Functiones et Approximatio
49.2 (2013), 251–267
doi: 10.7169/facm/2013.49.2.5

ON THE FOUR-DIMENSIONAL DIVISOR PROBLEM OF (a, b, c, c) TYPE

Xiaodong Cao, Wenguang Zhai

Abstract: In this paper we study the four-dimensional divisor problem of (a, b, c, c) type, where $1 \leqslant a \leqslant b < c$ are fixed integers. Our theorems improve some classical results. As an application, we study the error term of the summatory function of the exponential totient function.

Keywords: divisor function, exponential divisor, error term, exponential sum.

1. Introduction and main results

Let $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ be fixed positive integers. Without the loss of generality, we suppose $(a_1, a_2, a_3, a_4) = 1$. The divisor function $d(a_1, a_2, a_3, a_4; n)$ denotes the number of ways n can be written as a product $n_1^{a_1} n_2^{a_2} n_3^{a_3} n_4^{a_4} (n_j \in \mathbb{N}, j = 1, 2, 3, 4)$. The four-dimensional divisor problem is to study the properties of the error function

$$\Delta(a_1, a_2, a_3, a_4; x) := \sum_{n \leqslant x} d(a_1, a_2, a_3, a_4; n) - \sum_{j=1}^{4} \operatorname*{Res}_{s=a_j} \prod_{l=1}^{4} \zeta(a_l s) \frac{x^s}{s}. \qquad (1.1)$$

For the history and classical results about $\Delta(a_1, a_2, a_3, a_4; x)$, see for example, M. Vogts [28], E. Krätzel [14, 15, 16], A. Ivić [10, 11] and a recent survey article of A. Ivić, E. Krätzel, M. Kühleitner and W.G. Nowak [12].

The study of the four-dimensional divisor problem is very important in the analytic number theory. For example, the case $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$ is the well-known Piltz divisor problem of dimension 4, the case $(a_1, a_2, a_3, a_4) = (1, 2, 3, 4)$ is closely related to the number of finite abelian groups, the case $(a_1, a_2, a_3, a_4) = (1, 1, 2, 2)$ is closely related to the number of direct factors of finite abelian groups, etc.

This work is supported by the National Key Basic Research Program of China(Grant No. 2013CB834201), the National Natural Science Foundation of China(Grant No. 11171344) and the Natural Science Foundation of Beijing(Grant No. 1112010).

2010 Mathematics Subject Classification: primary: 11N37

The aim of this paper is to study the four-dimensional divisor problem for the case $(a_1, a_2, a_3, a_4) = (a, b, c, c)$, where a, b, c are fixed positive integers such that $a \leqslant b < c$ and $(a, b, c) = 1$. Let

$$A(a,b,c,c;x) = \sum_{n\leqslant x} d(a,b,c,c;n), \tag{1.2}$$

$$H(a,b,c,c;x) = \operatorname*{Res}_{s=\frac{1}{a}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} + \operatorname*{Res}_{s=\frac{1}{b}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} + \operatorname*{Res}_{s=\frac{1}{c}} \zeta(as)\zeta(bs)\zeta^2(cs)\frac{x^s}{s} \tag{1.3}$$

if $a \neq b$, otherwise an appropriate limit approach should be taken in the above sum.

From A. Ivić [11] we have

$$\Delta(a,b,c,c;x) = \Omega\left(x^{\max\left(\frac{1}{2(a+b)}, \frac{1}{a+b+c}\right)}\right). \tag{1.4}$$

One may conjecture that

$$\Delta(a,b,c,c;x) = O\left(x^{\max\left(\frac{1}{2(a+b)}, \frac{1}{a+b+c}\right)+\varepsilon}\right). \tag{1.5}$$

Now we introduce some notations for later use. Let $d(n)$ denote the Dirichlet divisor function. Let $x \geqslant 1$ be real and $\mathcal{L} = \log x$, $\{t\}$ denotes the fractional part of t, $\psi(t) = t - [t] - 1/2$, $e(t) = \exp(2\pi i t)$. ε is a fixed positive constant, not necessarily the same in all occurrences. $m \sim M$ means that $cM < m \leqslant CM$ for some constants $0 < c < C$. $d(a,b;n) = \sum_{m_1^a m_2^b = n} 1$ and $A(a,b;x) = \sum_{n\leqslant x} d(a,b;n)$. Let $\Delta(a,b;x)$ denote the error term of the asymmetric two-dimensional divisor problem,

$$\Delta(a,b;x) := \sum_{m^a n^b \leqslant x} 1 - \zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}} - \zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}} := A(a,b;x) - H(a,b;x), \tag{1.6}$$

if $a \neq b$. If $a = b$, then an appropriate limit approach should be taken in the above sum. Hence $H(1,1;x) = x\log x + (2\gamma - 1)x$, where γ is Euler's constant. For convenience, we also use notations $A(x)$ and $\Delta(x)$ to denote $A(1,1;x)$ and $\Delta(1,1;x)$, respectively. Krätzel [14] gives a series of classical results about the upper bound of $\Delta(a,b;x)$. Recently, the authors [31] studies the mean square of $\Delta(a,b;x)$ and obtains its asymptotic formula.

Theorem 6.8 of Krätzel [14] is an important theorem for the many-dimensional divisor problems. We first prove the following Theorem 1, which is a refined version of this theorem in our case.

Theorem 1. *Suppose a, b and c are fixed positive integers such that $1 \leqslant a < b < c$, $1 < y < x$ are two large real numbers, $0 < \alpha(a,b) < 1/c$ is a real number such that the estimate $\Delta(a, b; x) \ll x^{\alpha(a,b)}$ holds, then*

$$\Delta(a,b,c,c;x) = \sum_{m \leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) + \sum_{n \leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n)\Delta\left(a,b;\frac{x}{n^c}\right) \tag{1.7}$$
$$+ O\left(x^{\frac{1}{c}} y^{-\frac{1}{c}-\frac{1}{2(a+b)}} \log x + x^{-\frac{1}{4c}} y^{\frac{1}{a}+\frac{1}{4c}} + y^{\alpha(a,b)}\left(\frac{x}{y}\right)^{\frac{131}{416c}} \log^4 x\right).$$

Using the bounds $\Delta(t) \ll t^{131/416} \log^{26947/8320} t$ (see Huxley [9]) and $\Delta(a, b; x) \ll x^{\alpha(a,b)}$ in Theorem 1 directly, we get immediately the following Corollary 1.1.

Corollary 1.1. *Under the conditions of Theorem 1, we have*

$$\Delta(a,b,c,c;x) \ll x^{\frac{416-131a\alpha(a,b)}{285a+416c(1-a\alpha(a,b))}} (\log x)^4. \tag{1.8}$$

Remark 1. When we use Theorem 6.8 of Krätzel [14] to estimate $\Delta(a, b, c, c; x)$, we encounter some four-dimensional exponential sums, however if we use Theorem 1, we only need to estimate some three-dimensional exponential sums, which is much easier to handle. When the differences between the numbers a, b, c are comparatively large, the upper bound results obtained through Theorem 6.8 of Krätzel [14] are usually weak. In this paper, we shall get the following Theorem 2 through Theorem 1 with the help of an approach due to Heath-Brown [7]. We note that Theorem 2 can not be covered by Krätzel's results in [14, 16].

