function H. An alternative way of proceeding would be to avoid development in terms of functions u_n and to have recourse to the original equation (I). Guided by the results of the preceding sections, the solution u, as regards its main part, might be put into the form

$$u = \phi \cos(\lambda t + \epsilon) + \chi \cos 2(\lambda t + \epsilon),$$

λ and the two functions ϕ and χ to be determined by equating the coefficients of $\cos(\lambda t + \epsilon)$ and $\cos 2(\lambda t + \epsilon)$ separately. An extended problem analogous to (II) would result.

Noordwijk:
1934 *December* 29.

ON THE FORMAL COMPARISON OF MILNE'S KINEMATICAL SYSTEM WITH THE SYSTEMS OF GENERAL RELATIVITY.

A. G. Walker, Ph.D.

(Communicated by Professor E. A. Milne)

1. When Milne introduced his kinematical system for the description of the gravitational field of the universe, the question at once became important whether this model differed essentially from the variety of models already provided by general relativistic cosmology. The differences that may exist between such models can be investigated by two different methods. The first method is to find the relations between the various astronomical observables in one system and to compare these with the corresponding relations in the other systems.* The importance of this method is that it provides possible tests of the different theories, but apart from this, it is unsatisfactory owing to the restricted number of such observables. The second method is to make a *formal* comparison of the systems, and this is the object of the present paper. We shall construct in accordance with the general theory of relativity a material system which is identical with Milne's system, and then formally compare the gravitational fields of this system predicted by the two theories. That these fields differ was the conclusion of a previous paper † by the author. In that paper, Milne's equations of motion for a free particle in the presence of a certain distribution of matter in motion were expressed in the form of a principle of least action. The result at once indicated that the geometry appropriate for a representation of Milne's system is Finsler and not Riemannian geometry.

2. The first step is to find the metric which, according to the general theory of relativity, corresponds to the material system studied by Milne.

* This method was recently discussed by W. H. McCrea, *Zs. für Astrophys.*, 1935 January.

† *Proc. Roy. Soc.*, A, 147, 478, 1934.

This system consists of a set of fundamental particles attached to each of which is an observer, and these particles satisfy the cosmological principle, *i.e.* each fundamental observer sees the same sequence of world-pictures. From temporal experiences only, the observers provide themselves with clocks which may be said to be similar and synchronized. Each observer then determines the epoch and distance of a distant event from observations made on his own clock according to the following conventions. If a light signal sent by an observer A at time t_1 by his clock is reflected at an event E and received by A at time t_2, then A assigns to E an epoch T and distance X where

$$T = \tfrac{1}{2}(t_2 + t_1) \qquad X = \tfrac{1}{2}c(t_2 - t_1). \tag{1}$$

Here c is an arbitrary constant, chosen to be the same by each observer, and ultimately identified as the velocity of light reckoned in the chosen measures.

With the above definition of epoch and distance, the system considered by Milne is such that, relative to each observer, the other fundamental particles are moving with uniform velocities. Further, the epochs and distances assigned to an event by any two observers are connected by Lorentz formulæ.

The important fact that should be remembered is that, throughout the discussion of his system, Milne does not introduce any indefinable concepts, such as the concept of a rigid scale. His arguments on this subject have been very clearly stated elsewhere.* Bearing this in mind, we shall endeavour to construct and describe Milne's system by the methods of general relativity without introducing such concepts. We shall, in fact, not assume that the associated metric has any *a priori* physical significance, and in particular we shall not assume that an observer's "proper time" is the arc length of his world-line. We now have no rigid scale, and we shall define epoch and distance in accordance with Milne's conventions.

3. According to the general theory of relativity, events are so ordered that they may be represented by points of a Riemannian 4-space of signature -2 in such a way that Mach's principle is satisfied, and that the light paths correspond to null geodesics. Each particle of the system we are considering corresponds to a curve, or world-line, and there is one such curve through each point of the space. The cosmological principle at once indicates that each fundamental particle is a centre of spherical symmetry, whence the 4-space must exhibit spherical symmetry about each world-line. We are therefore seeking a 4-space striated by a system of curves about each of which the space has spherical symmetry, and it has been shown† that co-ordinates τ', r', θ', ϕ' can be chosen so that the metric of such a space takes the form

$$ds^2 = d\tau'^2 - \{R'(\tau')\}^2 \left\{ \frac{dr'^2}{1 - kr'^2} + r'^2(d\theta'^2 + \sin^2\theta' d\phi'^2) \right\}, \tag{2}$$

* An address delivered by Professor E. A. Milne to the British Institute of Philosophy, 1933 October 17; published in *Philosophy*, 1934 January.

