Ann. Henri Poincaré 2 (2001) 605 – 673

1424-0637/01/040605-69 $ 1.50+0.20/0

Annales Henri Poincaré

On the Formation of Singularities in Solutions of the Critical Nonlinear Schrödinger Equation

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Abstract. For the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity the Cauchy problem with initial data close to a soliton is considered. It is shown that for a certain class of initial perturbations the solution develops a self-similar singularity in finite time T^*, the profile being given by the ground state solitary wave and the limiting self-focusing law being of the form

$$\lambda(t) \sim (\ln|\ln(T^* - t)|)^{1/2}(T^* - t)^{-1/2}.$$

Introduction

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\triangle\psi - |\psi|^{2p}\psi, \quad x \in \mathbb{R}^d, \tag{1}$$

with initial data

$$\psi|_{t=0} = \psi_0 \in H^1.$$

It is well known that for $p \geq \frac{2}{d}$ the problem has solutions that blow up in finite time [G]. The case $p = \frac{2}{d}$ marks the transition between the global existence and the blowup phenomenon. In this paper we study the participation of nonlinear bound states in singularity formation in the one-dimensional critical case : $d = 1$, $p = 2$.

The NLS (1) has an important solution of special form- soliton : $e^{it}\varphi_0(x)$, where φ_0 is the "ground state solitary wave". Ground states are orbitally stable relative to small perturbations of initial data in the subcritical case and unstable in the critical and supercritical case. In fact for $p \geq \frac{2}{d}$ initial data arbitrary close to a ground state may give rise to a solution that blows up in finite time. In the critical case , however, a kind of orbital stability result is still valid provided one extends a definition of the ground state orbit taking dilation as well as translations into account. More precisely, any blowup solution ψ with L_2 norm close to L_2 norm of φ_0 is close (in L_2) to the set

$$\{e^{i\mu}\lambda^{1/2}\varphi_0(\lambda(x+b)),\, \mu, b \in \mathbb{R}, \lambda \in \mathbb{R}_+\}$$

for t close enough to the blowup time, see [MM], [W4]. Although giving some information on the spatial structure of the solutions near the blowup time this result does not answer the question of what the asymptotic behavior of the system is. Toward an understanding of this asymptotic behavior we have the following result. We consider the Cauchy problem for (1) ($p = 2$, $d = 1$) with even initial data close to a soliton :

$$\psi|_{t=0} = \varphi_0 + \chi_0, \tag{2}$$

where χ_0 is small in suitable sense. We show that for a certain set (open in $X = \{\chi_0 \in H^1, x\chi_0 \in L_2\}$) of initial perturbations the solution ψ blows up in finite time T^*, admitting the following asymptotic representation

$$\psi(t,x) \sim e^{i\mu(t)}\lambda^{1/2}(t)\varphi_0(\lambda(t)x), \quad t \to T^*, \tag{3}$$

$$\lambda(t) \sim (T^* - t)^{-1/2} \left(\ln|\ln(T^* - t)|\right)^{1/2}, \ \mu(t) \sim \ln(T^* - t)\ln|\ln(T^* - t)|. \tag{4}$$

Thus, up to a phase factor the formation of the singularity is self-similar with a profile given by the ground state.

In the multidimensional case the existence and stability of the blowup solutions with the asymptotic behavior (3), (4) have been conjectured and formally explained by several authors, see, for example, [DNPZ], [Fr], [KSZ], [LPSS], [LePSS1], [LePSS2], [M1], [M2], [M3], [SF], [SS1], [SS2].

The asymptotics (3), (4) clearly can not be true for all blowup solutions starting from data close to a ground state since there is a family of explicit blow up solutions with a different blowup rate :

$$\left(\frac{T^*}{T^* - t}\right)^{1/2} e^{i\frac{x^2}{4(t-T^*)} + i\frac{tT^*}{T^*-t}}\varphi_0\left(\frac{T^* x}{T^* - t}\right). \tag{5}$$

However it may be reasonable to expect the exceptional set of initial data to be a one-codimensional manifold and the corresponding solutions to behave (up to the invariances of the equation) like the explicit ones (5), see [BW]. This phenomenon is due to a certain degeneracy of the model and is unstable with respect to perturbations of the equation. For Zakharov equation (that can be considered as a physical refinement of (1)) the solutions with the blowup rate (4) disappear : the minimal blowup rate is given by that of the explicit solutions, see [GM], [Me3].

The structure of this article is briefly as follows. It consists of two sections fairly different in nature. The first contains a complete proof of the indicated result with reference to certain estimates for the linearized operators. The second contains a systematic treatment of the properties of the linearized operators, and, in particular, a proof of the estimates mentioned in Section 1. The expositions in the two sections are essentially independent up to the overlap concerning the estimates mentioned.

A brief variant of the present article containing a description of the main results was published in [P].

1. Asymptotic behavior of solutions of nonlinear equation

We start by devoting subsection 1.1 to a description of preliminary concepts and to the exact formulation of the results. Subsections 1.2 and 1.3 are devoted to the proof of (3) for the solution of the Cauchy problem(1) , (2). Up to some technical modifications the main line will repeat that of [BP1], [BP2].

1.1 Preliminary facts and formulation of the result

1.1.1 The nonlinear equation

We formulate here the necessary facts about the Cauchy problem for the equation

$$i\psi_t = -\psi_{xx} - |\psi|^4\psi \tag{1.1.1}$$

with initial data in H^1.

Proposition 1.1.1 *The Cauchy problem for equation* (1.1.1) *with initial data* $\psi(0,x) = \psi_0(x)$, $\psi_0 \in H^1$ *has a unique solution* ψ *in the space* $C([0,T^*) \to H^1)$ *with some* $T^* > 0$ *and*

(i) ψ *satisfies the conservation laws*

$$\int dx|\psi|^2 = const, \quad H(\psi) = \int dx[|\psi_x|^2 - \frac{1}{3}|\psi|^6] = const;$$

(ii) *if* $T^* < \infty$, *then* $\|\psi_x\|_2 \to \infty$ *as* $t \to T^*$ *and*

$$\|\psi_x\|_2 \geq c(T^* - t)^{-1/2};$$

(iii) *if* $H(\psi_0) < 0$ *then* $T^* < \infty$.

Suppose in addition that $x\psi_0 \in L_2$. *Then* $x\psi \in C([0,T^*) \to L_2)$ *and* ψ *satisfies the pseudo-conformal conservation law*

$$\int dx|(x + 2it\partial_x)\psi|^2 - \frac{4}{3}t^2 \int dx|\psi|^6 = const.$$

The assertions stated here can be found in [CW1], [OT], for example.

Equation (1.1.1) is invariant with respect to the transformations :

$$\psi(x,t) \to (a+bt)^{-1/2} e^{i\omega + i\frac{bx^2}{4(a+bt)}} \psi(\frac{x}{a+bt}, \frac{c+dt}{a+bt}), \tag{1.1.2}$$

where $\omega \in \mathbb{R}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$.

1.1.2 Exact blowup solutions

Equation (1.1.1) has a family of soliton solutions

$$e^{i\frac{\alpha^2}{4}t}\varphi_0(x,\alpha), \quad \alpha > 0,$$

where φ_0 is a positive even smooth decreasing function satisfying the equation

$$-\varphi_{0xx} + \frac{\alpha^2}{4}\varphi_0 - \varphi_0^5 = 0.$$

As $|x| \to \infty$, $\varphi_0 \sim \varphi_\infty(\alpha)e^{-\frac{\alpha}{2}|x|}$.

One has a relation

$$\varphi_0(x,\alpha) = \left(\frac{\alpha}{2}\right)^{1/2} \varphi_0(\frac{\alpha}{2}x), \tag{1.1.3}$$

where $\varphi_0(x)$ stands for $\varphi_0(x,2)$. One can give an explicit expression for φ_0 :

$$\varphi_0(x) = \frac{3^{1/4}}{\mathrm{ch}^{1/2} 2x}.$$

Applying the transformations (1.1.2) to (1.1.1) one gets a 3-parameter family of solutions

$$e^{i\mu(t)-i\beta(t)z^2/4}\lambda^{1/2}(t)\varphi_0(z), \; z = \lambda(t)x, \tag{1.1.4}$$

where μ, β, λ are given by

$$\lambda(t) = (a+bt)^{-1}, \;\; \beta(t) = -b(a+bt), \;\; \mu(t) = \frac{c+dt}{a+bt}.$$

Remark that $\lambda(t)$, $\beta(t)$, $\mu(t)$ satisfy the system

$$\lambda^{-3}\lambda_t = \beta, \; \lambda^{-2}\beta_t + \beta^2 = 0, \; \lambda^{-2}\mu_t = 1.$$

If $b \neq 0$, solution (1.1.4) blows up in finite time. It is known that equation (1.1.1) has no blowup solutions in the class

$$\{\psi \in H^1(\mathbb{R}), \|\psi\|_2 < \|\varphi_0\|_2\},$$

see [W3]. The solutions (1.1.4) are the only blowup solutions (up to Galilei invariance) with minimal mass, see [Me1], [Me2].

1.1.3 Extended manifold of blowup solutions

The 3-parameter family (1.1.4) can be considered as the boundary $a = 0$ of the 4-parameter family of formal solutions $w(x,\sigma(t))$,

$$w(x,\sigma) = e^{i\mu - i\beta z^2/4}\lambda^{1/2}\varphi(z,a), \; z = \lambda x,$$

$\sigma = (\frac{\mu}{2}, \lambda, \beta, a)$, $\lambda \in \mathbb{R}_+$, β, μ, $a \in \mathbb{R}$. Here

$$\varphi(z,a) = \sum_{n=0}^{\infty} a^n \varphi_n(z) \tag{1.1.5}$$

is a formal solution of the equation

$$-\varphi_{zz} + \varphi - \frac{az^2}{4}\varphi - \varphi^5 = 0, \tag{1.1.6}$$

Equation (1.1.6) is equivalent to the following system for φ_n :

$$L_{0+}\varphi_n = \frac{z^2}{4}\varphi_{n-1} + F_n, \quad n \geq 1,$$

where

$$L_{0+} = -\partial_z^2 + 1 - 5\varphi_0^4,$$

F_n being a homogeneous polynomial of $\varphi_k,\ k \leq n-1$ of degree 5. In particular, φ_1 is characterized by the equation :

$$L_{0+}\varphi_1 = \frac{z^2}{4}\varphi_0.$$

Since $L_{0+}\varphi_0' = 0$, the operator L_{0+} is invertible being restricted to the subspace of even functions. As a consequence, the above equations have a unique even solution decreasing as $|z| \to \infty$. More precisely,

$$|\varphi_n(z)| \leq c\,\langle z\rangle^{3n}\, e^{-|z|},\ \ z \in \mathbb{R}.$$

We use the notation $\langle z\rangle = (1+z^2)^{1/2}$.

Function $w(x, \sigma(t))$ is a formal solution of (1.1.1) if $\sigma(t)$ satisfies the system

$$\lambda^{-3}\lambda_t = \beta,\ \lambda^{-2}\beta_t + \beta^2 = a,\ \lambda^{-2}\mu_t = 1,\ a_t = 0, \tag{1.1.7}$$

which gives, in particular, $\lambda = (d_2t^2 + d_1t + d_0)^{-1/2}$, $a = d_1^2/4 - d_2d_1$. Here d_j are constant.

We shall use the notations $\varphi^N(z,a) = \sum_{k=0}^{N} a^k\varphi_k(z)$,

$$\varphi^N(z,\alpha,a) = \left(\frac{\alpha}{2}\right)^{1/2}\varphi^N(\frac{\alpha}{2}x, \frac{16a}{\alpha^4}).$$

1.1.4 Linearization of (1.1.1) on a soliton

Consider the linearization of (1.1.1) on the soliton $e^{it}\varphi_0(x)$:

$$i\chi_t = -\chi_{xx} - \varphi_0^4\chi - 2\varphi_0^4(\chi + e^{2it}\bar{\chi}).$$

Introduce the function $f : \chi = e^{it}f$. Then f satisfies the equation

$$i\vec{f}_t = H_0\vec{f}, \quad \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

$$H_0 = (-\partial_x^2 + 1)\sigma_3 + V(\varphi_0),\ V(\xi) = -3\xi^4\sigma_3 - 2i\xi^4\sigma_2,$$

σ_2, σ_3 being the standard Pauli matrices :

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

H_0 is considered as a linear operator in $L_2(\mathbb{R} \to \mathbb{C}^2)$ defined on the natural domain. In this section L_2 stands for the subspace of the standard L_2 consisting of even functions.

The operator H_0 satisfies the relations

$$\sigma_3 H_0 \sigma_3 = H_0^*, \quad \sigma_1 H_0 \sigma_1 = -H_0, \tag{1.1.8}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The continuous spectrum of H_0 consists of two semi-axes $(-\infty, -1]$, $[1, \infty)$ and is simple.

The point $E = 0$ is an eigenvalue of the multiplicity 4. By differentiating the solution w with respect to the parameters it is easy to distinguish an eigenfunction $\vec{\xi}_0$

$$\vec{\xi}_0 = i\varphi_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad H_0\vec{\xi}_0 = 0,$$

and three associated functions $\vec{\xi}_j$, $j = 1, 2, 3$,

$$H_0\vec{\xi}_j = i\vec{\xi}_{j-1},$$

where

$$\vec{\xi}_1(x) = \frac{1}{4}(1 + 2x\partial_x)\varphi_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{\xi}_2(x) = -i\frac{1}{8}x^2\varphi_0(x) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\vec{\xi}_3(x) = \frac{1}{2}\varphi_1(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

φ_1 being the second coefficient in the expansion (1.1.5).

Since

$$< \vec{\xi}_3, \sigma_3\vec{\xi}_0 >= -ie,\ e = \frac{\|x\varphi_0\|_2^2}{8} \neq 0,$$

vectors $\vec{\xi}_j,\ j = 0, \ldots, 4$, span the root subspace of H_0 corresponding to the eigenvalue $E = 0$.

It will be shown in Section 2 that $E = 0$ is the only eigenvalue of H_0 .

1.1.5 Main theorem

Consider the Cauchy problem for equation (1.1.1) with initial data

$$\psi|_{t=0} = \psi_0,\ \psi_0(x) = e^{-i\beta_0 x^2/4}(\varphi^N(x,\beta_0^2) + \chi_0(x)),\ \beta_0 > 0, \tag{1.1.9}$$

where $\chi_0(x) = \chi_0(-x)$ and χ_0 satisfies the estimate

$$\|\chi_0\|_X = O(\beta_0^{2N}). \tag{1.1.10}$$

Here $\|f\|_X = \|f\|_{H^1} + \|xf\|_{L_2}$.

Assume that

(i) N is sufficiently large;

(ii) β_0 is sufficiently small.

These conditions give, in particular,

$$H(\varphi^N(\beta_0^2) + \chi_0) = -2\beta_0^2 e + O(\beta_0^4) < 0,$$

which together with the conformal invariance implies that the solution ψ of the Cauchy problem (1.1.1), (1.1.9) blows up in finite time $T^* < \infty$.

Our main result is the following.

Theorem 1.1.1 *The solution ψ of the Cauchy problem* (1.1.1), (1.1.9) *blows up in finite time $T^* = \frac{1}{2\beta_0}(1+o(1))$, as $\beta_0 \to 0$, and there exist $\lambda(t)$, $\mu(t) \in C^1([0,T^*))$,*

$$\begin{aligned} &\lambda(t) = const(T^*-t)^{-1/2}(\ln|\ln(T^*-t)|)^{1/2}(1+o(1)),\\ &\mu(t) = const\ln(T^*-t)\ln|\ln(T^*-t)|(1+o(1)), \quad t \to T^*, \end{aligned} \tag{1.1.11}$$

such that ψ admits the representation

$$\psi(x,t) = e^{i\mu(t)}\lambda^{1/2}(t)\left(\varphi_0(z) + \chi(z,t)\right),\ \ z = \lambda(t)x,$$

where χ is small in $L_2 \cap L_\infty$ uniformly with respect to $t \in [0,T^*)$. Moreover, $\|\chi\|_\infty = o(1)$, as $t \to T^*$. The constants in (1.1.11) *are independent of initial data.*

Remark. Due to the conformal invariance the same result remains valid for initial data of the form

$$\tilde\psi_0(x) = e^{i\omega - ibz^2/4}\lambda^{1/2}\psi_0(z),\ z = \lambda x,$$

where $\omega \in \mathbb{R},\ \lambda \in \mathbb{R}_+,\ b > -\frac{1}{T^*}$.

Remark. In principle our approach makes it possible to obtain an explicit value of the constant assumed in the hypothesis (i). But this would make the calculations less transparent and the result would be very far from the optimal one (we expect the theorem be true for $N > 2$).

1.1.6 Outline of the proof

The proof contains two main ingredients : the ideas of the works [BP1], [BP2], [SW1], [SW2] where the asymptotic stability of solitary waves were considered and the asymptotic constructions of the works mentioned in the introduction, especially, that of [SF]. We shall now briefly describe the main steps of the proof.

Step 1. Splitting of the motion. Following [BP1], [BP2] we start by introducing some new coordinates for the description of the solution with initial data (1.1.9). The new coordinates posses an important property : they allow us to split the motion into two parts, the first part is a finite- dimensional dynamics on the manifold of formal solutions $\{w(\cdot, \sigma)\}$ and the second part remains small in some sense for all $t \in [0, T^*)$. To describe these coordinates we introduce a quasi-solution $\tilde{\varphi}(z, a)$ of (1.1.6). One of the principal difficulties in the description of the critical blow-up comes from the fact that (1.1.6) has no admissible solutions for $a > 0$, which explains the presence of a correction to the self similar blowup rate $(T^* - t)^{-1/2}$, see again [DNPZ], [Fr], [KSZ], [LPSS], [LePSS1], [LePSS2], [M1], [M2], [M3], [SF], [SS1], [SS2]. By admissible we mean a solution with the purely outgoing behavior at infinity

$$\varphi \sim const\, e^{i\frac{z^2 h}{4}} |z|^{-\frac{1}{2} - \frac{i}{h}}, \quad h = \sqrt{a},$$

as $|z| \to \infty$, which would give a finite energy blowup solution w of (1.1.1) with the blowup rate $(T^* - t)^{-1/2}$. To overcome this difficulty we follow the approach of [SF]. Instead of (1.1.6) we consider a modified equation where the quadratic potential $-\frac{az^2}{4}$ is replaced by zero outside the interval $h^{-1}[-2 + \delta_0, 2 - \delta_0]$ with some $\delta_0 > 0$. For a sufficiently small this modified equation has a solution $\tilde{\varphi}$ that decreases exponentially as $|z| \to \infty$. The obtained profile $\tilde{\varphi}$ almost satisfies (1.1.6) :

$$-\tilde{\varphi}_{zz} + \tilde{\varphi} - \frac{az^2}{4}\tilde{\varphi} - \tilde{\varphi}^5 = F_0(a),$$

the error F_0 is exponentially small (with respect to a). Choosing δ_0 sufficiently small we shall make F_0 to be almost of the same order as the effective small parameter of the problem $e^{-\frac{S_0}{h}}$, $S_0 = \int_0^2 ds\sqrt{1 - s^2/4}$ (we use this expression for S_0 instead of the explicit value in order to underline its obvious semi-classical meaning). The exact assertions related to the modified profile $\tilde{\varphi}$ as well as a description of the spectral properties of the corresponding linearized operator $\tilde{H}$ are given in subsubsection 1.2.1.

Using the profile $\tilde{\varphi}$ we decompose the solution ψ of (1.1.1), (1.1.9) as follows.

$$\psi(x, t) = e^{i\mu(t) - i\beta(t)z^2/4}\lambda^{1/2}(t)(\tilde{\varphi}(z, a) + f(z, t)),$$

the decomposition being fixed by some suitable orthogonality conditions that have a natural interpretation in terms of the spectral objects associated to $\tilde{H}$, see subsubsection 1.2.2. For the present the parameter δ_0 in the definition of $\tilde{\varphi}$ is arbitrary. We fix it only at the last steps of the proof.

The functions $\sigma(t) = (\frac{\mu(t)}{2}, \lambda(t), \beta(t), a(t))$ and f satisfy the system of coupled equations :

$$i\vec{f}_\tau = H(a)\vec{f} + N'(a, f), \tag{1.1.12}$$

$$\sigma_\tau = G'(a, f), \tag{1.1.13}$$

where $H(a) = (-\partial_z^2 + 1 - \frac{az^2}{4})\sigma_3 + V(\tilde{\varphi}(a))$, G', N' are some nonlinear functions, τ is a changed time variable : $\tau = \int_0^t ds\lambda^2(s)$, $\tau \to \infty$, as $t \to T^*$.

Step 2. Effective equations. Assuming that $a(\tau)$ is a small slowly varying function we single out the main order terms in N', G' and derive a model system that we expect to describe qualitatively the dynamics (1.1.12), (1.2.13). The model system has the form

$$if_\tau = (-\partial_z^2 + 1 - a\frac{z^2}{4})f + F_0(a),$$

$$\lambda^{-1}\lambda_\tau = \beta,\ \beta_\tau + \beta^2 = h^2,\ \mu_\tau = 1,\ h_\tau = -ch^{-1}e^{-\frac{S_0}{h}}(1 + O(h)),$$
$$f|_{\tau=0} = \chi_0, \quad \lambda(0) = 1,\ \beta(0) = h(0) = \beta_0,\ \mu(0) = 0,$$

where c is a positive constant. At this stage the constructions are formal and quite similar to those of [SF]. Solving the equation for h one gets $h \sim \ln^{-1}(\tau + \tau^*)$, $\tau^* \sim e^{\frac{2S_0}{\beta_0}}\beta_0^3$, which leads to (3), (4).

Step 3. Estimates of the solution. To prove that the complete dynamics (1.1.12), (1.1.13) is indeed close to the model one we employ the standard perturbation methods, the same methods were used in [BP1], [BP2]. To ensure that the correction terms in (1.1.12) can be treated perturbatively one requires suitable time-decay estimates (local in space) for the dispersive solutions of the linear equation

$$i\vec{f}_\tau = H(a(\tau))\vec{f}.$$

In our case this local decay is a consequence of the corresponding properties of the group $e^{-i\tau H(a)}$ restricted to the subspace of the "continuous" spectrum of $H(a)$, see proposition 1.2.7, and the fact that a depends on τ slowly.

1.2 Splitting of motions

1.2.1 Modified ground state

Consider the equation

$$-\tilde{\varphi}_{zz} + \frac{\alpha^2}{4}\tilde{\varphi} - \frac{az^2}{4}\theta(hz)\tilde{\varphi} - \tilde{\varphi}^5 = 0,\ \ h = \sqrt{|a|} > 0, \tag{1.2.1}$$

$\alpha,\ a \in \mathbb{R}$. Here $\theta \in C_0^\infty(\mathbb{R})$, $\theta(\xi) = \theta(-\xi)$, $\theta(\xi) \leq 1$,

$$\theta(\xi) = \begin{cases} 1, & |\xi| \leq 2 - \delta_0 \\ 0, & |\xi| > 2 - \delta_0/2 \end{cases},$$

$\delta_0 > 0$ is sufficiently small (θ can be considered as a family of cut-off functions parametrized by δ_0). One has the following proposition.

Proposition 1.2.1 *For α in some finite vicinity of 2 and for a sufficiently small,*[1] *equation (1.2.1) has a unique positive even smooth decreasing solution $\tilde{\varphi}(z, \alpha, a)$ which is close to $\varphi_0(z, \alpha)$. Moreover,*

(i) *as $a \to 0$, $\tilde{\varphi}(z, \alpha, a)$ admits the asymptotic expansion (1.1.5) in the sense*

$$|\tilde{\varphi} - \varphi^N| \leq c|a|^{N+1} < x >^{3(N+1)} e^{-\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)},$$

$\tilde{S}_{\alpha,a}(\xi) = \frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2 - (a)_+ s^2\theta(s)}$;

(ii) $\|e^{\frac{1}{h}S_{\alpha,a}(h|x|)}\tilde{\varphi}(\alpha, a)\|_\infty \leq c$, $S_{\alpha,a}(\xi) = \frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2 - \operatorname{sgn} as^2\theta(s)}$.

The similar formulas are valid for the derivatives of $\tilde{\varphi}$ with respect to z, α, a. Here $(a)_+$ stands for $\max(a, 0)$.

See subsection 2.2 for the proof.

Introduce a linearized operator $\tilde{H}(a)$ associated to the modified ground state $\tilde{\varphi}(z, a) = \tilde{\varphi}(z, 2, a)$

$$\tilde{H}(a) = (-\partial_x^2 + 1 - \frac{az^2}{4}\theta)\sigma_3 + V(\tilde{\varphi}(a)).$$

The continuous spectrum of $\tilde{H}(a)$ is the same as in the case of the operator H_0. The point $E = 0$ is an eigenvalue of $\tilde{H}(a)$ of the multiplicity 2. There are an eigenfunction $\tilde{\zeta}_0(a)$

$$\tilde{\zeta}_0(a) = i\tilde{\varphi}(a)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \tilde{H}\tilde{\zeta}_0 = 0,$$

and an associated function $\tilde{\zeta}_1(a)$

$$\tilde{\zeta}_1(a) = \partial_\alpha\tilde{\varphi}(\alpha, a)|_{\alpha=2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tilde{H}\tilde{\zeta}_1 = i\tilde{\zeta}_0,$$

$$\left\langle \tilde{\zeta}_1, \sigma_3\tilde{\zeta}_0 \right\rangle = i4ea + O(a^2).$$

A more detailed description of the discrete spectrum can be obtained by means of the standard perturbation methods. In particular, the following proposition is proved in subsubsection 2.3.2.