Theorem 2. *Suppose a, b, c are fixed positive integers such that $1 \leqslant a < b$ and $a + b \leqslant c < 2(a + b)$, $0 < \alpha(a, b) < 1/c$ is a real number such that the estimate $\Delta(a, b; x) \ll x^{\alpha(a,b)}$ holds. If the condition*

$$\alpha(a,b) \geqslant \frac{9c - 4a - 2b}{c(23a + 25b) - 18(a+b)^2} \tag{1.9}$$

is true, then

$$\Delta(a,b,c,c;x) \ll x^{\theta(a,b,c)+\varepsilon}, \tag{1.10}$$

where

$$\theta(a,b,c) = \frac{3 - 3\alpha(a,b)(a+b)}{3c + 2a - c(5a+3b)\alpha(a,b))}. \tag{1.11}$$

Remark 2. When dealing with exponential sums in the proof of Theorem 2, the main approach is due to Heath-Brown [7]. It is not difficult to improve Theorem 2 by using more precise estimates of the exponential sums(see for example, [1], [6], [8], [18], [23], [30]).

Corollary 1.2. *Suppose* $1 \leqslant a < b$, $a+b \leqslant c < 2(a+b)$, *and define*

$$\vartheta(a,b,c) = \begin{cases} \frac{15(a+b)}{29a^2+29ab-4ac+15bc} & \text{if } 11a \geqslant 8b \text{ and } \frac{113a^2+255ab+142b^2}{88a+107b} \leqslant c < \frac{29(a+b)}{19}, \\ \frac{21a+12b}{34a^2+28ab+ac+12bc} & \text{if } 2a \geqslant b > \frac{11}{8}a \text{ and } \frac{112a^2+270ab+152b^2}{77a+124b} \leqslant c < \frac{17a+14b}{10}, \\ \frac{9a+3b}{13a^2+9ab+2ac+3bc} & \text{if } 4a \geqslant b > 2a \text{ and } \frac{37a^2+95ab+54b^2}{22a+47b} \leqslant c < \frac{39a+27b}{21}. \end{cases}$$

Then

$$\Delta(a,b,c,c;x) \ll x^{\vartheta(a,b,c)+\varepsilon}. \tag{1.12}$$

From Corollary 1.2 we get immediately that

$$\Delta(1,3,5,5;x) \ll x^{18/95+\varepsilon}, \tag{1.13}$$

which will be used to prove the following Theorem 3.

Subbarao[24] introduced the definition of exponential convolution, which is closely related to the exponential divisors of natural numbers. Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_r^{a_r}$. An integer d is called an exponential divisor of n if $d = p_1^{b_1} \cdots p_r^{b_r}$ satisfies $b_j | a_j (1 \leqslant j \leqslant r)$, denoted by $d|_e n$. For convenience let $1|_e 1$. The exponential convolution is an analogue of the classical Dirichlet convolution, which is studied by several authors, see for example, [5, 13, 19, 20, 21, 22, 29]. The exponential totient function $\phi^{(e)}(n)$ denotes the number of divisors d of n such that d and n are exponentially coprime, namely, they don't have common exponential divisors. The function $\phi^{(e)}(n)$ was studied in J. Sándor[21], Tóth[25, 26, 27], Pétermann[20]. Tóth[25] proved that

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + O(x^{1/5+\varepsilon}), \tag{1.14}$$

where C_1, C_2 are computable constants. The estimate $O(x^{1/5+\varepsilon})$ was improved to $O(x^{1/5})$ in Pétermann[20], which is the best result up to date.

In this paper we prove the following

Theorem 3.

(i) *The asymptotic formula*

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + D_1 x^{1/5} \log x + D_2 x^{1/5} + O(x^{18/95+\varepsilon}) \tag{1.15}$$

holds, where D_1, D_2 *are computable constants.*

(ii) *We have*

$$\sum_{n \leqslant x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + D_1 x^{1/5} \log x + D_2 x^{1/5} + \Omega\left(x^{1/8}\right). \tag{1.16}$$

Remark 3. The asymptotic formula (1.15) is a substantial improvement to the formula (1.14). Numerically we have $\frac{18}{95} = 0.18947\ldots < \frac{1}{5}$.

2. The proof of Theorem 1 and Corollary 1.1

Lemma 2.1. *Let $x \geqslant 1$ and $\Delta(a,b;x)$ be defined by (1.6). We let*

$$E(a,b;x,s) := \Delta(a,b;x)x^{-s} - s\int_x^\infty \Delta(a,b;t)t^{-s-1}\,\mathrm{d}t \tag{2.1}$$

(i) *If $a<b$, $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ such that $\alpha(a,b) < \frac{1}{b}$, then*

$$\sum_{n\leqslant x} d(a,b;n)n^{-s} = \frac{\zeta\left(\frac{b}{a}\right)}{1-as}x^{\frac{1}{a}-s} + \frac{\zeta\left(\frac{a}{b}\right)}{1-bs}x^{\frac{1}{b}-s} + \zeta(as)\zeta(bs) + E(a,b;x,s) \tag{2.2}$$

holds for $s > \alpha(a,b)$. For $s=\frac{1}{a}$ or $s=\frac{1}{b}$ we take the limiting values.

(ii) *If $\Delta(x) \ll x^\alpha$ such that $\alpha < \frac{1}{3}$, then*

$$\sum_{n\leqslant x} d(n)n^{-s} = \left(\frac{\log x + 2\gamma - 1}{1-s} - \frac{s}{(1-s)^2}\right)x^{1-s} + \zeta^2(s) + E(1,1;x,s) \tag{2.3}$$

holds for $s>\alpha$. For $s=1$ we take the limiting value.

Proof. By partial summation formula and (1.6), we obtain

$$\begin{aligned}\sum_{n\leqslant x}\frac{d(a,b;n)}{n^s} &= A(a,b;x)x^{-s} + s\int_1^x A(a,b;t)t^{-s-1}\,\mathrm{d}t \qquad (2.4)\\ &= \frac{H(a,b;x)+\Delta(a,b;x)}{x^s} + s\int_1^x \frac{(H(a,b;t)+\Delta(a,b;t))}{t^{s+1}}\,\mathrm{d}t\\ &= \frac{\zeta\left(\frac{b}{a}\right)}{1-as}x^{\frac{1}{a}-s} + \frac{\zeta\left(\frac{a}{b}\right)}{1-bs}x^{\frac{1}{b}-s} - \frac{as\zeta\left(\frac{b}{a}\right)}{1-as} - \frac{bs\zeta\left(\frac{a}{b}\right)}{1-bs}\\ &\quad + s\int_1^\infty \Delta(a,b;t)t^{-s-1}\,\mathrm{d}t + \Delta(a,b;x)x^{-s}\\ &\quad - s\int_x^\infty \Delta(a,b;t)t^{-s-1}\,\mathrm{d}t.\end{aligned}$$