† A. G. Walker, *Quart. Journ. Maths.* (in press).

where $k=1$, 0 or -1. The world-lines of the fundamental particles are the geodesics r', θ', ϕ' = constants, and as the sub-spaces τ' = constant are orthogonal to these curves and are spaces of constant curvature, *i.e.* admit motion without deformation, the cosmological principle is satisfied.

The metric (2) is of course well known in relativistic cosmology and the above derivation is only one of many. We now, however, depart from the usual procedure, which is to assume that each observer is provided with a clock recording the co-ordinate τ' and can measure elements of distance $R'dr'$ by means of a rigid scale. It will be found convenient to replace the co-ordinate τ' by τ where $d\tau/\tau = d\tau'/R'(\tau')$, and writing $R(\tau)=R'(\tau')$, the metric (2) now takes the form

$$ds^2 = \{R(\tau)\}^2\left\{\frac{d\tau^2}{\tau^2} - \frac{dr'^2}{1-kr'^2} - r'^2(d\theta'^2 + \sin^2\theta' d\phi'^2)\right\}. \tag{3}$$

4. Consider now the observer A_0 at the origin $r'=0$ and another observer A_1 at a co-ordinate distance r_1' from the origin. Then from (3), the null geodesics connecting the two world-lines are given by

$$\frac{d\tau}{\tau} = \pm\frac{dr'}{(1-kr'^2)^{1/2}}, \tag{4}$$

whence, writing

$$\sigma(x) = \int_0^x \frac{dx}{(1-kx^2)^{1/2}}, \tag{5}$$

we see that if a light signal leaving A_0 when $\tau=\tau_1$ is reflected by A_1 and received by A_0 when $\tau=\tau_2$, then

$$\tau_2 = \tau_1 e^{2\sigma(r_1')}.$$

Hence, if A_0 has a clock recording τ and defines epoch and distance in terms of this clock by means of (1), he observes A_1 to have the constant radial velocity $c \tanh \sigma(r_1')$. The pair of observers A_0, A_1 is a typical pair, and hence each observer having a clock recording time τ observes the other fundamental particles to be moving with uniform velocities. These clocks are therefore similar to the clocks possessed by the observers in Milne's system.

Milne has shown * how a set of equivalent observers, starting with quite general clocks based merely on temporal experience, can choose clocks which may be said to be similar and synchronized as a result of an ideal set of experiments. There is a variety of such clocks and, in particular, we have shown that it is possible for the observers we are considering to select clocks in terms of which they appear to be in uniform relative motion. One such set of clocks consists of those recording time τ, and it can easily be verified that any other clocks leading to uniform relative motion must record time $b\tau^a$, where a, b are constants, the same for each observer. Hence, from observations only, the observers attached to the fundamental particles of the system corresponding to the metric (3) can provide themselves with

* In lectures delivered at Oxford, October term 1933; now in course of publication.

similar clocks in terms of which the particles are in uniform relative motion.* These clocks will record time $b\tau^a$ for some values of a and b not determined from the observations. The constant b is unimportant, for it depends only on the chosen unit of time, and we may therefore write $b=1$.

5. Each observer can now describe the system in terms of observable co-ordinates similar to those used by Milne's observers. We see from (4) that if the observer A_0 at the origin sends a signal when $\tau=\tau_1$ which is reflected at an event $E(\tau, r')$ and received by A_0 when $\tau=\tau_2$, then

$$\tau_1=\tau e^{-\sigma(r')}, \qquad \tau_2=\tau e^{\sigma(r')}. \tag{6}$$

The observer's clock is recording τ^a for some value of a, whence A_0 assigns to the event E a time t and a distance r where

$$t=\tfrac{1}{2}(\tau_2{}^a+\tau_1{}^a), \qquad r=\tfrac{1}{2}c(\tau_2{}^a-\tau_1{}^a).$$

Hence from (6) the transformation from co-ordinates τ, r' to the observable co-ordinates t, r used by A_0 is

$$t=\tau^a \cosh a\sigma(r'), \qquad r=c\tau^a \sinh a\sigma(r').$$

We have so far considered only measurements of time and distance, but the observers can also measure angles. The co-ordinates θ', ϕ' are essentially angular co-ordinates relative to a Euclidean frame of reference at the origin $r'=0$, and since the form (3) is unaltered by a rotation about the origin, we may say that θ', ϕ' are observable angular co-ordinates relative to a frame of reference chosen by the observer at the origin. Thus finally the transformation from co-ordinates τ, r', θ', ϕ' to observable co-ordinates t, r, θ, ϕ is

$$t=\tau^a \cosh a\sigma(r'), \qquad r=c\tau^a \sinh a\sigma(r'), \qquad \theta=\theta', \qquad \phi=\phi'. \tag{7}$$

Here r, θ, ϕ are the polar co-ordinates corresponding to the Cartesian co-ordinates x, y, z similar to those used by Milne, and t, r, θ, ϕ, or t, x, y, z, are the co-ordinates assigned by the observer at the origin to the event $(\tau, r', \theta', \phi')$.

