ON THE FORCE BETWEEN A DIELECTRIC CYLINDER IN A CONSTANT ELECTRIC FIELD AND A CONDUCTING HALF SPACE

By

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Abstract. The force between an infinitely long dielectric cylinder in a constant electric field and a conducting half space is determined using the separation of variables technique on the Laplace equation in bipolar coordinates. The force is obtained as a series containing the relative distance between the cylinder and the half space as a parameter. This series is not uniformly convergent for the cylinder approaching the half space and the corresponding force cannot be obtained by performing the limit term per term. A special asymptotic analysis is presented leading to an analytic expression for this limiting value of the force.

1. Introduction. The change of the electric field due to the introduction of a conducting cylinder in a constant electric field bordering a conducting half space has been treated earlier [1]. In this paper a homogeneous isotropic dielectric cylinder is considered (see also [2]). While the determination of the electric field and resulting force between the cylinder and the half space is straightforward if using the appropriate coordinate system, the calculation of the limiting value of the force for the cylinder approaching the half space is more intricate. Indeed, the series solution representing the force is not uniformly convergent in this limit. In [2] it was shown by a numerical calculation that the limit cannot be taken term per term. This pure mathematical problem is solved by a very careful asymptotic analysis, which could be useful for other similar series as well.

2. Basic theory. The problem is sketched in Fig. 1. An infinitely long homogeneous isotropic dielectric cylinder with dielectric constant ε is introduced in a half infinite space with dielectric constant ε_0 filled with a constant electric field $\vec{E}$. The remaining half space is a perfect conductor. The axis of the cylinder is parallel to the boundary between both half spaces. The distance between the center of the cylinder and the boundary is h. The resulting electric field and the force between the cylinder and the conducting half space are looked for.
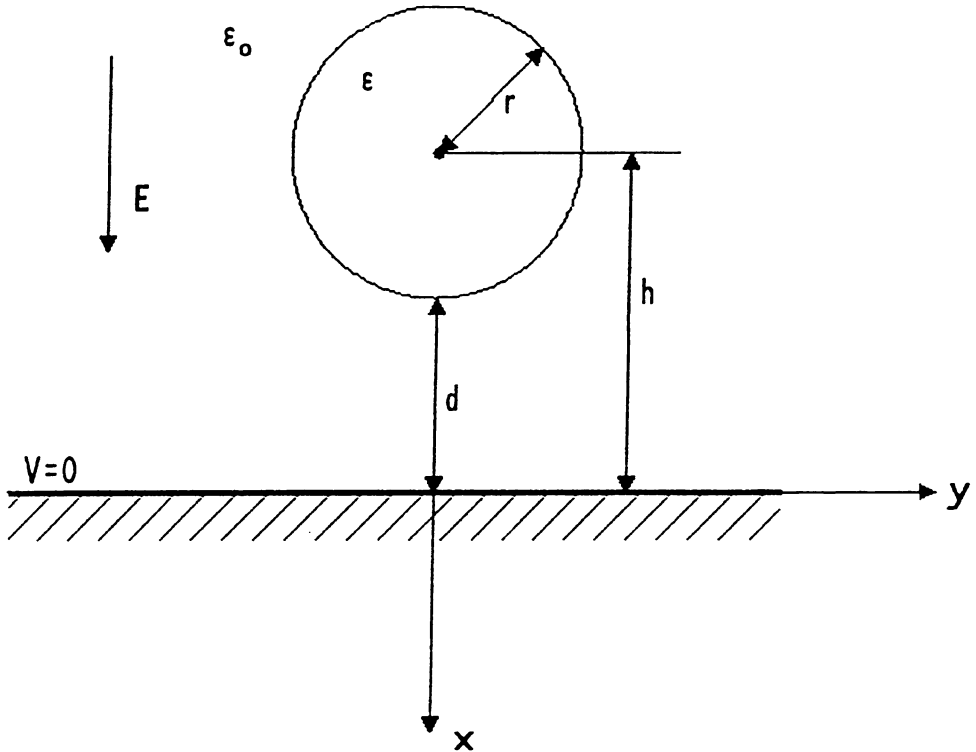