Clearly

$$E(a,b;x,s) \ll x^{\alpha(a,b)-s}. \tag{2.5}$$

Suppose that $s>1$, we have from (2.4) and the condition $\alpha(a,b) < \frac{1}{b}$, when $x\to\infty$

$$\zeta(as)\zeta(bs) = -\frac{as\zeta\left(\frac{b}{a}\right)}{1-as} - \frac{bs\zeta\left(\frac{a}{b}\right)}{1-bs} + s\int_1^\infty \Delta(a,b;t)t^{-s-1}\,\mathrm{d}t.$$

By analytic continuation this equation also holds for $\Re s > \alpha(a,b)$. Now we substitute this into (2.4), and this completes the proof of (2.2). The proof of (2.3) is similar to that of (2.2), we omit the details. ■

Lemma 2.2. *Let $x \geqslant 1$ and $\Delta(a,b;x)$ be defined by (1.6). Let*

$$E_1(a,b;x,s) := \Delta(a,b;x)x^{-s}\log x - \int_x^{\infty} \Delta(a,b;t)(s\log t - 1)t^{-s-1}\,\mathrm{d}t \tag{2.6}$$

If $a < b$, $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ such that $\alpha(a,b) < \frac{1}{b}$, then

$$\begin{aligned}\sum_{n\leqslant x} d(a,b;n)n^{-s}\log n = {}&\zeta\left(\frac{b}{a}\right)\left(\frac{\log x}{1-as} - \frac{a}{(1-as)^2}\right)x^{\frac{1}{a}-s}\\ &+\zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-bs} - \frac{b}{(1-bs)^2}\right)x^{\frac{1}{b}-s}\\ &-(\zeta(as)\zeta(bs))' + E_1(a,b;x,s)\end{aligned} \tag{2.7}$$

holds for $s > \alpha(a,b)$. For $s = \frac{1}{a}$ or $s = \frac{1}{b}$ we take the limiting values.

Proof. Similar to the proof of (2.4) we have

$$\begin{aligned}\sum_{n\leqslant x} \frac{d(a,b;n)}{n^s}\log n &= A(a,b;x)\frac{\log x}{x^s} + \int_1^x A(a,b;t)\frac{(s\log t-1)}{t^{s+1}}\,\mathrm{d}t \\ &= \zeta\left(\frac{b}{a}\right)\left(\frac{\log x}{1-as} - \frac{a}{(1-as)^2}\right)x^{\frac{1}{a}-s}\\ &\quad+\zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-bs} - \frac{b}{(1-bs)^2}\right)x^{\frac{1}{b}-s}\\ &\quad+\frac{a\zeta\left(\frac{b}{a}\right)}{(1-as)^2} + \frac{b\zeta\left(\frac{a}{b}\right)}{(1-bs)^2} + \Delta(a,b;x)x^{-s}\log x\\ &\quad+\int_1^{\infty} \Delta(a,b;t)\frac{(s\log t-1)}{t^{s+1}}\,\mathrm{d}t\\ &\quad-\int_x^{\infty} \Delta(a,b;t)(s\log t-1)t^{-s-1}\,\mathrm{d}t.\end{aligned} \tag{2.8}$$

Assuming that $s > 1$, we have, as $x \to \infty$

$$-(\zeta(as)\zeta(bs))' = \frac{a\zeta\left(\frac{b}{a}\right)}{(1-as)^2} + \frac{b\zeta\left(\frac{a}{b}\right)}{(1-bs)^2} + \int_1^{\infty} \Delta(a,b;t)\frac{(s\log t-1)}{t^{s+1}}\,\mathrm{d}t. \tag{2.9}$$

By analytic continuation (2.9) also holds for $\Re s > \alpha(a,b)$. Substituting (2.9) into (2.8) completes the proof of Lemma 2.2. ■

Lemma 2.3. *Let $a < b < c$ and $1 < y < x$. If $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ such that $\alpha(a,b) < \frac{1}{c}$, then*

$$\begin{aligned}\Delta(a,b,c,c;x) = &\sum_{m\leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) + \sum_{n\leqslant(\frac{x}{y})^{\frac{1}{c}}} d(n)\Delta\left(a,b;\frac{x}{n^c}\right) \\ &- \frac{x^{\frac{1}{c}}}{c}\int_y^\infty \left(\frac{\log x - \log t}{c} + 2\gamma\right)\Delta(a,b;t)t^{-\frac{1}{c}-1}\,\mathrm{d}t \\ &- \frac{c}{a}\zeta\left(\frac{b}{a}\right)x^{\frac{1}{a}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t)t^{-\frac{c}{a}-1}\,\mathrm{d}t \\ &- \frac{c}{b}\zeta\left(\frac{a}{b}\right)x^{\frac{1}{b}}\int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t)t^{-\frac{c}{b}-1}\,\mathrm{d}t - \Delta(a,b;y)\Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}}\right).\end{aligned} \tag{2.10}$$

Proof. By (1.2) and Dirichlet's splitting argument we write

$$A(a,b,c,c;x) = \sum_{mn^c\leqslant x} d(a,b;m)d(n) = S_1 + S_2 - S_3, \tag{2.11}$$

where

$$S_1 = \sum_{m\leqslant y} d(a,b;m)\sum_{n^c\leqslant x/m} d(n), \qquad S_2 = \sum_{n^c\leqslant x/y} d(n)\sum_{m\leqslant x/n^c} d(a,b;m),$$

$$S_3 = \sum_{m\leqslant y} d(a,b;m)\sum_{n^c\leqslant x/y} d(n).$$

By (1.6), (2.2) of Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned} S_1 = &\sum_{m\leqslant y} d(a,b;m)\left\{\left(\frac{\log x}{c} + (2\gamma-1) - \frac{\log m}{c}\right)\frac{x^{\frac{1}{c}}}{m^{\frac{1}{c}}} + \Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right)\right\} \\ = &\left(\frac{\log x}{c} + (2\gamma-1)\right)x^{\frac{1}{c}}\sum_{m\leqslant y}\frac{d(a,b;m)}{m^{\frac{1}{c}}} - \frac{x^{\frac{1}{c}}}{c}\sum_{m\leqslant y}\frac{d(a,b;m)\log m}{m^{\frac{1}{c}}} \\ &+ \sum_{m\leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) \\ = &\left(\frac{\log x}{c} + (2\gamma-1)\right)x^{\frac{1}{c}}\left\{\frac{\zeta\left(\frac{b}{a}\right)}{1-\frac{a}{c}}y^{\frac{1}{a}-\frac{1}{c}} + \frac{\zeta\left(\frac{a}{b}\right)}{1-\frac{b}{c}}y^{\frac{1}{b}-\frac{1}{c}} + \zeta\left(\frac{a}{c}\right)\zeta\left(\frac{b}{c}\right)\right. \\ &\left.+ E\Big(a,b;y,\frac{1}{c}\Big)\right\} - \frac{x^{\frac{1}{c}}}{c}\left\{\zeta\left(\frac{b}{a}\right)\left(\frac{\log y}{1-\frac{a}{c}} - \frac{a}{(1-\frac{a}{c})^2}\right)y^{\frac{1}{a}-\frac{1}{c}}\right. \\ &\left.+\zeta\left(\frac{a}{b}\right)\left(\frac{\log x}{1-\frac{b}{c}} - \frac{b}{(1-\frac{b}{c})^2}\right)y^{\frac{1}{b}-\frac{1}{c}}\right\} \\ &- \frac{x^{\frac{1}{c}}}{c}\left(-\left(\zeta(as)\zeta(bs)\right)'_{s=\frac{1}{c}} + E_1\Big(a,b;y,\frac{1}{c}\Big)\right) + \sum_{m\leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right). \end{aligned} \tag{2.12}$$