Proposition 1.2.2 *For a sufficiently small the discrete spectrum of the operator $\tilde{H}(a)$ in some finite vicinity of the point $E = 0$ consists of 0 and two simple eigenvalues $\pm\lambda(a)$, $\lambda(a) = i\sqrt{a}\lambda'(a)$, where λ' is a smooth real function of a. As $a \to 0$, $\lambda'(a) = 2 + O(a)$. Let $\tilde{\zeta}_2(a)$ be an eigenfunction corresponding to $\lambda(a)$ normalized by the condition*

$$\left\langle \tilde{\zeta}_2, \vec{\xi}_0 \right\rangle = \left\langle \tilde{\zeta}_0, \vec{\xi}_0 \right\rangle - \lambda^2 \left\langle \vec{\xi}_2, \vec{\xi}_0 \right\rangle.$$

[1]The constants here and below depend on δ_0.

Then $\tilde{\zeta}_2(a)$ *is a smooth function of* $a^{1/2}$ *admitting the following asymptotic expansion as* $a \to 0$

$$\tilde{\zeta}_2 = \tilde{\zeta}_0 - i\lambda\tilde{\zeta}_1 - \lambda^2\vec{\xi}_2 + i\lambda^3\vec{\xi}_3 + ia\lambda^2\begin{pmatrix}1\\-1\end{pmatrix}(h_0 + O(a)) + ia\lambda^3\begin{pmatrix}1\\1\end{pmatrix}(h_1 + O(a)),$$

where $h_i,\ i = 1, 2,$ *are some real even smooth exponentially decreasing functions.* $O(a)$ *corresponds to the* L_∞*-norm with the weight* $e^{\frac{1-\gamma}{h}\tilde{S}_a(h|x|)}$, $\tilde{S}_a(\xi) = \tilde{S}_{2,a}(\xi)$, $\gamma > 0$. *This asymptotic representation can be differentiated any number of times with respect to* x *and* a.

Let us mention that

$$\sigma_1\tilde{\zeta}_2 = \bar{\tilde{\zeta}}_2.$$

In the subspace generated by $\tilde{\zeta}_j(a),\ j = 0,\dots 3$, where $\tilde{\zeta}_3 = \sigma_1\tilde{\zeta}_2$ is an eigenfunction corresponding to the eigenvalue $-\lambda$, we introduce a new basis $\{\vec{e}_j(a)\}_{j=0}^3$:

$$\vec{e}_0 = \tilde{\zeta}_0, \quad \vec{e}_1 = \tilde{\zeta}_1,$$

$$\vec{e}_2 = \frac{1}{2\lambda^2}\left(-\tilde{\zeta}_2 + \tilde{\zeta}_3 + 2\tilde{\zeta}_0\right), \quad \vec{e}_3 = -\frac{i}{2\lambda^3}\left(\tilde{\zeta}_2 + \tilde{\zeta}_3 + i2\lambda\tilde{\zeta}_1\right),$$

$\vec{e}_2 = e_2\binom{1}{-1}$, $\vec{e}_3 = e_3\binom{1}{-1}$, $\bar{e}_j = (-1)^{j-1}e_j$.

It follows from proposition 1.2.2 that as $a \to 0$,

$$\vec{e}_2 = \vec{\xi}_2 - iah_0\begin{pmatrix}1\\-1\end{pmatrix} + O(a^2),$$

$$\vec{e}_3 = \vec{\xi}_3 + ah_1\begin{pmatrix}1\\1\end{pmatrix} + O(a^2).$$

1.2.2 Orthogonality conditions

Return to the Cauchy problem (1.1.1), (1.1.9). Using the profile $\tilde{\varphi}$ one can rewrite the initial data ψ_0 in the form : $\psi_0 = e^{-i\frac{\beta_0 x^2}{4}}(\tilde{\varphi}(\beta_0^2) + \chi_0')$, $\|\chi_0'\|_X = O(\beta_0^{2N})$. Below we shall omit " ′ " in the notation of χ_0'.

Write the solution ψ as the sum

$$\psi(x,t) = e^{i\Phi}\lambda^{1/2}(t)\left(\tilde{\varphi}(z, a(t)) + f(z,t)\right), \quad \Phi = \mu(t) - \frac{\beta}{4}z^2, \ \ z = \lambda(t)x, \quad (1.2.2)$$

where $\tilde{\varphi}(z,a) = \tilde{\varphi}(z,2,a)$, $\sigma(t) = (\frac{\mu(t)}{2}, \lambda(t), \beta(t), a(t))$ being an arbitrary curve in $\mathbb{R}_+ \times \mathbb{R}^3$, it is not a solution of (1.1.7) in general.

The decomposition can be fixed by the orthogonality conditions

$$\left\langle \vec{f}(t), \sigma_3\vec{e}_j(a(t))\right\rangle = 0, \quad j = 0,\dots,3. \qquad (1.2.3)$$

This means that σ has to satisfy the system

$$F_j(\psi,\sigma)=0,\quad j=0,\dots 3, \tag{1.2.4}$$

$$F_j(\psi,\sigma)=\lambda^{1/2}\left\langle\vec{\psi},\sigma_3 e^{i\Phi\sigma_3}\vec{e}_j(\lambda\cdot,a)\right\rangle-\langle\vec{e}_0(a),\vec{e}_j(a)\rangle=0,\quad \vec{\psi}=\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}.$$

The solvability of (1.2.4) for ψ in some small L_2- vicinity of φ_0 is guaranteed by the smoothness of the basis $\vec{e}_j(a),\ j=0,\dots,3$ and the non-degeneration of the corresponding Jacobi matrix

$$\mathcal{B}_0=\left\{\frac{\partial F_j}{\partial\sigma_k}\right\}\Big|_{\substack{\psi=\varphi_0\\ \sigma=(1,0,0,0)}}.$$

It is not difficult to check that

$$\mathcal{B}_0=-2\left\{\left\langle\vec{\xi}_k,\sigma_3\vec{\xi}_j\right\rangle\right\}_{k,j=0}^{3},\quad \det\mathcal{B}_0=\left|2\left\langle\vec{\xi}_1,\sigma_3\vec{\xi}_2\right\rangle\right|^4=(8e)^4\neq 0.$$

So, one can assume that the initial decomposition obeys (1.2.3) :

$$\left\langle\vec{\chi}_0,\sigma_3\vec{e}_j(\beta_0^2)\right\rangle=0,\quad j=0,\dots,3.$$

To prove the existence of a trajectory $\sigma(t)$ we need the following orbital stability result :

Proposition 1.2.3 *For any $\epsilon>0$ there exists $\delta>0$ such that for any ψ_0, $\|\psi_0-\varphi_0\|_{H^1}\leq\delta$, $E(\psi_0)<0$, there exists $\mu(t)\in C([0,T^*))$ such that the solution ψ corresponding to the initial data ψ_0 satisfies the inequality*

$$\|\psi(t)-\lambda^{1/2}(t)e^{i\mu(t)}\varphi_0(\lambda(t)\cdot)\|_2\leq\epsilon,\quad 0\leq t<T^*,$$

where $\lambda(t)$ is given by

$$\lambda(t)=\frac{\|\psi_x(t)\|_2}{\|\varphi_{0x}\|_2}.$$

See [LBSK], [W2], [W3] for the proof.

By (1.1.10), $\tilde{\psi}_0$, $\tilde{\psi}_0=\tilde{\varphi}(\beta_0^2)+\chi_0$ satisfies the conditions of the above proposition. Thus, the corresponding solution $\tilde{\psi}(t)$ admits the representation

$$\tilde{\psi}(x,t)=e^{i\tilde{\Phi}}\tilde{\lambda}^{1/2}(t)\left(\tilde{\varphi}(z,\tilde{a}(t))+\tilde{f}(z,t)\right),\quad \tilde{\Phi}=\tilde{\mu}(t)-\frac{\tilde{\beta}(t)}{4}z^2,\ \ z=\tilde{\lambda}(t)x,$$

where $\tilde{\sigma}(t)=(\frac{\tilde{\mu}(t)}{2},\tilde{\lambda}(t),\tilde{\beta}(t),\tilde{a}(t))$, $\tilde{\sigma}(0)=(0,1,0,\beta_0^2)$ is a continuous trajectory satisfying (1.2.4), $\|\tilde{f}\|_2$, $\tilde{\lambda}\frac{\|\varphi_{0x}\|_2}{\|\psi_x(t)\|_2}-1$, $\tilde{\beta}$, $\tilde{a}$ being small uniformly with respect to t.

By the conformal invariance we can write now the solution $\psi(t)$ of the Cauchy problem (1.1.1), (1.1.9) in the form (1.2.2) where

$$\mu(t)=\tilde{\mu}(\rho),\ \ \lambda(t)=(1-\beta_0 t)^{-1}\tilde{\lambda}(\rho),$$

$$\beta(t) = \beta_0(1-\beta_0 t)\tilde{\lambda}^{-2} + \tilde{\beta}(\rho), \quad a(t) = \tilde{a}(\rho), \ \ \rho = \frac{t}{1-\beta_0 t},$$

$f(z,t) = \tilde{f}(z,\rho)$ satisfying the orthogonality conditions (1.2.3).

By (i) of proposition 1.1.1, λ admits the estimate

$$\lambda(t) \geq c(T^* - t)^{-1/2}. \tag{1.2.5}$$

Remark that since $\psi(t) \in C^1([0,T^*) \to H^{-1})$ the trajectory $\sigma(t)$ belongs in fact, to C^1.

1.2.3 Differential equations

We write a system of equations for σ and f in explicit form. Introduce a new time variable τ :

$$\tau = \int_0^t ds\lambda^2(s).$$

By (1.2.5), $\tau \to \infty$ as $t \to T^*$.

In terms of f (1.1.1) takes the form

$$i\vec{f}_\tau = \tilde{H}(a)\vec{f} + N, \tag{1.2.6}$$

where

$$N = N_0(a,f) + N_1(\tilde{\varphi}, f) + l(\sigma)\left(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix} + \vec{f}\right) - ia_\tau\tilde{\varphi}_a\begin{pmatrix}1\\1\end{pmatrix},$$

$$N_0(a,f) = \frac{az^2}{4}(\theta(hz)-1)\sigma_3(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix} + \vec{f}), \tag{1.2.7}$$

$$N_1(\tilde{\varphi}, f) = -|\tilde{\varphi}+f|^4\sigma_3(\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix} + \vec{f}) + \tilde{\varphi}^5\begin{pmatrix}1\\-1\end{pmatrix} - V(\tilde{\varphi})\vec{f},$$

$$l(\sigma) = (\mu_\tau - 1)\sigma_3 + i(\beta - \frac{\lambda_\tau}{\lambda})(z\partial_z + \frac{1}{2}) + (a - \beta_\tau + \beta^2 - 2\beta\frac{\lambda_\tau}{\lambda})\frac{z^2}{4}\sigma_3.$$

Substitute the expression for $\vec{f_\tau}$ from (1.2.6), (1.2.7) into the derivative of the orthogonal conditions. The result can be written down as follows :

$$(\mathcal{A}_0(a) + \mathcal{A}_1(a,f))\vec{\eta} = \vec{g}(a,f). \tag{1.2.8}$$

Here

$$\vec{\eta} = (\frac{\mu_\tau - 1}{2}, \frac{\lambda_\tau}{\lambda} - \beta, \beta_\tau - \beta^2 + 2\beta\frac{\lambda_\tau}{\lambda} - a, a_\tau),$$

$$\mathcal{A}_0 = 2\begin{pmatrix} 0 & 0 & 0 & -(\tilde{\varphi}_a, \tilde{\varphi}) \\ 2(\tilde{\varphi}, \tilde{\varphi}_\alpha) & 0 & -(\frac{z^2}{4}\tilde{\varphi}, \tilde{\varphi}_\alpha) & 0 \\ 0 & -i((z\partial_z + \frac{1}{2})\tilde{\varphi}, e_2) & 0 & -i(\tilde{\varphi}_a, e_2) \\ 2(\tilde{\varphi}, e_3) & 0 & -(\frac{z^2}{4}\tilde{\varphi}, e_3) & 0 \end{pmatrix},$$

$$(\mathcal{A}_1\vec{\eta})_j = \left\langle l(\sigma)\vec{f}, \sigma_3\vec{e}_j \right\rangle + ia_\tau \left\langle \vec{f}, \sigma_3\vec{e}_{ja} \right\rangle,$$
$$g_j(a,f) = -\left\langle N_0 + N_1, \sigma_3\vec{e}_j \right\rangle.$$

By propositions 1.2.1, 1.2.2,

$$\mathcal{A}_0(a) = i\mathcal{B}_0 + O(a), \tag{1.2.9}$$

as $a \to 0$.

In principle (1.2.8) can be solved with respect to the derivatives η and together with equation (1.2.6) constitutes a complete system for $\sigma, \vec{f}$:

$$i\vec{f}_\tau = H(a)\vec{f} + N'(a,f), \tag{1.2.10}$$

$$\vec{\eta} = G(a,f), \tag{1.2.11}$$

$$f|_{\tau=0} = \chi_0, \quad \sigma|_{\tau=0} = (0,1,\beta_0,\beta_0^2).$$

Here $H(a) = (-\partial_z^2 + 1 - \frac{az^2}{4})\sigma_3 + V(\tilde{\varphi}(a))$, $N' = N - a\frac{z^2}{4}(\theta-1)\sigma_3\vec{f}$.

1.2.4 Effective equations

In order to derive a system of effective equations consider the main nonlinear terms of (1.2.10), (1.2.11). Below it will become clear that the function a depends slowly on τ . More precisely,

$$a \sim \ln^{-2}(\tau + \tau^*), \tag{1.2.12}$$

with some $\tau^* = O(e^{\frac{2S_0}{\beta_0}}\beta_0^3)$. We shall also see that the contribution f of the continuous spectrum asymptotically is of the order $e^{-\frac{S_0}{h}}$, $h = \sqrt{a}$, (in the uniform norm) and of the order $e^{-\frac{2S_0}{h}}$ for z not too large. In its turn the vector η also has the order $e^{-\frac{2S_0}{h}}$. We shall use these facts while deriving the equations. At this stage we are not worrying about formal justification.

The main terms of N are generated by the expression

$$N \sim F_0(a)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad F_0(a) = a\frac{z^2}{4}(\theta - 1)\tilde{\varphi}. \tag{1.2.13}$$

Thus, it is clear that in the region $|z| \geq const\, h^{-1}$ the main order term of f is given by the expression

$$f \sim -(l(a) + 1 - i0)^{-1}F_0(a), \tag{1.2.14}$$

where $l(a) = -\partial_z^2 - a\frac{z^2}{4}$.

The sign "-" (in $-i0$) is essential : it means that $e^{-i\frac{hz^2}{4}}(l(a)+1-i0)^{-1}F_0(a)$ has finite energy.

For the following it is convenient to write $f = f^0 + f^1$, $f^0 = -(l(a) + 1 - i0)^{-1}F_0(a)$. It will become clear later that in the region $|z| \geq const\, h^{-1}$ f^0 and

f^1 are of the order $e^{-\frac{S_0}{h}}$ and $e^{-\frac{2S_0}{h}}$ respectively while for $|z| \sim 1$ both f^0 and f^1 have the order $e^{-\frac{2S_0}{h}}$.

Consider (1.2.11). The main term of G is given by the expression

$$G \sim \mathcal{A}_0^{-1}(a)\vec{g}^0(a),$$

where $g_j^0 = -\langle N_0(a, f_0), \sigma_3 \vec{e}_j \rangle$. So we rewrite (1.2.11) in the form

$$\vec{\eta} = G_0(a) + G_R(a, f). \tag{1.2.15}$$

Here $G_0(a) = -\mathcal{A}_0^{-1}(a)\vec{g}^0(a)$, G_R being the remainder.

The behavior of $f^0(a)$, $G_0(a)$ in the limit $a \to 0$ is described by the following proposition.

Proposition 1.2.4 *For $a > 0$ sufficiently small, $f^0(a)$, $G_0(a)$ satisfy the estimates*

$$\|f^0(a)\|_\infty \le ce^{-(1-\epsilon)\frac{S_0}{h}}, \quad \|\tilde{\varphi}(a) f^0(a)\|_\infty \le ce^{-(2-\epsilon)\frac{S_0}{h}},$$

$$\|\widehat{e^{-i\frac{hz^2}{4}} f^0}\|_1,\ \|\widehat{(z\partial_z + \frac{1}{2})e^{-i\frac{hz^2}{4}} f^0}\|_1,\ \|\partial_h \widehat{e^{-i\frac{hz^2}{4}} f^0}\|_1 \le ce^{-(1-\epsilon)\frac{S_0}{h}},$$

$$\|G_0(a)\| \le ce^{-(2-\epsilon)\frac{S_0}{h}}.$$

Moreover, G_0^3 admits the following representation

$$G_0^3(a) = -2\nu_0 e^{-\frac{2S_0}{h}}(1 + O(a)), \quad \nu_0 = \frac{\varphi_\infty^2}{e}.$$

This asymptotic estimate can be differentiated any number of times with respect to a. Here $\hat{f}$ stands for the Fourier transform of f :

$$\hat{f}(p) = (2\pi)^{-1/2} \int dx e^{-ipx} f(x).$$

Here and in what follows the letter ϵ is used as a general notation for small positive constants that depend on the choice of the cut off function θ and tend to zero as $\delta_0 \to 0$. They may change from line to line.

The proof of this proposition is given in appendix 2.

In order to estimate qualitatively the behavior of a, consider the last equation of (1.2.15) neglecting the remainder G_R :

$$a_\tau = G_0^3(a).$$

We denote by $a_0(\tau)$ the solution of this equation with initial data $a_0(0) = \beta_0^2$. It is easy to check that $h_0 = \sqrt{a_0}$ admits the representation

$$h_0^{-1}(\tau) = \frac{1}{2S_0}(\ln \nu_1(\tau + \tau^*) + 3 \ln \ln \nu_1(\tau + \tau^*)) + O(\frac{\ln \ln(\tau + \tau^*)}{\ln(\tau + \tau^*)}), \tag{1.2.16}$$

as $\tau + \tau^* \to +\infty$, $\nu_1 = \frac{\nu_0}{4S_0^2}$, $\tau^* = \frac{\beta_0^3}{2S_0\nu_0} e^{\frac{2S_0}{\beta_0}}(1 + O(\beta_0))$.

1.2.5 Spectral properties of the operator H(a)

To study the behavior of solutions to (1.2.10), (1.2.11) we need some information about spectral properties of $H(a)$, $a > 0$, in the limit $a \to 0$. The necessary facts are collected in this subsubsection, the proofs being given in Section 2.

We renormalize $H(a)$ to make the principal part independent of the parameters :

$$H(a) = a^{1/2} T(a^{1/4}) \hat{H}(a) T^*(a^{1/4}), \ (T(a)f)(z) = a^{1/2} f(az), \ a > 0.$$

The operator $\hat{H}(a)$ has the form

$$\hat{H}(a) = (-\partial_z^2 + \hat{E}_0 - \frac{z^2}{4})\sigma_3 + \hat{W}(a), \quad \hat{E}_0 = a^{-1/2},$$

where $\hat{W}(a) = a^{-1/2} T^*(a^{1/4}) V(\tilde{\varphi}(a)) T(a^{1/4})$.

We consider $\hat{H}(a)$ as a linear operator in $L_2(\mathbb{R} \to \mathbb{C}^2)$ defined on the domain where the operator $(-\partial_z^2 - \frac{z^2}{4})\sigma_3$ is self-adjoint. The continuous spectrum of $\hat{H}(a)$ coincides with $\mathbb{R}$. Because of the exponential decrease of the potential $\hat{W}$ at infinity the point spectrum contains only finitely many eigenvalues, and the corresponding root subspaces are finite-dimensional. $\hat{H}(a)$ satisfies the same relations (1.1.8) as H_0. As a consequence the spectrum is symmetric with respect to transformations $E \to -E$ and $E \to \bar{E}$.

Consider the equation

$$(\hat{H} - E)\psi = 0. \tag{1.2.17}$$

One can find a basis of solutions $\hat{f}_j(z, E)$, $j = 1, \dots, 4$, with the following properties. The solutions f_j are holomorphic functions of E , $E \in \mathbb{C}$, admitting the following asymptotic representations as $z \to +\infty$

$$\hat{f}_1(z, E) = e^{i\frac{z^2}{4}} z^{\hat{\nu}(E)} [\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)],$$

$$\hat{f}_2(z, E) = e^{-i\frac{z^2}{4}} z^{\overline{\hat{\nu}(\bar{E})}} [\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)],$$

$$\hat{f}_3(z, E) = e^{-i\frac{z^2}{4}} z^{\overline{\hat{\nu}(-\bar{E})}} [\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)],$$

$$\hat{f}_4(z, E) = e^{i\frac{z^2}{4}} z^{\hat{\nu}(-E)} [\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)],$$

where $\hat{\nu}(E) = -\frac{1}{2} + i(E - \hat{E}_0)$.

We introduce the solutions $\hat{g}_j(z, E)$, $j = 1, \dots, 4$, with standard behavior at $-\infty$ by

$$\hat{g}_j(z, E) = \hat{f}_j(-z, E).$$

Consider the matrix solutions

$$\hat{F}_1 = (\hat{f}_1, \hat{f}_3),\ \ \hat{F}_2 = (\hat{f}_2, \hat{f}_4),\ \ \hat{G}_1 = (\hat{g}_1, \hat{g}_3),\ \ \hat{G}_2 = (\hat{g}_2, \hat{g}_4).$$

One can express $\hat{F}_1$ in terms of $\hat{G}_j,\ j = 1, 2$:

$$\hat{F}_1 = \hat{G}_2 \hat{A} + \hat{G}_1 \hat{B},$$

$\hat{A} = \hat{A}(E),\ \hat{B} = \hat{B}(E)$ are holomorphic functions of E , $E \in \mathbb{C}$.

The eigenvalues of the operator $\hat{H}$ lying in the upper half plane $\{\operatorname{Im} E > 0\}$ are characterized by the equation

$$\det \hat{A}(E) = 0.$$

The solutions of this equation in lower half plane $\{\operatorname{Im} E \le 0\}$ are called resonances. One can prove the following result.

Proposition 1.2.5 *For $a > 0$ sufficiently small ,*

(i) *the point spectrum of $\hat{H}(a)$ restricted to the subspace of even functions consists of four simple purely imaginary eigenvalues $\pm i\hat{E}_{1,2}(a)$, $\hat{E}_j > 0$,*

$$\hat{E}_1 = O(e^{-(1-\epsilon)S_0/h}), \quad |\hat{E}_2(a) - \lambda'(a)| = O(e^{-(2-\epsilon)S_0/h}),$$

(ii) *there exists $C_0 > 0$, independent of a, such that in the strip $\{E : -C_0 < \operatorname{Im} E \le 0\}$ the operator $\hat{H}(a)$ has only one simple resonance $i\hat{E}_R(a)$, $\hat{E}_R < 0$.*

Moreover, $\hat{E}_R$ admits the asymptotic estimates

$$\hat{E}_R = O(e^{-(1-\epsilon)S_0/h}), \quad \hat{E}_R + \hat{E}_1 = O(a^{-2}e^{-2S_0/h}).$$

Let $\hat{\zeta}_j,\ j = 1, \dots 4,$ be eigenfunctions corresponding to the eigenvalues $\pm i\hat{E}_j$, $j = 1, 2$:

$$\hat{H}\hat{\zeta}_j = i\hat{E}_j\hat{\zeta}_j, \quad \hat{H}\hat{\zeta}_{j+2} = -i\hat{E}_j\hat{\zeta}_{j+2}, \quad j = 1, 2.$$

Let $\hat{\zeta}_R$ be a resonant function corresponding to $i\hat{E}_R$:

$$\hat{H}\hat{\zeta}_R = i\hat{E}_R\hat{\zeta}_R,$$

$$\hat{\zeta}_R \sim e^{\frac{iz^2}{4}\sigma_3} |z|^{-\frac{1}{2} - \hat{E}_R - i\hat{E}_0\sigma_3} \vec{c},$$

as $|z| \to \infty$. Here $\vec{c}$ is a constant vector.

Let $\hat{P}(a)$ stand for the spectral projection onto eigenspace corresponding to the eigenvalues $i\hat{E}_1, \pm i\hat{E}_2$ and to the resonance $i\hat{E}_R$:

$$\hat{P}(a)f = n_1^{-1}\hat{\zeta}_1 \left\langle f, \sigma_3\hat{\zeta}_3 \right\rangle + n_2^{-1}\hat{\zeta}_2 \left\langle f, \sigma_3\hat{\zeta}_4 \right\rangle$$
$$+ \bar{n}_2^{-1}\hat{\zeta}_4 \left\langle f, \sigma_3\hat{\zeta}_2 \right\rangle + n_R^{-1}\hat{\zeta}_R \left\langle f, \sigma_3\bar{\hat{\zeta}}_R \right\rangle.$$

The normalization constants n_1, n_2, n_R are given by

$$n_R = \left\langle \hat{\zeta}_R, \sigma_3 \bar{\hat{\zeta}}_R \right\rangle, \quad n_j = \left\langle \hat{\zeta}_j, \sigma_3 \hat{\zeta}_{j+2} \right\rangle, \quad j = 1, 2.$$

The spectral projection $P(a)$ of the operator $H(a)$ corresponding to the eigenvalues $iE_1, \pm iE_2$ and to the resonance iE_R is given by

$$P(a) = T(a^{1/4})\hat{P}(a)T^*(a^{1/4}).$$

Introduce the operator $Q(a)$:

$$Q(a) = (I - \tilde{P}(a))P(a)(I - \tilde{P}(a)),$$

where $\tilde{P}(a)$ is the spectral projection of the operator $\tilde{H}(a)$ onto the subspace corresponding to the eigenvalues $E = \pm\lambda(a)$ and $E = 0$:

$$\tilde{P}(a)f = \tilde{n}_1^{-1}\tilde{\zeta}_0 \left\langle f, \sigma_3\tilde{\zeta}_1 \right\rangle - \tilde{n}_1^{-1}\tilde{\zeta}_1 \left\langle f, \sigma_3\tilde{\zeta}_0 \right\rangle$$

$$+\tilde{n}_2^{-1}\tilde{\zeta}_2 \left\langle f, \sigma_3\tilde{\zeta}_3 \right\rangle - \tilde{n}_2^{-1}\tilde{\zeta}_3 \left\langle f, \sigma_3\tilde{\zeta}_2 \right\rangle,$$

$$\tilde{n}_1 = \left\langle \tilde{\zeta}_0, \sigma_3\tilde{\zeta}_1 \right\rangle, \tilde{n}_2 = \left\langle \tilde{\zeta}_2, \sigma_3\tilde{\zeta}_3 \right\rangle.$$

The following proposition is proved in subsubsection 2.4.4.