FIG. 1. Problem configuration

The problem is essentially two-dimensional. The potential field inside (V_i) and outside (V_0) the cylinder can be written as the sum of the potential field V due to the constant electric field and a disturbance due to the introduction of the dielectric cylinder:

$$V_i = V + V_{di}, \tag{1}$$

$$V_o = V + V_{do} \tag{2}$$

where (∇ is the nabla vector differential operator):

$$\vec{E} = -\nabla V. \tag{3}$$

Since the cylinder is homogeneous, isotropic and free from external charges, the basic equation inside and outside the cylinder is the Laplace equation [3]:

$$\Delta V_{di} = 0, \tag{4}$$

$$\Delta V_{do} = 0. \tag{5}$$

The boundary conditions amount to:
 1. The field must be symmetric about the x-axis.
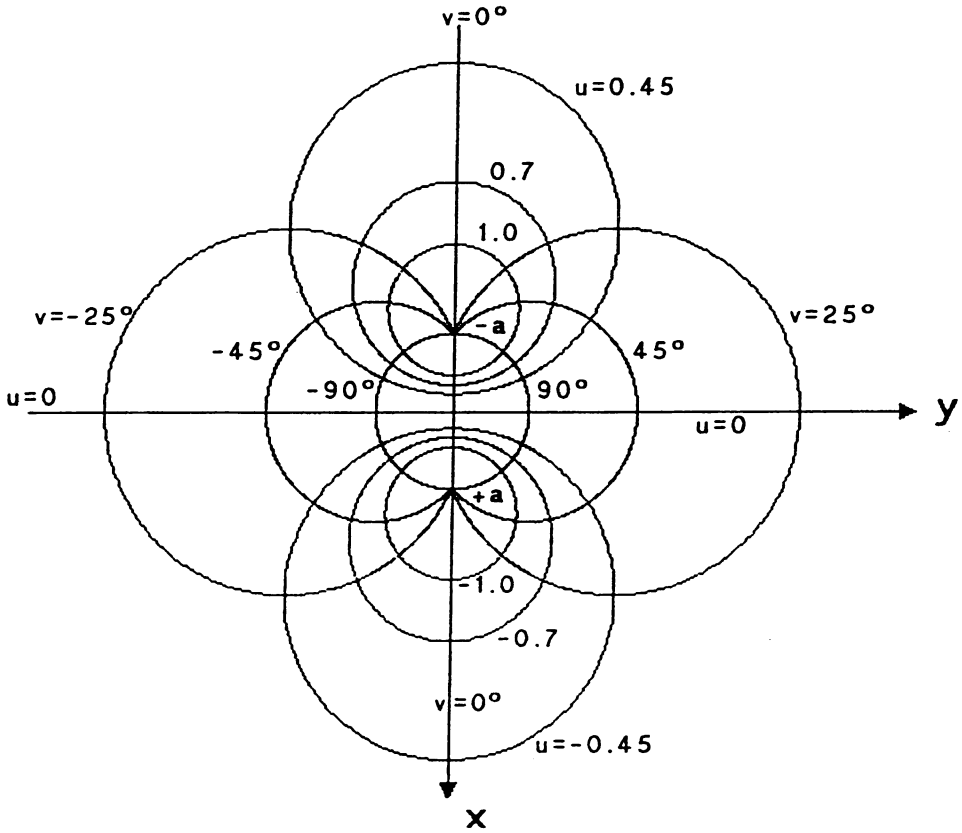 2. V_{di} must remain finite.

FIG. 2.   Bipolar coordinate system

3. $V_o = 0$ on the boundary between both half spaces.
4. The potential is continuous on the surface of the cylinder:

$$V_i = V_o. \tag{6}$$

5. The normal component of the electric displacement is continuous on the surface of
    the cylinder:

$$\varepsilon \frac{\partial V_i}{\partial n} = \varepsilon_0 \frac{\partial V_o}{\partial n}. \tag{7}$$

For the present geometry the use of bipolar coordinates (u, v) is very well suited (Fig. 2,
[4]). The cartesian coordinates (x, y) can be expressed as

$$x(u, v) = -a \frac{\sinh u}{\cosh u - \cos v}, \tag{8}$$

$$y(u, v) = a \frac{\sin v}{\cosh u - \cos v}. \tag{9}$$

Taking the surface of the cylinder as $u = u_0$, h and r amount to (Figs. 1, 2):

$$h = \frac{1}{2}[x(u_0, v = 0) + x(u_0, v = \pi)] = \frac{a \cosh u_0}{\sinh u_0}, \tag{10}$$

$$r = \frac{1}{2}[x(u_0, v = 0) - x(u_0, v = \pi)] = \frac{a}{\sinh u_0}. \tag{11}$$

The metric tensor components are ([5]):

$$g_{uu} = g_{vv} = \frac{a^2}{(\cosh u - \cos v)^2}, \tag{12}$$

$$g_{uv} = g_{vu} = 0, \tag{13}$$

leaving the cartesian form of the Laplace equation unchanged, e.g., for V_{di}:

$$\frac{\partial^2 V_{di}}{\partial u^2} + \frac{\partial^2 V_{di}}{\partial v^2} = 0. \tag{14}$$