By (1.6) and (2.3) of Lemma 2.1 we obtain

$$
\begin{aligned}
S_2 &= \sum_{n \leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n) \left\{ \zeta\left(\frac{b}{a}\right) \frac{x^{\frac{1}{a}}}{n^{\frac{c}{a}}} + \zeta\left(\frac{a}{b}\right) \frac{x^{\frac{1}{b}}}{n^{\frac{c}{b}}} + \Delta\left(a, b; \frac{x}{n^c}\right) \right\} \qquad (2.13) \\
&= \zeta\left(\frac{b}{a}\right) x^{\frac{1}{a}} \Bigg\{ \left(\frac{\frac{1}{c}\log\frac{x}{y} + 2\gamma - 1}{1 - \frac{c}{a}} - \frac{ac}{(c-a)^2} \right) \left(\frac{x}{y}\right)^{\frac{1}{c} - \frac{1}{a}} \\
&\quad + \zeta^2\left(\frac{c}{a}\right) + E\left(1, 1; \left(\frac{x}{y}\right)^{\frac{1}{c}}, \frac{c}{a}\right) \Bigg\} \\
&\quad + \zeta\left(\frac{a}{b}\right) x^{\frac{1}{b}} \Bigg\{ \left(\frac{\frac{1}{c}\log\frac{x}{y} + 2\gamma - 1}{1 - \frac{c}{b}} - \frac{bc}{(c-b)^2} \right) \left(\frac{x}{y}\right)^{\frac{1}{c} - \frac{1}{b}} \\
&\quad + \zeta^2\left(\frac{c}{b}\right) + E\left(1, 1; \left(\frac{x}{y}\right)^{\frac{1}{c}}, \frac{c}{b}\right) \Bigg\} + \sum_{n \leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n) \Delta\left(a, b; \frac{x}{n^c}\right).
\end{aligned}
$$

From (1.6) we also have

$$
\begin{aligned}
S_3 &= \left\{ \zeta\left(\frac{b}{a}\right) y^{\frac{1}{a}} + \zeta\left(\frac{a}{b}\right) y^{\frac{1}{b}} + \Delta(a, b; y) \right\} \qquad (2.14) \\
&\quad \times \left\{ \left(\frac{1}{c}\log\frac{x}{y} + 2\gamma - 1 \right) \left(\frac{x}{y}\right)^{\frac{1}{c}} + \Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}} \right) \right\}.
\end{aligned}
$$

Combining (2.11)–(2.14), (2.1) and (2.6) we get

$$
\begin{aligned}
A(a, b, c, c; x) &= \zeta\left(\frac{b}{a}\right) \zeta^2\left(\frac{c}{a}\right) x^{\frac{1}{a}} + \zeta\left(\frac{a}{b}\right) \zeta^2\left(\frac{c}{b}\right) x^{\frac{1}{b}} \qquad (2.15) \\
&\quad + \left(\left(\frac{\log x}{c} + (2\gamma - 1) \right) \zeta\left(\frac{a}{c}\right) \zeta\left(\frac{b}{c}\right) + \frac{1}{c} \left(\zeta(as)\zeta(bs)\right)'_{s=\frac{1}{c}} \right) x^{\frac{1}{c}} \\
&\quad + \sum_{m \leqslant y} d(a, b; m) \Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}} \right) + \sum_{n \leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n) \Delta\left(a, b; \frac{x}{n^c}\right) \\
&\quad - \frac{x^{\frac{1}{c}}}{c} \int_y^\infty \left(\frac{\log x - \log t}{c} + 2\gamma \right) \Delta(a, b; t) t^{-\frac{1}{c} - 1} \,\mathrm{d}t \\
&\quad - \frac{c}{a} \zeta\left(\frac{b}{a}\right) x^{\frac{1}{a}} \int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t) t^{-\frac{c}{a} - 1} \,\mathrm{d}t \\
&\quad - \frac{c}{b} \zeta\left(\frac{a}{b}\right) x^{\frac{1}{b}} \int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t) t^{-\frac{c}{b} - 1} \,\mathrm{d}t - \Delta(a, b; y) \Delta\left(\left(\frac{x}{y}\right)^{\frac{1}{c}} \right).
\end{aligned}
$$

Finally, Lemma 2.3 follows from (1.1)–(1.3) and (2.15) at once. ∎

Proof of Theorem 1. Applying Lemma 5.7 of Krätzel [14] (or Theorem 3 of Cao [3]) and integrating by parts we have

$$x^{\frac{1}{c}} \int_y^\infty \left(\frac{\log x - \log t}{c} + 2\gamma \right) \Delta(a,b;t) t^{-\frac{1}{c}-1}\, \mathrm{d}t \ll x^{\frac{1}{c}} y^{-\frac{1}{c}-\frac{1}{2(a+b)}} \log x. \tag{2.16}$$

Similarly, we also have

$$x^{\frac{1}{a}} \int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t) t^{-\frac{c}{a}-1}\, \mathrm{d}t \ll x^{-\frac{1}{4c}} y^{\frac{1}{a}+\frac{1}{4c}}, \tag{2.17}$$

and

$$x^{\frac{1}{b}} \int_{(\frac{x}{y})^{\frac{1}{c}}}^\infty \Delta(t) t^{-\frac{c}{b}-1}\, \mathrm{d}t \ll x^{-\frac{1}{4c}} y^{\frac{1}{b}+\frac{1}{4c}}. \tag{2.18}$$

M. N. Huxley[9] showed that $\Delta(1,1;x) \ll x^{\frac{131}{416}} (\log x)^{\frac{26947}{8320}}$. Now combining Lemma 2.3, (2.16)–(2.18) finishes the proof of Theorem 1. ■

The proof of Corollary 1.1. By Theorem 1 and the estimate $\Delta(1,1;x) \ll x^{\frac{131}{416}} (\log x)^4$, then applying partial summation formula and (1.6) we have

$$\begin{aligned} \Delta(a,b,c,c;x) x^{\varepsilon} &\ll \mathcal{L}^4 \sum_{m \leqslant y} d(a,b;m) \left(\left(\frac{x}{m} \right)^{\frac{1}{c}} \right)^{\frac{131}{416}} + \sum_{n \leqslant (\frac{x}{y})^{\frac{1}{c}}} d(n) \left(\frac{x}{n^c} \right)^{\alpha(a,b)} \\ &\ll x^{\frac{131}{416c}} y^{\frac{1}{a}-\frac{131}{416c}} \mathcal{L}^4 + x^{\frac{1}{c}} y^{\alpha(a,b)-\frac{1}{c}} \mathcal{L}. \end{aligned} \tag{2.19}$$

Taking $y = x^{\frac{285a}{285a+416c(1-a\alpha(a,b))}}$, we obtain $\Delta(a,b,c,c;x) \ll x^{\frac{416-131a\alpha(a,b)}{285a+416c(1-a\alpha(a,b))}} \mathcal{L}^4$, and this completes the proof of Corollary 1.1. ■

3. The proof of Theorem 2

In the proof of Theorem 2 we will use the following lemmas. (3.1) in Lemma 3.1 is well-known, (3.2) is Theorem 5.1 in [14]. Lemma 3.2 is Lemma 4 of Cao [4] (also see (2.1) in Wu [30], Heath-Brown's method), Lemma 3.3 is Lemma 10 in [4] (Process B, then Heath-Brown's method).

