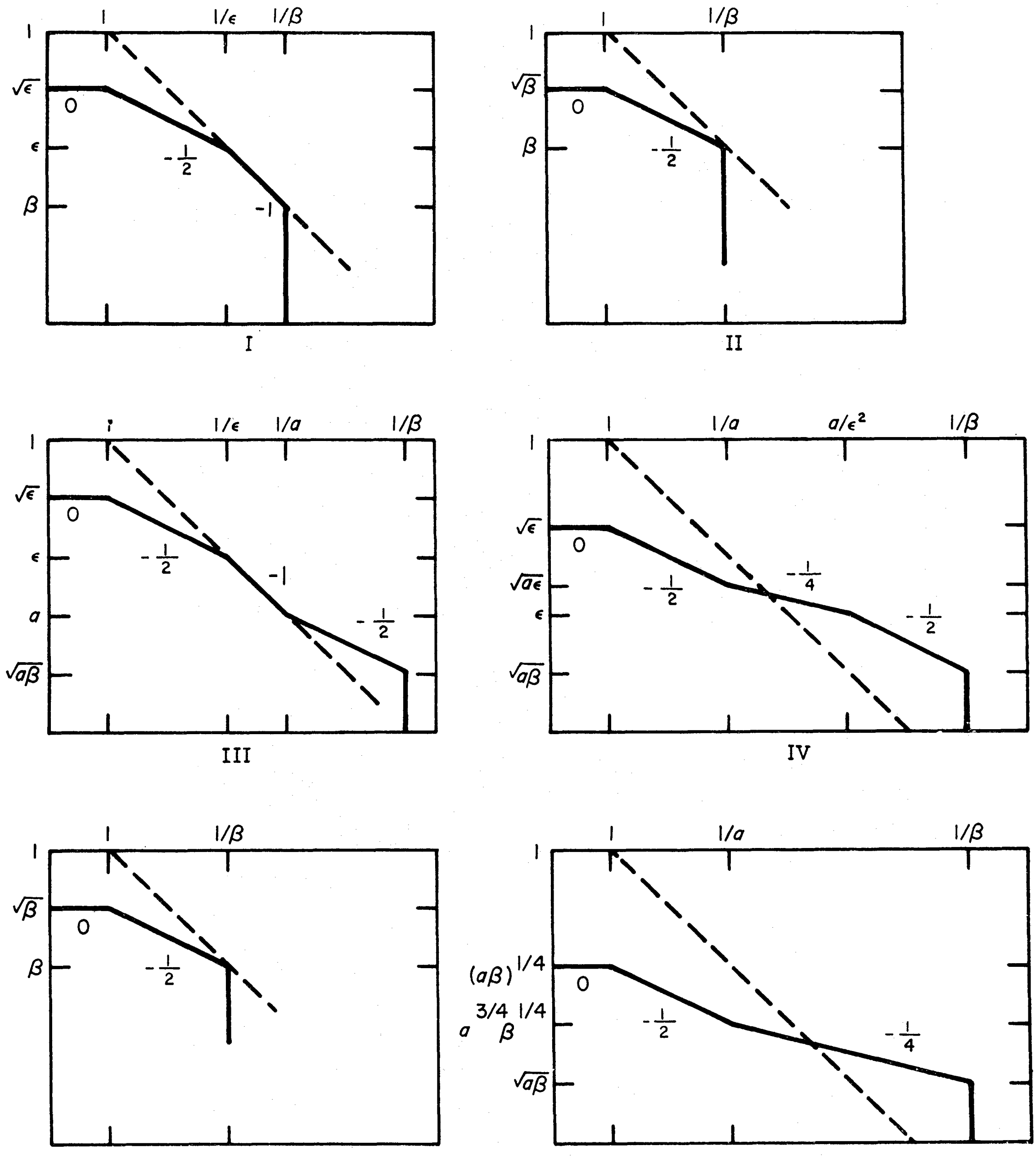

Fig. 2. *Sketches of the dependence of the line force on line optical depth for six cases of the parameters* ϵ, β, *and* a *defined in the text. The ordinate is approximately the force in units of* $(4\pi\kappa_L\Delta\nu_D/c)B$. *The dashed line in each panel gives the relation*

$$f = (4\pi\kappa_L\Delta\nu_D/c)B/\tau.$$

The values of $d \log f/d \log \tau$ *are indicated next to each segment of the curves.*