The electric field $\vec{E}_{di}$ takes the form

$$\vec{E}_{di} = -\nabla V_{di} = -\left\{ \frac{1}{\sqrt{g_{uu}}} \frac{\partial V_{di}}{\partial u} \vec{e}_u + \frac{1}{\sqrt{g_{vv}}} \frac{\partial V_{di}}{\partial v} \vec{e}_v \right\} \tag{15}$$

where

$$\vec{e}_u = \frac{1}{\sqrt{g_{uu}}} \left\{ \frac{\partial x}{\partial u} \vec{i}_x + \frac{\partial y}{\partial u} \vec{i}_y \right\}, \tag{16}$$

$$\vec{e}_v = \frac{1}{\sqrt{g_{vv}}} \left\{ \frac{\partial x}{\partial v} \vec{i}_x + \frac{\partial y}{\partial v} \vec{i}_y \right\}. \tag{17}$$

$\vec{e}_u, \vec{e}_v$ are unit vectors in the bipolar coordinate system, $\vec{i}_x, \vec{i}_y$ in the cartesian system. Using separation of variables the solution of Eq. (14) can be written as

$$V_{di} = \sum_{n=0}^{\infty} (A'_n e^{nu} + B'_n e^{-nu})(C'_n \cos nv + D'_n \sin nv) \tag{18}$$

and similarly for V_{do}. The symmetry about the x-axis and the finiteness of V_{di} lead to

$$V_i = Ea \frac{\sinh u}{\cosh u - \cos v} + \sum_{n=0}^{\infty} C_n e^{-nu} \cos nv, \tag{19}$$

$$V_0 = Ea \frac{\sinh u}{\cosh u - \cos v} + \sum_{n=0}^{\infty} (A_n e^{nu} + B_n e^{-nu}) \cos nv, \tag{20}$$

where the first term in each expression is due to the original constant electric field (expressed in bipolar coordinates). A zero potential on the boundary of both half spaces requires

$$A_n = -B_n, \quad n = 0, 1, 2, \ldots \tag{21}$$

whereas Eq. (6) leads to

$$C_n = A_n e^{2nu_0} + B_n, \quad n = 0, 1, 2, \ldots .$$ (22)

Equation (7) can be rewritten as

$$\varepsilon \frac{\partial V_i}{\partial u} = \varepsilon_0 \frac{\partial V_o}{\partial u}$$ (23)

yielding

$$C_n = \frac{\varepsilon_0}{\varepsilon}(B_n - A_n e^{2nu_0}) + \left(\frac{\varepsilon_0}{\varepsilon} - 1\right) 2Ea, \quad n = 1, 2, 3, \ldots .$$ (24)

To get Eq. (24) , V has been rewritten as ([6], formula 1.461.2):

$$V = Ea \frac{\sinh u}{\cosh u - \cos v} = Ea \left(1 + 2 \sum_{n=1}^{\infty} e^{-nu} \cos nv\right), \quad u > 0.$$ (25)

The unknown constants can be solved from Eqs. (21), (22), and (24):

$$A_n = -B_n = \frac{(\varepsilon_0 - \varepsilon)2Ea}{\varepsilon(e^{2nu_0} - 1) + \varepsilon_0(e^{2nu_0} + 1)}, \quad n = 1, 2, 3, \ldots,$$

$$C_n = \frac{(e^{2nu_0} - 1)(\varepsilon_0 - \varepsilon)2Ea}{\varepsilon(e^{2nu_0} - 1) + \varepsilon_0(e^{2nu_0} + 1)}, \quad n = 1, 2, 3, \ldots,$$ (26)

$$C_0 = 0,$$

leading to

$$V_i = Ea \left[1 + 4\varepsilon_0 \sum_{n=1}^{\infty} \frac{e^{-nu} \cos nv}{\varepsilon(1 - e^{-2nu_0}) + \varepsilon_0(1 + e^{-2nu_0})}\right],$$ (27)

$$V_0 = Ea \left[1 + 2 \sum_{n=1}^{\infty} \frac{(\varepsilon_0 - \varepsilon)e^{nu}e^{-2nu_0} + (\varepsilon_0 + \varepsilon)e^{-nu}}{\varepsilon(1 - e^{-2nu_0}) + \varepsilon_0(1 + e^{-2nu_0})} \cos nv\right].$$ (28)

Taking the limit for $\varepsilon \to \infty$, and switching the limit with the summation sign (allowed due to the uniform convergence of the above series) leads to the expressions obtained by H. Jenss [1] for the conducting cylinder.