Lemma 3.1. *Let $\varepsilon > 0$ and $\Delta(a,b;x)$ be defined by (1.6). Then for $1 \leqslant Y < x$*

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\sqrt{2}\pi} \sum_{m \leqslant Y} \frac{\tau(m)}{m^{\frac{3}{4}}} \cos(4\pi\sqrt{mx} - \frac{\pi}{4}) + O\left(x^{\frac{1}{2}+\varepsilon} Y^{-\frac{1}{2}} + x^{\varepsilon} \right) \tag{3.1}$$

and

$$\Delta(a,b;x) = -\sum_{n^{a+b} \leqslant x} \psi\left(\left(\frac{x}{x^b} \right)^{\frac{1}{a}} \right) + \psi\left(\left(\frac{x}{x^a} \right)^{\frac{1}{b}} \right) + O(1). \tag{3.2}$$

Lemma 3.2. *Let $x \geqslant 2$, α, β, γ be given real numbers with $\alpha(\alpha-1)\beta\gamma \neq 0$, $|a(m)| \leqslant 1$, $|b(n_1, n_2)| \leqslant 1$. Suppose $G = xM^\alpha N_1^\beta N_2^\gamma$, (κ, λ) is an exponent pair and*

$$T(M, N_1, N_2) = \sum_{m\sim M} \sum_{n_1\sim N_1} \sum_{n_2\sim N_2} a(m)b(n_1, n_2)e(xm^\alpha n_1^\beta n_2^\gamma). \tag{3.3}$$

Then

$$\begin{aligned} T(M, N_1, N_2)\mathcal{L}^{-2} \ll \left(G^\kappa M^{1+\lambda+\kappa}(N_1N_2)^{2+\kappa}\right)^{\frac{1}{2+2\kappa}} + M^{\frac12}N_1N_2 \\ + M(N_1N_2)^{\frac12} + G^{-\frac12}MN_1N_2. \end{aligned} \tag{3.4}$$

Lemma 3.3. *Let u, v, w be positive numbers. Suppose $x \geqslant 2$, $|a(m)| \leqslant 1$, $G = \left(\frac{x}{M^uN^v}\right)^{\frac1w}$ and*

$$T(M, N) = \sum_{m\sim M} \sum_{\substack{n\sim N \\ m^u n^{v+w} \leqslant x}} a(m)\psi\left(\left(\frac{x}{m^un^v}\right)^{\frac1w}\right). \tag{3.5}$$

Then

$$\begin{aligned} T(M, N)\mathcal{L}^{-6} \ll \left(G^{1+\kappa}M^{2+\lambda+\kappa}N^{1+\kappa}\right)^{\frac{1}{3+2\kappa}} + G^{\frac13}M^{\frac23}N^{\frac13} \\ + MN^{\frac12} + G^{-\frac12}MN. \end{aligned} \tag{3.6}$$

Lemma 3.4. *Let $u > 0$, $v > 0$ and $\max(u, v) \leqslant c$, define*

$$T_{(u,v)}(x; N, M) := \sum_{N<n\leqslant 2N} d(n) \sum_{\substack{m\sim M \\ m^{u+v}n^c \leqslant x}} \psi\left(\left(\frac{x}{m^un^c}\right)^{\frac1v}\right). \tag{3.7}$$

(i) *If $u \geqslant v$ and $MN \gg x^\theta$, then*

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll \left(x^{1-(u-v)\theta}N^{-c+u+v}\right)^{\frac{3}{8v}} + x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \tag{3.8}$$

(ii) *If $u < v$, then*

$$x^{-\varepsilon}T_{(u,v)}(x; N, M) \ll x^{\frac{3}{4(u+v)}}N^{\frac34\left(1-\frac{c}{u+v}\right)} + x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \tag{3.9}$$

Proof. By Lemma 3.3 with $(u,v,w)=(c,u,v)$, we get

$$\begin{aligned} x^{-\varepsilon}T_{(u,v)}(x;N,M) &\ll \left(\left(\frac{x}{M^uN^c}\right)^{\frac{1+\kappa}{v}}N^{2+\lambda+\kappa}M^{1+\kappa}\right)^{\frac{1}{3+2\kappa}} \\ &\quad+\left(\frac{x}{M^uN^c}\right)^{\frac{1}{3v}}N^{\frac{2}{3}}M^{\frac{1}{3}}+NM^{\frac{1}{2}}+\left(\frac{x}{M^uN^c}\right)^{-\frac{1}{2v}}NM \\ &\ll x^{\frac{1+\kappa}{v(3+2\kappa)}}N^{\frac{2v-c+v\lambda+(v-c)\kappa}{v(3+2\kappa)}}M^{\frac{(v-u)(1+\kappa)}{v(3+2\kappa)}}+x^{\frac{1}{3v}}N^{\frac{2v-c}{3v}}M^{\frac{v-u}{3v}} \\ &\quad+NM^{\frac{1}{2}}+x^{-\frac{1}{2v}}N^{1+\frac{c}{2v}}M^{1+\frac{u}{2v}}. \end{aligned} \tag{3.10}$$

If $u\geqslant v$, taking $(\kappa,\lambda)=(\frac{1}{2},\frac{1}{2})$, then applying $MN\gg x^{\theta}$ and $N^cM^{u+v}\ll x$ it follows from (3.10) that

$$\begin{aligned} x^{-\varepsilon}T_{(u,v)}(x;N,M) &\ll \left(\frac{x}{N^{c-u-v}}\frac{1}{(MN)^{u-v}}\right)^{\frac{3}{8v}}+\left(\frac{x}{N^{c-u-v}(NM)^{u-v}}\right)^{\frac{1}{3v}} \\ &\quad+N^{1-\frac{c}{2(u+v)}}\left(N^cM^{u+v}\right)^{\frac{1}{2(u+v)}} \\ &\quad+x^{-\frac{1}{2v}}N^{1-\frac{c}{2(u+v)}}\left(N^cM^{u+v}\right)^{\frac{u+2v}{2v(u+v)}} \\ &\ll \left(\frac{x^{1-(u-v)\theta}}{N^{c-u-v}}\right)^{\frac{3}{8v}}+\left(\frac{x^{1-(u-v)\theta}}{N^{c-u-v}}\right)^{\frac{1}{3v}}+x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \end{aligned} \tag{3.11}$$

Since $\frac{3}{8}>\frac{1}{3}$, then the second term in the above expression is less than the first term, hence (3.8) holds.

