On the Fourier analysis of operators on the torus

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Abstract. Basic properties of Fourier integral operators on the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ are studied by using the global representations by Fourier series instead of local representations. The results can be applied in studying hyperbolic partial differential equations.

1. Introduction

In this paper we will discuss the version of the Fourier analysis and pseudo-differential operators on the torus. Using the toroidal Fourier transform we will show several simplifications of the standard theory. We will also discuss the corresponding toroidal version of Fourier integral operators. To distinguish them from those defined using the Euclidean Fourier transform, we will call them *Fourier series operators*. The use of discrete Fourier transform will allow to use global representation of these operators, thus eliminating a number of topological obstructions known in the standard theory. We will prepare the machinery and describe how it can be further used in the calculus of Fourier series operators and applications to hyperbolic partial differential equations. In fact, the form of the required discrete calculus is not a-priori clear, for example, the form of the discrete Taylor's theorem best adopted to the calculus. We will develop the corresponding version of the periodic analysis similar in formulations to the standard Euclidean theory.

It was realised already in the 1970s that on the torus, one can study pseudodifferential operators globally using Fourier series expansions, in analogy to Euclidean pseudodifferential calculus. These *periodic pseudodifferential operators* were treated e.g. by Agranovich [1, 2]. Contributions have been made by many authors, and the following is a non-comprehensive list of the research on the torus: Agranovich, crediting the idea to Volevich, proposed the Fourier series representation of pseudodifferential operators. Later, he proved the equivalence of the Fourier series representation and Hörmander's definition for $(1,0)$-symbol classes; the case of classical pseudodifferential operators on the circle had been treated by Saranen

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and Wendland [15]; McLean [9] proved the equivalence of these approaches for Hörmander's general (ρ,δ)-classes, by using charts; in [17], the equivalence for the case of $(1,0)$-classes is proven by studying iterated commutators of pseudodifferential operators and smooth vector fields. Elschner [5] and Amosov [3] constructed asymptotic expansions for classical pseudodifferential operators; these results were generalized for (ρ,δ)-classes in [18]. There are plenty of papers considering applications and numerical computation of pseudodifferential equations on torus, e.g. spline approximations by Prössdorf and Schneider [10], physical applications by e.g. Vainikko and Lifanov [19, 20], and many others.

On the other hand, the use of operators which are discrete in the frequency variable allows one to weaken regularity assumptions on symbols with respect to ξ. Symbols with low regularity in x have been under intensive study for many years, e.g. Kumano-go and Nagase [8], Sugimoto [16], Boulkhemair [4], Garello and Morando [6], and many others. However, in these papers one assumes symbols to be smooth or sufficiently regular in ξ. The discrete approach in this paper will allow us to reduce regularity assumptions with respect to ξ. For example, no regularity with respect to ξ is assumed for L^2 estimates, and for elements of the calculus. Moreover, one can consider scalar hyperbolic equations with C^1 symbols with respect to ξ. For example, this allows to construct parametrices to certain hyperbolic systems with variable multiplicities. Details of such constructions will appear in our forthcoming paper [14].

Let us now fix the notation. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz test function space with its usual topology, and let $\mathcal{S}'(\mathbb{R}^n)$ be its dual, the space of tempered distributions. Let $\mathcal{F}_E : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be the Euclidean Fourier transform (hence the subscript $_E$) defined by

$$(\mathcal{F}_E f)(\xi) = \widehat{f}_E(\xi) := \int_{\mathbb{R}^n} f(x)\ \mathrm{e}^{-\mathrm{i}x\cdot\xi}\ \tilde{\mathrm{d}}x,$$

where $\tilde{\mathrm{d}}x = (2\pi)^{-n}\ \mathrm{d}x$. Then $\mathcal{F}_E$ is a bijection and

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}_E(\xi)\ \mathrm{e}^{\mathrm{i}x\cdot\xi}\ \mathrm{d}\xi,$$

and this Fourier transform can be uniquely extended to $\mathcal{F}_E : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

The main symbol class in the sequel consists of Hörmander's (ρ,δ)-symbols of order m: Let $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. For $\xi \in \mathbb{R}^n$ define $\langle\xi\rangle := (1 + \|\xi\|^2)^{1/2}$, where $\|\xi\|^2 := \sum_{j=1}^n |\xi_j|^2$. Then $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ consists of those functions $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for which

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\right| \le C_{\sigma\alpha\beta m}\ \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{1.1}$$

for every $x \in \mathbb{R}^n$ and for every $\alpha, \beta \in \mathbb{N}^n$.