Proposition 1.2.6 *The operators P, Q admit the estimates*

$$|(Pf)(z)| \le c < z >^{-1/2+\hat{E}_R} \|e^{-i\frac{z^2h}{4}\sigma_3} f\|_{H^1},$$

$$|(Qf)(z)| \le c < z >^{-1/2+\hat{E}_R} e^{\frac{1}{h}S(h|z|)} e^{-(3-\epsilon)S_0/h} \|e^{-i\frac{z^2h}{4}\sigma_3} f\|_{H^1},$$

where $S(\xi) = \int_0^\xi ds\sqrt{(1-s^2/4)_+}$.

Let us introduce the operators $\hat{\mathbb{F}}$, $\hat{\mathbb{G}} : L_2(\mathbb{R} \to \mathbb{C}^2) \to L_2(\mathbb{R} \to \mathbb{C}^2)$:

$$(\hat{\mathbb{F}}\Phi)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dE \hat{\mathcal{F}}(z, E)\Phi(E),$$

$$(\hat{\mathbb{G}}\Phi)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dE \hat{\mathcal{G}}(z, E)\Phi(E).$$

Here $\hat{\mathcal{F}}$, $\hat{\mathcal{G}}$ are solutions of the scattering problem :

$$\hat{\mathcal{F}} = \hat{F}_1\hat{A}^{-1}, \quad \hat{\mathcal{G}} = \hat{G}_1\hat{A}^{-1},$$

$$\hat{\mathcal{F}}(z, E) \sim e^{\frac{iz^2}{4}\sigma_3} z^{-\frac{1}{2}+i(E-\hat{E}_0\sigma_3)}\hat{A}^{-1}, \quad z \to +\infty,$$

$$\hat{\mathcal{F}}(z, E) \sim e^{-\frac{iz^2}{4}\sigma_3}|z|^{-\frac{1}{2}-i(E-\hat{E}_0\sigma_3)} + e^{\frac{iz^2}{4}\sigma_3}|z|^{-\frac{1}{2}+i(E-\hat{E}_0\sigma_3)}\hat{B}\hat{A}^{-1}, \quad z \to -\infty.$$

The action of the adjoint operators $\hat{\mathbb{F}}^*$, $\hat{\mathbb{G}}^*$ is given by

$$(\hat{\mathbb{F}}^*\psi)(E) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz \hat{\mathcal{F}}^*(z,E)\psi(z),$$

$$(\hat{\mathbb{G}}^*\psi)(E) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz \hat{\mathcal{G}}^*(z,E)\psi(z).$$

It is not difficult to show that $\hat{\mathbb{F}}$, $\hat{\mathbb{G}}$ are bounded in L_2 and satisfy the relations

$$\hat{\mathbb{E}}\hat{\sigma}_3\hat{\mathbb{E}}^*\sigma_3 = P^c, \quad \hat{\mathbb{E}}^*\sigma_3\hat{\mathbb{E}}\hat{\sigma}_3 = I,$$

where $\hat{\mathbb{E}} : L_2(\mathbb{R} \to \mathbb{C}^2) \times L_2(\mathbb{R} \to \mathbb{C}^2) \to L_2(\mathbb{R} \to \mathbb{C}^2)$,

$$\hat{\mathbb{E}}\vec{\Phi} = \hat{\mathbb{F}}\Phi_1 + \hat{\mathbb{G}}\Phi_2, \quad \vec{\Phi} = (\Phi_1, \Phi_2),$$

$\hat{\sigma}_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, P^c being the spectral projection onto the subspace of the continuous spectrum. Moreover, one can prove the following proposition.

Proposition 1.2.7 *For $a > 0$ sufficiently small, there exists b_0, $\frac{1}{2} > b_0 > 0$, independent of a, such that*

(i) *for $e^{-i\frac{z^2}{4}\sigma_3} f \in H^1$, $(\hat{\mathbb{F}}^* f)(E)$ is a meromorphic function of E in the strip $-b_0 \leq \operatorname{Im} E \leq 0$ with the only pole in $-i\hat{E}_1$ and satisfies the estimate*

$$\|\hat{\mathbb{F}}^* f\|_{L_2(\mathbb{R}-ib)}, \ \|\partial_h \hat{\mathbb{F}}^* f\|_{L_2(\mathbb{R}-ib)} \leq c h^{-K_1} \|e^{-i\frac{z^2}{4}\sigma_3} f\|_{H^1},$$

$h^L \leq b \leq b_0$;

(ii) *let us introduce the operators $\hat{\mathbb{F}}_b$:*

$$(\hat{\mathbb{F}}_b \Phi)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dE \hat{\mathcal{F}}(z, E - ib)\Phi(E).$$

For $h^L \leq b \leq b_0$, they satisfy the inequality.

$$\|(1+|z|)^{-\nu_2} \hat{\mathbb{F}}_b \Phi\|_2 \leq c h^{-K_2} \|\Phi\|_2, \quad \nu_2 > 1/2,$$

the same being true for $\hat{\mathbb{F}}$ replaced by $\hat{\mathbb{G}}$.

Here K_j, $j = 1, 2$, depend only on L.

1.2.6 Equations on the finite interval

Following [BP1], [BP2] we consider the system (1.2.10), (1.2.11) on some finite interval $[0, \tau_1]$ and later investigate the limit $\tau_1 \to \infty$.

On the interval $[0, t_1]$, $t_1 = t(\tau_1)$ we approximate the trajectory $\sigma(t)$ by $\sigma_1(t)$ where $\sigma_1(t) = (\frac{\mu(t)}{2}, \lambda_1(t), \beta_1(t), a_1(t))$ is the solution of the following Cauchy problem

$$\lambda_1^{-3}\lambda_1' = \beta_1, \ \lambda_1^{-2}\beta_1' + \beta_1^2 = a_1, \ a_1' = 0,$$

$$\lambda_1(t_1) = \lambda(t_1), \;\; \beta_1(t_1) = a^{1/2}(t_1), \;\; a_1(t_1) = a(t_1).$$

We associate to the trajectory σ_1 a new function g

$$g(y, \rho) = e^{iy^2 \triangle} r^{1/2} f(ry, \tau),$$

where $\triangle = \frac{1-\beta r^2}{4}$, $r = \frac{\lambda}{\sqrt{\beta_1}\lambda_1}$, $\rho = \int_0^\tau ds r^{-2}$.

Equation (1.2.10) in terms of g takes the form

$$i\vec{g}_\rho = \hat{H}(a)\vec{g} + \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3, \tag{1.2.18}$$

where

$$\mathcal{N}_0 = e^{iy^2\triangle\sigma_3} r^{5/2} F_0(a) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathcal{N}_1 = e^{iy^2\triangle\sigma_3} r^{5/2} N_1,$$

$$\mathcal{N}_2 = e^{iy^2\triangle\sigma_3} r^{5/2} \left(l(\sigma)\tilde{\varphi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - ia_\tau \tilde{\varphi}_a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \tag{1.2.19}$$

$$\mathcal{N}_3 = e^{iy^2\triangle\sigma_3} r^{5/2} V(\tilde{\varphi}(a))\vec{f} - \hat{W}(a)\vec{g} + (\mu_\rho - h^{-1})\sigma_3\vec{g}.$$

Since a depends slowly on τ it is natural to rewrite the above equation in terms of the spectral representation of $\hat{H}(a)$. Write $\vec{g}$ as the sum

$$\vec{g} = \vec{h} + \vec{k} \tag{1.2.20}$$

of the projections on the subspaces corresponding to the discrete and continuous spectra of $\hat{H}(a)$. More precisely, set

$$\vec{k} = \hat{P}(a)\vec{g}.$$

Then

$$\vec{h} = (\hat{\mathbb{F}}_b + \hat{\mathbb{G}}_b)\sigma_3 \Phi(\cdot - ib), \quad \Phi(E) = (\hat{\mathbb{F}}^* \sigma_3 \vec{g})(E),$$

where $-\hat{E}_R < b \le b_0$. Let us remark that due to the orthogonality conditions (1.2.3) the four dimensional component k is controlled by h (or equivalently by Φ).

Projecting (1.2.18) on the subspace of the continuous spectrum of $\hat{H}(a)$ one gets an equation for Φ :

$$i\Phi_\rho = E\Phi + D, \tag{1.2.21}$$

where $D = D_0 + D_1 + D_2$,

$$D_0 = \hat{\mathbb{F}}^* \sigma_3 \mathcal{N}_0, \quad D_1 = i\hat{\mathbb{F}}^*_\rho \sigma_3 \vec{g}, \quad D_2 = \sum_{j=1}^{3} \hat{\mathbb{F}}^* \sigma_3 \mathcal{N}_j. \tag{1.2.22}$$

Consider (1.2.21) on the line $\operatorname{Im} E = -b$ with some b, $0 < b \le b_0$, that will be fixed later, rewriting it as an integral equation :

$$\Phi(\rho) = e^{-iE\rho} \hat{\mathbb{F}}^*(0)\sigma_3 \vec{g}_0 - i \int_0^\rho ds e^{-iE(\rho - s)} D(s), \; \operatorname{Im} E = -b. \tag{1.2.23}$$

Here

$$\hat{\mathbb{F}}(0)=\hat{\mathbb{F}}(a(0)),\ g_0(y)=e^{iy^2\triangle_0}r_0^{1/2}\chi_0(r_0y),$$
$$\triangle_0=\frac{1-\beta_0r_0^2}{4},\ r_0=(\beta_1\lambda_1^2(0))^{-1/2}.$$

The relations (1.2.3), (1.2.15), (1.2.20), (1.2.23) make up the final form of the equations which is used to investigate the dynamical system on the interval $[0,\tau_1]$.

It follows from (1.2.13), (1.2.14) that the main part of D is given by D_0. The contribution of D_0 in (1.2.22) allows some asymptotic simplifications. After a natural integration by parts one gets

$$\Phi=\Phi_0+\Phi_1,\quad \Phi_0=-\frac{1}{E}D_0,$$
$$\Phi_1(\rho)=e^{-iE\rho}\sigma_3\Phi_{10}-i\int_0^\rho dse^{-iE(\rho-s)}D'(s). \tag{1.2.24}$$

Here $\Phi_{10}=\mathbb{F}^*(0)\sigma_3\vec{g}_0+\frac{1}{E}D_0(0)$, $D'=D_1+D_2+i\frac{D_{0\rho}}{E}$. In accordance with (1.2.13) the main order term of Φ is given by Φ_0.

1.3 Estimates of the solution

Here we prove that the new coordinates indeed admit only small (in suitable sense) deviations from their initial values. As in [BP1], [BP2], for this purpose we use the method of majorants.

1.3.1 Estimates of soliton parameters

Introduce a natural system of norms for the components of the solution ψ :

$$s_0(\tau)=\sup_{s\le\tau}|h(s)-h_0(s)|h_0^{-2}(s),$$
$$s_1(\tau)=\sup_{s\le\tau}|\beta(s)-h(s)|h_0^{-2}(s)p^{-1}(s;\kappa_1,r_1),$$
$$s_2(\tau)=\sup_{\tau\le s\le\tau_1}|\beta(s)-r^{-2}|h_0^{-2}(s)p^{-1}(s;\kappa_2,r_2),$$
$$M_0(\tau)=\sup_{s\le\tau}\|f(s)\|_\infty p^{-1}(s;\kappa_0,r_0),$$
$$M_1(\tau)=\sup_{s\le\tau}\|\langle z\rangle^{-\nu_3}f^1(s)\|_\infty p^{-1}(s;\kappa_3,r_3),\ \nu_3\ge 2,$$
$$M_2(\tau)=\sup_{s\le\tau}\|\rho_\delta f(s)\|_2 p^{-1}(s;\kappa_4,r_4),$$

where

$$p(\tau;\kappa,r)=e^{-\kappa\int_0^\tau dsh_0(s)}+e^{-r\frac{S_0}{h_0(\tau)}},\ \rho_\delta=e^{-\frac{(1-\delta)}{h_0}\int_0^{h_0|z|}ds\sqrt{1-\frac{s^2}{4}\theta(s)}},$$

$\kappa_4 = \frac{b_0}{4}$, $\kappa_0 = \kappa_3 = \frac{7}{8}\kappa_4$, $\kappa_1 = \frac{3}{2}\kappa_4$, $\kappa_2 = \frac{5}{4}\kappa_4$, $r_0 = \frac{3}{4}$, $r_1 = \frac{15}{8}$, $r_2 = \frac{7}{4}$, $r_3 = \frac{4}{3}$, $r_4 = \frac{3}{2}$, $\delta > 0$ is supposed to be a sufficiently small fixed number.

At last, set

$$\hat{s}_j = s_j(\tau_1),\ j = 0,1, \quad \hat{s}_2 = s_2(0), \quad \hat{M}_j = M_j(\tau_1).$$

Consider equation (1.2.15). It follows immediately from (1.2.7), (1.2.9) and from proposition 1.2.4 that

$$|\eta| \le W(M,s)[e^{-(2-\epsilon)\frac{S_0}{h_0(\tau)}} + e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}}\| \langle z\rangle^{-\nu_3} f^1\|_\infty + \|\rho_\delta f\|_2^2 + \|\rho_\delta f\|_2 \|f\|_\infty^4],$$
$$|G_R| \le W(M,s)[e^{-(4-\epsilon)\frac{S_0}{h_0(\tau)}} + e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}}\| \langle z\rangle^{-\nu_3} f^1\|_\infty + \|\rho_\delta f\|_2^2 + \|\rho_\delta f\|_2 \|f\|_\infty^4].$$

We use $W(M,s)$ as a general notation for functions of M_j, $j = 0,1,2$, s_k, $k = 0,1,2$, defined on $\mathbb{R}^6$, which are bounded in some finite neighborhood of 0 and may acquire the infinite value $+\infty$ outside some larger neighborhood. While depending on δ_0, δ, W does not depend on β_0. In all the formulas where W appear it would be possible to replace them by some explicit expressions but such expressions are useless for our aims.

In terms of majorants the above inequalities take the form

$$|\eta| \le W(M,s)\left[\Psi_0(M)e^{-2\kappa_3\int_0^\tau ds h_0(s)} + e^{-(2-\epsilon)\frac{S_0}{h_0(\tau)}}\right], \tag{1.3.1}$$

$$|G_R| \le W(M,s)\Psi_1(M)\left[e^{-\frac{3\kappa_3}{2}\int_0^\tau ds h_0(s)} + e^{-\frac{3r_4}{2}\frac{S_0}{h_0(\tau)}}\right], \tag{1.3.2}$$

where

$$\Psi_0(M) = M_2 M_0^4 + \beta_0^4 M_1^2 + M_2^2,$$
$$\Psi_1(M) = e^{-\gamma/\beta_0} + M_2 M_0^4 + M_2^2,$$

with some $\gamma > 0$.

Using (1.3.1), (1.3.2) and proposition 1.2.4 it is not difficult to get the following inequalities

$$s_0 \le W(M,s)\left(s_0^2 + \beta_0^{-4}\Psi_1(M)\right),$$
$$s_1 \le W(M,s)\left(\beta_0 s_1^2 + e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-4}\Psi_0(M)\right),$$
$$s_2 \le W(\hat{M},\hat{s})\left(\hat{s}_1 + \beta_0 s_2^2 + e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-3}\Psi_0(\hat{M})\right).$$

See appendix 3 for the proof.

Changing if necessary, functions W one can simplify these inequalities :

$$s_0 \le W(M,s)\beta_0^{-4}\Psi_1(M),$$
$$s_1 \le W(M,s)\left(e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-4}\Psi_0(M)\right), \tag{1.3.3}$$
$$s_2 \le W(\hat{M},\hat{s})\left(e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-4}\Psi_0(\hat{M})\right),\ \gamma > 0.$$

1.3.2 Estimates of D_j

Consider (1.2.24). Using propositions 1.2.4, 1.2.7 one gets for D_0

$$\|D_0\|_{L_2(\mathbb{R}-ib)} \leq W(\hat{M},\hat{s})e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}}, \tag{1.3.4}$$

$$\begin{aligned}\|\frac{D_{0\rho}}{E}\|_{L_2(\mathbb{R}-ib)} &\leq W(\hat{M},\hat{s})e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}}[|a_\rho|+|\beta_\rho|+|r_\rho|]\\ &\leq W(\hat{M},\hat{s})e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}}[|\eta|+|\beta-h|+|\beta-r^{-2}|].\end{aligned} \tag{1.3.5}$$

In a similar manner

$$\|D_1\|_{L_2(\mathbb{R}-ib)} \leq W(\hat{M},\hat{s})h_0^{-K}\left(|\eta|+|\beta-h|+|\beta-r^{-2}|\right)\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1}. \tag{1.3.6}$$

In this subsubsection and the next one we use letter K as a general notation for nonnegative numbers independent of parameters that may change from line to line.

Consider D_2. It is not difficult to show that

$$\begin{aligned}\|e^{-i\frac{y^2}{4}\sigma_3}\mathcal{N}_1\|_{H^1} &\leq W(\hat{M},\hat{s})h_0^{-3}\|e^{-i\frac{\beta z^2}{4}\sigma_3}N_1\|_{H^1}\\ &\leq W(\hat{M},\hat{s})h_0^{-3}(1+\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1})\big[e^{-(2-\epsilon)\frac{S_0}{h_0(\tau)}}\\ &+\|<z>^{-\nu_3} f^1\|_\infty(\|\partial_z(e^{-i\frac{hz^2}{4}}f)\|_2+\|\rho_\delta f\|_2)+\|\rho_\delta f\|_2^2+\|f\|_\infty^4\big],\\ \|e^{-i\frac{y^2}{4}\sigma_3}\mathcal{N}_2\|_{H^1} &\leq W(\hat{M},\hat{s})h_0^{-K}|\eta|,\end{aligned} \tag{1.3.7}$$

$$\begin{aligned}\|e^{-i\frac{y^2}{4}\sigma_3}\mathcal{N}_3\|_{H^1} &\leq W(\hat{M},\hat{s})h_0^{-K}\left(|\mu_\rho-r^2|+|\triangle|+|r^{-2}-h|\right)\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1}\\ &\leq W(\hat{M},\hat{s})h_0^{-K}\left(|\eta|+|\beta-r^{-2}|+|\beta-h|\right)\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1}.\end{aligned}$$

Combining the inequalities (1.3.5)-(1.3.7) one obtains

$$\begin{aligned}\|D'\|_{L_2(\mathbb{R}-ib)} &\leq W(\hat{M},\hat{s})h_0^{-K}(1+\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1})\\ &\times\big[|\eta|+|\beta-h|+|\beta-r^{-2}|+e^{-(2-\epsilon)\frac{S_0}{h_0(\tau)}}\\ &+\|<z>^{-\nu_3} f^1\|_\infty(\|\partial_z(e^{-i\frac{hz^2}{4}}f)\|_2+\|\rho_\delta f\|_2)\\ &+\|\rho_\delta f\|_2^2+\|f\|_\infty^4\big].\end{aligned} \tag{1.3.8}$$

It follows directly from the conservation laws that

$$\begin{aligned}\|f\|_2 &\leq W(\hat{M},\hat{s}),\\ \|\partial_z(e^{-i\frac{hz^2}{4}}f)\|_2 &\leq W(\hat{M},\hat{s})[\lambda^{-1}\beta_0^N+|h-\beta|^{1/2}\end{aligned} \tag{1.3.9}$$

$$+e^{-(1-\epsilon)\frac{S_0}{h_0(\tau)}} + \|\rho_\delta f\|_2^{1/2} + \|f\|_\infty^2].$$

In the last inequality we also made use of the obvious asymptotic estimate

$$|H(e^{-i\frac{hz^2}{4}}\tilde{\varphi}(a))| = O(e^{-(2-\epsilon)\frac{S_0}{h}}).$$

The inequalities (1.3.8), (1.3.9) lead to the estimate

$$\|D'\|_{L_2(\mathbb{R}-ib)} \leq W(\beta_0^{-1}\hat{M}, \hat{s})h_0^{-K}[\beta_0^{2N} + \Psi_2(M)]p(\tau; \kappa_2, r_2), \qquad (1.3.10)$$

$\Psi_2(M) = M_1 M_2^{1/2} + (M_0 + M_1)^2 + M_2^2$. Here we have also used (1.3.1), (1.3.3) and the obvious inequality

$$\lambda^{-1} \leq W(\beta_0^{-1}\hat{M}, \hat{s})e^{-\gamma\int_0^\tau ds h_0(s)}, \quad \gamma < 1.$$

1.3.3 Estimates of f in L_2

To estimate f we represent it as the sum

$$f = f_0 + f_1 + f_2,$$

$$\vec{f_j} = (I - \tilde{P}(a))T(r^{-1})e^{-iy^2\triangle\sigma_3}\vec{h}_j, \;\; j = 0, 1,$$

where

$$\vec{h}_j = (\hat{\mathbb{F}}_b + \hat{\mathbb{G}}_b)\Phi_j(\cdot - ib).$$

At last,

$$\vec{f_3} = (I - \tilde{P}(a))T(r^{-1})e^{-iy^2\triangle\sigma_3}\vec{k}.$$

Consider f_0. Using the representation

$$\vec{h}_0 = -(\hat{H} - i0)^{-1}(I - \hat{P})\mathcal{N}_0,$$

one can get the following estimate (see appendix 5)

$$\|\rho_\delta f_2\|_2 \leq W(\hat{M}, \hat{s})e^{-(2-\epsilon)\frac{S_0}{h}}. \qquad (1.3.11)$$

Here and in what follows ϵ depends on both δ_0 and δ and tends to zero as δ_0, $\delta \to 0$.

It follows from proposition 1.2.7 and (1.2.24), (1.3.4), (1.3.10) that

$$\|\rho_\delta f_1\|_2 \leq W(\beta_0^{-1}\hat{M}, \hat{s})h_0^{-K}[\beta_0^{2N} + \Psi_2(M)]p(\tau; \kappa_2, r_2), \qquad (1.3.12)$$

provided $b > \kappa_2$.

Using proposition 1.2.6 one can easily prove the following estimate

$$\|\rho_\delta f_2\|_2 \leq W(\hat{M}, \hat{s})h_0^{-K}[e^{-(2-\epsilon)\frac{S_0}{h}} + |\beta - h| + |\beta - r^{-2}|]\|e^{-i\frac{\beta z^2}{4}}f\|_{H^1}. \quad (1.3.13)$$

Combining (1.3.11)-(1.3.13) and taking into account (1.3.3) one gets finally

$$\begin{aligned}\|\rho_\delta f\|_2 &\leq W(\beta_0^{-1}\hat{M}, \hat{s})h_0^{-K_0}[\beta_0^{2N} + \Psi_2(M)]p(\tau; \kappa_2, r_2)\\ &\leq W(\beta_0^{-1}\hat{M}, \hat{s})\beta_0^{-K_0}[\beta_0^{2N} + \Psi_2(M)]p(\tau; \kappa_4, r_4) \qquad (1.3.14)\end{aligned}$$

with some $K_0 \geq 0$.

1.3.4 Estimates of f in L_∞

We represent f by the sum $\vec{f} = e^{i\frac{\beta z^2}{4}\sigma_3}(\tilde{f}^0 + \tilde{f}^1)$, where $\tilde{f}^0 = e^{-i\frac{hz^2}{4}\sigma_3}\vec{f}^0(a)$. Then $\tilde{f}^1$ satisfies the equation

$$i\tilde{f}^1_\tau = (-\partial_z^2 + \mu_\tau)\sigma_3\tilde{f}^1 - i\frac{\lambda_\tau}{\lambda}(\frac{1}{2} + z\partial_z)\tilde{f}^1 + \mathcal{H}_0 + \mathcal{H}_1, \tag{1.3.15}$$

where $\mathcal{H}_0 = \mathcal{H}_{00} + \mathcal{H}_{01} + \mathcal{H}_{02}$,

$$\mathcal{H}_{00} = -i\tilde{f}^0_\tau + (\mu_\tau - 1)\sigma_3\tilde{f}^0 + i(h - \frac{\lambda_\tau}{\lambda})(\frac{1}{2} + z\partial_z)\tilde{f}^0,$$

$$\mathcal{H}_{01} = e^{-i\frac{\beta z^2}{4}\sigma_3}N_1,$$

$$\mathcal{H}_{02} = e^{-i\frac{\beta z^2}{4}\sigma_3}\left(l(\sigma)\tilde{\varphi}\begin{pmatrix}1\\1\end{pmatrix} - ia_\tau\tilde{\varphi}_a\begin{pmatrix}1\\1\end{pmatrix}\right)$$
$$+(e^{-i\frac{\beta z^2}{4}\sigma_3} - e^{-i\frac{hz^2}{4}\sigma_3})F_0(a)\begin{pmatrix}1\\-1\end{pmatrix}.$$

At last, $\mathcal{H}_1 = e^{-i\frac{\beta z^2}{4}\sigma_3}V(\tilde{\varphi}(a))\vec{f}$.