6. There is still another condition to be satisfied in order that the system corresponding to the metric (3) shall be equivalent to Milne's system. Each observer can assign to a given event co-ordinates similar to t, x, y, z, and it is required that the co-ordinates assigned to the same event by any two observers shall be connected by Lorentz formulæ. It can easily be verified that the necessary and sufficient conditions for this requirement to be satisfied are that the expressions

$$t^2-(x^2+y^2+z^2)/c^2, \qquad dt^2-(dx^2+dy^2+dz^2)/c^2, \tag{8}$$

shall be invariant under the transformation from the co-ordinates of any one observer to those of any other observer of the system. We see at once from

* It must be remembered that the relative motion here considered is not necessarily the motion given by measurements of Doppler shift, etc. It is not assumed that the observers finally provide themselves with atomic clocks, although these would probably constitute a set of similar clocks. The observers are selecting only those clocks in terms of which they observe uniform motion according to the given conventions.

(7) that the first condition is already satisfied, for we have $t^2 - r^2/c^2 = \tau^{2a}$, and since the value of a is the same for all observers, the expression $t^2 - r^2/c^2$ must be invariant.

To find an invariant corresponding to the second expression of (8), we transform from τ, r', θ', ϕ' to t, r, θ, ϕ in the metric (3) by means of (7). It can easily be verified that the metric becomes

$$ds^2 = \{F(X)\}^2\left[dt^2 - \frac{1}{c^2}\{dr^2 + \rho^2(d\theta^2\ \sin^2\theta d\phi^2)\}\right], \tag{9}$$

where $\quad X = t^2 - r^2/c^2 = \tau^{2a}, \qquad F(X) = R(\tau)/a\tau^a, \qquad \rho = car'\tau^a.$ (10)

Hence, since ds is invariant under the transformations we are considering and X has been shown to be invariant, the expression

$$dt^2 - \frac{1}{c^2}\{dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\}$$

must also be invariant, ρ being expressed in terms of t and r. This expression evidently corresponds to the second expression of (8), and is identical with it only if $\rho = r$. From (10), (7) and (5), $\rho = r$ if

$$\sinh^{-1} ar' = a\sigma(r') = a\int_0^{r'} \frac{dx}{(1 - kx^2)^{1/2}},$$

and this equation is satisfied if, and only if,

$$k = -1, \qquad a = 1.$$

Hence the systems of general relativity which are equivalent to Milne's system are those given by the metric (3) with $k = -1$, *i.e.* the hyperbolic systems.

Further, since a is required to have a determinate value ($a = 1$), we see that the clocks to be chosen by the observers in each of the hyperbolic systems in order that they may identify themselves with Milne's observers are unique. In the variety of clocks, given by different values of a, that can be chosen by each observer, only one will satisfy the required conditions, and this clock can be selected by comparing observations with observations made by the other observers.

7. It is clear from the above analysis that Milne has restricted his system by assuming the Lorentz transformation, for the systems of general relativity given by (3) with $k = 1$ or 0 satisfy the cosmological principle but require transformations other than those of the Lorentz type. For the systems given by $k = 1$ or 0 the corresponding transformations are very complicated, and it may be suggested that the observable co-ordinates chosen to give uniform relative motion are not the most convenient co-ordinates to be used in describing these systems. It is possible that more convenient co-ordinates can be found by assuming the observers to choose new clocks in accordance with observations so that they no longer have uniform relative motions. For example, it is probable that the most simple description of the systems given by $k = 0$ is to be found by assuming the fundamental observers to be relatively at rest.

8. We have shown that the material system considered by Milne is, according to the general theory of relativity, given by the metric (3) with $k = -1$, the form of $R(\tau)$ being arbitrary. We can now investigate the gravitational field of this system according to each of the two theories, and to do this we calculate the equations of motion of free test-particles in the presence of the system.

Milne derives the equations of motion of free particles from general arguments based on the cosmological principle,* and obtains

$$\frac{d^2x}{dt^2} = \left(x - t\frac{dx}{dt}\right)\frac{Y}{X}G(X, \xi), \tag{11}$$

etc., where

$$X = t^2 - \frac{1}{c^2}\sum x^2, \quad Y = 1 - \frac{1}{c^2}\sum\left(\frac{dx}{dt}\right)^2, \quad Z = t - \frac{1}{c^2}\sum x\frac{dx}{dt}, \quad \xi = \frac{Z^2}{XY}. \tag{12}$$

The form of $G(X, \xi)$ is not further determined by these arguments, but Milne restricts the form after a consideration of physical dimensions. We see that in equation (11) G must be dimensionless. Now ξ has no physical dimensions but X has dimensions T^2, whence if X is included in G there must exist a dimensional world-constant. This at once introduces an empirical element contrary to Milne's theory of the universe, and Milne consequently reasons that G must only contain ξ.