Let $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ denote the n-dimensional torus. We may identify $\mathbb{T}^n$ with the hypercube $[0,2\pi[^n \subset \mathbb{R}^n$ (or $[-\pi,\pi[^n$). Functions on $\mathbb{T}^n$ may be thought as those functions on $\mathbb{R}^n$ that are 2π-periodic in each of the coordinate directions. Let $\mathcal{D}(\mathbb{T}^n)$ be the vector space $C^\infty(\mathbb{T}^n)$ endowed with the usual test function topology, and let $\mathcal{D}'(\mathbb{T}^n)$ be its dual, the space of distributions on $\mathbb{T}^n$. Inclusion $\mathcal{D}(\mathbb{T}^n) \subset \mathcal{D}'(\mathbb{T}^n)$ is interpreted by

$$\phi(\psi) := \int_{\mathbb{T}^n} \phi(x)\ \psi(x)\ \mathrm{d}x,$$

where we identify the measure on torus with the corresponding restriction of the Euclidean measure on the hypercube. Let $\mathcal{S}(\mathbb{Z}^n)$ denote the space of *rapidly decaying functions* $\mathbb{Z}^n \to \mathbb{C}$. Let $\mathcal{F}_T : \mathcal{D}(\mathbb{T}^n) \to \mathcal{S}(\mathbb{Z}^n)$ be the *toroidal Fourier transform* (hence the subscript $_T$) defined by

$$(\mathcal{F}_T f)(\xi) = \widehat{f}_T(\xi) := \int_{\mathbb{T}^n} f(x)\ \mathrm{e}^{-\mathrm{i}x\cdot\xi}\ \tilde{\mathrm{d}}x,$$

where $\tilde{\mathrm{d}}x = (2\pi)^{-n}\ \mathrm{d}x$. Then $\mathcal{F}_T$ is a bijection and

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} \widehat{f}_T(\xi)\ \mathrm{e}^{\mathrm{i}x\cdot\xi}.$$

This Fourier transform is extended uniquely to $\mathcal{F}_T : \mathcal{D}'(\mathbb{T}^n) \to \mathcal{S}'(\mathbb{Z}^n)$. Notice that $\mathcal{S}'(\mathbb{Z}^n)$ consists of those functions $\mathbb{Z}^n \to \mathbb{C}$ growing at infinity at most polynomially. Any continuous linear operator $A : \mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ can be presented by a formula

$$(Af)(x) = \sum_{\xi\in\mathbb{Z}^n} \sigma_A(x,\xi)\ \widehat{f}(\xi)\ \mathrm{e}^{\mathrm{i}x\cdot\xi},$$

where the unique function $\sigma_A \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ is called the *symbol of* A:

$$\sigma_A(x,\xi) = \mathrm{e}^{-\mathrm{i}x\cdot\xi} A e_\xi(x),$$

where $e_\xi(x) := \mathrm{e}^{\mathrm{i}x\cdot\xi}$. Notice that when $\widehat{s_A(x)}_T(\xi) = \sigma_A(x,\xi)$, the Schwartz kernel K_A of A satisfies

$$K_A(x,y) = s_A(x)(x-y)$$

in the sense of distributions.

Next, in analogy to the classical differential calculus, we discuss difference calculus, which is needed when dealing with Fourier series operators.

2. Difference calculus

Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$. Let $v_j \in \mathbb{N}^n$, $(v_j)_j = 1$ and $(v_j)_i = 0$ if $i \neq j$. Let us define the partial difference operator $\triangle_{\xi_j}$ by

$$\triangle_{\xi_j}\sigma(\xi) := \sigma(\xi + v_j) - \sigma(\xi),$$

and define

$$\triangle_\xi^\alpha := \triangle_{\xi_1}^{\alpha_1} \cdots \triangle_{\xi_n}^{\alpha_n}$$

for $\alpha = (\alpha_j)_{j=1}^n \in \mathbb{N}^n$.