We rewrite (1.2.15) as an integral equation

$$\tilde{f}^1 = U(\tau,0)\vec{\chi}_1 - i\int_0^\tau dsU(\tau,s)(\mathcal{H}_0(s) + \mathcal{H}_1(s)), \tag{1.3.16}$$

where $\chi_1 = e^{-i\frac{\beta_0 z^2}{4}}(\chi_0 - f^0(\beta_0^2))$, $U(\tau,s)$ being the propagator corresponding to the equation $if_\tau = (-\partial_z^2 + \mu_\tau)\sigma_3 f - i\frac{\lambda_\tau}{\lambda}(\frac{1}{2} + z\partial_z)f$.

It follows from (1.3.16) that

$$\|\tilde{f}^1\|_\infty \le c[\lambda^{-1/2}(\tau)\|\hat{\chi}_0\|_1 + \int_0^\tau ds\left(\frac{\lambda(s)}{\lambda(\tau)}\right)^{1/2}\|\hat{\mathcal{H}}_0\|_1 \tag{1.3.17}$$
$$+\int_0^\tau ds\frac{\lambda^{-\frac{1}{2}}(\tau)\lambda^{-\frac{1}{2}}(s)}{\sqrt{t(\tau) - t(s)}}\|\mathcal{H}_1\|_1].$$

Here we made use of the obvious estimates

$$\|U(\tau,s)f\|_\infty \le c\begin{cases}\left(\frac{\lambda(s)}{\lambda(\tau)}\right)^{1/2}\|\hat{f}\|_1,\\ \frac{\lambda^{-\frac{1}{2}}(\tau)\lambda^{-\frac{1}{2}}(s)}{\sqrt{t(\tau)-t(s)}}\|f\|_1\end{cases}.$$

The first term in the right hand side of (1.3.17) can be estimated as follows

$$\lambda^{-1/2}(\tau)\|\hat{\chi}_1\|_1 \le W(\beta_0^{-1}M, s)\beta_0^{2N}p(\tau;\kappa_3, r_3). \tag{1.3.18}$$

Consider $\mathcal{H}_0$. Using proposition 1.2.4 and (1.3.1), (1.3.7) one gets

$$\|\hat{\mathcal{H}}_0\|_1 \le c(\|\hat{\mathcal{H}}_{00}\|_1 + \|\mathcal{H}_{01}\|_{H^1} + \|\mathcal{H}_{02}\|_{H^1})$$
$$\le W(\beta_0^{-1}M, s)[\beta_0^{2N} + \beta_0^{L_0}s_1 + \Psi_2(M)]p(\tau;\kappa_2, r_2).$$

Thus, the contribution of $\mathcal{H}_0$ in the right hand side of (1.3.17) admits the estimate

$$\int_0^\tau ds \left(\frac{\lambda(s)}{\lambda(\tau)}\right)^{\frac{1}{2}} \|\hat{\mathcal{H}}_0\|_1 \le W(\beta_0^{-1}M, s)\beta_0^{-1} \times[\beta_0^{2N} + \beta_0^{L_0}s_1 + \Psi_2(M)]p(\tau;\kappa_3, r_3). \tag{1.3.19}$$

The third term of (1.3.17) can be estimated as follows :

$$\int_0^\tau ds \frac{\lambda^{-\frac{1}{2}}(\tau)\lambda^{-\frac{1}{2}}(s)}{\sqrt{t(\tau)-t(s)}}\|\mathcal{H}_1\|_1 \le W(M,s)M_2 \int_0^\tau ds \frac{\lambda^{-\frac{1}{2}}(\tau)\lambda^{-\frac{1}{2}}(s)}{\sqrt{t(\tau)-t(s)}} p(s;\kappa_4, r_4)$$
$$\le W(M,s)M_2\beta_0^{-1}p(\tau;\kappa_3, r_3),$$

which together with proposition 1.2.4 and (1.3.3), (1.3.17)-(1.3.19) gives

$$M_0 + M_1 \le W(\beta_0^{-1}M, s)\beta_0^{-1}[\beta_0^{2N} + M_2 + (M_0+M_1)^2 + \beta_0^{-2}M_2^2 + \beta_0^{-2}(M_0+M_1)^4]. \tag{1.3.20}$$

1.3.5 Estimates of majorants

Consider the system of inequalities (1.3.3), (1.3.14), (1.3.20). Introduce new scales :

$$\hat{M}_j = \beta_0 \hat{\mathbb{M}}_j,\ j = 0, 1, \quad \hat{M}_2 = \beta_0^{2K_0+2}\hat{\mathbb{M}}_2.$$

Remark that one can choose the function W to be spherically symmetric and monotone. Then in terms of $\hat{\mathbb{M}}_j$ the inequalities (1.3.3), (1.3.14), (1.3.20) can be written in the form

$$\hat{s}_0,\ \hat{s}_1,\ \hat{s}_2 \le W(\hat{\mathbb{M}}, \hat{s}) \left[e^{-\frac{\gamma}{\beta_0}} + \beta_0^2(\hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1)^2 + \beta_0^{4K_0}\hat{\mathbb{M}}_2^2\right], \tag{1.3.21}$$

$$\hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1 \le W(\hat{\mathbb{M}}, \hat{s}) \left[\beta_0^{2N-2} + +\beta_0^{2K_0}\hat{\mathbb{M}}_2\right], \tag{1.3.22}$$

$$\hat{\mathbb{M}}_2 \le W(\hat{\mathbb{M}}, \hat{s})\left[\beta_0^{2N-3K_0-2} + \beta_0^{-2K_0}\hat{\mathbb{M}}_2^{\frac{1}{2}}(\hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1) + \beta_0^{-3K_0}(\hat{\mathbb{M}}_0 + \hat{\mathbb{M}}_1)^2\right].$$

Taking into account the second inequality one can rewrite the third one as follows.

$$\hat{\mathbb{M}}_2 \le W(\hat{\mathbb{M}}, \hat{s})\beta_0^{2N-3K_0-2}. \tag{1.3.23}$$

Choosing $N > 1 + \frac{3K_0}{2}$ one gets that for β_0 sufficiently small the solution of (1.3.21)-(1.3.23) can belong either to a small neighborhood of 0 or to some domain whose distance from 0 is bounded uniformly with respect to β_0. Since all $\hat{\mathbb{M}}_j$, s_j are continuous functions of τ_1 and for $\tau_1 = 0$ are small only the first possibility can be realized.

As a consequence, one finally obtains

$$M_0, M_1 \leq c\beta_0^{2N-K_0-1}, \quad M_2 \leq c\beta_0^{2N-K_0}, \tag{1.3.24}$$

$$s_0,\, s_1 \leq c\beta_0^{4N-2K_0-4}, \;\; \tau \leq \tau_1. \tag{1.3.25}$$

The constant c here does not depend either on β_0 or on τ_1. Since τ_1 is arbitrary these estimates are valid, in fact, for $\tau \in \mathbb{R}$.

1.3.6 Asymptotic behavior of the solution as $t \to T^*$

The statement of theorem 1.1.1 is a simple consequence of the inequalities (1.3.1), (1.3.24), (1.3.25). Indeed, proposition 1.2.1 and the estimates (1.3.24), (1.3.25) ensure that

$$\psi(x,t) = e^{i\mu(t)}\lambda^{1/2}(t)\left(\varphi_0(z) + \chi(z,t)\right), \;\; z = \lambda(t)x,$$

where χ admits the estimate

$$\|\chi\|_\infty \leq ch_0.$$

Consider $\lambda = e^{\int_0^\tau ds(\beta+\eta_2)}$. By (1.3.1), (1.3.24), (1.3.25),

$$|\beta + \eta_2 - h_0| \leq ch_0^2. \tag{1.3.26}$$

So, one gets for λ

$$\lambda = e^{\int_0^\tau ds(h_0+O(h_0^2))} = e^{\frac{2S_0\tau}{\ln\tau}(1+o(1))}, \quad \tau \to +\infty. \tag{1.3.27}$$

In the last equality we have made use of (1.2.16).

Consider the relation

$$T^* - t = \int_\tau^\infty ds\frac{1}{\lambda^2} = \frac{1}{2h_0\lambda^2} - \int_\tau^\infty ds\frac{1}{h_0\lambda^2}\big(\beta + \eta_2 - h_0 + \frac{h_0'}{2h_0}\big).$$

By (1.3.26), this identity implies

$$\lambda = (2h_0(T^*-t))^{-1/2}(1+O(h_0)) = \left(\frac{4S_0(T^*-t)}{\ln\tau}\right)^{-1/2}(1+o(1)), \tag{1.3.28}$$

as $t \to T^*$, which together with (1.3.27) gives

$$\ln\tau e^{-\frac{4S_0\tau}{\ln\tau}(1+o(1))} = 4S_0(T^*-t)(1+o(1)), \quad t \to T^*.$$

As a consequence, one gets

$$\tau = \frac{1}{4S_0}|\ln(T^*-t)|\ln(|\ln(T^*-t)|)(1+o(1)). \tag{1.3.29}$$

Combining (1.3.28), (1.3.29), one obtains finally

$$\lambda = \left(\frac{4S_0(T^* - t)}{\ln|\ln(T^* - t)|} \right)^{-1/2} (1 + o(1)).$$

Consider $\mu = \tau + 2\int_0^\tau ds\eta_1$. By (1.3.1), (1.3.24), (1.3.25),

$$\mu = \tau(1 + o(1)),$$

which together with (1.3.29) implies

$$\mu = \frac{1}{4S_0}|\ln(T^* - t)|\ln(|\ln(T^* - t)|)(1 + o(1)).$$

2. Properties of the linearized equations

As mentioned in the introduction, this section has a technical value : it contains a detailed description of the spectral properties of the operators $\tilde{H}(a)$, $H(a)$ in the limit $a \to 0$. In particular, we prove here the propositions 1.2.1, 1.2.2 and 1.2.4-1.2.7. The present section consists of four subsection. In the first subsection we collect some elementary properties of the soliton linearization H_0 [1] that will be used in what follows (most of them were proved in [BP1].) In subsection 2.2 we construct the modified ground state $\tilde{\varphi}(a)$ and prove proposition 1.2.1. Subsection 2.3 contains a proof of proposition 1.2.2. In subsection 2.4 we prove the estimates related to the operator $H(a)$. Finally, we have five appendices where some technical details are removed.

2.1 Operator H_0

2.1.1 Standard solutions

Consider the equation

$$H_0 f = Ef, \tag{2.1.1}$$

Since $\sigma_1 H_0 = -H_0\sigma_1$, it suffices to consider the solutions for $\operatorname{Re} E \geq 0$. In [BP1] a basis of solutions $f_j,\ j = 1, \ldots, 4$ with the standard behavior $e^{\pm ikx}\binom{1}{0}$, $e^{\pm\mu x}\binom{0}{1}$, $k = \sqrt{E-1}$, $\mu = \sqrt{E+1}$, as $x \to +\infty$ was constructed. We collect here some properties of these solutions that we shall need later :

(i) the decreasing solution $f_i^0(x,k)$, $i = 1, 3$, and its derivatives with respect to x are holomorphic functions of $k \in \Omega_i$, $i = 1, 3$, where

$$\Omega_3 = \{k,\ \operatorname{Re}\mu - |\operatorname{Im} k| > -\delta_1\},$$

$$\Omega_1 = \{k,\ k \in \Omega_3,\ \operatorname{Im} k > -\delta_1\},$$

[1]Here we consider H_0 as an operator on the whole $L_2(\mathbb{R} \to \mathbb{C}^2)$.

$\mu = \sqrt{k^2+2}$, the root being defined on the plane with the cuts $(-i\infty, -i\sqrt{2}]$, $[i\sqrt{2}, i\infty)$, $\operatorname{Re}\mu > 0$. Here δ_1 is a small positive number determined by the rate of decrease of the potential $V(\varphi_0)$.

(ii) $f_i^0, i = 1, 3$, have the following asymptotics as $x \to +\infty$

$$f_1^0(x,k) = e^{ikx}[\begin{pmatrix}1\\0\end{pmatrix} + O((1+|k|)^{-1}e^{-\gamma x})], \quad k \in \Omega_{11}$$

$$f_1^0(x,k) = e^{ikx}\begin{pmatrix}1\\0\end{pmatrix} + c(k)e^{-\mu x}\begin{pmatrix}0\\1\end{pmatrix} + O((1+|k|)^{-1}e^{-\operatorname{Im} kx - \gamma x}),\ k \in \Omega_{12}$$

$$f_3^0(x,k) = e^{-\mu x}[\begin{pmatrix}0\\1\end{pmatrix} + O((1+|k|)^{-1}e^{-\gamma x})], \quad k \in \Omega_3. \tag{2.1.2}$$

Here γ is some positive number, Ω_{11} and Ω_{12} are two subsets of $\Omega_1 = \Omega_{11} \cup \Omega_{12}$, $\Omega_{11} = \{k, \operatorname{Re}\mu - \operatorname{Im} k > \delta_2\}$, $\Omega_{12} = \{k, \operatorname{Re}\mu - \operatorname{Im} k \le \delta_2\}$, $\delta_2 > 0$ being a small positive number, $c(k)$ is a holomorphic function of k admitting the estimate $c(k) = O((1+|k|)^{-1})$.

(iii) The increasing solutions f_i^0, $i = 2, 4$, are holomorphic functions of $k \in \Omega_2 = \{k, |\operatorname{Im} k| < \delta_1\}$, with the following asymptotic behavior as $x \to \infty$

$$f_2^0(x,k) = e^{-ikx}[\begin{pmatrix}1\\0\end{pmatrix} + O((1+|k|)^{-1}e^{-\gamma x})],$$

$$f_4^0(x,k) = e^{\mu x}[\begin{pmatrix}0\\1\end{pmatrix} + O((1+|k|)^{-1}e^{-\gamma x})], \tag{2.1.3}$$

uniformly with respect to k, $k \in \Omega_2$.

The asymptotic representations (2.1.2), (2.1.3) can be differentiated with respect to x and k any number of times.

(iv) One can choose f_j^0 in such a way that

$$f_1^0(x,-k) = f_2^0(x,k), \quad f_{3,4}^0(x,-k) = f_{3,4}^0(x,k),$$

$$\overline{f_1^0(x,k)} = f_2^0(x,k), \quad \overline{f_{3,4}^0(x,k)} = f_{3,4}^0(x,k), \quad k \in \mathbb{R}. \tag{2.1.4}$$

The Wronskian

$$w(f,g) = < f', g >_{\mathbb{R}^2} - < f, g' >_{\mathbb{R}^2}$$

does not depend on x if f and g are solutions of (2.1.1).

(v) The system of Wronskians for f_j^0 has the form

$$w(f_1^0, f_2^0) = 2ik,\ w(f_1^0, f_3^0) = 0,$$

$$w(f_1^0, f_4^0) = 0,\ w(f_3^0, f_4^0) = -2\mu,\ \ k \in \Omega_2. \tag{2.1.5}$$

The solutions with standard behavior as $x \to -\infty$ can be obtained by using the fact that the operator H_0 is invariant under the change of variable $x \to -x$. Let

$$g_j^0(x,k) = f_j^0(-x,k),\ j = 1,\dots,4.$$

In addition to scalar Wronskian we shall also use matrix Wronskian

$$W(F,G) = F^{t'}G - F^tG',$$

where F and G are 2×2 matrices composed of pairs of solutions. The matrix Wronskian do not depend on x.

We introduce the concrete matrix solutions

$$F_1^0 = (f_1^0, f_3^0),\ F_2^0 = (f_2^0, f_4^0), \qquad G_1^0 = (g_1^0, g_3^0),\ G_2 = (g_2^0, g_4^0).$$

Since V decays exponentially H_0 cannot have more than a finite number of the eigenvalues, all of them being of finite multiplicity. It was shown in [BP1] that

Proposition 2.1.1 *The eigenvalues of the operator H_0 in the domain $\operatorname{Re} E \geq 0$ and its resonances at the boundary point $E = 1$ of the continuous spectrum* [1] *are characterized by the equation*

$$\det D_0 = 0,$$

where $D_0 = W(G_1^0, F_1^0)$.

Remark. Let us mentioned that the most rapidly decreasing solution f_3^0 is simply defined by means of the integral equation

$$f_3^0(y) = e^{\mu x}\begin{pmatrix}0\\1\end{pmatrix} - \int_x^\infty dy \begin{pmatrix} \frac{\sin k(x-y)}{k} & 0 \\ 0 & \frac{\operatorname{sh}\mu(x-y)}{\mu} \end{pmatrix} \sigma_3 V(\varphi_0(y)) f_3^0(y).$$

For E in some small vicinity of zero one can use the similar equations to construct a complete set of solutions. Indeed, consider the equation

$$w_1^0(x) = e^{ikx}\begin{pmatrix}1\\0\end{pmatrix} - \int_x^\infty dy \begin{pmatrix} \frac{\sin k(x-y)}{k} & 0 \\ 0 & \frac{\operatorname{sh}\mu(x-y)}{\mu} \end{pmatrix} \sigma_3 V(\varphi_0(y)) w_1^0(y). \quad (2.1.6)$$

The potential $V(\varphi_0)$ decreases exponentially :

$$|V(\varphi_0(x))| \leq ce^{-4|x|},$$

so, for E in a sufficiently small vicinity of zero (for ex., for $|E| \leq 2$) the integral operator in (2.1.6) reproduces the behavior of the free term. Thus, omitting standard details we get the existence of solution $w_1^0(x,k)$ of (2.1.1) that is holomorphic

[1] Generically the equation $H_0 f = \pm f$ does not have solutions bounded at infinity. If, nevertheless such bounded solutions exist the points ± 1 are called resonances.

function of $k \in \Omega_0$, $\Omega_0 = \{k,\ |k^2+1| < 2\}$, with the following asymptotic behavior as $x \to +\infty$:

$$w_1^0 = e^{ikx}\left[\begin{pmatrix}1\\0\end{pmatrix} + O(e^{-4x})\right], \tag{2.1.7}$$

uniformly with respect to k. This asymptotic formula can be differentiated with respect to x, k any number of times. The constructed solution satisfies the relation

$$f_3^0(x,k) = \sigma_1 w_1^0(x, i\mu).$$

On the set Ω_0 with the cuts along the intervals $(-i\sqrt{3}, -i\sqrt{2}]$, $[i\sqrt{2}, i\sqrt{3})$ introduce the basis of solutions $\{w_j^0\}_{j=1}^4$,

$$w_2^0(x,k) = w_1^0(x,-k), \quad w_3^0(x,k) = \sigma_1 w_1^0(x, i\mu),$$

$$w_4^0(x,k) = \sigma_1 w_1^0(x,-i\mu), \quad \operatorname{Re}\mu > 0.$$

w_j^0 satisfy the same set of relations (2.1.4), (2.1.5) as f_j^0.

Consider the Wronskian :

$$\hat{D}_0 = W(\mathcal{U}^0, \mathcal{W}^0),$$

where $\mathcal{W}^0 = (w_1, w_3)$, $\mathcal{U}^0(x,k) = \mathcal{W}^0(-x,k)$. Clearly, the zeros of $\det \hat{D}_0$ coincide with those of $\det D_0$ (in $\Omega_0 \cap \Omega_1$).

Since H_0 is invariant under the change of variable $x \to -x$, the matrices D_0, $\hat{D}_0$ can be factorized :

$$D_0 = -2D_0^+ D_0^-, \quad D_0^-(k) = (F_1^0(0,k))^t, \ \ D_0^+(k) = F_{1x}^0(0,k).$$

$$\hat{D}_0 = -2\hat{D}_0^+ \hat{D}_0^-, \quad \hat{D}_0^-(k) = (\mathcal{W}^0(0,k))^t, \ \ \hat{D}_0^+(k) = \mathcal{W}_x^0(0,k).$$

2.1.2 Discrete spectrum

Taking into account the special structure of the perturbation $V(\varphi_0)$ one can get a more precise description of the discrete spectrum. The structure of the root subspace of H_0 restricted to the subspace of even functions corresponding to the eigenvalue $E = 0$ has already been described in Section 1. Taking into account also the Galilei invariance of the equation (1.1.1) one can get the complete description : corresponding to the point $E = 0$ are two eigenvectors $\vec{\eta}_0$, $\vec{\xi}_0$ and four associated functions $\vec{\eta}_1$, $\vec{\xi}_i, i = 1,2,3$,

$$H_0\vec{\xi}_0 = H_0\vec{\eta}_0 = 0, \quad H\vec{\eta}_1 = i\vec{\eta}_0, \ H_0\vec{\xi}_i = i\vec{\xi}_{i-1}, \ i = 1,2,3,$$

$$\vec{\xi}_i = \begin{pmatrix}\xi_i\\ \bar{\xi}_i\end{pmatrix}, \quad \vec{\eta}_i = \begin{pmatrix}\eta_i\\ \bar{\eta}_i\end{pmatrix},$$

$$\xi_0 = i\varphi_0, \ \xi_1 = \frac{1}{4}(1 + 2x\partial_x)\varphi_0, \ \xi_2 = -i\frac{1}{8}x^2\varphi_0,$$

$$\xi_3 = \frac{1}{2}\varphi_1, \ \ \eta_0 = \varphi_0', \ \ \eta_1 = -\frac{i}{2}x\varphi_0.$$

Since

$$\left\langle \vec{\xi}_3, \sigma_3 \vec{\xi}_0 \right\rangle = -\left\langle \vec{\xi}_2, \sigma_3 \vec{\xi}_1 \right\rangle = -i\frac{\|x\varphi_0\|_2^2}{8},$$

$$\langle \vec{\eta}_1, \sigma_3 \vec{\eta}_0 \rangle = i\frac{\|\varphi_0\|_2^2}{2},$$

the vectors $\vec{\xi}_i$, $i = 0$, $\vec{\eta}_j$, $i = 0, \ldots, 3$, $j = 0, 1$, span the root subspace corresponding to the point $E = 0$.

Let us pass to a new basis in the matrix representation of H_0 :

$$\mathcal{L}_0 = W H_0 W^{-1}, \quad W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The operator $\mathcal{L}_0$ has the form

$$\mathcal{L}_0 = \begin{pmatrix} 0 & L_{0-} \\ L_{0+} & 0 \end{pmatrix},$$

where

$$L_{0+} = -\partial_x^2 + 1 - 5\varphi_0^4, \quad L_{0-} = -\partial_x^2 + 1 - \varphi_0^4.$$

The operators $L_{0\pm}$ are self-adjoint in L_2, the continuous spectra lie on the half-axe $E \geq 1$. L_{0-} has the only eigenvalue $E = 0$ with the eigenfunction φ_0. L_{0+} has two eigenvalues E_0, 0, $E_0 < 0$, with the eigenfunctions φ_0^3, φ_0' respectively. Both L_{0-} and L_{0+} have no resonances at the end point of the continuous spectrum.

Remark that

$$\mathcal{L}_0^2 = \begin{pmatrix} T_0 & 0 \\ 0 & T_0^* \end{pmatrix}, \quad T_0 = L_{0-}L_{0+}.$$

The spectra of the operators T_0 and T_0^* are connected in a canonical way, i.e., are complex conjugated and the corresponding root subspaces are finite-dimensional and have the same structure.

Consider T_0. Obviously, $T_0\xi_1 = T_0\eta_0 = 0$. The spectrum of T_0 is real, the minimal eigenvalue being equal to zero (see [BP1], for example). Moreover, one has the following proposition.

Proposition 2.1.2 *Zero is the only eigenvalue of the operator T_0 in the interval $(-\infty, 1]$.*

Proof. We prove it by a contradiction. Let $1 \geq E > 0$ be an eigenvalue of T_0 with an eigenfunction $\psi : T_0\psi = E\psi$. Then $(\psi, \varphi_0) = 0, (L_{0-}^{-1}\psi, \xi_1) = (L_{0-}^{-1}\psi, \eta_0) = 0$.

Consider the self-adjoint operator $A = PL_{0+}P$, P being the projection orthogonal to φ_0. The direct calculations show that

$$\frac{(Au, u)}{(u, u)} \leq \frac{(L_{0-}^{-1}Pu, Pu)}{(Pu, Pu)} < 1,$$

provided $u \in F$, $F = \mathcal{L}\{\psi, \eta_0, \xi_j,\ j = 0, 1\}$. Obviously, $\dim F = 4$, which implies that the number of the eigenvalues of A in $(-\infty, 1)$ counted with their multiplicities is greater or equal than four. On the other hand the only eigenvalue of A in the interval $(-\infty, 1)$ is the point $E = 0$, η_0, ξ_j, $j = 0, 1$, being the corresponding eigenfunctions. Indeed, let $E \neq 0$ be an eigenvalue of $PL_{0+}P$, then $E > E_0$ and there exists $u, (u, \varphi_0) = 0$, such that $L_{0+}u = Eu + \varphi_0$. Consequently, $u = (L_{0+} - E)^{-1}\varphi_0$, which implies

$$((L_{0+} - E)^{-1}\varphi_0, \varphi_0) = 0. \tag{2.1.8}$$

Consider the function $g(\lambda) = ((L_{0+} - \lambda)^{-1}\varphi_0, \varphi_0)$, assuming that $\lambda \in (E_0, 1)$. The function g has the following obvious properties :

1) $g(\lambda)$ is monotonically increasing, because $g'(\lambda) = \|(L_{0+} - \lambda)^{-1}\varphi_0\|_2^2$;

2) $g(0) = -(\xi_1, \varphi_0) = 0$.

Thus, (2.1.8) is impossible for $E \neq 0$.

Proposition 2.1.2. extends immediately to the operators $\mathcal{L}_0$ and H_0 :

Corollary 2.1.3 *$E = 0$ is the unique point in the discrete spectrum of the operator H_0.*

A slight modification of the arguments used in the proof of proposition 2.1.2 allows us to get

Proposition 2.1.4 *The operator H_0 has no resonances at the end points of the continuous spectrum.*

See appendix 1 for the proof.