If $u\leqslant v$, similar to the proof of (3.11) we also have

$$\begin{aligned} x^{-\varepsilon}T_{(u,v)}(x;N,M) &\ll \left(xN^{2v\left(1-\frac{c}{u+v}\right)}\left(N^cM^{u+v}\right)^{\frac{v-u}{u+v}}\right)^{\frac{3}{8v}} \\ &\quad+\left(xN^{2v\left(1-\frac{c}{u+v}\right)}\left(N^cM^{u+v}\right)^{\frac{v-u}{u+v}}\right)^{\frac{1}{3v}} \\ &\quad+N^{1-\frac{c}{2(u+v)}}\left(N^cM^{u+v}\right)^{\frac{1}{2(u+v)}} \\ &\quad+x^{-\frac{1}{2v}}N^{1-\frac{c}{2(u+v)}}\left(N^cM^{u+v}\right)^{\frac{u+2v}{2v(u+v)}} \\ &\ll x^{\frac{3}{4(u+v)}}N^{\frac{3}{4}\left(1-\frac{c}{u+v}\right)}+x^{\frac{2}{3(u+v)}}N^{\frac{2}{3}\left(1-\frac{c}{u+v}\right)} \\ &\quad+x^{\frac{1}{2(u+v)}}N^{1-\frac{c}{2(u+v)}}. \end{aligned} \tag{3.12}$$

Since $\frac{3}{4}>\frac{2}{3}$, and this completes the proof of (3.9). ∎

Lemma 3.5. *Let $a\leqslant b\leqslant c$, $M\leqslant\frac{x}{4}$ and define*

$$S_{(a,b)}(x;M):=\sum_{M<m\leqslant 2M}d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right). \tag{3.13}$$

If $M\leqslant\frac{1}{3}x^{\frac{a}{a+c}}$, then

$$x^{-\varepsilon}S_{(a,b)}(x;M)\ll x^{\frac{1}{3c}}M^{\frac{5c-2a}{6ac}}+x^{\frac{1}{4c}}M^{\frac{1}{a}-\frac{1}{4c}}. \tag{3.14}$$

Proof. From (3.1) and (1.6), we have for any $1 \leqslant Y \leqslant \left(\frac{x}{4M}\right)^{\frac{1}{c}}$

$$\begin{aligned} S_{(a,b)}(x;M) &\ll x^{\frac{1}{4c}} \left| \sum_{m\sim M} \frac{d(a,b;m)}{m^{\frac{1}{4c}}} \sum_{n\leqslant Y} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m}\right)^{\frac{1}{2c}}\right) \right| \\ &\quad + \sum_{M<m\leqslant 2M} d(a,b;m)\left(\left(\frac{x}{m}\right)^{\frac{1}{2c}+\varepsilon} Y^{-\frac{1}{2}} + x^{\varepsilon}\right) \\ &\ll x^{\frac{1}{4c}} \left| \sum_{m_1^a m_2^b \sim M} \frac{1}{(m_1^a m_2^b)^{\frac{1}{4c}}} \sum_{n\leqslant Y} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m_1^a m_2^b}\right)^{\frac{1}{2c}}\right) \right| \\ &\quad + x^{\frac{1}{2c}+\varepsilon} M^{\frac{1}{a}-\frac{1}{2c}} Y^{-\frac{1}{2}} + M^{\frac{1}{a}} x^{\varepsilon}. \end{aligned} \tag{3.15}$$

By Lemma 3.2 with $(\alpha,\beta,\gamma) = (\frac{1}{2}, -\frac{a}{2c}, -\frac{b}{2c})$, we have

$$\begin{aligned} &x^{-\varepsilon} \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ m_1^a m_2^b \sim M}} \frac{1}{(m_1^a m_2^b)^{\frac{1}{4c}}} \sum_{n\sim N} \frac{\tau(n)}{n^{\frac{3}{4}}} e\left(2\left(\frac{n^c x}{m_1^a m_2^b}\right)^{\frac{1}{2c}}\right) \\ &\ll M^{-\frac{1}{4c}} N^{-\frac{3}{4}} \left\{ \left(\left(\frac{N^c x}{M}\right)^{\frac{\kappa}{2c}} N^{1+\lambda+\kappa} (M_1M_2)^{2+\kappa} \right)^{\frac{1}{2+2\kappa}} + N^{\frac{1}{2}} M_1 M_2 \right. \\ &\quad \left. + N(M_1M_2)^{\frac{1}{2}} + \left(\frac{N^c x}{M}\right)^{-\frac{1}{4c}} N M_1 M_2 \right\} \\ &\ll x^{\frac{\kappa}{4c(1+\kappa)}} N^{\frac{2\lambda-1}{4(1+\kappa)}} M^{\frac{4c+2(c-a)\kappa-a}{4ac(1+\kappa)}} + N^{-\frac{1}{4}} M^{\frac{1}{a}-\frac{1}{4c}} + N^{\frac{1}{4}} M^{\frac{1}{2a}-\frac{1}{4c}} + x^{-\frac{1}{4c}} M^{\frac{1}{a}}, \end{aligned} \tag{3.16}$$

here we use $M_1M_2 \ll (M_1^a M_2^b)^{\frac{1}{a}} \ll M^{\frac{1}{a}}$.

Now we choose $Y = x^{\frac{1}{3c}} M^{\frac{2}{3a}-\frac{1}{3c}}$ and $(\kappa,\lambda) = (\frac{1}{2}, \frac{1}{2})$. Note that if $M \leqslant \frac{1}{3} x^{\frac{a}{a+c}}$, then $1 \leqslant Y \leqslant \left(\frac{x}{4M}\right)^{\frac{1}{c}}$. By a simple splitting argument, $M^{\frac{1}{a}} \ll x^{\frac{1}{4c}} M^{\frac{1}{a}-\frac{1}{4c}}$ and $x^{\frac{1}{3c}} M^{\frac{2}{3a}-\frac{1}{3c}} \ll x^{\frac{1}{3c}} M^{\frac{5c-2a}{6ac}}$, Lemma 3.5 follows from (3.15) and (3.16) at once. ■

Proof of Theorem 2. We choose $y = x^{\frac{4a(a+b)(1-c\alpha)}{(2a+2b-c)(3c+2a-c(5a+3b)\alpha)}}$, and let $N^* = x^{\frac{3-(5a+3b)\alpha}{3c+2a-c(5a+3b)\alpha}}$, where $\theta = \theta(a,b,c)$ and $\alpha = \alpha(a,b)$ are defined by Theorem 2.