3. Force acting on the cylinder. The mechanical stress vector at some point of the cylinder amounts to [7]:

$$\vec{t} = \frac{1}{2}(\varepsilon_0 - \varepsilon)(\vec{E}_i \cdot \vec{E}_0)\vec{e}_u$$ (29)

leading to the force

$$\vec{K} = \frac{1}{2}(\varepsilon_0 - \varepsilon) \int_{v=0}^{2\pi} (\vec{E}_i \cdot \vec{E}_0) \vec{e}_u \sqrt{g_{vv}} \, dv \tag{30}$$

for the complete cylinder, per unit length. Using Eq. (15), the cartesian components of the force yield

$$K_x = \frac{1}{2a}(\varepsilon_0 - \varepsilon) \int_0^{2\pi} \left[\frac{\varepsilon_0}{\varepsilon} \left(\frac{\partial V_o}{\partial u} \bigg|_{u=u_0} \right)^2 + \left(\frac{\partial V_o}{\partial v} \bigg|_{u=u_0} \right)^2 \right] (\cosh u_0 \cos v - 1) \, dv, \tag{31}$$

$$K_y = \frac{1}{2a}(\varepsilon_0 - \varepsilon) \int_0^{2\pi} \left[\frac{\varepsilon_0}{\varepsilon} \left(\frac{\partial V_o}{\partial u} \bigg|_{u=u_0} \right)^2 + \left(\frac{\partial V_o}{\partial v} \bigg|_{u=u_0} \right)^2 \right] (- \sinh u_0 \sin v) \, dv. \tag{32}$$

Finally, substitution of (27) and (28) into (31) and (32) leads to

$$K_x = 2\pi \varepsilon_0 (\varepsilon_0^2 - \varepsilon^2) E^2 r \sinh u_0$$
$$\cdot \sum_{n=1}^{\infty} \frac{n(n+1)\cosh u_0(\varepsilon \sinh nu_0 + \varepsilon_0 \cosh nu_0) - n^2(\varepsilon \sinh(n+1)u_0 + \varepsilon_0 \cosh(n+1)u_0)}{(\varepsilon \sinh nu_0 + \varepsilon_0 \cosh nu_0)^2(\varepsilon \sinh(n+1)u_0 + \varepsilon_0 \cosh(n+1)u_0)}, \tag{33}$$

$$K_y = 0. \tag{34}$$

Equation (34) is a natural consequence of the symmetry about the x-axis. In Table 1 the dimensionless force K_x^* defined by

$$K_x^* \doteq \frac{K_x}{E^2 \varepsilon_0 r} \tag{35}$$

has been calculated for several values of u_0 and $\varepsilon_r \doteq \varepsilon/\varepsilon_0$. Since

$$d/r = \cosh u_0 - 1, \tag{36}$$

u_0 is a measure of d for constant r, i.e., $d \to 0$ for $u_0 \to 0$ and r constant. Table 1 shows that for finite ε_r and $u_0 \to 0$ (i.e., the cylinder approaches the conducting half space) the force remains finite, whereas for $\varepsilon_r \to \infty, u_0 \to 0$ the values of K_x^* seem to blow up like π/u_0. The latter case corresponds to a conducting cylinder approaching the half space and will be considered first since it is the simpler case.

TABLE 1. Numerical values for the dimensionless force
(the symbol "-" signifies that no value was calculated).

u_0	d/r	K_x^* ($\varepsilon_r = 5$)	K_x^* ($\varepsilon_r = 10$)	K_x^* ($\varepsilon_r = 100$)	$K_x^* u_0/\pi$ ($\varepsilon_r \to \infty$)
10.0	$1.1 \ 10^4$	$5.30 \ 10^{-13}$	$7.96 \ 10^{-13}$	$1.14 \ 10^{-12}$	$3.79 \ 10^{-12}$
5.0	73.21	$1.71 \ 10^{-6}$	$2.57 \ 10^{-6}$	$3.69 \ 10^{-6}$	$6.11 \ 10^{-6}$
2.0	2.762	$1.34 \ 10^{-2}$	$2.04 \ 10^{-2}$	$2.94 \ 10^{-2}$	0.0195
1.5	1.352	$5.75 \ 10^{-2}$	$8.81 \ 10^{-2}$	0.1287	0.0641
1.0	0.543	0.2331	0.3688	0.5587	0.1863
0.5	0.128	0.8807	1.5650	2.7260	0.4630
0.25	$3.14 \ 10^{-2}$	1.6507	3.4730	-	-
0.10	$5.00 \ 10^{-3}$	2.2444	5.7813	21.916	0.8651
0.05	$1.25 \ 10^{-3}$	-	6.6587	-	0.9305
0.04	$8.00 \ 10^{-4}$	-	-	-	0.9441
0.03	$4.50 \ 10^{-4}$	-	-	56.149	0.9579
0.01	$5.00 \ 10^{-5}$	2.4413	7.0757	86.784	0.9858
0.005	$1.25 \ 10^{-5}$	-	-	95.915	-
0.001	$5.00 \ 10^{-7}$	2.44357	7.0956	100.226	0.9986
0.0005	$1.25 \ 10^{-7}$	-	7.09574	100.381	-
0.0001	$5.00 \ 10^{-9}$	2.44360	7.09578	100.431	0.9998