2.1.3 Embedded eigenvalues

In this subsubsection we prove the absence of embedded eigenvalues. Consider equation (2.1.1) with $E > 1$. After a change of variables

$$f(x) = v(z),\ \ z = \operatorname{th} 2x,$$

(2.1.1) takes the form

$$\left(-\partial_z^2 + \frac{2z}{1 - z^2}\partial_z + \frac{1}{4(1 - z^2)^2}\right) v$$
$$-\frac{9}{4(1 - z^2)}v - \frac{3}{2(1 - z^2)}\sigma_1 v = \frac{E}{4(1 - z^2)^2}\sigma_3 v. \tag{2.1.9}$$

The only singular points of this system (considered on the whole plane $z \in \mathbb{C}$) are $z_\pm = \pm 1$ and $z_\infty = \infty$. It is easy to check that they are regular. In particular, in a vicinity of $z_\pm$ one can find a basis of solutions of the form

$$(z - z_j)^{ik/4}e_{j1}(z),\ (z - z_j)^{-ik/4}e_{j2}(z),\ (z - z_j)^{\mu/4}e_{j3}(z),$$

$$\begin{cases} (z-z_j)^{-\mu/4}e_{j4}, & \text{if } \mu/2 \notin \mathbb{Z}, \\ \ln(z-z_j)(z-z_j)^{\mu/4}e_{j3}(z) + (z-z_j)^{-\mu/4}e_{j4}, & \text{if } \mu/2 \in \mathbb{Z}, \end{cases}$$

where e_{jl}, $l = 1,\dots,4$, $j = \pm$, are holomorphic non vanishing functions in some vicinity of z_j, k and μ being the same as in subsubsection 2.1.2. Thus, if $E > 1$ is an eigenvalue of H_0 there exists a nontrivial solution v of (2.1.9) such that

$$v(z) = (1-z^2)^{\mu/4}\tilde{v}(z),$$

where $\tilde{v}$ is an entire function. Since z_∞ is a regular singular point of (2.1.9) $\tilde{v}$ has at most polynomial growth at infinity, which means that $\tilde{v}$ is polynomial. Moreover, it is easy to check that the roots of the characteristic equation at infinity are given by $-\frac{1}{2} \pm 2$, $-\frac{1}{2} \pm 1$, which implies

$$n = 0,$$

where n is the degree of $\tilde{v}$. The direct calculation shows that (2.1.9) has no nontrivial solution of the form $(1-z^2)^{\mu/4}a$, where a is a constant vector.

Combining these results with the results of the previous subsection one gets the proposition.

Proposition 2.1.5

$$\det D_0(k) \neq 0, \quad k \in \Omega_1, \ \operatorname{Im} k \geq 0,$$

provided $k \neq i$.

2.2 Profile $\tilde{\varphi}$

Consider (1.2.1). We are looking for a real even solution of (1.2.1). Write $\tilde{\varphi}$ as the sum

$$\tilde{\varphi}(x,\alpha,a) = \varphi_0(x,\alpha) + \chi(x,\alpha,a).$$

Then χ satisfies the equation

$$\chi = \tilde{L}_+^{-1}\chi_0 + \mathcal{J}(\chi), \tag{2.2.1}$$

where

$$\chi_0 = \frac{ax^2}{4}\theta(hx)\varphi_0(x,\alpha),$$

$$\mathcal{J}(\chi) = \tilde{L}_+^{-1}\left[(\varphi_0+\chi)^5 - \varphi_0^5 - 5\varphi_0^4\chi\right],$$

$$\tilde{L}_+ = -\partial_x^2 + \frac{\alpha^2}{4} - \frac{ax^2}{4}\theta(hx) - 5\varphi_0^4.$$

$\tilde{L}_+$ is a self-adjoint operator in L_2. It follows from the corresponding properties of L_{+0} that the restriction of $\tilde{L}_+$ to the subspace of even functions has a bounded inverse. Moreover, one has the estimate

$$|\tilde{G}_+(x,y)| \leq ce^{-\frac{1}{h}|S_{\alpha,a}(hx)-S_{\alpha,a}(hy)|}, \quad x \geq 0, y \geq 0, \tag{2.2.2}$$

$S_{\alpha,a}(\xi) = \frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2 - \operatorname{sgn} as^2\theta(s)}$. Here $\tilde{G}_+$ is the kernel of $\tilde{L}_+^{-1}$, if we consider $\tilde{L}_+$ as an operator on the half-line $x \geq 0$ with the Neumann boundary condition at $x = 0$. This estimate can be obtained as an immediate consequence of the constructions developed in the next subsection.

It follows from (2.2.2) that

$$\left|\left(\tilde{L}_+^{-1}\chi_0\right)(x)\right| \leq c|a|\,\langle x\rangle^3\, e^{-\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}, \tag{2.2.3}$$

$$\|e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}\tilde{L}_+^{-1}f\|_\infty \leq c\|e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}f\|_1. \tag{2.2.4}$$

Here $\tilde{S}_{\alpha,a}(\xi) = \frac{1}{2}\int_0^\xi ds\sqrt{\alpha^2 - (a)_+ s^2\theta(s)}$.

Consider (2.2.1). The basis idea is to view this equation as a mapping of the space of continuous functions equipped with the norm

$$|||\chi|||_p = \|\,\langle x\rangle^{-p}\, e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}\chi\|_\infty,$$

with some $p \geq 0$, to itself and to seek for a fixed point. Using (2.2.4) it is not difficult to check that the nonlinear operator $\mathcal{J}$ maps this space into itself :

$$|||\mathcal{J}(\chi)|||_p \leq c[|||\chi|||_p^2 + |||\chi|||_p^5]. \tag{2.2.5}$$

Moreover,

$$|||\mathcal{J}(\chi_1)-\mathcal{J}(\chi_2)|||_p \leq c|||\chi_1-\chi_2|||_p[|||\chi_1|||_p+|||\chi_2|||_p+(|||\chi_1|||_p+|||\chi_2|||_p)^4]. \tag{2.2.6}$$

The estimates (2.2.3), (2.2.5), (2.2.6) mean that for a sufficiently small the mapping $\chi \to \chi_0 + \mathcal{J}(\chi)$ is a contraction of the ball $|||\chi|||_3 \leq \eta$ into itself with some $\eta > 0$, and, consequently, has a unique fixed point which satisfies the estimate

$$|||\chi|||_3 \leq c|a|. \tag{2.2.7}$$

In the same manner one can prove the asymptotic expansion (1.1.5). Write $\tilde{\varphi} = \varphi^N + \chi_N$. The function χ^N satisfies the equation

$$-\partial_x^2\chi_N + \frac{\alpha^2}{4}\chi_N - \frac{ax^2}{4}\theta(hx)\chi_N - (\varphi^N + \chi^N)^5 + (\varphi^N)^5 - R_N = 0,$$

where R_N admits the estimate

$$|R_N(x)| \leq c\left[a^{N+1}\langle x\rangle^{3N+2} + (1-\theta(hx))\sum_{k=0}^{N-1}|a|^{k+1}|x|^{3k+2}\right]e^{-\frac{\alpha}{2}|x|}.$$

We rewrite this equation in the form similar to (2.2.1) :

$$\chi^N = \chi_0^N + \mathcal{J}_N(\chi^N), \tag{2.2.8}$$

$$\chi_0^N = \tilde{L}_+^{-1} R_N, \ \ \mathcal{J}_N(\chi) = \tilde{L}_+^{-1} F_N(\chi),$$

where

$$F_N(\chi) = (\varphi^N + \chi)^5 - (\varphi^N)^5 - 5\varphi_0^4\chi.$$

By (2.2.2), (2.2.4),

$$\| < x >^{-3(N+1)} \, e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)} \chi_0^N \|_\infty \leq c|a|^{N+1},$$

$$|||\mathcal{J}_N(\chi)|||_p \leq c(|a||||\chi|||_p + |||\chi|||_p^2 + |||\chi|||_p^5),$$

which together with (2.2.7) implies

$$\| < x >^{-3(N+1)} \, e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)} \chi_N \|_\infty \leq c|a|^{N+1},$$

provided a is sufficiently small.

By (2.2.7), $\tilde{\varphi}$ admits the estimate

$$\| \langle x \rangle^{-3} \, e^{\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)} \tilde{\varphi} \|_\infty \leq c. \tag{2.2.9}$$

Plugging this inequality into right hand side of the representation

$$\tilde{\varphi} = (-\partial_x^2 + \frac{\alpha^2}{4} - \frac{ax^2}{4}\theta)^{-1} \tilde{\varphi}^5$$

and using the corresponding estimate of the free resolvent one gets an improved version of (2.2.9) :

$$c_2(1 + O(e^{-\frac{4}{h}S_{\alpha,a}(h|x|)})) \leq e^{\frac{1}{h}S_{\alpha,a}(h|x|)} \tilde{\varphi} \leq c_1, \ \ x \in \mathbb{R}, \tag{2.2.10}$$

with some c_1, $c_2 > 0$ independent of α, a, which together with (2.2.7) implies the positivity of $\tilde{\varphi}$ provided a is sufficiently small. We can now formulate the final assertion with respect to $\tilde{\varphi}$.

Proposition 2.2.1 *For α in some finite vicinity of* 2 *and for a sufficiently small, equation* (2.2.1) *has a unique positive even decreasing solution $\tilde{\varphi}(z, \alpha, a)$ which is close to $\varphi_0(z, \alpha)$. Moreover, as $a \to 0$, $\tilde{\varphi}(z, \alpha, a)$ admits the asymptotic expansion* (1.1.5) *in the sense*

$$|\tilde{\varphi} - \varphi^N| \leq c|a|^{N+1} < x >^{3(N+1)} \, e^{-\frac{1}{h}\tilde{S}_{\alpha,a}(h|x|)}. \tag{2.2.11}$$

Remark. It is not difficult to check that

(i) the solution $\tilde{\varphi}$ is a smooth function of its arguments and the asymptotic representation (2.2.11) can be differentiated with respect to x, α and a any number of times;

(ii) $\tilde{\varphi}$ "almost" satisfies the scaling law

$$\tilde{\varphi}(x, \alpha, a) \sim \left(\frac{\alpha}{2}\right)^{1/2} \tilde{\varphi}(\frac{\alpha}{2}x, 2, \frac{16a}{\alpha^4}). \tag{2.2.12}$$

More precisely,

$$\left| \tilde{\varphi}(x, \alpha, a) - \left(\frac{\alpha}{2}\right)^{1/2} \tilde{\varphi}(\frac{\alpha}{2}x, 2, \frac{16a}{\alpha^4}) \right| \leq ce^{-\gamma_1/h} e^{-\gamma_2|x|},$$

with some γ_1, $\gamma_2 > 0$.

2.3 Operator $\tilde{H}(a)$

In this subsection we establish the spectral properties of the operator $\tilde{H}(a)$ (in the limit $a \to 0$) that were announced and used in Section 1.

2.3.1 Standard solutions

Consider the equation

$$(\tilde{H}(a) - E)\psi = 0. \tag{2.3.1}$$

For E in some small but fixed vicinity of zero we introduce a basis of solutions ψ_j, $j = 1, \dots, 4$, of (2.3.1) with the standard behavior at $+\infty$ by means of the integral equations

$$\psi_j(x, E) = \psi_{0j}(x, E) - \int_x^\infty dy \tilde{K}(x, y, E)\sigma_3 V(\tilde{\varphi}(y))\psi_j(y, E), \tag{2.3.2}$$

$j = 1, \dots, 4$, where $\psi_{0j}(x, E) = \sigma_1 \psi_{0j+2}(x, -E)$,

$$\psi_{01}(x, E) = u_1(x, \lambda_1)\begin{pmatrix}1\\0\end{pmatrix}, \quad \psi_{02}(x, E) = u_2(x, \lambda_1)\begin{pmatrix}1\\0\end{pmatrix}, \quad \lambda_1 = E - 1,$$

$$\tilde{K}(x, y, E) = \begin{pmatrix} \tilde{k}(x, y, \lambda_1) & 0 \\ 0 & \tilde{k}(x, y, \lambda_2) \end{pmatrix}, \quad \lambda_2 = -E - 1,$$

$$\tilde{k}(x, y, \lambda) = \frac{1}{w(u_1, u_2)}(u_1(x, \lambda)u_2(y, \lambda) - u_1(y, \lambda)u_2(x, \lambda)),$$

$w(u_1, u_2) = u_1' u_2 - u_2' u_1$, $u_2(x, \lambda) = u_1(-x, \lambda)$, u_1 being a decreasing (as $x \to +\infty$) solution of the equation

$$-u_{xx} - \frac{ax^2}{4}\theta(hx)u = \lambda u.$$

We normalize u_1 by the condition

$$u_1 = \frac{1}{(-\lambda - \frac{ax^2}{4}\theta(hx))^{1/4}} e^{-\frac{1}{h}\int_0^{hx} ds \sqrt{-\lambda - \operatorname{sgn} a \frac{s^2}{4}\theta(s)}}, \quad x \to +\infty. \tag{2.3.3}$$

The roots here are defined on the complex plane with the cut along the negative semi-axis. They are positive for the positive values of the argument.

For λ in some finite vicinity of -1, $x \in \mathbb{R}$, the asymptotics of u_1 as $a \to 0$ is given by the standard WKB formulas

$$u_1(x, \lambda) = e^{-\frac{1}{h}\int_0^{hx} ds \sqrt{-\lambda - \operatorname{sgn} a \frac{s^2}{4}\theta(s)}} \sum_{j=0}^{\infty} h^j u_1^j(hx, \lambda), \tag{2.3.4}$$

where

$$u_1^0(\xi,\lambda) = \frac{1}{(-\lambda - \operatorname{sgn} a \frac{\xi^2\theta(\xi)}{4})^{1/4}},$$

$$u_1^j(\xi,\lambda) = -\frac{1}{2(-\lambda - \operatorname{sgn} a \frac{\xi^2\theta(\xi)}{4})^{1/4}} \int_\xi^\infty ds \frac{u_{1ss}^{j-1}}{(-\lambda - \operatorname{sgn} a \frac{s^2\theta(s)}{4})^{1/4}}.$$

As a consequence, one gets

$$w(u_1,u_2) = -2 + O(h),$$

$$|\tilde{k}(x,y,\lambda)| \le c e^{\frac{1}{h}\int_{hx}^{hy} ds \operatorname{Re}\sqrt{-\lambda - \operatorname{sgn} a \frac{s^2}{4}\theta(s)}}, \quad x \le y,$$

uniformly with respect to λ in some finite vicinity of -1. The potential $V(\tilde{\varphi})$ decreases exponentially :

$$|V(\tilde{\varphi}(x))| \le c e^{-\frac{4}{h} S_a(h|x|)},$$

$S_a(\xi) = S_{2,a}(\xi)$, so for E in some finite vicinity of zero we get the existence of a solution ψ_j of (2.3.1) that has the following asymptotic behavior as $x \to +\infty$:

$$\psi_j(x,E) = u_j(x,E-1)\left[\begin{pmatrix}1\\0\end{pmatrix} + O(e^{-\frac{4}{h}S_a(hx)})\right], \ j=1,2, \tag{2.3.5}$$

$$\psi_j(x,E) = u_{j-2}(x,-E-1)\left[\begin{pmatrix}0\\1\end{pmatrix} + O(e^{-\frac{4}{h}S_a(hx)})\right], \ j=3,4, \tag{2.3.6}$$

uniformly with respect to a, E. In this formulation and in subsequent ones we omit phrases of the following type : the solutions ψ_j and its derivatives with respect to x are holomorphic functions of E and the asymptotic representations can be differentiated with respect to x and E any number of times.

Clearly,

$$\psi_{j+2}(x,E) = \sigma_1 \psi_j(x,-E), \quad \psi_j(x,E) = \overline{\psi_j(x,\bar{E})}, \tag{2.3.7}$$

$$\begin{array}{ll} w(\psi_1,\psi_2) = w(\psi_{10},\psi_{20}), & w(\psi_1,\psi_{3,4}) = 0, \\ w(\psi_3,\psi_4) = w(\psi_{30},\psi_{40}), & w(\psi_{3,4},\psi_2) = 0. \end{array} \tag{2.3.8}$$

One can use $\psi_j(-x,E)$, $j=1,\dots,4$ as a basis of solutions with the standard behavior at $-\infty$.

We shall describe now the behavior of the decreasing solutions $\psi_{1,3}$ in the limit $a \to 0$. By (2.3.7), it is sufficient to consider ψ_1. We represent it as the sum

$$\psi_1 = e^{-ikx} u_1(x,\lambda_1) w_1^0(x,k) + r_1, \quad k = \sqrt{E - E_0}, \ \operatorname{Im} k > 0. \tag{2.3.9}$$

One can write down the following integral equation for r_j

$$r_1(x,E) = -\int_x^\infty dy \tilde{K}(x,y,E)[\mathcal{R}_1 + \sigma_3 V(\tilde{\varphi}(y)) r_1(y,E)].$$

Here

$$\mathcal{R}_1 = (V(\tilde{\varphi}) - V(\varphi_0))e^{-ikx}u_1(x,\lambda_1)w_1^0(x,k)$$
$$-2e^{ikx}(e^{-ikx}u_1(x,\lambda_1))_x\sigma_3(e^{-ikx}w_1^0(x,k))_x.$$

By (2.1.7), (2.3.4),

$$|\mathcal{R}_1| \leq ch|u_1(x,\lambda_1)|\, \langle x\rangle^3\, e^{-\frac{4}{h}\tilde{S}_{\alpha,a}(hx)},$$

which leads to the following asymptotic estimate for r_1 :

$$r_1 = O(hu_1(x,\lambda_1)e^{-\frac{\gamma}{h}\tilde{S}_a(hx)}),\ \ \gamma < 4. \tag{2.3.10}$$

For x not too large the representation (2.3.9), (2.3.10) can be simplified :

$$\psi_1 = d_0 w_1^0 + O(he^{-\frac{1-\gamma}{h}\tilde{S}_a(hx)}), \quad d_0 = (-\lambda_1)^{-1/4}, \tag{2.3.11}$$

with some $\gamma > 0$[1], uniformly with respect to E in some finite vicinity of zero.

In a similar way one can get a complete asymptotic expansion of ψ_1 in powers of h. Without dwelling on the derivation we describe the result. Let us introduce a formal solution w,

$$w(x,E,a) = \sum_{n=0}^{\infty} a^n w^n(x,E), \tag{2.3.12}$$

of the equation

$$\left[(-\partial_x^2 + 1 - \frac{ax^2}{4})\sigma_3 + V(\varphi(a))\right]\psi = E\psi. \tag{2.3.13}$$

Equation (2.3.13) is equivalent to the following recurrent system for w^n :

$$(H_0 - E)w^0 = 0,$$

$$(H_0 - E)w^n - \frac{x^2}{4}\sigma_3 w^{n-1} + \sum_{k=1}^{n} V^k w^{n-k} = 0, \quad n \geq 1,$$

where V^k are the coefficients of the expansion $V(\varphi(a)) = \sum_{k\geq 0} a^k V^k$. It is easy to check that this system admits a solution with the following asymptotic behavior

$$w^n = e^{ikx}\left[P_n(x,E)\begin{pmatrix}1\\0\end{pmatrix} + O(\langle x\rangle^{3n} e^{-4x})\right],\ \ x \to +\infty,$$

$k = \sqrt{E-1}$, $\operatorname{Im} k > 0$, P_n being polynomial of x of the degree $3n$. The coefficients w^n can be fixed uniquely by the condition $P_n(0,E) = 0$ for $n > 0$, $P_0 = 1$. Then $w^0(x,E) = w_1^0(x,k)$, $w(x,E,a) = \overline{w(x,\bar{E},a)}$.

[1] γ can be made arbitrary small by choosing a sufficiently small vicinity of the point $E = 0$.

One can show that after a renormalization the solution ψ_1 admits the asymptotic expansion (2.3.12). More precisely, there exists a formal series $d(E,a) = \sum_{n\geq 0} h^n d_n(E,\hat{a})$, $\hat{a} = a/|a|$, (d_0 being the same as in (2.3.11)), such that

$$\psi_1 = dw, \tag{2.3.14}$$

in the sense

$$|\psi_1(x,E,a) - \sum_{n\leq N} h^n \psi_{1n}(x,E,\hat{a})| \leq c h^{N+1} e^{-\frac{(1-\gamma)}{h}\tilde{S}_a(hx)}, \quad x \geq 0, \tag{2.3.15}$$

uniformly with respect to E in some finite vicinity of zero. Here ψ_{1n} are the coefficients of the series dw, γ is the same as in (2.3.11).

It is worth mentioning that d can be found from the formal relation

$$u_1(x,\lambda_1,a) = e^{ikx} d(E,a) \sum_{n\geq 0} a^n P_n(x,E).$$

In particular,

$$d_1 = \frac{1}{2(-\lambda_1)^{1/4}} \int_0^\infty ds \left(\frac{\partial}{\partial s}(-\lambda_1 - \hat{a}\frac{s^2}{4}\theta(s))^{-1/4} \right)^2 .$$

By (2.3.7), an expansion similar to (2.3.14), (2.3.15) is valid for ψ_3 :

$$\psi_3(x,E,a) = \sum_{n\geq 0} h^n \psi_{3n}(x,E,\hat{a}), \tag{2.3.16}$$

where $\psi_{3n}(x,E,\hat{a}) = \sigma_1 \psi_{1n}(x,-E,\hat{a})$.

2.3.2 Spectral properties of the operator $\tilde{H}(a)$

The operator $\tilde{H}(a)$ has the same continuous spectrum as H_0. In addition, $\tilde{H}(a)$ can have only finitely many eigenvalues of finite multiplicity. $\tilde{H}(a)$ satisfies the relations similar to (1.1.8) :

$$\sigma_3 \tilde{H}(a) \sigma_3 = \tilde{H}^*(a), \quad \sigma_1 \tilde{H}(a) \sigma_1 = -\tilde{H}(a), \tag{2.3.17}$$

which leads to a clear symmetry in the structure of the spectrum of $\tilde{H}(a)$.

The point $E = 0$ is an eigenvalue : there is an eigenfunction $\tilde{\zeta}_0$ and an associated function $\tilde{\zeta}_1$,

$$\tilde{H}(a)\tilde{\zeta}_0 = 0, \quad \tilde{H}(a)\tilde{\zeta}_1 = i\tilde{\zeta}_0, \tag{2.3.18}$$

$$\tilde{\zeta}_0(a) = i\tilde{\varphi}(a)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \tilde{\zeta}_1(a) = \partial_\alpha \tilde{\varphi}(\alpha,a)|_{\alpha=2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\left\langle \tilde{\zeta}_1(a), \sigma_3\tilde{\zeta}_0(a) \right\rangle = 4ia(\varphi_0, \varphi_1) + O(a^2) = 4iea + O(a^2). \tag{2.3.19}$$

The eigenvalues of $\tilde{H}(a)$ lying in some finite vicinity of zero can be characterized by the equation

$$\det D(E) = 0,$$

where $D = W(\Xi_1, \Psi_1)$, $\Psi_1 = (\psi_1, \psi_3)$, $\Xi_1(x, E) = \Psi_1(-x, E)$, D is a holomorphic function of E in some finite vicinity of the point $E = 0$. In the same manner as D_0, the matrix D can be factorized :

$$D = -2D^-D^+, \quad D^-(E, a) = \Psi_1^t(0, E, a), \;\; D^+(E, a) = \Psi'_{1x}(0, E, a),$$

the zeros of $\det D^+$ ($\det D^-$) (counted with their multiplicity) corresponds to the eigenvalues of $\tilde{H}(a)$ restricted to the subspace of even (odd) functions.

By (2.3.7),

$$\sigma_1 D^{\pm}(E)\sigma_1 = D^{\pm}(-E), \quad \overline{D^{\pm}(\bar{E})} = D^{\pm}(E). \tag{2.3.20}$$

It follows from (2.3.18), (2.3.19) that the point $E = 0$ is a root of $\det D^+$ of the multiplicity two :

$$\det D^+ = \kappa(a)E^2 + O(E^4). \tag{2.3.21}$$

As $a \to 0$, κ admits the asymptotic representation of the form :

$$\kappa(a) = d^2(0, a)\hat{\kappa}(a), \tag{2.3.22}$$

where $\hat{\kappa}(a)$ is a formal series in powers of a, in particular,

$$\hat{\kappa}(a) = \kappa_0 a + O(a^2), \quad \kappa_0 = \frac{(\varphi_0^4(0) - 1)e}{\varphi_\infty^2} > 0. \tag{2.3.23}$$

where $\varphi_\infty = \varphi_\infty(2)$.