According to the general theory of relativity, the paths of free particles correspond to the geodesics of space-time. Referring to co-ordinates t, x, y, z, the metric (9) becomes, when $k = -1$, $a = 1$,

$$ds^2 = \{F(X)\}^2\left\{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)\right\}, \tag{13}$$

and the geodesic equations can be readily calculated. Varying x, y, z as functions of t, the equation $\delta\int ds = 0$ at once gives

$$\frac{d^2x}{dt^2} = 2Y\frac{F'}{F}x - \left(\frac{1}{F}\frac{dF}{dt} - \frac{1}{2Y}\frac{dY}{dt}\right)\frac{dx}{dt},$$

etc., where Y is defined as in (12). Hence, writing

$$\frac{d^2x}{dt^2} = \mathrm{P}x - \mathrm{Q}\frac{dx}{dt}, \tag{14}$$

etc., we have

$$\mathrm{P} = 2Y\frac{F'}{F}, \qquad \mathrm{Q} = \frac{1}{F}\frac{dF}{dt} - \frac{1}{2Y}\frac{dY}{dt}, \tag{15}$$

and substituting (14) in (15), we find $\mathrm{Q} = t\mathrm{P}$. The geodesic equations can therefore, by (14) and (15), be written in the form

$$\frac{d^2x}{dt^2} = \left(x - t\frac{dx}{dt}\right)\frac{Y}{X}G(X), \tag{16}$$

* In a lecture delivered before the London Mathematical Society, 1934 May 17, entitled "World-Gravitation by Kinematical Methods," and in lectures delivered at the Mathematical Colloquium, St. Andrews, 1934 July; now in course of publication.

etc., where

$$G(X) = 2X\frac{F'(X)}{F(X)}. \tag{17}$$

These then are the equations of motion of free particles according to general relativity, expressed in terms of the co-ordinates used by Milne.

Comparing equations (16) and (11), we see at once that Milne's analysis gives a variety of possible systems of paths which includes the systems resulting from general relativity. In (16) G contains only X, and the exclusion of ξ is evidently a consequence of the assumption that the paths correspond to geodesics of a Riemannian space. In order to include ξ, either it must be allowed that the paths are not geodesics, or a more general type of space (Finsler space) must be adopted.

Thus, Milne's analysis gives a variety of possible systems, described by $G(X, \xi)$. Milne then adopts one particular form, namely, $G \equiv G(\xi)$, as a consequence of a fundamental physical argument; general relativity leads to another form, namely, $G \equiv G(X)$, for reasons which have no physical basis and are purely geometrical. General relativity also postulates tacitly the existence of dimensional world-constants.

9. We see that a system of paths of general relativity for which $G(X) \equiv$ constant is equivalent to a particular case of Milne's system, given by $G(\xi) \equiv$ constant. Returning to the form (2), it can be shown that

$$G(X) = \frac{d}{d\tau'}\{R'(\tau')\} - 1;$$

whence, if $G(X) =$ constant, $R'(\tau')$ must be of the form $K\tau'$ (omitting a constant that can be transformed away).

The particular form * given by $R'(\tau') \equiv \tau'$ has been discussed by Kermack and McCrea,† Robertson ‡ and McVittie.§ These writers show that the fundamental particles in this system are equivalent to Milne's particles, but, as we have seen, this is true for any form of $R'(\tau')$. McVittie also considers the geodesic equations, and finds the acceleration of a stationary free particle near the origin to be $-r^3/c^2t^4$ in terms of Milne's co-ordinates; he concludes that the discrepancy between this formula and Milne's formula for the acceleration is to be traced to Milne's method of calculating the acceleration. It appears, however, that McVittie is at fault, for when $R'(\tau') \equiv \tau'$, $G \equiv 0$ from (18), whence from (16) there is no acceleration and all free particles move with uniform velocities relative to each of the fundamental observers.

My thanks are due to Professor E. A. Milne for giving me access to the MSS. of his lectures and for critical discussion of the present paper.

Oxford:
1935 *January.*

* This form was chosen in order that the time kept by an observer should be the arc length of his world-line.

† *M.N.*, **93**, 519, 1933. ‡ *Zs. für Astrophys.*, **7**, 153, 1933.

§ *M.N.*, **94**, 476, 1934. McVittie considers Milne's statistical system, but the results of the present paper are formally the same for this system.