4. Limit value of the force for a conducting cylinder approaching the half space.

In this section the asymptotic behaviour of K_x^* for $\varepsilon_r \to \infty$ and $u_0 \to 0$ is looked at. For the conducting cylinder the nondimensional form of Eq. (33) yields

$$K_x^* = -2\pi \sinh u_0 \sum_{n=1}^{\infty} \frac{n(n+1) \cosh u_0 \sinh nu_0 - n^2 \sinh(n+1)u_0}{\sinh^2 nu_0 \sinh(n+1)u_0}. \tag{37}$$

The statement at the end of the last section amounts to

$$\lim_{u_0 \to 0} \frac{-K_x^*}{2\pi} u_0 = -\frac{1}{2}, \tag{38}$$

which has to be proven. The main problem in taking the limit of Eq. (37) for $u_0 \to 0$ is that the limit and summation sign cannot be switched. Indeed, it is not difficult to check that the limit for each single term in the series for $u_0 \to 0$ tends to zero (for n fixed). This phenomenon can be better understood if one realizes that the series is not uniformly convergent in u_0 over the interval $[0, \alpha]$ where α is some positive real number. No matter how small u_0 is chosen it is always possible to find a value of n big enough to yield a significant contribution of nu_0. The key in the present asymptotic approach is that, although $u_0 \to 0$, no assumption is made about the size of nu_0. Looking at the expression

$$I \doteq \frac{-K_x^*}{2\pi} u_0 = u_0 \sinh u_0 \sum_{n=1}^{\infty} \frac{n(n+1) \cosh u_0 \sinh nu_0 - n^2 \sinh(n+1)u_0}{\sinh^2 nu_0 \sinh(n+1)u_0}$$

$$\doteq u_0 \sinh u_0 \sum_{n=1}^{\infty} \frac{N_n}{D_n}, \tag{39}$$

one can write

$$N_n = n \cosh u_0 \sinh n u_0 - n^2 \cosh n u_0 \sinh u_0$$

$$\sim n \sinh n u_0 - n^2 u_0 \cosh n u_0, \quad u_0 \to 0,$$

$$\sim n \sum_{k=0}^{\infty} (n u_0)^{2k+1} \left\{ \frac{1}{(2k+1)!} - \frac{1}{(2k)!} \right\}, \quad u_0 \to 0, \tag{40}$$

$$\sim -n \sum_{k=1}^{\infty} (n u_0)^{2k+1} \frac{2k}{(2k+1)!}, \quad u_0 \to 0,$$

$$D_n \sim \sinh^2 n u_0 [e^{n u_0}(1 + u_0) - e^{-n u_0}(1 - u_0)]/2, \quad u_0 \to 0,$$

$$\sim \sinh^3 n u_0, \quad u_0 \to 0. \tag{41}$$

Hence

$$I \sim -u_0^2 \sum_{k=1}^{\infty} u_0^{2k+1} \frac{2k}{(2k+1)!} \sum_{n=1}^{\infty} \frac{n^{2k+2}}{\sinh^3 n u_0}, \quad u_0 \to 0. \tag{42}$$

Since according to [8]

$$\frac{1}{\sinh^3 n u_0} = \frac{8}{e^{3 n u_0}(1 - e^{-2 n u_0})^3}$$

$$= 8 \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} e^{-(2i+3)n u_0}, \tag{43}$$

Eq. (42) is equivalent to

$$I \sim -8 u_0^2 \sum_{k=1}^{\infty} u_0^{2k+1} \frac{2k}{(2k+1)!} \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} \sum_{n=1}^{\infty} n^{2k+2} (e^{-(2i+3)u_0})^n, \quad u_0 \to 0. \tag{44}$$

Defining

$$S_k \doteq \sum_{n=1}^{\infty} n^k q^n, \tag{45}$$

it is not difficult to prove that

$$S_k = q \frac{dS_{k-1}}{dq}. \tag{46}$$

Using this expression and

$$S_0 = \frac{q}{1-q}, \tag{47}$$

it is possible to find an explicit expression for S_k:

$$S_k = \frac{\sum_{i=1}^{k} a_i^{(k)} q^i}{(1-q)^{k+1}}, \tag{48}$$

where the coefficients $a_i^{(k)}$ are determined by a recursive relation:

$$a_1^{(k)} = a_k^{(k)} = 1,$$
$$a_j^{(k)} = j a_j^{(k-1)} + (k+1-j) a_{j-1}^{(k-1)}, \quad j = 2, \ldots, k-1. \tag{49}$$