In terms of the matrix solution Ψ_1 (2.3.14), (2.3.16) take form

$$\Psi_1 = \mathcal{W}\Lambda, \tag{2.3.24}$$

where $\mathcal{W}$ is the formal matrix solution of (2.3.9)

$$\mathcal{W}(x, E, a) = \sum_{n\geq 0} a^n \mathcal{W}^n(x, E), \;\; \mathcal{W}^n(x, E) = (w_n(x, E), \sigma_1 w_n(x, -E)), \tag{2.3.25}$$

$$\Lambda(E, a) = \begin{pmatrix} d(E, a) & 0 \\ 0 & d(-E, a) \end{pmatrix}.$$

Let us note the obvious relation

$$\mathcal{W}^0(x, 0) = \frac{1}{\sqrt{2}\varphi_\infty}(\vec{\eta}_0, -\vec{\xi}_0)W. \tag{2.3.26}$$

The formulas (2.3.24), (2.3.25) imply the following asymptotic expansion of $D^{\pm}$:

$$D^{+} = \hat{D}^{+}\Lambda, \quad D^{-} = \Lambda\hat{D}^{-}, \tag{2.3.27}$$

where $\hat{D}^{+}$ is a formal series in powers of a :

$$\hat{D}^{\pm}(E,a) = \sum_{n\geq 0} \hat{D}^{\pm}_{n}(E)a^{n},$$

$$\hat{D}^{-}_{n}(E) = (\mathcal{W}^{n}(0,E))^{t}, \quad \hat{D}^{+}_{n}(E) = \mathcal{W}^{n}_{x}(0,E).$$

Consider $\hat{D}^{+}_{0}(E)$. Taking into account the structure of the root subspace of H_0 corresponding to the zero eigenvalue one can get the following relation :

$$\hat{D}^{+}_{0}(E)W = \hat{D}^{+}_{0}(E)\begin{pmatrix} 1 & m_1(E) \\ 1 & m_1(E) \end{pmatrix} + E^4\gamma_0 \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + O(E^5),$$

$$\hat{D}^{+}_{0}(0)W = \gamma_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad m_1(E) = m_{10}E + m_{11}E^3, \tag{2.3.28}$$

m_{1k}, $k = 0,1$, γ_k, $k = 0,1$, are some constants, all of them can be calculated explicitly but in what follows we shall need only γ_k, $k = 0,1$

$$\gamma_1 = -\frac{\varphi_{0xx}(0)}{\sqrt{2}\varphi_{\infty}}, \quad \gamma_0 = \frac{e}{4\sqrt{2}\varphi_0(0)\varphi_{\infty}}.$$

These formulas imply :

$$\det \hat{D}^{+}_{0} = \frac{\kappa_0}{4}E^4 + O(E^6). \tag{2.3.29}$$

In a similar manner one can get

$$\det \hat{D}^{-}_{0} = \kappa_1 E^2 + O(E^4), \quad \kappa_1 = \frac{\|\varphi_0\|_2^2}{2\varphi_{\infty}(1-\varphi_0^4)}. \tag{2.3.30}$$

It follows from (2.3.27) that asymptotically (as $a \to 0$), the eigenvalues of $\tilde{H}(a)$ restricted to the subspace of even (odd) functions are characterized by the equation $\Phi_{+}(E,a) = 0$ ($\Phi_{-}(E,a) = 0$) where $\Phi_{\pm} = \det \hat{D}^{\pm}$ is a formal series in powers of a :

$$\Phi_{\pm}(E,a) = \sum_{n\geq 0} a^n \Phi^{\pm}_{n}(E), \quad \Phi^{\pm}_{0} = \det \hat{D}^{\pm}_{0}. \tag{2.3.31}$$

By (2.3.20),

$$\Phi^{\pm}_{n}(E) = \Phi^{\pm}_{n}(-E) = \overline{\Phi^{\pm}_{n}(\bar{E})},$$

and by (2.3.21), as $E \to 0$,

$$\Phi^{+}_{n}(E) = O(E^2).$$

One can show

$$\Phi^{-}_{1}(E) = \kappa_1 + O(E^2), \quad \Phi^{+}_{1}(E) = \kappa_0 E^2 + O(E^4). \tag{2.3.32}$$

The formulas (2.3.30)-(2.3.32) show that for a sufficiently small $\det D^+(E,a)$ and $\det D^-(E,a)$ have two simple roots $\pm\lambda(a)$ and $\pm\mu(a)$ respectively, $\lambda(a) = i\sqrt{a}\lambda'(a)$, $\mu(a) = i\sqrt{a}\mu'(a)$ where $\lambda'(a)$, $\mu'(a)$ are smooth real functions,

$$\lambda'(a) = 2 + O(a), \quad \mu'(a) = 1 + O(a).$$

Since for a sufficiently small the number of the roots of $\det D^-$ ($\det D^+$) counted with their multiplicity in some finite vicinity of the point $E = 0$ is equal two (four), there are no roots except for $\pm\mu$ (zero and $\pm\lambda$).

Let $\tilde{\zeta}_2(a)$ be an eigenfunction of $\tilde{H}(a)$ corresponding to the eigenvalue $\lambda(a)$. By (2.3.26), (2.3.28), $\vec{\zeta}_2(a)$ can be normalized in such a way that

$$\left\langle \tilde{\zeta}_2, \vec{\xi}_0 \right\rangle = \left\langle \tilde{\zeta}_0, \vec{\xi}_0 \right\rangle - \lambda^2 \left\langle \vec{\xi}_2, \vec{\xi}_0 \right\rangle. \tag{2.3.33}$$

Then

$$\tilde{\zeta}_2 = \tilde{\zeta}_0 + O(h).$$

A little bit more detailed consideration of the series $\mathcal{W}$, $\hat{D}^+$ allows us to get the following refinement of the above representation :

$$\tilde{\zeta}_2 = \tilde{\zeta}_0 - i\lambda\tilde{\zeta}_1 - \lambda^2\vec{\xi}_2 + i\lambda^3\vec{\xi}_3 + \sum_{k\geq 4} i\lambda^k h_k \begin{pmatrix} 1 \\ (-1)^{k-1} \end{pmatrix}, \tag{2.3.34}$$

where h_k are even smooth real exponentially decreasing functions of x, $(h_{2k}, \varphi_0) = 0$. This asymptotic expansion holds in the sense of the L_∞-norm with the weight $e^{\frac{(1-\gamma)}{h}\tilde{S}_a(h|x|)}$, $\gamma > 0$:

$$|\tilde{\zeta}_2 - \tilde{\zeta}_0 + i\lambda\tilde{\zeta}_1 + \lambda^2\vec{\xi}_2 - i\lambda^3\vec{\xi}_3 - \sum_{k\geq 4}^{N} i\lambda^k h_k| \leq c|a|^{N+1} e^{-\frac{(1-\gamma)}{h}\tilde{S}_a(h|x|)}. \tag{2.3.35}$$

The results of this subsubsection implies in particular the following proposition.

Proposition 2.3.1 *For a sufficiently small, the discrete spectrum of the operator $\tilde{H}(a)$ (restricted on the subspace of even functions) in some finite vicinity of the point $E = 0$ consists of 0, the corresponding root subspace being described by* (2.3.18)*, and two simple eigenvalues $\pm\lambda(a)$, $\lambda(a) = i\sqrt{a}\lambda'(a)$, where $\lambda'(a)$ is a smooth real function of a, $\lambda'(a) = 2+O(a)$. The eigenfunction $\tilde{\zeta}_2(a)$ corresponding to the eigenvalue $\lambda(a)$, normalized by the condition* (2.3.33) *is a smooth function of $\sqrt{a}$, admitting the asymptotic expansion* (2.3.34) *as $a \to 0$ in the sense* (2.3.35).

2.4 Operator $H(a)$

In this subsection we establish the estimates related to the operator $H(a)$, $a > 0$, that were announced and used in Section 1.

2.4.1 Standard solutions

Consider the equation

$$(H(a) - E)\psi = 0. \tag{2.4.1}$$

We introduce a basis of solutions $f_j(x,E)$, $j = 1, \dots 4$, of (2.4.1) with the following asymptotic behavior as $x \to +\infty$:

$$\begin{aligned}
f_1(x,E) &= v(x,\lambda_1)\left[\begin{pmatrix}1\\0\end{pmatrix} + O_{E,a}(e^{-\frac{4}{h}S_a(hx)})\right],\\
f_2(x,E) &= v^*(x,\lambda_1)\left[\begin{pmatrix}1\\0\end{pmatrix} + O_{E,a}(e^{-\frac{4}{h}S_a(hx)})\right],\\
f_3(x,E) &= v^*(x,\lambda_2)\left[\begin{pmatrix}0\\1\end{pmatrix} + O_{E,a}(e^{-\frac{4}{h}S_a(hx)})\right],\\
f_4(x,E) &= v(x,\lambda_2)\left[\begin{pmatrix}0\\1\end{pmatrix} + O_{E,a}(e^{-\frac{4}{h}S_a(hx)})\right],
\end{aligned} \tag{2.4.2}$$

where $v^*(x,\lambda) = \overline{v(x,\bar{\lambda})}$,

$$v(x,\lambda) = C_\nu e^{i\frac{hx^2}{4}} H_\nu\left(e^{-\frac{i\pi}{4}}\left(\frac{h}{2}\right)^{1/2} x\right), \quad \nu = -\frac{1}{2} + i\frac{\lambda}{h},\ C_\nu = e^{\frac{i\nu\pi}{4}}(2h)^{-\frac{\nu}{2}},$$

H_ν being the Hermite function. The function v is a holomorphic function of $\lambda \in \mathbb{C}$ satisfying the equation

$$-v_{xx} - \frac{ax^2}{4}v = \lambda v. \tag{2.4.3}$$

As $x \to +\infty$,

$$v = e^{i\frac{hx^2}{4}} x^\nu \left(1 + O_\nu(< hx^2 >^{-1})\right).$$

The solutions f_j can be characterized by the appropriate integral equations. In particular, one can write for f_1 the following one.

$$f_1(x,E) = v(x,\lambda_1)\begin{pmatrix}1\\0\end{pmatrix} - \int_x^\infty dy K(x,y,E)\sigma_3 V(\tilde{\varphi}(y)) f_1(x,E),$$

where

$$K(x,y,E) = \begin{pmatrix} k(x,y,\lambda_1) & 0\\ 0 & k(x,y,\lambda_2)\end{pmatrix},$$

$$k(x,y,\lambda) = \frac{1}{w(v,v^*)}(v(x,\lambda)v^*(y,\lambda) - v(y,\lambda)v^*(x,\lambda)).$$

By standard arguments one gets from this equation the existence of a solution f_1 with the asymptotic behavior (2.4.2) as $x \to \infty$, f_1 being a entire function of E.

The solutions f_j, $j = 1, \dots, 4$, satisfy the relations :

$$f_2(x, E) = \overline{f_1(x, \bar{E})}, \quad f_3(x, E) = \sigma_1 \overline{f_1(x, -\bar{E})}, \quad f_4(x, E) = \sigma_1 f_1(x, -E),$$

$$w(f_1, f_2) = ih, \quad w(f_{1,2}, f_{3,4}) = 0, \quad w(f_3, f_4) = -ih.$$

Let us introduce the solutions $g_j(z, E)$, $j = 1, \dots, 4$, with standard behavior at $-\infty$ by

$$g_j(x, E) = f_j(-x, E).$$

Consider the matrix solutions

$$F_1 = (f_1, f_3), \;\; F_2 = (f_2, f_4), \;\; G_1 = (g_1, g_3), \;\; G_2 = (g_2, g_4).$$

One can express F_1 in terms of G_j, $j = 1, 2$:

$$F_1 = G_2 A + G_1 B,$$

$A = A(E)$, $B = B(E)$ are holomorphic functions of E , $E \in \mathbb{C}$. One can get the Wronskian representations for A and B :

$$A = ih^{-1}\sigma_3 W(G_1, F_1), \quad B = -ih^{-1}\sigma_3 W(G_2, F_1),$$

A admitting a factorization on the even and odd parts :

$$A = -2ih^{-1}\sigma_3 A^- A^+, \quad A^- = F_1^t(0, E), \;\; A^+ = F_{1x}(0, E).$$

The solutions F_j, G_j satisfy the following orthogonal relations

$$\int_{\mathbb{R}} dx F_1^t(x, E)\sigma_3 G_1(x, E') = 2\pi h \sigma_3 A(E)\delta(E - E'),$$

$$\int_{\mathbb{R}} dx F_2^t(x, E)\sigma_3 G_1(x, E') = 0. \tag{2.4.4}$$

2.4.2 Asymptotics of the standard solutions as $a \to 0$

In this subsubsection we describe the asymptotic behavior of the solutions f_j in the limit $a \to 0$. We formulate the results and outline the proofs omitting some technical details of the calculations.

Consider f_3 on the set $\mathcal{D} = \{E, \operatorname{Re} E \ge 0, \operatorname{Im} E \ge -\delta_3 h\}$, where δ_3 is a small positive number. It is not difficult to check that on this set f_3 admits the following asymptotic representation.

Lemma 2.4.1 *As $x \to \infty$,*

$$f_3(x, E) = v^*(x, \lambda_2)\left[e^{\mu x} f_3^0(x, k) + O(h(1 + |E|)^{-1/2} e^{-\frac{\gamma}{h} S_a(hx)})\right],$$

$0 < \gamma < 4$, *uniformly with respect to h in some small vicinity of zero, and $E \in \mathcal{D}$. Here $\mu = \sqrt{E+1}$, $\operatorname{Re}\mu > 0$, $k = \sqrt{E-1}$.*

By the way of explanation we remark that the assertions of Lemma 2.4.1 can be got by combining the standard WKB description of $v(x,\lambda)$ (see appendix 4) and the following representation :

$$f_3(x,E) = v^*(x,\lambda_2)e^{\mu x} f_3^0(x,k) + f_3^1(x,E),$$
$$f_3^1(x,E) = -\int_x^\infty dy K(x,y,E)\sigma_3[\mathcal{R} + V(\tilde{\varphi}(y)) f_3^1(x,E)],$$

where

$$\mathcal{R} = (V(\tilde{\varphi}) - V(\varphi_0))v^*(x,\lambda_2)e^{\mu x} f_3^0 - 2(v^*(x,\lambda_2)e^{\mu x})_x \sigma_3 (f_{3x}^0 + \mu f_3^0),$$
$$|\mathcal{R}| \le ch < x >^3 e^{-4/hS_a(hx)} |v^*(x,\lambda_2)|,$$

uniformly with respect to $E \in \mathcal{D}$, $x \in \mathbb{R}_+$, and h sufficiently small.

To describe the behavior of f_1 we must single out three subsets on the set $\mathcal{D}$:

$$\mathcal{D} = \mathcal{D}_{0,R} \cup \mathcal{D}_{1,R} \cup \mathcal{D}_{2,R},$$
$$\mathcal{D}_{0,R} = \{E,\, |E-1| \ge Rh,\, \arg(1-E) \in (-\delta_4, \delta_4)\} \cap \mathcal{D},$$
$$\mathcal{D}_{1,R} = \{E,\, |E-1| \le Rh\} \cap \mathcal{D}, \quad \mathcal{D}_{2,R} = \mathcal{D} \setminus (\mathcal{D}_{0,R} \cup \mathcal{D}_{1,R}),$$

where δ_4 is a small fixed number, $R > 0$. Proceeding in the same manner as in lemma 2.4.1 one can get the following result.

Lemma 2.4.2 *The solution f_1 admits the following estimates :*

(i) *if $E \in \mathcal{D}_{0,R}$ then*

$$f_1(x,E) = v(x,\lambda_1)\left[e^{ikx} w_1^0(x,k) + O(\frac{h}{|k|} e^{-\frac{\gamma}{h}S_a(hx)})\right],$$

where $k = \sqrt{E-1}$, $\operatorname{Im} k > 0$, provided R is sufficiently large, h is sufficiently small;

(ii) *if $E \in \mathcal{D}_{1,R}$ then*

$$f_1(x,E) = v(x,\lambda_1)\left[w_1^0(x,0) + O_R(h^{1/2} e^{-\frac{\gamma}{h}S_a(hx)})\right].$$

Here γ is the same as in lemma 2.4.1.

To describe the behavior of f_1 on the set $\mathcal{D}_{2,R}$ we use the standard substitution reducing the order of the system (2.4.1) :

$$f_1 = z_0 f_3 + z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.4.5}$$

Setting $z_2 = z_0' f_{32}$ where $f_3 = \binom{f_{31}}{f_{32}}$ we get

$$-z_1'' - (E-1)z_1 - \frac{ax^2}{4} z_1 + V_{11} z_1 + V_{12} z_2 = 0, \tag{2.4.6}$$

$$-z_2' - z_2\frac{v^{*\prime}(x,\lambda_2)}{v^*(x,\lambda_2)} + V_{21}z_1 + V_{22}z_2 = 0.$$

Here

$$V_{11} = V_1(\tilde\varphi) - V_2(\tilde\varphi)\frac{\chi_1}{\chi_2}, \quad V_{12} = \frac{2}{\chi_2^2}(\chi_2'\chi_1 - \chi_1'\chi_2),$$

$$V_{21} = V_2(\tilde\varphi), \quad V_{22} = -\frac{\chi_2'}{\chi_2},$$

$\chi_j = \frac{f_{3j}}{v*(x,\lambda_2)}$, $j = 1,2$, V_1 and V_2 being the components of the potential V : $V = V_1\sigma_3 + iV_2\sigma_2$.

By lemma 2.4.1, this system has smooth coefficients for $x \geq M$ (M sufficiently large) which are holomorphic functions of $E \in \mathcal{D}$.

Let $\vec z^0 = \binom{z_1^0}{z_2^0}$ be the most rapidly decreasing solution of the unperturbed system

$$-z_1'' - k^2 z_1 + V_{11}^0 z_1 + V_{12}^0 z_2 = 0,$$

$$-z_2' + \mu z_2 + V_{21}^0 z_1 + V_{22}^0 z_2 = 0,$$

where

$$V_{11}^0 = V_1(\varphi_0) - V_2(\varphi_0)\frac{\chi_1^0}{\chi_2^0}, \quad V_{12}^0 = \frac{2}{(\chi_2^0)^2}(\chi_2^{0\prime}\chi_1^0 - \chi_1^{0\prime}\chi_2^0),$$

$$V_{21}^0 = V_2(\varphi_0), \quad V_{22}^0 = -\frac{\chi_2^{0\prime}}{\chi_2^0},$$

$\chi_0 = \binom{\chi_1^0}{\chi_2^0}$ being defined by $\chi_0 = e^{\mu x} f_3^0(x,k)$. The solution $\vec z_0$ can be characterized by the following integral equation

$$\vec z^0 = e^{ikx}\begin{pmatrix}1\\0\end{pmatrix} - \int_x^\infty dy \begin{pmatrix} \frac{\sin k(x-y)}{k} & 0 \\ 0 & e^{-\mu(y-x)}\end{pmatrix} \mathbb{V}_0 \vec z^0(y),$$

where $\mathbb{V}_0 = \begin{pmatrix} V_{11}^0 & V_{12}^0 \\ V_{21}^0 & V_{22}^0\end{pmatrix}$. If $k \in \Omega_1$ then for sufficiently large $x \geq M$ a solution $\vec z^0$ is defined that depends smoothly on x, holomorphically on k and admits the asymptotic representation

$$\vec z^0 = e^{ikx}\left[\begin{pmatrix}1\\0\end{pmatrix} + O((1+|k|)^{-1}e^{-4x})\right].$$

It is worth mentioning that the function $z_0^0 f_3^0 + z_1^0\binom{1}{0}$ where $z_0^0 = \int_M^x dy \frac{z_2^0}{f_{32}^0}$ satisfies (2.1.1) and in fact coincides with the solution f_1^0.

Let us return to the complete system (2.4.6). Write $\vec z$ as the sum

$$\vec z = v(x,\lambda_1)e^{-ikx}\vec z^0(x,k) + \vec z^1,$$

where $k = \sqrt{E-1}$, the square root being defined on the complex plane with the cut along negative semi- axes, $\mathrm{Re}\, k > 0$. Then for $\vec{z}^1$ one can write down the following equation

$$\vec{z}^1 = -\int_x^\infty dy \begin{pmatrix} k(x,y,\lambda_1) & 0 \\ 0 & t(x,y,\lambda_2) \end{pmatrix} \left[\mathcal{R}' + \mathbb{V}\vec{z}^1(y)\right],$$

where $t(x,y,\lambda) = \frac{v^*(y,\lambda)}{v^*(x,\lambda)}$, $\mathbb{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$,

$$\mathcal{R}' = (\mathbb{V} - \mathbb{V}_0)e^{-ikx}v(x,\lambda_1)\vec{z}^0(x,k) -$$

$$\begin{pmatrix} 2e^{ikx}(v(x,\lambda_1)e^{-ikx})_x(e^{-ikx}z_1^0)_x \\ v(x,\lambda_1)e^{-ikx}z_2^0(\frac{v_x(x,\lambda_1)}{v(x,\lambda_1)} - ik + \frac{v_x^*(x,\lambda_2)}{v^*(x,\lambda_2)} - \mu) \end{pmatrix}.$$

By lemma 2.4.1, $\mathcal{R}'$ admits the estimate

$$|\mathcal{R}'| \le ch|k|^{-1}(1+|k|)e^{-\frac{\gamma}{h}S_a(hx)}|v|,$$

provided $x \ge M$, $\gamma < 4$.

Using the standard arguments one checks that a solution $\vec{z}^1$ is defined, depends smoothly on x, $x \ge M$, depends holomorphically on $E \in \mathcal{D}_{2,R}$ and admits the estimate

$$|\vec{z}^1| \le ch|k|^{-1}e^{-\frac{\gamma_1}{h}S_a(hx)}|v|.$$

Here R is supposed again to be sufficiently large.

Thus, f_1 admits a representation of the form (2.4.5), where

$$z_0 = -\int_x^\infty dy \frac{z_2}{f_{32}}, \tag{2.4.7}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{-ikx}v\left[\begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} + O(\frac{h}{|k|}e^{-\frac{\gamma}{h}S_a(hx)})\right],$$

provided $x \ge M$.

As a direct consequence of lemmas 2.4.1, 2.4.2 and (2.4.5), (2.4.7) one gets the following asymptotic representations of the matrices $A_\pm$.

For $E \in \mathcal{D}_{0,R}$:

$$\begin{aligned} A^-(E) &= \mathbf{a}(E)\left(\hat{D}_0^-(k) + O(\tfrac{h}{|k|})\right), \\ A^+(E) &= \left(\hat{D}_0^+(k) + O(\tfrac{h}{|k|})\right)\mathbf{a}(E), \ \mathrm{Im}\, k > 0, \end{aligned} \tag{2.4.8}$$

where $\mathbf{a}(E) = \begin{pmatrix} a(\lambda_1) & 0 \\ 0 & a^*(\lambda_2) \end{pmatrix}$, $a(\lambda) = v(0,\lambda)$. Here R is the same as in the first part of lemma 2.4.2.

For $a(\lambda)$ one can write down an explicit expression :

$$a(\lambda) = \left(\frac{2}{h}\right)^{\nu/2} e^{\frac{i\pi\nu}{4}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}.$$

Thus, $a(\lambda)$ has no zeros except for the points $\lambda = -ih(\frac{3}{2} + 2n),\ n = 0, -1, \ldots$.

For $E \in \mathcal{D}_{1,R}$:

$$\begin{aligned} A^-(E) &= \mathbf{a}(E)\left(\hat{D}_0^-(0) + O_R(h^{1/2})\right), \\ A^+(E) &= \left(\hat{D}_0^+(0) + O_R(h^{1/2})\right)\mathbf{a}(E). \end{aligned} \tag{2.4.9}$$

Here we made use of the obvious estimate

$$|\frac{v_x(0,\lambda_1)}{v(0,\lambda_1)}| \le ch^{1/2},$$

provided $E \in \mathcal{D}_{1,R},\ \delta_3 < 3/2$.

It follows from lemma 2.4.1, (2.1.2), (2.4.5), (2.4.7) that

(i) as $|E| \to \infty$, $E \in \mathcal{D}_{2,R}$

$$\begin{aligned} A^-(E) &= \mathbf{a}_1^t(E)\left(I + O(|E|^{-1/2})\right), \\ A^+(E) &= \left(I + O(|E|^{-1/2})\right)(ik\mathbf{p} - \mu\mathbf{q})\mathbf{a}_1(E), \end{aligned} \tag{2.4.10}$$

$\mathbf{p} = \binom{1\,0}{0\,0}$, and $\mathbf{q} = \binom{0\,0}{0\,1}$, uniformly with respect to h sufficiently small;

(ii)

$$\begin{aligned} A^-(E) &= \mathbf{a}_2^t(E)\left(D_0^-(k) + O(\tfrac{h}{|k|})\right), \\ A^+(E) &= \left(D_0^+(k) + O(\tfrac{h}{|k|})\right)\mathbf{a}_2(E) \end{aligned} \tag{2.4.11}$$

uniformly with respect to E in any compact subset of $\mathcal{D}_{2,R}$.

Here $\operatorname{Re} k > 0$,

$$\mathbf{a}_j = \begin{pmatrix} a(\lambda_1) & 0 \\ a_j(E) & a^*(\lambda_2) \end{pmatrix}, \quad j = 1, 2,$$

a_j being holomorphic functions of $E \in \mathcal{D}_{2,R}$.

2.4.3 The point spectrum of $H(a)$

Since H satisfies (2.3.14) the spectrum is invariant under transformations $E \to -E$, $E \to \bar{E}$. It follows from (2.4.2) that the eigenvalues of H lie outside the continuous spectrum. In the upper half plane they are characterized by the equation

$$\det A = 0,$$

zeros of $\det A_+$ ($\det A_-$) corresponding to the eigenvalues of H restricted on the subspace of even (odd) functions.

The zeros of $\det A$ in the closed lower half plane $\{\operatorname{Im} E \leq 0\}$ are called resonances.

It follows directly from (2.4.8)-(2.4.11) and proposition 2.1.5 that for h sufficiently small the number of zeros of $\det A$ in the half plane $\operatorname{Im} E \geq -\delta_3 h$ is finite and if there are any, they belongs to a small vicinity of the point $E = 0$. Moreover, one has the following proposition.

Proposition 2.4.3 *For $a > 0$ sufficiently small, in the half plane $\operatorname{Im} E > -\delta_3 h$ ($\delta_3 > 0$ sufficiently small)*

(i) $\det A_+$ *has only three zeros : $iE_{1,2}(a)$, $iE_R(a)$. They are simple purely imaginary, $E_{1,2} > 0$, $E_R < 0$, and admit the following asymptotic estimates as $a \to 0$:*

$$|iE_2(a) - \lambda(a)| = O(e^{-(2-\epsilon)S_0/h}), \quad E_1, E_R = O(e^{-(1-\epsilon)S_0/h}),$$

$$E_R + E_1 = O(a^{-3/2}e^{-2S_0/h}).$$

(ii) $\det A_-$ *has only one zero which is simple purely imaginary and belongs to a $O(e^{-(2-\epsilon)S_0/h})$ vicinity of $\mu(a)$.*

Here $\lambda(a)$ ($\mu(a)$) is the corresponding eigenvalue of $\tilde{H}(a)$ restricted to the subspace of even (odd) functions :

$$\lambda(a) = i\sqrt{a}(2 + O(a)), \quad \mu(a) = i\sqrt{a}(1 + O(a)), \quad \sqrt{a} > 0.$$

Before starting the proof we mention the following obvious consequence of the above proposition :

(i) the discrete spectrum of $H(a)$ restricted to the subspace of even functions consists of four simple purely imaginary eigenvalues $\pm iE_{1,2}(a)$;

(ii) in the strip $\{E : -\delta_3 h < \operatorname{Im} E \leq 0\}$ the operator $H(a)$ has only one simple resonance $iE_R(a)$.