Lemma 2.1. *By the binomial theorem,*

$$\triangle_\xi^\alpha \sigma(\xi) = \sum_{\beta\le\alpha} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \sigma(\xi+\beta).$$

By induction, one can show:

Lemma 2.2 (Leibnitz formula for differences)**.** *Let $\phi, \psi : \mathbb{Z}^n \to \mathbb{C}$. Then*

$$\triangle_\xi^\alpha (\phi\psi)(\xi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \left(\triangle_\xi^\beta \phi(\xi)\right)\ \triangle_\xi^{\alpha-\beta} \psi(\xi+\beta).$$

"Integration by parts" has the discrete analogy "summation by parts"

$$\sum_{\xi\in\mathbb{Z}^n} \phi(\xi)\ (\triangle_\xi^\alpha \psi)(\xi) = -\sum_{\xi\in\mathbb{Z}^n} ((\triangle_\xi{}^\alpha)^t \phi)(\xi)\ \psi(\xi),$$

where $((\triangle_{\xi_j})^t \phi)(\xi) = \phi(\xi) - \phi(\xi - v_j)$, provided that the series converge absolutely.

For $\xi \in \mathbb{Z}^n$ and $\gamma \in \mathbb{Z}^n$, let us define

$$\xi^{(\gamma)} = \prod_{j=1}^n \xi_j^{(\gamma_j)},$$

where

$$\xi_j^{(\gamma_j)} := \begin{cases} \prod_{i=0}^{\gamma_j-1} (\xi_j - i), & \gamma_j > 0, \\ 1, & \gamma_j = 0, \\ \prod_{i=\gamma_j+1}^{0} (\xi_j - i)^{-1}, & \gamma_j < 0. \end{cases}$$

Then

$$\triangle_\xi^\alpha \xi^{(\gamma)} = \gamma^{(\alpha)}\ \xi^{(\gamma-\alpha)},$$

in analogy to $\partial_\xi^\alpha \xi^\gamma = \gamma^{(\alpha)} \xi^{\gamma-\alpha}$.

Let us now discuss the discrete version of the Taylor's theorem. For simplicity, let us consider the one dimensional case first.

Theorem 2.3 (Discrete Taylor's theorem)**.** *For a function $\sigma : \mathbb{Z} \to \mathbb{C}$,*

$$\sigma(\xi+\eta) = \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!}\ (\triangle_\xi^\alpha \sigma)(\xi)\ \eta^{(\alpha)} + R_N(\xi,\eta), \quad (\xi,\eta \in \mathbb{Z},\ N \in \mathbb{N}),$$

where

$$\begin{aligned} |\triangle_\xi^\alpha R_N(\xi,\eta)| \ &\le \ \begin{cases} \frac{1}{N!} \eta^{(N)} \max_{0\le\kappa<\eta} \left|\triangle_\xi^{N+\alpha} \sigma(\xi+\kappa)\right|, & \eta \ge N, \\ 0, & 0 \le \eta < N, \\ \frac{1}{N!} \left|\eta^{(N)}\right| \max_{\eta\le\kappa<0} \left|\triangle_\xi^{N+\alpha} \sigma(\xi+\kappa)\right|, & \eta < 0, \end{cases} \\ &\le \ \frac{1}{N!} \left|\eta^{(N)}\right| \max_{\kappa\in\{0,\ldots,\eta\}} \left|\triangle_\xi^{N+\alpha} \sigma(\xi+\kappa)\right|. \end{aligned}$$

Notice that the estimate above resembles closely the Lagrange form of the error term in the traditional Taylor theorem:

$$\begin{cases} f(x+y) = \sum_{j=0}^{N-1} \frac{1}{j!} f^{(j)}(x) y^j + R_N(x,y), \\ R_N(x,y) = \frac{1}{N!} f^{(N)}(x+\theta) y^N, \qquad \theta \in [\min\{0,y\}, \max\{0,y\}]. \end{cases}$$

Proof. First assume that $\eta \geq 0$. Then, by the binomial formula,

$$\sigma(\xi+\eta) = (I+\triangle_\xi)^\eta \sigma(\xi) = \sum_{\alpha=0}^{\eta} \binom{\eta}{\alpha} \triangle_\xi^\alpha \sigma(\xi) = \sum_{\alpha=0}^{\eta} \frac{1}{\alpha!} \triangle_\xi^\alpha \sigma(\xi)\, \eta^{(\alpha)}. \tag{2.1}$$