It is easy to verify that if $\alpha(a,b) \geqslant \frac{3c-2b}{c(9a+7b)-6(a+b)^2}$, then $M \leqslant \frac{y}{2} \leqslant \frac{1}{3} x^{\frac{a}{a+c}}$. Applying a splitting argument and Lemma 3.5, we get

$$x^{-\varepsilon} \sum_{m\leqslant y} d(a,b;m) \Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) \ll x^{\frac{1}{3c}} y^{\frac{5c-2a}{6ac}} + x^{\frac{1}{4c}} y^{\frac{1}{a}-\frac{1}{4c}}. \tag{3.17}$$

By the condition $\alpha(a,b) \geqslant \frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$, one can check

$$x^{\frac{1}{4c}} y^{\frac{1}{a}-\frac{1}{4c}} \ll x^{\theta+\varepsilon}. \tag{3.18}$$

Similarly, by the conditions $a+b \leqslant c < 2(a+b)$ and $\alpha(a,b) \geqslant \frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$, we may also check that both $\alpha(a,b) \geqslant \frac{3c-2b}{c(9a+7b)-6(a+b)^2}$ and

$$x^{\frac{1}{3c}} y^{\frac{5c-2a}{6ac}} \ll x^{\theta+\varepsilon} \tag{3.19}$$

hold.

From (3.17)–(3.19) we obtain

$$\sum_{m\leqslant y} d(a,b;m)\Delta\left(\left(\frac{x}{m}\right)^{\frac{1}{c}}\right) \ll x^{\theta+\varepsilon}. \tag{3.20}$$

Now we write

$$T_{(a,b)}(x;N) := \sum_{N<n\leqslant 2N} d(n)\Delta\left(a,b;\frac{x}{n^c}\right). \tag{3.21}$$

For $1 \ll N \ll \left(\frac{x}{y}\right)^{\frac{1}{c}}$, combining Theorem 1 and (3.20), an estimate

$$T_{(a,b)}(x;N) \ll x^{\theta+\varepsilon} \tag{3.22}$$

would suffice to complete the proof Theorem 2.

Now we consider two cases.

Case i. If $1 \ll N \ll N^*$, by the condition $\Delta(a,b;x) \ll x^{\alpha(a,b)}$ we easily check

$$T_{(a,b)}(x;N) \ll \sum_{N<n\leqslant 2N} d(n)\left(\frac{x}{n^c}\right)^{\alpha(a,b)} \ll x^{\alpha(a,b)}N^{1-\alpha(a,b)c}\log x \ll x^{\theta+\varepsilon}. \tag{3.23}$$

Case ii. If $N^* \ll N \ll \left(\frac{x}{y}\right)^{\frac{1}{c}}$, from (3.2) we have

$$\begin{aligned} T_{(a,b)}(x;N) &= -\sum_{N<n\leqslant 2N} d(n) \sum_{m^{a+b}\leqslant \frac{x}{n^c}} \psi\left(\left(\frac{x}{m^b n^c}\right)^{\frac{1}{a}}\right) \\ &\quad - \sum_{N<n\leqslant 2N} d(n) \sum_{m^{a+b}\leqslant \frac{x}{n^c}} \psi\left(\left(\frac{x}{m^a n^c}\right)^{\frac{1}{b}}\right) + O(N) \\ &:= -T^{(1)}_{(a,b)}(x;N) - T^{(2)}_{(a,b)}(x;N) + O(N). \end{aligned} \tag{3.24}$$

Applying a splitting argument and (3.8) in Lemma 3.4 with $(u,v)=(b,a)$, we get

$$\begin{aligned} x^{-\varepsilon}T^{(1)}_{(a,b)}(x;N) &\ll \left(\frac{x^{1-(b-a)\theta}}{N^{*(c-a-b)}}\right)^{\frac{3}{8a}} + x^{\frac{1}{2(a+b)}}(xy^{-1})^{\frac{1}{c}(1-\frac{c}{2(a+b)})} + x^{\theta} \\ &\ll x^{\theta} + x^{\frac{1}{c}} y^{-\frac{1}{c}+\frac{1}{2(a+b)}} = 2x^{\theta}. \end{aligned} \tag{3.25}$$

Here if $MN \ll x^{\theta}$, we use a trivial estimate.

Similarly, from (3.9) with $(u,v)=(a,b)$ and the condition $\alpha(a,b)<\frac{1}{c}\leqslant\frac{1}{a+b}$, we also have

$$x^{-\varepsilon}T^{(2)}_{(a,b)}(x;N)\ll x^{\frac{3}{4(a+b)}}N^{*\frac{3}{4}\left(1-\frac{c}{a+b}\right)}+x^{\frac{1}{2(a+b)}}(xy^{-1})^{\frac{1}{c}\left(1-\frac{c}{2(a+b)}\right)}\ll x^{\theta}. \quad (3.26)$$

Finally, from (3.24)–(3.26), (3.22) holds in this case. This completes the proof of Theorem 2. ■

Proof of Corollary 1.2. From Theorem 5.12 of Krätzel [14], we have

$$\Delta(a,b;x)\ll\begin{cases} x^{\frac{19}{29(a+b)}}\log^2 x, & \text{if } 11a\geqslant 8b\ ,\\ x^{\frac{10}{17a+14b}}\log^2 x, & \text{if } 2a\geqslant b>\frac{11}{8}a,\\ x^{\frac{21}{39a+27b}}\log^2 x, & \text{if } 4a\geqslant b>2a\end{cases} \quad (3.27)$$

By (3.27), the condition $\alpha(a,b)\geqslant\frac{9c-4a-2b}{c(23a+25b)-18(a+b)^2}$ of Theorem 2 is equivalent to

$$\begin{cases} c\geqslant\frac{113a^2+255ab+142b^2}{88a+107b}, & \text{if } 11a\geqslant 8b\ ,\\ c\geqslant\frac{112a^2+270ab+152b^2}{77a+124b}, & \text{if } 2a\geqslant b>\frac{11}{8}a,\\ c\geqslant\frac{37a^2+95ab+54b^2}{22a+47b}, & \text{if } 4a\geqslant b>2a\end{cases} \quad (3.28)$$

Now Corollary 1.2 follows from (3,27), (3.28) and Theorem 2 at once. ■

4. The proof of Theorem 3

The exponential totient function $\phi^{(e)}(n)$ is multiplicative and for each prime power p^a one has $\phi^{(e)}(p^a)=\phi(a)$, where ϕ is the Euler function. Hence for $\Re s>1$, by Euler product we get

$$\sum_{n=1}^{\infty}\frac{\phi^{(e)}(n)}{n^s}=\prod_p\left(1+\sum_{m=1}^{\infty}\frac{\phi^{(e)}(m)}{p^{ms}}\right) \quad (4.1)$$

Applying the product representation of Riemann zeta-function

$$\zeta(s)=\prod_p(1+p^{-s}+p^{-2s}+p^{-3s}+\ldots)=\prod_p(1-p^{-s})^{-1},\Re s>1, \quad (4.2)$$

we have for $\Re s>1$

$$\zeta(s)\zeta(3s)\zeta^2(5s)\zeta^4(7s)=\prod_p(1-p^{-s})^{-1}(1-p^{-3s})^{-1}(1-p^{-5s})^{-2}(1-p^{-7s})^{-4}. \quad (4.3)$$