Proof of proposition 2.4.3. For E in some small vicinity of zero and for $h|x| \leq 2 - \delta_0$ the solution F_1 of (2.4.1) can be expressed in terms of the solutions Ψ_1, Ψ_2, $\Psi_2 = (\psi_2, \psi_4)$ of (2.3.1)

$$F_1 = \Psi_1 T_1 + \Psi_2 T_2, \tag{2.4.12}$$

$$W(\Psi_2, F_1) = W(\Psi_2, \Psi_1)T_1, \quad W(\Psi_1, F_1) = -W(\Psi_2, \Psi_1)T_2.$$

It follows directly from lemmas 2.4.1, 2.4.2 that for $E \in \Upsilon = \{|E| \leq \delta_4, \operatorname{Im} E > -h\delta_3\}$, $\delta_4 > 0$ sufficiently small,

$$\begin{aligned} T_1(E) &= \mathbf{t}_1(E) + O(e^{-\frac{4-\epsilon-O(E)}{h}S_0})\mathbf{a(E)}, \\ T_2(E) &= \mathbf{t}_2(E) + O(e^{-\frac{6-\epsilon-O(E)}{h}S_0})\mathbf{a(E)}, \end{aligned} \tag{2.4.13}$$

where

$$\mathbf{t}_i(E) \begin{pmatrix} t_i(E) & 0 \\ 0 & \overline{t_i(-\bar{E})} \end{pmatrix},$$

$$t_1(E) = \frac{w(u_2(\lambda_1), v(\lambda_1))}{w(u_2(\lambda_1), u_1(\lambda_1))} = (-\lambda_1)^{1/4} a(\lambda_1)(1 + O(h)),$$

$$t_2(E) = -\frac{w(u_1(\lambda_1), v(\lambda_1))}{w(u_2(\lambda_1), u_1(\lambda_1))}$$

$$= \frac{a}{4w(u_2(\lambda_1), u_1(\lambda_1)} \int_0^\infty dx x^2 (1-\theta) v(\lambda_1) u_1(\lambda_1)$$

$$= O(e^{-\frac{2-\epsilon-O(E)}{h} S_0}) a(\lambda_1). \tag{2.4.14}$$

Here we used the WKB representation (2.3.4) of u_1 and a similar one of v (see appendix 4).

The above representation imply the equivalence between the equations

$$\det A_+ = 0$$

and

$$\Phi(E) = \det[D^+(E) + \Psi_{2x}(0, E) T_0(E)] = 0, \quad T_0 = T_2 T_1^{-1}.$$

By (2.4.13), (2.4.14),

$$T_0 = \mathbf{t}_0 + O(e^{-\frac{6-\epsilon-O(E)}{h} S_0}), \quad \mathbf{t}_0(E) = \begin{pmatrix} t_0(E) & 0 \\ 0 & \overline{t_0(-\bar{E})} \end{pmatrix}, \tag{2.4.15}$$

where $t_0(E) = t_2(E) t_1^{-1}(E)$. The zeros of $\det A_-$ are characterized by a similar equation, D^+ being replaced by D^- and Ψ_{2x} by Ψ_2^t.

The asymptotic estimates (2.4.13)-(2.4.15) together with the analytic properties of $D^\pm$ implies directly that in Υ

(i) $\det A_+(E)$ has only three zeros (counted with their multiplicity) : one (E_2) is in a $O(e^{-\frac{2-\epsilon}{h} S_0})$ vicinity of $\lambda(a)$, two others belong to a $O(e^{-\frac{1-\epsilon}{h} S_0})$ vicinity of the point $E = 0$;

(ii) $\det A_-(E)$ has only one zero E_3 which belongs to a $O(e^{-\frac{2-\epsilon}{h} S_0})$ vicinity of $\mu(a)$.

Since

$$A_\pm(E) = \sigma_1 \overline{A_\pm(-\bar{E})} \sigma_1, \tag{2.4.16}$$

$E_{2,3}$ are purely imaginary.

Clearly, the zeros of $\det A_+$ that are exponentially close to the point $E = 0$ can be characterized (asymptotically) by the equation :

$$\det[D^+(E) + \Psi_{2x}(0, 0) \mathbf{t}_0(0)] = 0.$$

Taking into account the structure of the root subspace of $\tilde{H}(a)$ corresponding to the zero eigenvalue one can rewrite (again asymptotically) the above equation as follows :

$$\kappa E^2 + 2\gamma_2 \gamma_3 \mathrm{Re}\, t_0(0) = 0, \tag{2.4.17}$$

where κ have been introduced in subsubsection 2.3.2 and $\gamma_{2,3}$ are defined by the relations :

$$\gamma_2\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\Psi_{1x}(0,0)\begin{pmatrix}1\\1\end{pmatrix},\quad \gamma_3\begin{pmatrix}1\\-1\end{pmatrix}=\frac{1}{\sqrt{2}}\Psi_{2x}(0,0)\begin{pmatrix}1\\-1\end{pmatrix}.$$

It follows from (2.4.13), (2.4.14) and the WKB representations of u_i, $i=1,2$, and v that

$$\operatorname{Re} t(0)\geq c\int_0^\infty dy(1-\theta(hy))e^{-\frac{1}{h}\mathcal{S}(hy)}, \tag{2.4.18}$$

with some positive constant c. Here

$$\mathcal{S}(\xi)=\int_0^\xi ds(\sqrt{1-s^2\theta(s)/4}+\sqrt{(1-s^2/4)_+}).$$

By (2.3.7), (2.3.21), (2.3.23), (2.3.25),

$$\gamma_2=d(0,a)(\gamma_1+O(a)),\quad \gamma_3=\frac{\sqrt{2}d_0(0)\varphi_\infty}{\varphi_0(0)}+O(h). \tag{2.4.19}$$

The formulas (2.4.15), (2.4.17)-(2.4.19) imply the existence of two simple zeros $iE_1,\ iE_R$ of $\det A_+$,

$$E_1,\,E_R=\pm\sqrt{\frac{2\operatorname{Re}t_0\gamma_2\gamma_3}{\kappa}}+O(e^{(2-\epsilon)S_0/h})=\pm\sqrt{\frac{2\operatorname{Re}t_0}{ea}}\varphi_\infty(1+O(h)). \tag{2.4.20}$$

By (2.4.16), they are purely imaginary.

The expression E_1+E_R can be calculated as follows.

$$E_1+E_R=i\frac{\Phi'(0)}{\Phi''(0)}+O(e^{(3-\epsilon)S_0/h}).$$

By (2.3.18), (2.4.13)-(2.4.15),

$$\Phi''(0)=2\kappa(a)+O(e^{(2-\epsilon)S_0/h}). \tag{2.4.21}$$

For $\Phi'(0)$ the direct calculations give

$$\Phi'(0)=-i\frac{S_0\gamma_2\gamma_3}{2h}e^{-2S_0/h}(1+O(h)). \tag{2.4.22}$$

Combining (2.3.19), (2.3.20) and (2.4.19), (2.4.21), (2.4.22) one gets

$$E_1+E_R=\frac{\kappa_2}{h^3}e^{-2S_0/h}(1+O(h)),$$

$$\kappa_2=\frac{S_0\varphi_\infty^2}{4e}.$$

Let $\zeta_1(x,a)$ and $\zeta_2(x,a)$ be eigenfunctions corresponding to the eigenvalues iE_1 and iE_2 respectively. Let $\zeta_R(z,a)$ be a resonant function associated to the resonance iE_R :

$$H\zeta_R = iE_R\zeta_R,$$
$$\zeta_R \sim e^{\frac{ihx^2}{4}\sigma_3}|x|^{-\frac{1}{2}-\frac{E_R+i\sigma_3}{h}}\vec{c},$$

as $|x| \to \infty$. Here $\vec{c}$ is a constant vector. Clearly ζ_j, $j = 1, 2, R$, can be normalized by the conditions :

$$\left\langle \zeta_j, \tilde{\zeta}_0 \right\rangle = \left\langle \tilde{\zeta}_0, \tilde{\zeta}_0 \right\rangle, \quad j = 1, 2, R.$$

The following lemma is an immediate consequence of (2.4.12)-(2.4.14), lemmas 2.4.1, 2.4.2 and (2.3.23)-(2.3.25).

Lemma 2.4.4 ζ_j, $j = 1, 2, R$, *admit the estimates*

$$|\zeta_j - \tilde{\zeta}_0 - E_j\tilde{\zeta}_1| \le ce^{-(2-\epsilon)S_0/h}e^{\frac{1}{h}\int_0^{h|x|} ds\sqrt{(1-s^2/4)_+}} < x >^{-\frac{1}{2}-\frac{E_j}{h}}, \quad j = 1, R,$$

$$|\zeta_2 - \tilde{\zeta}_2| \le ce^{-(2-\epsilon)S_0/h}e^{\frac{1}{h}\int_0^{h|x|} ds\sqrt{(1-s^2/4)_+}} < x >^{-\frac{1}{2}-\frac{E_2}{h}},$$

where $\tilde{\zeta}_2 = \tilde{\zeta}_2(a)$ *is the eigenfunction of* $\tilde{H}(a)$ *corresponding to the eigenvalue* $\lambda(a)$*, normalized by the condition*

$$\left\langle \tilde{\zeta}_2, \tilde{\zeta}_0 \right\rangle = \left\langle \tilde{\zeta}_0, \tilde{\zeta}_0 \right\rangle.$$

Let us mention that $\tilde{\zeta}_2(a)$ introduced here differs a little bit from that of subsection 2.3.

As a consequence of lemmas 2.4.1, 2.4.2, 2.4.4 and the representations (2.4.12), (2.4.13) one can get the estimates of the operators $P(a)$, $Q(a)$ announced in proposition 1.2.6.

2.4.4 The resolvent of $H(a)$

The resolvent $R(E) = (H - E \cdot I)^{-1}$, $\operatorname{Im} E > 0$, of H is an integral operator with 2×2 matrix kernel

$$G(x,y,E) = \begin{cases} F_1(x,E)\mathcal{D}^{-1}G_1^t(y,E)\sigma_3, & y \le x, \\ G_1(x,E)\mathcal{D}^{t^{-1}}F_1^t(y,E)\sigma_3, & x \le y, \end{cases}$$

where $\mathcal{D} = W(G_1, F_1) = -ih\sigma_3 A$, the resolvent kernel in the lower half plane $\operatorname{Im} E < 0$ being given by $\overline{G(x,y,\bar{E})}$.

The kernel G is a meromorphic function of E on the complex plane and its poles in the upper (lower) half plane coincide with the zeros of $\det A$, i.e., with the eigenvalues (resonances) of H. It follows from the estimates (2.4.2) for the solutions F_1 and G_1 that for $\operatorname{Im} E > 0$ and away from the zeros of A the kernel G determines a bounded operator in L_2.

The formula for the resolvent makes it easy to construct on the continuous spectrum a complete system of generalized eigenfunctions. Let $\mathcal{F}$, $\mathcal{G}$ be solutions of the scattering problem :

$$\mathcal{F} = F_1 A^{-1}, \quad \mathcal{G} = G_1 A^{-1},$$

$$\mathcal{F}(x,E) \sim e^{\frac{ihx^2}{4}\sigma_3} x^{-\frac{1}{2}+\frac{i}{h}(E-\sigma_3)} A^{-1}, \quad x \to +\infty,$$

$$\mathcal{F}(x,E) \sim e^{-\frac{ihx^2}{4}\sigma_3} |x|^{-\frac{1}{2}-\frac{i}{h}(E-\sigma_3)} + e^{\frac{ihx^2}{4}\sigma_3} |x|^{-\frac{1}{2}+\frac{i}{h}(E-\sigma_3)} BA^{-1}, \quad x \to -\infty.$$

By proposition 2.4.3, $\mathcal{F}$, $\mathcal{G}$ are meromorphic functions in the strip $-h\delta_3 < \operatorname{Im} E < h\delta_3$ with the only poles at iE_R, iE_2 which are simple.

The relations (2.4.4) imply the orthonormality of the scattering problem solutions :

$$\begin{aligned} &\frac{1}{2\pi h}\int_{\mathbb{R}} dx \mathcal{F}^*(x,E)\sigma_3 \mathcal{F}(x,E') = \delta(E-E')\sigma_3, \\ &\frac{1}{2\pi h}\int_{\mathbb{R}} dx \mathcal{G}^*(x,E)\sigma_3 \mathcal{G}(x,E') = \delta(E-E')\sigma_3, \\ &\frac{1}{2\pi h}\int_{\mathbb{R}} dx \mathcal{F}^*(x,E)\sigma_3 \mathcal{G}(x,E') = 0. \end{aligned} \tag{2.4.23}$$

It is easy to express the jump of the resolvent on the continuous spectrum in terms of the solutions $\mathcal{F}$, $\mathcal{G}$:

$$\frac{1}{2\pi i}\left(G(x,y,E+i0) - G(x,y,E-i0)\right) =$$

$$\frac{1}{2\pi h}[\mathcal{F}(x,E)\sigma_3\mathcal{F}^*(y,E) + \mathcal{G}(x,E)\sigma_3\mathcal{G}^*(y,E)]\sigma_3. \tag{2.4.24}$$

Introduce the operators $\mathbb{F}$, $\mathbb{G}$: $L_2(\mathbb{R}\to\mathbb{C}^2) \to L_2(\mathbb{R}\to\mathbb{C}^2)$:

$$(\mathbb{F}\vec{\Phi})(x) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} dE \mathcal{F}(x,E)\Phi(E),$$

$$(\mathbb{G}\vec{\Phi})(x) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} dE \mathcal{G}(x,E)\Phi(E).$$

The action of the adjoint operators $\mathbb{F}^*$, $\mathbb{G}^*$ is given by

$$(\mathbb{F}^*\psi)(E) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} dx \mathcal{F}^*(x,E)\psi(x),$$

$$(\mathbb{G}^*\psi)(E) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} dx \mathcal{G}^*(x,E)\psi(x).$$

Proposition 2.4.5 $\mathbb{F}$ *is a bounded operator. Moreover,*

(i) *for* $e^{-ih\frac{x^2}{4}\sigma_3} f \in H^1$, $(\mathbb{F}^* f)(E)$ *is a meromorphic function of* E *in the strip* $-b_0 h < \operatorname{Im} E \le 0$ *with the only pole in* $-iE_2$ *and satisfies the estimate*

$$\|\mathbb{F}^* f\|_{L_2(\mathbb{R}-ibh)} \le ch^{-K_1} \|e^{-i\frac{hx^2}{4}\sigma_3} f\|_{H^1}, \;\; h^L \le b < b_0,$$

(ii) *let us introduce the operators* $\mathbb{F}_b$ *:*

$$(\mathbb{F}_b \Phi)(x) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} dE \mathcal{F}(x, E - ibh)\Phi(E).$$

For $h^L \le b < b_0$, *they satisfy the inequality*

$$\| \langle x \rangle^{-\nu_2} \mathbb{F}_b \Phi\|_2 \le ch^{-K_2} \|\Phi\|_2, \quad \nu_2 > 1/2,$$

provided b_0 *is sufficiently small. Here* K_j, $j = 1, 2$, *depend on* L *but do not depend on* a.

The same is true for the operator $\mathbb{G}$.

Proof. This proposition is a direct consequence of the similar estimates related to the unperturbed operator $H^0(a)$, $H^0(a) = (-\partial_x^2 + 1 - \frac{ax^2}{4})\sigma_3$, lemmas 2.4.1 and 2.4.2, the representation (2.4.5), (2.4.7) and proposition 2.4.3. To illustrate the arguments used we prove here the estimates for $\mathbb{F}$, the part (i) can be obtained in a similar manner. We start by remarking that in the free case ($V = 0$) the above proposition is an immediate consequence of the explicit factorization of the corresponding operators $\mathbb{F}_0$, $\mathbb{F}_0^*$ in terms of the Fourier transform :

$$\mathcal{F}_0(x, E, a) = \frac{1}{\sqrt{\pi}} \left(\frac{h}{2}\right)^{\frac{1}{4} + i\frac{E-\sigma_3}{2h}} e^{i\frac{hx^2}{4}\sigma_3 + i\frac{\pi}{4}\sigma_3} \int_0^\infty d\rho e^{i\rho^2\sigma_3 + i\sqrt{2h}x\rho\sigma_3} \rho^{-\frac{1}{2} - i\frac{E-\sigma_3}{h}}. \tag{2.4.25}$$

Here $\mathcal{F}_0$ is the solution of the scattering problem associated to the operator $H^0(a)$. This representation implies, in particular, the unitary property of $\mathbb{F}_0$ and the estimates

$$\| \langle x \rangle^{-\nu_2} \mathbb{F}_{0b}\Phi\|_2 \le ch^{b/2} \|\Phi\|_2, \tag{2.4.26}$$

$$\|\mathbb{F}_0^* f\|_{L_2(\mathbb{R}-ibh)} \le ch^{-b/2} \|e^{-ih\frac{x^2}{4}\sigma_3} f\|_{H^1},$$

provided $0 \le b < \frac{1}{2}$.

To take into account the perturbation V we use the representation

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1, \quad \mathcal{F}_1 = -(H^0 - E)_+^{-1} V\mathcal{F}.$$

Here $(H^0 - E)_+^{-1}$ stands for the meromorphic continuation of the resolvent $(H^0 - E)^{-1}$ from the upper half-plane into the lower half-plane.

Using lemmas 2.4.1 and 2.4.2, the representation (2.4.5), (2.4.7) and proposition 2.4.3 it is not difficult to prove the estimate

$$e^{-\gamma|x|}|\mathcal{F}(x,E,a)|,\ e^{-\gamma|x|}|\mathcal{F}_E(x,E,a)| \le ch^{-K}e^{-\gamma_1|x|}(1+|E|)^{-\frac{1}{4}+\frac{\operatorname{Im}E}{2h}}, \quad (2.4.27)$$

$h^L \le |\operatorname{Im} E| \le h\delta_3$,

$$e^{-\gamma|x|}|\mathcal{F}(x,E,a)| \le c(h)e^{-\gamma_1|x|}(1+|E|)^{-1/4}, \quad E\in\mathbb{R}. \quad (2.4.28)$$

Here $\gamma > \gamma_1 > 0$, K is a positive constant depending only on L.

Combining (2.4.27) with the obvious estimates of the free operator :

$$\|(H^0-E)_+^{-1}f\|_\infty \le ch^{-1/2}(1+|E|)^{-1/2}\|\langle x\rangle^M f\|_\infty, \quad |\operatorname{Im}E| \le \frac{h}{2}, \quad (2.4.29)$$

where M is a positive constant independent of h and λ, one gets

$$\|\int_{\mathbb{R}} dE\mathcal{F}_1(x,E-ibh)\Phi(E)\|_\infty \le ch^{-K-1/2}\|\Phi\|_2, \quad (2.4.30)$$

$h^{L-1} \le b \le \min(1/2,\delta_3)$,

$$\|\int_{\mathbb{R}} dE\mathcal{F}_1(x,E)\Phi(E)\|_\infty \le c(h)\|\Phi\|_2, \quad (2.4.31)$$

The inequalities (2.4.26), (2.4.30) lead to the desired estimate for $\mathbb{F}_b$.

To estimate L_2-norm of the integral $\int_{\mathbb{R}} dE\mathcal{F}_1(x,E)\Phi(E)$ the following refinement of (2.3.29) is needed :

$$\|(l(a)-\lambda-i0)^{-1}f\|_\infty \le c|\lambda|^{-1}\|\langle x\rangle^M f\|_\infty, \quad \lambda\le -1, \quad (2.4.32)$$

$$\left|(l(a)-\lambda-i0)^{-1}f(x)+\frac{v(x,\lambda)}{2v(0,\lambda)v_x(0,\lambda)}\int_{\mathbb{R}} dyv(-y,\lambda)f(y)\right| \le ch^{-1/2}\langle\lambda\rangle^{-1/2}\langle x\rangle^{-\alpha}\|\langle y\rangle^M f\|_\infty, \quad (2.4.33)$$

$l(a) = -\partial_x^2 - \frac{ax^2}{4}$. In the second estimate $hx \ge 2(-\lambda)_+^{1/2},\ \lambda\in\mathbb{R}$, α is arbitrary, M depends on α. By the way of the explanation we remark that these estimates as well as (2.4.29) can be got easily by combining the explicit representation of the resolvent $(l(a)-\lambda-i0)^{-1}$ in terms of $v(x,\lambda)$ with the corresponding properties of the Weber functions, see [B] and appendix 4.

Since $\mathcal{F}(x,E) = \sigma_1\overline{\mathcal{F}(x,-E)}\sigma_1$, $E\in\mathbb{R}$, it is sufficient to consider the integral

$$I = \int_0^\infty dE\mathcal{F}_1(x,E)\Phi(E).$$

By (2.4.31),

$$\int_{h|x|\le 4} dx|I|^2 \le c(h)\|\Phi\|_2^2. \quad (2.4.34)$$

To estimate I in the region $h|x| \geq 4$ we break it into two terms :

$$I = I_1 + I_2, \quad I_1 = \mathbf{p}I, \quad I_2 = \mathbf{q}I.$$

Consider I_1. Using (2.4.28), (2.4.33), the boundedness of $\mathbb{F}_0$ and the obvious estimate (see appendix 4, (A4.1), (A4.4))

$$|v(0,\lambda)v_x(0,\lambda)| \leq c(h), \quad \lambda \geq -1,$$

one gets immediately

$$\int\limits_{hx\geq 4} dx|I_1|^2 \leq c(h)\|\Phi\|_2^2.$$

The same estimate is valid in the region $hx \leq -4$. Thus,

$$\int\limits_{h|x|\geq 4} dx|I_1|^2 \leq c(h)\|\Phi\|_2^2. \tag{2.4.35}$$

Consider I_2. We represent it as the sum $I_2 = I_{21} + I_{22}$, $I_{21} = \int_0^{\frac{h^2x^2}{16}-1} dE$, $I_{22} = \int_{\frac{h^2x^2}{16}-1}^{\infty} dE$.

By (2.4.28), (2.4.32),

$$|\mathbf{q}\mathcal{F}_1(x,E)| \leq c(h)\langle E\rangle^{-5/4}, \quad E \geq 0, \ \ x \in \mathbb{R},$$

which allows us to get for I_{22}

$$\int\limits_{h|x|\geq 4} dx|I_{22}|^2 \leq c(h)\|\Phi\|_2^2. \tag{2.4.36}$$

To estimate I_{21} in the region $hx \geq 4$ we combine (2.4.33) with the following estimate of v (see appendix 4, (A4.1))

$$|v(x,\lambda) - e^{i\frac{hx^2}{4}}x^{-1/2+i\lambda/h}| \leq c|\lambda|^2h^{-3}x^{-5/2},$$

$\lambda \leq -1$, $hx \geq |\lambda|^{1/2}(2+\delta)$, $\delta > 0$. As a result, one gets the representation

$$\mathbf{q}\mathcal{F}_1(x,E) = e^{-i\frac{hx^2}{4}}x^{-1/2+i(E+1)/h}\mu(E) + \mathcal{R}_2,$$

where

$$\mu(E) = \int dy \frac{\overline{v(-y,\lambda_2)}}{2v(0,\lambda_2)v_x(0,\lambda_2)}\mathbf{q}V\mathcal{F}(y,E),$$

$\mathcal{R}_2$ admits the estimate

$$|\mathcal{R}_2| \leq c(h)x^{-5/2}[(E+1)^{-3/4} + (E+1)^2|\mu(E)|], \tag{2.4.37}$$

provided $hx \geq 4(E+1)^{1/2}$, $E \geq 0$.

The function μ can be estimated as follows.

$$|\mu(E)| \leq c(h)e^{-\gamma\frac{(E+1)^{1/2}}{h}}, \tag{2.4.38}$$

with some $\gamma > 0$. Here we have used (2.4.28) and the following estimate of v

$$e^{-\gamma|x|}\left|\frac{v(x,\lambda)}{v(0,\lambda)v_x(0,\lambda)}\right| \leq ce^{-\gamma'\frac{|\lambda|^{1/2}}{h}}, \tag{2.4.39}$$

$-\lambda \geq \delta > 0$, γ' is a positive constant depending only on δ and γ. (2.4.39) is an immediate consequence of the WKB representations of v, see appendix 4, (A4.2)-(A4.4).

It follows from (2.4.35), (2.4.36) that for $hx \geq 4$,

$$I_{21} = e^{-i\frac{hx^2}{4}}x^{-1/2+i/h}\int_0^\infty dEx^{iE/h}\mu(E)\Phi(E) + O_h(\|\Phi\|_2 x^{-5/2}).$$

As a consequence,

$$\int_{hx\geq 4} dx|I_{21}|^2 \leq c(h)\|\Phi\|_2^2. \tag{2.4.40}$$

In a similar way one can obtain

$$\int_{hx\leq -4} dx|I_{21}|^2 \leq c(h)\|\Phi\|_2^2. \tag{2.4.41}$$

Combining (2.4.34)- (2.4.36), (2.4.40), (2.4.41) one gets finally :

$$\|I\|_2 \leq c(h)\|\Phi\|_2,$$

which implies the boundedness of the operator $\mathbb{F}$.

Since

$$\hat{\mathcal{F}}(z,E,a) = \mathcal{F}(h^{-1/2}z, hE, a)h^{-\frac{1}{4}-\frac{i}{2}(E-\hat{E}_0\sigma_3)}, \tag{2.4.42}$$

proposition 2.4.4 implies immediately the corresponding inequalities of proposition 1.2.7.