Thus $R_N(\xi,\eta) = 0$ for $0 \leq \eta < N$. Therefore

$$\triangle_\eta^\alpha R_N(\xi,\eta)|_{\eta=0} = 0,$$

when $0 \leq \alpha < N$. Now let η be an arbitrary integer. We notice that $\triangle_\eta^N \eta^{(\alpha)} = \alpha^{(N)}\ \eta^{(\alpha-N)} = 0$ for $0 \leq \alpha < N$, so that when we apply $\triangle_\eta^N$, we get

$$\triangle_\eta^N \sigma(\xi+\eta) = \triangle_\eta^N R_N(\xi+\eta).$$

We have hence the Cauchy problem

$$\begin{cases} \triangle_\eta^N R_N(\xi,\eta) = \triangle_\eta^N \sigma(\xi+\eta), \\ \triangle_\eta^\alpha R_N(\xi,\eta)\big|_{\eta=0} = 0, \qquad 0 \leq \alpha \leq N-1. \end{cases}$$

It is enough to prove the estimate for $|R_N(\xi,\eta)|$ (i.e. $\alpha = 0$). Let us define $\sigma(\eta) := \eta^{(N)}/N!$. Then $\triangle_\eta^N \sigma(\eta) = N^{(N)} \eta^{(N-N)}/N! = 1$, and $\triangle_\xi^\alpha \sigma(\xi)|_{\xi=0} = 0$ when $0 \leq \alpha < N$, so by the uniqueness of the solution of the Cauchy problem it has to be

$$\begin{cases} \sum_{\kappa_N=0}^{-1} \sum_{\kappa_{N-1}=1}^{\kappa_N-1} \cdots \sum_{\kappa_1=1}^{\kappa_2-1} 1 = \frac{1}{N!} \eta^{(N)}, & \eta \geq N, \\ \sum_{\kappa_N=m}^{-1} \sum_{\kappa_{N-1}=\kappa_N}^{-1} \cdots \sum_{\kappa_1=\kappa_2}^{-1} 1 = \frac{1}{N!} \left|\eta^{(N)}\right|, & \eta < 0, \end{cases}$$

completing the proof.

Let us now deal with discrete Taylor polynomial -like expansions for a function $f : \mathbb{Z}^n \to \mathbb{C}$. For $b \in \mathbb{N}$, let us denote

$$I_k^b := \sum_{0 \leq k < b} \quad \text{and} \quad I_k^{-b} := -\sum_{-b \leq k < 0}. \tag{2.2}$$

It is useful to think of $I_\xi^\theta \cdots$ as of a discrete version of the one-dimensional integral $\int_0^\theta \cdots \mathrm{d}\xi$. In this discrete context, the difference $\triangle_\xi$ takes the role of the differential operator $\mathrm{d}/\mathrm{d}\xi$.

In the sequel, we adopt the notational conventions

$$I_{k_1}^\theta I_{k_2}^{k_1} \cdots I_{k_\alpha}^{k_{\alpha-1}} 1 = \begin{cases} 1, & \text{if } \alpha = 0, \\ I_k^\theta 1, & \text{if } \alpha = 1, \\ I_{k_1}^\theta I_{k_2}^{k_1} 1, & \text{if } \alpha = 2, \end{cases}$$

and so on.

Lemma 2.4. *If $\theta \in \mathbb{Z}$ and $\alpha \in \mathbb{N}$ then*

$$I_{k_1}^{\theta} I_{k_2}^{k_1} \cdots I_{k_\alpha}^{k_{\alpha-1}} 1 = \frac{1}{\alpha!}\ \theta^{(\alpha)}. \tag{2.3}$$

Proof. The result follows step-by-step from the observations $k^{(0)} \equiv 1$, $\triangle_k k^{(i)} = i\ k^{(i-1)}$ and $I_k^b \triangle_k k^{(i)} = b^{(i)}$.

Remark 2.5. This is like applying a discrete trivial version of the fundamental theorem of calculus: $\int_0^\theta f'(\xi)\ \mathrm{d}\xi = f(\theta) - f(0)$ for smooth enough $f : \mathbb{R} \to \mathbb{C}$ corresponds to $I_\xi^\theta \triangle_\xi f(\xi) = f(\theta) - f(0)$ for $f : \mathbb{Z} \to \mathbb{C}$.

Corollary 2.6. *If $\theta \in \mathbb{Z}^n$ and $\alpha \in \mathbb{N}^n$ then*

$$\prod_{j=1}^{n} I_{k(j,1)}^{\theta} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)} 1 = \frac{1}{\alpha!}\ \theta^{(\alpha)}, \tag{2.4}$$

where $\prod_{j=1}^n I_j$ means $I_1 I_2 \cdots I_n$, where $I_j := I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)}$.