Let

$$\begin{aligned} f_{\phi^{(e)}}(z) &:= 1+\sum_{m=1}^{\infty}\phi^{(e)}(m)z^m \quad (4.4)\\ &= 1+z+z^2+2z^3+2z^4+4z^5+2z^6+6z^7+4z^8+6z^9\\ &\quad +4z^{10}+\sum_{m=11}^{\infty}q_r(m)z^m. \end{aligned}$$

By a simple calculation one get for $|z|<1$

$$(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4 = 1 - z - z^3 + z^4 - 2z^5 + 2z^6 - 4z^7 + 6z^8 - 2z^9 + 5z^{10} + \ldots + z^{42},$$

$$f_{\phi^{(e)}}(z)(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4 = 1 - 3z^6 - 4z^8 + 4z^9 - 9z^{10} + \sum_{m=11}^{\infty} c_m z^m,$$

and

$$\left(1 + z^6 + z^{12} + \ldots\right)^3 \left(1 + z^8 + z^{16} + \ldots\right)^4 = 1 + 3z^6 + 4z^8 + 6z^{12} + \ldots.$$

From the above two relations, we easily obtain for $|z|<1$

$$f_{\phi^{(e)}}(z)(1-z)(1-z^3)(1-z^5)^2(1-z^7)^4(1-z^6)^{-3}(1-z^8)^{-4} = 1 + 4z^9 + \sum_{m=10}^{\infty} C_m z^m. \quad (4.5)$$

From (4.1)–(4.5) we get that

$$\Phi^{(e)}(s) := \sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta(3s)\zeta^2(5s)\zeta^4(7s)}{\zeta^3(6s)\zeta^4(8s)} V(s), \qquad \Re s > 1, \qquad (4.6)$$

where $V(s)$ is absolutely convergent for $\Re s > \frac{1}{9}$.

Now the asymptotic formula (1.15) follows from (4.6) and (1.13) via the well-known convolution method.

The Ω-estimate (1.16) follows from Theorem 2 of Küleitner and Nowak [17] (or by Balasubramanian, Ramachandra and Subbarao's method in [2]). This completes the proof of Theorem 3.

References

[1] R.C. Baker, *The square-free divisor problem II*, Quart. J. Oxford **47**(2) (1996), 133–146.

[2] R. Balasubramanian, K. Ramachandra and M.V. Subbarao, *On the error function in the asymptotic formula for the counting function of k-full numbers*, Acta Arith. **50** (1988), 107–118.

[3] Xiaodong Cao, *On the general divisor problem* (in Chinese), Acta Math. Sin. **36**(5) (1993), 644–653.

[4] Xiaodong Cao, *The distribution of square-full integers*, Period. Math. Hung. **34**(3) (1997), 169–175.

[5] Xiaodong Cao, Wenguang Zhai, *Some arithmetical functions involving exponential divisors*, Journal of Integer Sequences **13** (2010), Article 10.3.7.

[6] E. Fouvry, H. Iwaniec, *Exponential sums with monomials*, J. Number Theory **33**(3) (1989), 311–333 .
[7] D.R. Heath-Brown, *The Pjateckii-Sapiro prime number theorem*, J. Number Theory **16** (1983), 242–266.
[8] M.N. Huxley, *Area, Lattice Points, and Exponential Sums*, Oxford University Press, New York, 1996.
[9] M.N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. **87**(3) (2003), 591–609.
[10] A. Ivić, *The Riemann Zeta-Function*, John. Wiley & Sons, 1985.
[11] A. Ivić, *The general divisor problem*, J. Number Theory **27** (1987), 73–91.
[12] A. Ivić, E. Krätzel, M. Kühleitner, W.G. Nowak, *Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic* (English summary), Elementare und analytische Zahlentheorie, 89–128, Schr.Wiss. Ges. Johann Wolfgang Goethe Univ.Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006.
[13] I. Kátai, M.V. Subbarao, *On the distribution of exponential divisors*, Ann. Univ. Sci. Budapest. Sect. Comput. **22** (2003), 161–180.
[14] E. Krätzel, *Lattice Points*, Kluwer, Dordrecht-Boston-London, 1988.
[15] E. Krätzel, *The four-dimensional divisor problem*, Unif. Distrib. Theory **3**(1) (2008), 93–103.
[16] E. Krätzel, *New estimates in the four-dimensional divisor problem with applications*, Acta Math. Hungar. **126**(3) (2010), 258–278.
[17] M. Küleitner, W.G. Nowak, *An omega theorem for a class of arithmetic functions*, Math. Nachr. **165** (1994), 79–98.
[18] A. Mercier, W.G. Nowak, *Problème des diviseurs pour des valeurs polynômiales II*, Ann. Sci. Math. Québec **16**(1) (1992), 81–94.
[19] Y.-F.S. Pétermann, J. Wu, *On the sum of exponential divisors of an integer*, Acta Math. Hungar. **77**(1-2) (1997), 159–175.
[20] Y.-F.S. Pétermann, *Arithmetical functions involving exponential divisors: Note on two papers by L. Tóth*, Ann. Univ. Sci. Budapest. Sect. Comput. **32** (2010), 143–149.
[21] J. Sándor, *On an exponential totient function*, Sudia Univ. Babes-Bolyai Math. **41** (1996), 91–94.
[22] E.G. Straus and M.V. Subbarao, *On exponential divisors*, Duke Math. J. **41** (1974), 465–471.
[23] P. Sargos and J. Wu, *Mutiple exponential sums with monomials and their applications in number theory*, Acta Math. Hungar. **87**(4) (2000), 333–354.
[24] M.V. Subbarao, *On some arithmetic convolutions*, in The Theory of arithmetic functions, Lecture Notes in Mathematics, no. 251, 247–271, Springer, 1972.
[25] L. Tóth, *On certain arithmetic functions involving exponential divisors*, Ann. Univ. Sci. Budapest. Sect. Comput. **24** (2004), 285–294.
[26] L. Tóth, *On certain arithmetical functions involving exponential divisors II*, Ann. Univ. Sci. Budapest. Sect. Comput. **27** (2007), 155–166.
[27] L. Tóth, *An order result for the exponential divisor function*, Publ. Math. Debrecen **71**(1-2) (2007), 165–171.

[28] M. Vogts, *Many-dimensional generalized divisor problems*, Math. Nachr. **124** (1985), 103–121.

[29] Jie Wu, *Problème de diviseurs exponentiels et entiers exponentiellement sans facteur carré*, J. Théor. Nombres Bordeaux **7**(1) (1995), 133–141.

[30] Jie Wu, *On the distribution of square-full integers*, Arch. Math. (Basel) **77**(3) (2001), 233–240.

[31] Wenguang Zhai and Xiaodong Cao, *On the mean square of the error term for the asymmetric two-dimensional divisor problem (I)*, Monatsh. Math. **159** (2010), 185–209.

Addresses: Xiaodong Cao: Department of Mathematics and Physics, Beijing Institute of Petro-Chemical Technology, Beijing, 102617, P.R. China;
Wenguang Zhai: Department of Mathematics, China University of Mining and Technology, Beijing 100083, P.R. China.

E-mail: caoxiaodong@bipt.edu.cn, zhaiwg@hotmail.com

Received: 10 July 2012; **revised:** 7 March 2013