Equation (49) also yields

$$\sum_{i=1}^{k} a_i^{(k)} = k \sum_{i=1}^{k-1} a_i^{(k-1)} \tag{50}$$

leading to

$$\sum_{i=1}^{k} a_i^{(k)} = k! \tag{51}$$

Using Eq. (48) in Eq. (44) leads to

$$I \sim -8u_0^2 \sum_{k=1}^{\infty} u_0^{2k+1} \frac{2k}{(2k+1)!} \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} \frac{\sum_{j=1}^{2k+2} a_j^{(2k+2)} [e^{-(2i+3)u_0}]^j}{[(2i+3)u_0]^{2k+3}}, \quad u_0 \to 0, \tag{52}$$

or, using Eq. (51),

$$I \sim -8 \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{(2i+3)^3} \sum_{k=1}^{\infty} \frac{k(2k+2)}{[(2i+3)^2]^k}, \quad u_0 \to 0. \tag{53}$$

The expression in k is again of the form (45). Summing explicitly yields

$$I \sim -\frac{1}{2} \sum_{i=0}^{\infty} \frac{2i+3}{(i+1)^2(i+2)^2}, \quad u_0 \to 0. \tag{54}$$

After splitting into partial fractions the calculation is complete:

$$I \sim -\frac{1}{2} \sum_{i=0}^{\infty} \left\{ \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2} \right\}, \quad u_0 \to 0, \tag{55}$$

$$\sim -\frac{1}{2}, \quad u_0 \to 0.$$

It demonstrates that the first term in the asymptotic expansion of K_x^* (Eq. (37)) in u_0 is indeed π/u_0.

5. Limit value of the force for a dielectric cylinder. For a dielectric cylinder, Eq. (33) can be written in the form (using $\varepsilon_r \doteq \varepsilon/\varepsilon_0$ and (35)):

$$K_x^* = 2\pi(1 - \varepsilon_r^2) \sinh u_0$$

$$\cdot \sum_{n=1}^{\infty} \frac{[n(n+1)\cosh u_0(\varepsilon_r \sinh nu_0 + \cosh nu_0) - n^2(\varepsilon_r \sinh(n+1)u_0 + \cosh(n+1)u_0)]}{(\varepsilon_r \sinh nu_0 + \cosh nu_0)^2[\varepsilon_r \sinh(n+1)u_0 + \cosh(n+1)u_0]} \tag{56}$$

$$\doteq 2\pi(1 - \varepsilon_r^2) \sinh u_0 \sum_{n=1}^{\infty} \frac{N_n}{D_n}.$$

N_n and D_n now take the form

$$N_n \sim \varepsilon_r n[\sinh nu_0 - (nu_0)\cosh nu_0] + n[\cosh nu_0 - (nu_0)\sinh nu_0] + O(u_0^2), \quad u_0 \to 0, \tag{57}$$

$$D_n \sim (\varepsilon_r \sinh nu_0 + \cosh nu_0)^3 \left[1 + u_0 \frac{\varepsilon_r \cosh nu_0 + \sinh nu_0}{\varepsilon_r \sinh nu_0 + \cosh nu_0} \right], \quad u_0 \to 0. \tag{58}$$

Consequently the leading term of $K_x^*/[2\pi(1 - \varepsilon_r^2)]$, $u_0 \to 0$, yields

$$J_{-1} \doteq \sinh u_0 \sum_{n=1}^{\infty} \frac{\varepsilon_r n[\sinh nu_0 - (nu_0)\cosh nu_0] + n[\cosh nu_0 - (nu_0)\sinh nu_0]}{[\varepsilon_r \sinh nu_0 + \cosh nu_0]^3}. \quad (59)$$

The numerator of a typical term in Eq. (59) can be written as

$$\sinh u_0 \left\{ -\varepsilon_r n \sum_{k=0}^{\infty} (nu_0)^{2k+1} \frac{2k}{(2k+1)!} - n \sum_{k=0}^{\infty} (nu_0)^{2k} \frac{2k-1}{(2k)!} \right\} \quad (60)$$

and the denominator yields

$$\frac{1}{(\varepsilon_r \sinh nu_0 + \cosh nu_0)^3} = \frac{8}{(\varepsilon_r + 1)^3} \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i e^{-(2i+3)nu_0} \quad (61)$$

using the same strategy as for the conducting cylinder, J_{-1} is transformed into (comparable to Eq. (54))

$$\begin{aligned}
J_{-1} \sim \frac{1}{u_0} \Bigg\{ &-\frac{\varepsilon_r}{2(\varepsilon_r + 1)^3} \sum_{i=0}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i \left[\frac{1}{(i+1)^2} - \frac{1}{(i+2)^2} \right] \\
&- \frac{1}{2(\varepsilon_r + 1)^3} \sum_{i=0}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i \left[\frac{1}{(i+1)^2} + \frac{1}{(i+2)^2} \right] \\
&+ \frac{1}{(\varepsilon_r + 1)^3} \sum_{i=0}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i \Bigg\}, \quad u_0 \to 0,
\end{aligned} \quad (62)$$