We now have

Theorem 2.7. *Let $p : \mathbb{Z}^n \to \mathbb{C}$ and*

$$r_M(\xi,\theta) := p(\xi+\theta) - \sum_{|\alpha|<M} \frac{1}{\alpha!}\ \theta^{(\alpha)} \triangle_\xi^\alpha p(\xi).$$

Then

$$\left|\triangle_\xi^\omega r_M(\xi,\theta)\right| \le c_M \max_{|\alpha|=M,\ \nu\in Q(\theta)} \left|\theta^{(\alpha)}\ \triangle_\xi^{\alpha+\omega} p(\xi+\nu)\right|, \tag{2.5}$$

where $Q(\theta) := \{\nu \in \mathbb{Z}^n :\ \min(0,\theta_j) \le \nu_j \le \max(0,\theta_j)\}$.

Proof. For $0 \neq \alpha \in \mathbb{N}^n$, let us denote $m_\alpha := \min\{j :\ \alpha_j \neq 0\}$. For $\theta \in \mathbb{Z}^n$ and $i \in \{1,\ldots,n\}$, let us define $\nu(\theta,i,k) \in \mathbb{Z}^n$ by

$$\nu(\theta,i,k) := (\theta_1,\ldots,\theta_{i-1},k,0,\ldots,0),$$

i.e.

$$\nu(\theta,i,k)_j = \begin{cases} \theta_j, & \text{if } 1 \le j < i, \\ k, & \text{if } j = i, \\ 0, & \text{if } i < j \le n. \end{cases}$$

We claim that the remainder can be written in the form

$$r_M(\xi,\theta) = \sum_{|\alpha|=M} r_\alpha(\xi,\theta), \tag{2.6}$$

where for each α,

$$r_\alpha(\xi,\theta) = \prod_{j=1}^{n} I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)}\ \triangle_\xi^\alpha p(\xi + \nu(\theta, m_\alpha, k(m_\alpha, \alpha_{m_\alpha}))); \tag{2.7}$$

recall (2.2) and (2.4). The proof of (2.7) is by induction: The first remainder term r_1 is of the claimed form, since

$$r_1(\xi,\theta) = p(\xi+\theta) - p(\xi) = \sum_{i=1}^{n} r_{v_i}(\xi,\theta),$$

where

$$r_{v_i}(\xi,\theta) = I_k^{\theta_i} \triangle_\xi^{v_i} p(\xi + \nu(\theta,i,k));$$

here r_{v_i} is of the form (2.7) for $\alpha = v_i$, $m(\alpha) = i$ and $\alpha_{m_\alpha} = 1$. So suppose that the claim (2.7) is true up to order $|\alpha| = M$. Then

$$\begin{aligned}
r_{M+1}(\xi,\theta) &= r_M(\xi,\theta) - \sum_{|\alpha|=M} \frac{1}{\alpha!}\,\theta^{(\alpha)}\,\triangle_\xi^\alpha p(\xi) \\
&= \sum_{|\alpha|=M} \left(r_\alpha(\xi,\theta) - \frac{1}{\alpha!}\,\theta^{(\alpha)}\,\triangle_\xi^\alpha p(\xi) \right) \\
&= \sum_{|\alpha|=M} \prod_{j=1}^{n} I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)} \\
&\quad \triangle_\xi^\alpha \left[p(\xi + \nu(\theta, m_\alpha, k(m_\alpha, \alpha_{m_\alpha}))) - p(\xi) \right],
\end{aligned}$$

where we used (2.7) and (2.4) to obtain the last equality. Combining this to the observation

$$p(\xi + \nu(\theta, m_\alpha, k)) - p(\xi) = \sum_{i=1}^{m_\alpha} I_\ell^{\nu(\theta,m_\alpha,k)_i}\ \triangle_\xi^{v_i} p(\xi + \nu(\theta,i,\ell)),$$

we get

$$\begin{aligned}
r_{M+1}(\xi,\theta) &= \sum_{|\alpha|=M} \prod_{j=1}^{n} I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)} \sum_{i=1}^{m_\alpha} I_{\ell(i)}^{\nu(\theta,m_\alpha,k(m_\alpha,\alpha_{m_\alpha}))_i} \\
&\quad \triangle_\xi^{\alpha+v_i} p(\xi + \nu(\theta,i,\ell(i))) \\
&= \sum_{|\beta|=M+1} \prod_{j=1}^{n} I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\beta_j)}^{k(j,\beta_j-1)} \\
&\quad \triangle_\xi^\beta p(\xi + \nu(\theta, m_\beta, k(m_\beta, \beta_{m_\beta})));
\end{aligned}$$