In order to prove the estimates for the derivative $\hat{\mathbb{F}}_a$ one can use the following representation

$$(\hat{\mathbb{F}}_a^*\vec{g})(E) = -\hat{E}_{0a}\frac{d}{dE}(\hat{\mathbb{F}}^*\sigma_3\vec{g})(E) + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dy\mathcal{F}_2^*(y,E)\vec{g}(y),$$

where

$$\mathcal{F}_2(E) = -(\hat{H}-E)^{-1}\left[\hat{E}_{0a}[\sigma_3,\hat{W}]\hat{\mathcal{F}}_E(E) + \hat{W}_a\hat{\mathcal{F}}(E)\right], \mathrm{Im}\, E > 0.$$

The desired inequalities follows then directly from proposition 2.4.4, the estimate (2.4.27) and (2.4.42).

Introduce the operator $\mathbb{E} : L_2(\mathbb{R} \to \mathbb{C}^2) \times L_2(\mathbb{R} \to \mathbb{C}^2) \to L_2(\mathbb{R} \to \mathbb{C}^2)$:

$$\mathbb{E}\vec{\Phi} = \mathbb{F}\Phi_1 + \mathbb{G}\Phi_2, \quad \vec{\Phi} = (\Phi_1, \Phi_2).$$

In terms of $\mathbb{E}$ the orthonormality conditions (2.4.23) mean

$$\mathbb{E}^*\sigma_3\mathbb{E}\hat{\sigma}_3 = I.$$

The formula for the jump in the resolvent leads to a relation meaning that the scattering problem solutions form a complete system of eigenfunctions of the continuous spectrum of H :

$$\mathbb{E}\hat{\sigma}_3\mathbb{E}^*\sigma_3 = P^c,$$

where $\hat{\sigma}_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, P^c being the spectral projection onto the subspace of the continuous spectrum.

The operator $\mathbb{E}$ realizes a linear equivalence between the restriction of H to the continuous spectrum and the multiplication by E :

$$HP^c = \mathbb{E}E\hat{\sigma}_3\mathbb{E}^*\sigma_3.$$

Moreover, for any bounded continuous function φ we have

$$\varphi(H)P^c = \mathbb{E}\varphi(E)\hat{\sigma}_3\mathbb{E}^*\sigma_3.$$

Appendix 1

Here we prove proposition 2.1.2. By (1.1.8) it suffices to consider the point $E = 1$. Let the equation $(\mathcal{L}_0 - 1)\psi = 0$ have a bounded solution ψ, $\psi \notin L_2$. Then the same is true for the operator T_0 : there exists ψ_0 such that

$$T_0\psi_0 = \psi_0, \quad \psi_0 = C_\pm(1 + O(e^{\mp\gamma x})), \ x \to \pm\infty, \tag{A1.1}$$

where $\gamma > 0$, $|C_-| + |C_+| > 0$. Obviously, $(\psi_0, \varphi_0) = 0$. One can consider ψ_0 be real and either odd or even. We normalize ψ_0 in such a way that $C_+ = 1$.

Introduce a truncated resonant function ψ_0^ϵ :

$$\psi_0^\epsilon(x) = \Theta(\epsilon x)\psi_0 + \mu(\epsilon)\varphi_0, \quad \mu(\epsilon) = -\frac{(\Theta\psi_0, \varphi_0)}{\|\varphi_0\|_2^2},$$

where $\epsilon > 0$ is small, Θ is even, $\Theta \in C_0^\infty$, $\Theta(\xi) = 1$ in some vicinity of zero. Clearly,

$$(\psi_0^\epsilon, \varphi_0) = 0, \ |\mu(\epsilon)| \le ce^{-\gamma/\epsilon}, \ \gamma > 0.$$

The direct calculations show

$$\|\psi_0^\epsilon\|_2^2 = \epsilon^{-1}M_0 + M_1 + O(e^{-\gamma/\epsilon}), \quad M_0 = \|\Theta\|_2^2,\ M_1 = \int_{\mathbb{R}} dx(|\psi_0|^2 - 1),$$

$$(L_{0+}\psi_0^\epsilon, \psi_0^\epsilon) = \epsilon^{-1}M_0 + M_1 + M_2 + O(\epsilon), \quad M_2 = ((L_+ - 1)\psi_0, \psi_0). \qquad \text{(A1.2)}$$

As in the proof of proposition 2.1.2 we consider the quotient $\frac{(Au,u)}{(u,u)}$, $A = PL_{0+}P$, $u \in F$, $F = \mathcal{L}\{\psi_0^\epsilon, \eta_0, \xi_j,\ j = 0, 1\}$. It is clear that $\dim F = 4$.

It follows from (A1.2) that

$$\frac{(Au,u)}{(u,u)} \le (1 + \epsilon\frac{M_2}{M_0} + O(\epsilon^2)) \max_{x\in\mathbb{C}^3} \frac{|x_1|^2}{< (I+B)x, x >_{\mathbb{C}^3}},$$

where

$$B = \begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & 0 & 0 \\ b_2 & 0 & 0 \end{pmatrix},\ b_j = \frac{(\psi_0^\epsilon, e_j)}{\|\psi_0^\epsilon\|_2}, \quad e_j = \left\{ \begin{array}{ll} \frac{\eta_0}{\|\eta_0\|_2}, & j = 1 \\ \frac{\xi_1}{\|\xi_1\|_2}, & j = 2 \end{array} \right. ,$$

$$b_j = \epsilon^{1/2}(M_0^{-1/2}(\psi_0, e_j) + O(\epsilon)), \quad j = 1, 2.$$

It is easy to check that

$$\max_{x\in\mathbb{C}^3} \frac{|x_1|^2}{< (I+B)x, x >_{\mathbb{C}^3}} = \frac{1}{1 - b_j^2}, \quad j = \left\{ \begin{array}{l} 1 \text{ if } \psi_0 \text{ is odd,} \\ 2 \text{ if } \psi_0 \text{ is even} \end{array} \right. .$$

Thus,

$$\frac{(Au,u)}{(u,u)} \le (1 + \epsilon\frac{\kappa_j}{M_0} + O(\epsilon^2)), \quad \kappa_j = M_2 + (\psi_0, e_j)^2.$$

Consider κ_j. Clearly,

$$\kappa_j = (f, \psi_0) + (f, e_j)^2 \le (f, \psi_0 + f),$$

where $f = (PL_{0+} - 1)\psi_0$, f is a real smooth function decreasing exponentially as $|x| \to \infty$, $(f, \varphi_0) = 0$. By (A1.1),

$$(f, \psi_0 + f) = -((L_{0-} - 1)^{-1}f, f).$$

Since L_{0-} has no resonances at the end point $E = 1$ of the continuous spectrum the expression $((L_{0-} - 1)^{-1}f, f)$ is well defined and positive since $(f, \varphi_0) = 0$. Thus,

$$\kappa_j < 0,\ j = 1, 2.$$

This means that for ϵ sufficiently small

$$\frac{(Au,u)}{(u,u)} < 1,$$

provided $u \in F$, which contradicts to the fact that the number of the eigenvalues of A counted with their multiplicity is equal three.

Appendix 2

Here we prove proposition 1.2.6. Using the obvious estimate

$$|(l(a)+1-i0)^{-1}(x,y)| \le ch^{-1/3}e^{-\frac{1}{h}|S(hx)-S(hy)|},$$

and the inequality (ii) of proposition 1.2.1. one gets immediately

$$\|f^0(a)\|_\infty \le ce^{-(1-\epsilon)\frac{S_0}{h}}, \quad \|\tilde\varphi(a)f^0(a)\|_\infty \le ce^{-(2-\epsilon)\frac{S_0}{h}}.$$

By (1.2.9), the expression $G_0(a)$ can be estimated as follows.

$$\|G_0(a)\| \le c\sum_{j=0,\dots,3} a\int dx x^2(1-\theta(hx))|\vec e_j|(\tilde\varphi+|f^0(a)|) \le ce^{-(2-\epsilon)\frac{S_0}{h}}.$$

Here we also made use of propositions 1.2.1, 1.2.2.

Consider G_0^3 :

$$G_0^3 = \frac{1}{2(\tilde\varphi_a,\tilde\varphi)}\left\langle \frac{az^2}{4}(\theta-1)\bar f^0, \tilde\varphi\begin{pmatrix}1\\-1\end{pmatrix}\right\rangle =$$

$$\frac{1}{(\tilde\varphi_a,\tilde\varphi)}\mathrm{Im}\,(\frac{az^2}{4}(\theta-1)f^0,\tilde\varphi) = -\frac{1}{(\tilde\varphi_a,\tilde\varphi)}\lim_{R\to+\infty}\mathrm{Im}\int_R^R dy f^0\cdot\overline{l(a)f^0} =$$

$$\frac{2}{(\tilde\varphi_a,\tilde\varphi)}\lim_{R\to+\infty}\mathrm{Im}\,\left(\bar f^{0\prime}(R)f^0(R)\right) = -\frac{h}{(\tilde\varphi_a,\tilde\varphi)}|\kappa|^2, \tag{A2.1}$$

where κ can be characterized by the asymptotic representation

$$f^0 = e^{\frac{ihz^2}{4}}|z|^{-1/2-i/h}(\kappa+o(1)), \quad z\to\infty.$$

It is not difficult to check that

$$\kappa = \frac{1}{w(\psi_-,\psi_+)}\int_{\mathbb{R}} dy\psi_-(y)\frac{ay^2}{4}(1-\theta(hy))\tilde\varphi(y) = \frac{1}{w(\psi_-,\psi_+)}\int_{\mathbb{R}} dy\psi_-(y)\tilde\varphi^5(y).$$

Here $\psi_\pm$ is a solution of the equation $(l(a)+1)\psi=0$, characterized by the following behavior at $\pm\infty$:

$$\psi_\pm = e^{\frac{ihz^2}{4}}|z|^{-1/2-i/h}(1+o(1)), \quad z\to\pm\infty.$$

Using the standard WKB descriptions of $\psi_\pm$, see appendix 4, and proposition 1.2.1 one can easily check that as $a\to 0$, $\Gamma^{-1}h|\kappa|^2$ admits an asymptotic expansion in powers of a :

$$|\kappa|^2 = h^{-1}e^{-\frac{2S_0}{h}}\sum_{n\ge 0}k_na^n, \quad k_0 = \frac{1}{2}\left(\int dye^y\varphi_0^5(y)\right)^2 = 2\varphi_\infty^2. \tag{A2.2}$$

Combining (A2.1) and (A2.2) one gets the following asymptotic (as $a \to +0$) representation of G_0^3 :

$$G_0^3 = e^{-\frac{2S_0}{h}} \sum_{n\geq 0} G_{0k}^3 a^k, \quad G_{00}^3 = -\frac{k_0}{(\varphi_0, \varphi_1)} = -\frac{2\varphi_\infty^2}{e} < 0.$$

This asymptotic expansion can be differentiated any numbers of time with respect to a.

To estimate the Fourier transform $\hat{\tilde{f}}^0$ of $\tilde{f}^0 = e^{-\frac{ihz^2}{4}} f^0$ we use the representation :

$$\hat{\tilde{f}}^0(p) = -\frac{i}{h} \int_{|p|}^{\infty} ds e^{\frac{i}{2h}(p^2-s^2)} \left|\frac{p}{s}\right|^{i/h} \frac{\hat{\tilde{F}}_0(s)}{|p|^{1/2}|s|^{1/2}}, \tag{A2.3}$$

where

$$\hat{\tilde{F}}_0 = \widehat{e^{-\frac{ihz^2}{4}} F_0}.$$

This representation gives immediately

$$\|\hat{\tilde{f}}^0\|_1 \leq ch^{-1} \|\hat{\tilde{F}}_0\|_1 \leq ch^{-1} \|e^{-\frac{ihz^2}{4}} F_0\|_{H^1} \leq ce^{-(1-\epsilon)\frac{S_0}{h}}.$$

Consider $(z\partial_z + \frac{1}{2})\tilde{f}^0$. Using the representation $\widehat{(z\partial_z + \frac{1}{2})\tilde{f}^0} = -\frac{i}{h}[(p^2+1)\hat{\tilde{f}}^0 + \hat{\tilde{F}}_0]$, and taking into account (A2.3) one gets

$$\|\widehat{(z\partial_z + \frac{1}{2})\tilde{f}^0}\|_1 \leq ch^{-2} \| \langle p \rangle^2 \hat{\tilde{F}}_0\|_1$$

$$\leq ch^{-2} \|e^{-\frac{ihz^2}{4}} F_0\|_{H^3} \leq ce^{-(1-\epsilon)\frac{S_0}{h}}.$$

At last, the expression $\widehat{\partial_h \tilde{f}^0}$ can be estimated as follows.

$$\|\widehat{\partial_h \tilde{f}^0}\|_1 \leq ch^{-1} \left[\|\partial_h \hat{\tilde{F}}_0\|_1 + \|(p\partial_p + \frac{1}{2})\hat{\tilde{f}}^0\|_1 \right] \leq ce^{-(1-\epsilon)\frac{S_0}{h}}.$$

Appendix 3

Here we prove the inequalities (1.3.3). We start by estimating s_0. Write h as the sum $h = h_0 + h_1$. Then h_1 admits the representation

$$h_1(\tau) = \int_0^\tau ds e^{\int_s^\tau du \Lambda_0(h_0(u))} \Lambda_1(s),$$

where

$$\Lambda_0(h) = \frac{1}{2} \frac{d}{dh} h^{-1} G_0^3(h),$$

$$\Lambda_1 = \frac{1}{2h} G_0^3(h) - \frac{1}{2h_0} G_0^3(h_0) - \Lambda_0 h_1 + \frac{1}{2h} G_R^3.$$

Taking into account proposition 1.2.4 one can estimate Λ_j as follows.

$$\Lambda_0(\tau) \le -c h_0'(\tau) h_0^{-2}, \;\; c > 0,$$

$$|\Lambda_1(\tau)| \le W(M,s)[\Psi_1(M) h_0^{-1}(\tau)(e^{-\frac{3\kappa_3}{2}\int_0^\tau ds h_0(s)} + e^{-\frac{3r_4}{2}\frac{S_0}{h_0(\tau)}}) + s^2 h_0^{-1}(\tau) e^{-2S_0/h_0(\tau)}],$$

which implies the inequality

$$|h_1| \le W(M,s)\left[\Psi_1(M)(I_1 + I_2) + s^2 I_3\right].$$

Here

$$I_1 = \int_0^\tau ds e^{c(h_0^{-1}(s) - h_0^{-1}(\tau))} h_0^{-1}(s) e^{-\frac{3\kappa_3}{2}\int_0^s du h_0(u)} \le c h_0^2(\tau)\beta_0^{-4},$$

$$I_2 = \int_0^\tau ds e^{c(h_0^{-1}(s) - h_0^{-1}(\tau))} h_0^{-1}(s) e^{-\frac{3r_4}{2}\frac{S_0}{h_0(s)}} \le c h_0^2(\tau) e^{-\gamma/\beta_0},$$

with some $\gamma > 0$,

$$I_3 = \int_0^\tau ds e^{c(h_0^{-1}(s) - h_0^{-1}(\tau))} h_0^{-1}(s) e^{-2\frac{S_0}{h_0(s)}} \le c h_0^2(\tau).$$

Combining these inequalities one gets

$$s_0 \le W(M,s)\left(s_0^2 + \beta_0^{-4}\Psi_1(M)\right).$$

Consider s_1. Set $\beta_2 = h - \beta$. For β_2 one can write down the following equation

$$\beta_2 = \int_0^\tau ds e^{-2\int_s^\tau du h(u)} \Lambda_3(s),$$

$$\Lambda_3 = \beta_2^2 + 2\beta\eta_1 - \eta_2 + \frac{1}{2h}\eta_3.$$

Taking into account (1.3.1) one can estimate Λ_3 as follows

$$|\Lambda_3| \le W(M,s)\left[s_1^2 h_0^4 p(\tau;\kappa_1,r_1) + e^{-(2-\epsilon)\frac{S_0}{h_0}} + h_0^{-1}\Psi_0(M) p(\tau; 2\kappa_3, 2r_3)\right].$$

As a consequence, one obtains the following estimate of β_2 :

$$|\beta_2(\tau)| \le W(M,s)\left(\beta_0 s_1^2 + e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-4}\Psi_0(M)\right) h_0^2(\tau) p(\tau;\kappa_1,r_1).$$

Here we made use of the obvious estimates

$$\int_0^\tau ds e^{-\int_s^\tau du h_0(u)} h_0^M(s) e^{-\gamma/h_0(s)} \le c h_0^{M-1}(\tau) e^{-\gamma/h_0(\tau)},$$

$$\int_0^\tau dse^{-\int_s^\tau duh_0(u)} h_0^M(s) e^{-\alpha \int_0^s duh_0(u)} \le ch_0^{M-1}(\tau) e^{-\alpha \int_0^\tau duh_0(u)},$$

provided $\alpha < 1$.

Consider $\beta_3 = \beta - r^{-2}$. It satisfies the equation

$$\beta_{3\tau} = 2\beta\beta_3 + \Lambda_4,$$

$$\Lambda_4 = 2\beta_3^2 - 2\beta_3\eta_1 + \eta_2 + a - \beta^2.$$

By (1.3.1),

$$|\Lambda_4| \le W(M,s)[s_2^2 h_0^4 p(\tau,\kappa_2,r_2) + e^{-(2-\epsilon)\frac{S_0}{h_0}} + \Psi_0(M)p(\tau, 2\kappa_3, 2r_3) + s_1 h_0^3 p(\tau,\kappa_1,r_1)].$$

Since

$$|\beta_3| \le \int_\tau^{\tau_1} dse^{3\int_s^\tau duh_0(u)} |\Lambda_4(s)|,$$

one finally gets

$$s_2 \le W(\hat{M},\hat{s}) \left(\hat{s}_1 + \beta_0 s_2^2 + e^{-\frac{\gamma}{\beta_0}} + \beta_0^{-3}\Psi_0(\hat{M}) \right).$$

Appendix 4

In this appendix we collect some results related to the behavior of the function $v(x,\lambda)$ in the limit $\frac{h}{|\lambda|} \to 0$, which corresponds to the semi-classical regime for the equation (2.4.3). The necessary results can be obtained by the WKB method (see, e.g., [F]). Since the subject is so well-known we just formulate them.

For $\arg\lambda \in [0, \pi-\delta]$, where δ is a small positive number, the asymptotics of v as $\epsilon \equiv \frac{h}{|\lambda|} \to 0$ is given by the standard WKB formula (uniformly with respect to $x \in \mathbb{R}$) :

$$v(x,\lambda) = C_0(\lambda,h) e^{\frac{i}{\epsilon}\Omega_0(y,\omega)} (\omega + y^2/4)^{-1/4} \big[1 + O(\frac{\epsilon}{1+(y)_+^2})\big], \tag{A4.1}$$

Here $\omega = \frac{\lambda}{|\lambda|}$, $y = \frac{hx}{|\lambda|^{1/2}}$, $C_0(\lambda,h) = \frac{1}{\sqrt{2}}\left(\frac{h}{|\lambda|^{1/2}}\right)^{-\nu}$,

$$\Omega_0(y,\omega) = y^2/4 + \omega \ln y - \int_y^\infty ds\big(\sqrt{\omega + s^2/4} - s/2 - \omega/s\big).$$

The roots are defined on the complex plane with the cut along the negative semi-axis. They are positive for the positive values of the argument. A similar representation (with the appropriate change of the signs in the phases) is valid for v^* on the semi-bounded intervals $y \ge const$ provided $\frac{\operatorname{Im}\lambda}{h}$ is sufficiently small.

Consider the case $\arg\lambda \in (\pi-\delta, \pi]$. For $y \ge \operatorname{Re} y_1 + \delta'$, $y_1 = 2\sqrt{-\omega}$, $\delta' > 0$ fixed, (A4.1) is still valid. To describe the behavior of the solutions on a finite vicinity of the turning point y_1 one can use so called Olver type asymptotic representations, see [F]. Let b be an interval of the form $b = (-\mathrm{Re}\, y_1 + \delta', +\infty)$. For $y \in b$ the function v has the following asymptotic behavior as $\epsilon \to 0$

$$v(x,\lambda) = C_1(\lambda,h)\left[\epsilon^{-1/6}A(y,\epsilon)w_1(-\epsilon^{-2/3}\zeta(y)) + \epsilon^{1/6}B(y,\epsilon)w_1'(-\epsilon^{-2/3}\zeta(y))\right], \tag{A4.2}$$

Here $C_1(\lambda,h) = C_0(\lambda,h)e^{i\frac{\lambda}{2h}(\ln(-\omega)+S_1)}$, $S_1 = \int_2^\infty ds\big(\sqrt{s^2-4} - s + 2/s\big) - 2 + 2\ln 2$, $w_1(z)$ is the solution of the Airy equation $w_1'' - zw_1 = 0$ with the following asymptotic behavior as $z \to -\infty$

$$w_1(z) = e^{i2/3(-z)^{3/2}}(-z)^{-1/4}[1 + O((-z)^{-3/2})].$$

As $z \to +\infty$,

$$w_1(z) = e^{2/3z^{3/2} - i\pi/4}z^{-1/4}[1 + O((z)^{-3/2})].$$

The new slow variable $\zeta(y)$ is given by

$$\zeta(y) = \left(\frac{3}{2}\int_{y_1}^{y}\sqrt{\omega + s^2/4}\,ds\right)^{2/3}.$$

$\zeta(y)$ is a holomorphic function of y in some finite vicinity of y_1 and it is real for real ω and y. As $y \to y_1$, $\zeta(y) \sim (-\omega)^{1/3}(y - y_1)$. Note that $\zeta(y)$ is a solution of the equation

$$(\zeta')^2\zeta = \omega + \frac{y^2}{4}.$$

At last,

$$A = (\zeta_y)^{-1/2}(1 + O(\epsilon)), \quad B = O(\epsilon\,\langle y\rangle^{-5/6}), \tag{A4.3}$$

uniformly with respect to $y \in b$.

The solution v^* admits a similar representation (with w_1 replaced by $w_2 = w_1^*$).

In the limit $\epsilon^{-2/3}(y - \mathrm{Re}\, y_1) \to +\infty$ the representation (A4.2), (A4.3) takes the simpler form (A4.1). When $\epsilon^{-2/3}(y - \mathrm{Re}\, y_1) \to -\infty$ (A4.2), (A4.3) can be again simplified and one gets the standard WKB formula (now with a real phase for $\lambda \in \mathbb{R}$) :

$$v(x,\lambda) = C_2(\lambda,h)e^{-\frac{1}{\epsilon}\Omega_1(y,\omega)}(-\omega - y^2/4)^{-1/4}\big[1 + O(\epsilon)\big], \tag{A4.4}$$

uniformly with respect to y, $|y| \le \mathrm{Re}\, y_1 - \delta'$. Here $C_2 = C_1 e^{-i\frac{\pi}{4} - \frac{\lambda}{h}S_0}$, $\Omega_1(y,\omega) = \int_0^y dy\sqrt{-\omega - s^2/4}$. The solution v^* admits a similar description.

Appendix 5

Here we outline the proof of the estimate (1.3.11) for $\vec{f_0}$,

$$\vec{f_0} = -a^{1/2}(I - \tilde{P}(a))e^{-iz^2\triangle r^{-2}\sigma_3}T^*(ra^{1/4})\tilde{h}_0,$$

where

$$\tilde{h}_0 = (H(a) - i0)^{-1}(I - P(a))T(a^{1/4})\mathcal{N}_0.$$

Clearly,

$$\|\rho_\delta \vec{f_0}\|_2 \leq W(\hat{M}, \hat{s})\|\rho_{2\delta}\tilde{h}_0\|_2. \tag{A5.1}$$

To estimate $\tilde{h}_0$, we rewrite it in the form

$$\tilde{h}_0 = \frac{1}{2\pi i}\oint\limits_{|E|=a} \frac{dE}{E}(I - P)(H - E)^{-1}_{+}T(a^{1/4})\mathcal{N}_0. \tag{A5.2}$$

Using lemmas 2.4.1, 2.4.2, proposition 2.4.3 and the WKB representations of the solutions of (2.4.3) one can prove the following estimate for the kernel of $(H-E)^{-1}_{+}$

$$|G(x, y, E)| \leq ca^{-K}e^{-\frac{1}{h}|S(hx)-S(hy)|}, \quad |E| = a,$$

with some $K > 0$. As a consequence, one gets the inequality

$$\|\rho_{2\delta}(H - E)^{-1}_{+}T(a^{1/4})\mathcal{N}_0\|_2 \leq W(\hat{M}, \hat{s})e^{-(2-\epsilon)\frac{S_0}{h_0}}. \tag{A5.3}$$

Here we have also used proposition 2.2.1.

Consider the expression $\oint\limits_{|E|=a} \frac{dE}{E}P(H - E)^{-1}_{+}T(a^{1/4})\mathcal{N}_0$. Using propositions 1.2.6, 2.3.1 and lemma 2.4.4 it is not difficult to show that it admits an estimate similar to (A5.3) :

$$\|\rho_{2\delta}\oint\limits_{|E|=a} \frac{dE}{E}P(H - E)^{-1}_{+}T(a^{1/4})\mathcal{N}_0\|_2 \leq W(\hat{M}, \hat{s})e^{-(2-\epsilon)\frac{S_0}{h_0}}. \tag{A5.4}$$

Combining (A5.1)-(A5.4) one gets the desired result :

$$\|\rho_\delta \vec{f_0}\|_2 \leq W(\hat{M}, \hat{s})e^{-(2-\epsilon)\frac{S_0}{h_0}}.$$

Acknowledgment

It is a pleasure to thank V.S. Buslaev, F. Merle and J. Sjöstrand for numerous helpful discussions.

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Communicated by Bernard Helffer
submitted 20/10/00, accepted 08/03/01