Case II. $a < \epsilon < \beta < 1$. Thermalization occurs in the continuum, and damping wings are still negligible. The surface source function is now $\sqrt{\beta}B$, so the force is like that in Case I with ϵ replaced by β. The region between the thermalization depth and unit depth in the continuum has vanished, however.

Case III. $\beta < a < \epsilon < 1$. The damping constant is large enough to influence part of the region between the thermalization depth and unit depth in the continuum, but not large enough to affect the thermalization process, which occurs by means of collisions, as in Case I. At the optical depth $1/a$, where the wings come in, we switch from equation (2.27) to equation (2.28) in equation (2.13).

Case IV. $\sqrt{a\beta} < \epsilon < a < 1$. The damping constant a is larger yet than in Case III, so that the damping wings are important at the thermalization depth, which becomes a/ϵ^2. We switch from equation (2.19) to equation (2.25) at the depth $1/a$, and use equations (2.13) and (2.28) between the thermalization depth and unit depth in the continuum.

Case V. $\epsilon < a < \beta < 1$. This case is the same as Case II, since neither ϵ nor a affects the force.

Case VI. $\epsilon < \sqrt{a\beta} < a < 1$. Here thermalization occurs in the continuum, but the wings become important before unit depth in the continuum is reached. The surface source function is given by equations (3.12), (3.17) and (3.45). For optical depths less than $1/a$ equation (2.19) is used, and for optical depths between $1/a$ and unit depth in the continuum equation (2.25) is used; in each case ϵ is replaced by $\sqrt{a\beta}$.

For all the cases with constant ϵ, β, a, and B the force may be obtained exactly using the method of discrete ordinates (*cf.* Avrett & Hummer 1965; Hummer 1968) and equation (3.34) once the kernel functions have been approximated by sums of exponentials. This has been done for the case $\beta = a = 0$, which is a limiting form of Cases I and III. The calculated force is shown as a function of optical depth for several values of ϵ in Fig. 3. The three sections of the curves, with slopes 0, $-\frac{1}{2}$, and -1, can be seen as in the first panel of Fig. 2, but the unexpected feature is that the part of the curve with slope $-\frac{1}{2}$ overshoots the -1 line by quite a bit, and then approaches that line from *above* with increasing τ. The reason for this is that the integral of the force over optical depth must give the radiation pressure, which has its LTE value $(1/3)aT^4$ at great depth, independent of ϵ. Since, for small ϵ, the force is reduced at the surface compared with $\epsilon = 1$, it must be greater than for $\epsilon = 1$ at larger optical depths in order to yield the same value of the integral. This figure indicates that simply using the relevant asymptotic formula for the force in any region of depth may give errors of an order of magnitude.

In Fig. 2, a dashed line has been used to indicate the force on the hypothesis of LTE and no damping wings, when the force varies as $1/\tau$ for τ greater than unity (*cf.* equations (2.13) and (2.27)). Bearing in mind that the most interesting values of τ are about 10^{-2} or 10^{-3} times $1/\beta$, corresponding to continuum optical depth 10^{-2} or 10^{-3}, we see that the LTE result is usually within an order of magnitude of the actual result. This suggests the very rough general formula

$$f_{\mathrm{rad,\,L}} \approx \frac{4\pi\kappa_{\mathrm{L}}\Delta\nu_{\mathrm{D}}}{c} B \min(1, 1/\tau). \tag{5.1}$$

In the case of a rapidly expanding atmosphere, the force is evaluated much more simply. Since, as we have seen, the contribution to the force due to the