which vanishes for a finite ε_r (can be checked through appropriate combination of terms). This shows that the leading term of K_x^* remains finite. The next term of $K_x^*/[2\pi(1 - \varepsilon_r^2)]$ yields

$$J_0 \doteq -u_0 \sinh u_0 \sum_{n=1}^{\infty} \frac{N_n[\varepsilon_r \cosh nu_0 + \sinh nu_0]}{[\varepsilon_r \sinh nu_0 + \cosh nu_0]^4} \quad (63)$$

where N_n has the form (57). The numerator after the summation sign amounts to

$$\begin{aligned}
N_n[\varepsilon_r \cosh nu_0 + \sinh nu_0] = &\, n \left\{ \frac{\varepsilon_r^2 + 1}{2} - \varepsilon_r(nu_0) \right\} \sinh(2nu_0) \\
&+ n \left\{ \varepsilon_r - (nu_0)\left(\frac{\varepsilon_r^2 + 1}{2} \right) \right\} \cosh(2nu_0) + n(nu_0)\left(\frac{1 - \varepsilon_r^2}{2} \right).
\end{aligned} \quad (64)$$

Substitution of Eq. (64) into Eq. (63) yields

$$\begin{aligned}
J_0 = -u_0 \sinh u_0 \Bigg\{ &\left(\frac{\varepsilon_r^2 + 1}{2} \right) L(0,1) - \varepsilon_r L(1,1) \\
&+ \varepsilon_r L(0,0) - \left(\frac{\varepsilon_r^2 + 1}{2} \right) L(1,0) + \left(\frac{1 - \varepsilon_r^2}{2} \right) \sum_{n=1}^{\infty} \frac{n(nu_0)}{[\varepsilon_r \sinh nu_0 + \cosh nu_0]^4} \Bigg\}
\end{aligned}$$

$$(65)$$

where

$$L(\beta, \gamma) \doteq \sum_{n=1}^{\infty} n(nu_0)^{\beta} \frac{1}{(\varepsilon_r \sinh nu_0 + \cosh nu_0)^4} \sum_{k=0}^{\infty} \frac{(2nu_0)^{2k+\gamma}}{(2k+\gamma)!}. \tag{66}$$

Analogously as for the conducting cylinder one can show that

$$L(\beta, \gamma) \sim \frac{4}{u_0^2 (\varepsilon_r + 1)^4}$$
$$\cdot \sum_{i=0}^{\infty} \frac{(i+1)(i+3)}{6(i+2)} \left\{ \frac{1}{2^{\beta}(i+2)^{\beta+\gamma}} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{\beta+1}(2k+\gamma+j)}{[(i+2)^2]^k} \right\} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i, \quad u_0 \to 0 \tag{67}$$

and

$$\sum_{n=1}^{\infty} \frac{n(nu_0)}{(\varepsilon_r \sinh nu_0 + \cosh nu_0)^4} \sim \frac{4}{u_0^2 (\varepsilon_r + 1)^4} \sum_{i=0}^{\infty} \frac{(i+1)(i+3)}{6(i+2)} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i \frac{1}{i+2}, \tag{68}$$
$$u_0 \to 0,$$

leading to the following expression for J_0:

$$J_0 \sim -\frac{4}{(\varepsilon_r + 1)^4} \sum_{i=0}^{\infty} \frac{(i+1)(i+3)}{6(i+2)} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^i M, \quad u_0 \to 0, \tag{69}$$

where

$$M = \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{1}{i+2} \sum_{k=0}^{\infty} \frac{2k+2}{(i+2)^{2k}} - \frac{\varepsilon_r}{2(i+2)^2} \sum_{k=0}^{\infty} \frac{(2k+3)(2k+2)}{(i+2)^{2k}}$$
$$+ \varepsilon_r \sum_{k=0}^{\infty} \frac{2k+1}{(i+2)^{2k}} - \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{1}{2(i+2)} \sum_{k=0}^{\infty} \frac{(2k+2)(2k+1)}{(i+2)^{2k}} + \frac{1}{i+2} \left(\frac{1 - \varepsilon_r^2}{2} \right). \tag{70}$$