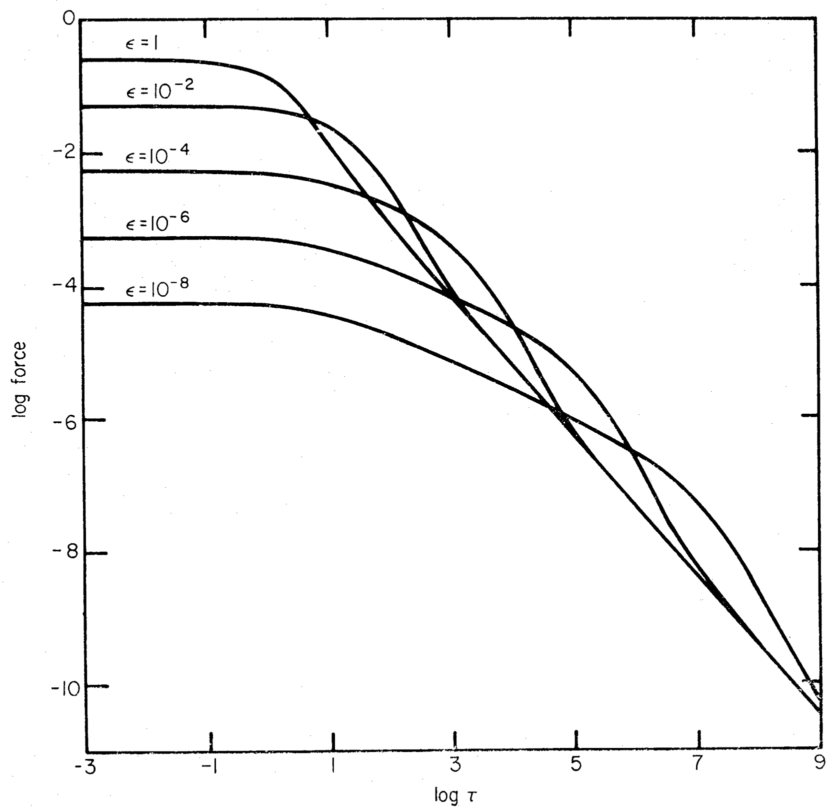

FIG. 3. *The line force as a function of line optical depth as calculated accurately for the case* $\beta = a = 0$, $B =$ *constant, and for the values* 1, 10^{-2}, 10^{-4}, 10^{-6}, *and* 10^{-8} *of* ϵ. *The force is expressed in units of* $(4\pi\kappa_L\Delta\nu_D/c)B$.

line from that part of the radiation field created by line emissions within the expanding envelope of the star is negligible, the parameters ϵ and a do not enter the expression for the force. For optical depths less than unity in the continuum, β also does not enter. The force in this case is given by equation (4.12), which can be cast in the following approximate form

$$f_{\mathrm{rad,\,L}} \approx \frac{\kappa_L \Delta\nu_D}{c} F_c \min(1, 1/\tau), \tag{5.2}$$

where F_c is the local continuum flux and τ is a modified optical depth variable,

$$\tau = \kappa_L \rho \frac{v_{\mathrm{th}}}{(dv_1/dl)_{\mathrm{av}}}. \tag{5.3}$$

This is a local quantity, and is related not to the total column density of absorbing atoms, but to the number of atoms in a cylinder whose height is the distance in which the velocity changes by a thermal unit.

It is interesting to note that equations (5.1) and (5.2), expressing the force in static and expanding atmospheres, are essentially the same. The main difference is the significance attached to τ in the two cases. Since τ in the moving atmosphere counts only a fraction of the atoms included within τ in the static atmosphere, the force is significantly larger in the moving atmosphere.

Equation (5.2) may be used to find the total force produced by an ensemble of lines, which in turn yields an estimate of the rate of mass loss associated with a stellar wind which is driven by this process. These results will be described in another communication.

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