Through summation M yields

$$M = \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{1}{j} \left(\frac{2j^4}{(j^2 - 1)^2} \right) - \varepsilon_r \frac{j^2(3j^2 + 1)}{(j^2 - 1)^3} + \varepsilon_r \frac{j^2(j^2 + 1)}{(j^2 - 1)^2}$$
$$- \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{1}{j} \left(\frac{j^4(j^2 + 3)}{(j^2 - 1)^3} \right) + \left(\frac{1 - \varepsilon_r^2}{2} \right) \frac{1}{j} \tag{71}$$

where $j = i + 2$. Consequently,

$$J_0 \sim -\frac{4}{(\varepsilon_r + 1)^4} \sum_{j=2}^{\infty} \frac{1}{6} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^{j-2} \left\{ \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{2j^2}{(j^2 - 1)} - \varepsilon_r \frac{j(3j^2 + 1)}{(j^2 - 1)^2} \right.$$
$$\left. + \varepsilon_r \frac{j(j^2 + 1)}{(j^2 - 1)} - \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{j^2(j^2 + 3)}{(j^2 - 1)^2} + \left(\frac{1 - \varepsilon_r^2}{2} \right) \frac{j^2 - 1}{j^2} \right\}, \quad u_0 \to 0. \tag{72}$$

Splitting into partial fractions yields:

$$\frac{2j^2}{j^2 - 1} = 2 + \frac{1}{j-1} - \frac{1}{j+1}, \tag{73}$$

$$\frac{j(3j^2 + 1)}{(j^2 - 1)^2} = \frac{-1 + 3j}{2(j-1)^2} + \frac{1 + 3j}{2(j+1)^2}, \tag{74}$$

$$\frac{j(j^2 + 1)}{(j^2 - 1)} = j + \frac{1}{j-1} + \frac{1}{j+1}, \tag{75}$$

$$\frac{j^2(j^2 + 3)}{(j^2 - 1)^2} = 1 + \frac{-1 + 3j}{2(j-1)^2} - \frac{1 + 3j}{2(j+1)^2}. \tag{76}$$

Substitution of Eqs. (73)–(76) into Eq. (72) reveals that

$$\sum_{j=2}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^{j-2} \left\{ \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{2j^2}{(j^2 - 1)} - \varepsilon_r \frac{j(3j^2 + 1)}{(j^2 - 1)^2} \right.$$

$$\left. + \varepsilon_r \frac{j(j^2 + 1)}{(j^2 - 1)} - \left(\frac{\varepsilon_r^2 + 1}{2} \right) \frac{j^2(j^2 + 3)}{(j^2 - 1)^2} \right\} = -2 \sum_{j=2}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^{j-2} \left(\frac{1 - \varepsilon_r^2}{2} \right) \left(\frac{j^2 - 1}{j^2} \right) \tag{77}$$

and, consequently,

$$J_0 \sim \frac{1 - \varepsilon_r^2}{3(\varepsilon_r + 1)^4} \sum_{j=2}^{\infty} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^{j-2} \left(1 - \frac{1}{j^2} \right), \quad u_0 \to 0, \tag{78}$$

$$\sim \frac{1 - \varepsilon_r^2}{3(\varepsilon_r + 1)^4} \left\{ \frac{\varepsilon_r + 1}{2} - \sum_{j=2}^{\infty} \frac{1}{j^2} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^{j-2} \right\}, \quad u_0 \to 0. \tag{79}$$

The series in Eq. (79) is related to S_{-2} which can be obtained from S_0 through repeated application of the inverse operation of Eq. (46):

$$S_{k-1} = \int_0^q \frac{S_k}{\xi} \, d\xi. \tag{80}$$

This yields, starting from S_0 in Eq. (47):

$$S_{-1} = -\ln(1 - q), \tag{81}$$

$$S_{-2} = -\int_0^q \frac{\ln(1 - \xi)}{\xi} \, d\xi, \tag{82}$$

which cannot be expressed in elementary functions ([6], formula 2.728.2). So the final expression for K_x^* is

$$K_x^* \sim \frac{2\pi}{3} \frac{(\varepsilon_r^2 - 1)^2}{(\varepsilon_r + 1)^4} \left\{ \frac{\varepsilon_r + 1}{2} - \sum_{j=0}^{\infty} \frac{1}{(j + 2)^2} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right)^j \right\}, \quad u_0 \to 0. \tag{83}$$

Evaluation of Eq. (83) agrees with the values in Table 1 for $u_0 \ll 1$.

6. Conclusions. The force between a dielectric cylinder in a constant electric field and a conducting half space has been determined using the separation of variables technique on the Laplace equation in bipolar coordinates. The force has been obtained as a series containing the relative distance between the cylinder and the half space as a parameter. Due to the nonuniform convergence of the series in this parameter, a special asymptotic analysis was needed to find the limiting value of the force as the cylinder approaches the half space. It was shown that for a dielectric cylinder the force remains finite whereas for a conducting cylinder the force is inversely proportional to the relative distance.

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