the last step here is just simple tedious book-keeping. Thus the induction proof of (2.7) is complete. Finally, let us prove estimate (2.5). By (2.7), we obtain

$$\begin{aligned}\left|\triangle_\xi^\omega r_M(\xi,\theta)\right| &= \left|\sum_{|\alpha|=M} \triangle_\xi^\omega r_\alpha(\xi,\theta)\right| \\ &= \left|\sum_{|\alpha|=M} \prod_{j=1}^n I_{k(j,1)}^{\theta_j} I_{k(j,2)}^{k(j,1)} \cdots I_{k(j,\alpha_j)}^{k(j,\alpha_j-1)}\right. \\ &\qquad \left.\triangle_\xi^{\alpha+\omega} p(\xi+\nu(\theta,m_\alpha,k(m_\alpha,\alpha_{m_\alpha})))\right| \\ &\leq \sum_{|\alpha|=M} \frac{1}{\alpha!}\left|\theta^{(\alpha)}\right| \max_{\nu\in Q(\theta)} \left|\triangle_\xi^{\alpha+\omega} p(\xi+\nu)\right|,\end{aligned}$$

where in the last step we used (2.4). The proof is complete.

Remark 2.8. If $n \geq 2$, there are many alternative forms for remainders $r_\alpha(\xi,\theta)$. This is due to the fact that there may be many different shortest discrete step-by-step paths in the space $\mathbb{Z}^n$ from ξ to $\xi+\theta$. In the proof above, we chose just one such path, traveling via the points

$$\xi, \quad \xi+\theta_1 v_1, \quad \ldots, \quad \xi+\sum_{i=1}^j \theta_i v_i, \quad \ldots, \quad \xi+\theta.$$

But if $n = 1$, there is just one shortest discrete path from $\xi \in \mathbb{Z}$ to $\theta \in \mathbb{Z}$, and in that case

$$r_M(\xi,\theta) = I_{k_1}^\theta I_{k_2}^{k_1} \cdots I_{k_M}^{k_{M-1}} \triangle_\xi^M p(\xi+k_M).$$

Notice also that the discrete Taylor theorem presented above implies the following smooth Taylor result:

Corollary 2.9. *Let $p \in C^\infty(\mathbb{R}^n)$ and*

$$r_M(\xi,\theta) := p(\xi+\theta) - \sum_{|\alpha|<M} \frac{1}{\alpha!}\, \theta^\alpha \left(\frac{\partial}{\partial\xi}\right)^\alpha p(\xi).$$

Then

$$\left|\partial_\xi^\omega r_M(\xi,\theta)\right| \leq c_M \max_{|\alpha|=M,\ \nu\in Q_{\mathbb{R}^n}(\theta)} \left|\theta^\alpha\ \partial_\xi^{\alpha+\omega} p(\xi+\nu)\right|, \tag{2.8}$$

where $Q_{\mathbb{R}^n}(\theta) := \{\nu \in \mathbb{R}^n :\ \min(0,\theta_j) \leq \nu_j \leq \max(0,\theta_j)\}$.

Remark 2.10. We see that in the remainder estimates above, the cubes $Q(\theta) \subset \mathbb{Z}^n$ and $Q_{\mathbb{R}^n}(\theta) \subset \mathbb{R}^n$ could be replaced by (discrete, resp. continuous) paths from 0 to θ; especially, $Q_{\mathbb{R}^n}(\theta)$ could be replaced by the straight line from 0 to θ.

3. Pseudodifferential operators on the torus

Let $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$. Then $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ consists of those functions $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ for which

$$\left|\triangle_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\right| \leq C_{\sigma\alpha\beta m}\ \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{3.1}$$

for every $x \in \mathbb{T}^n$, for every $\alpha, \beta \in \mathbb{N}^n$. If $\sigma_A \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, we denote $A \in \mathrm{Op}S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$. The class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$ consists of the functions $a \in C^\infty(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$ such that

$$\left|\triangle_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi)\right| \leq C_{a\alpha\beta\gamma m}\ \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta+\gamma|} \tag{3.2}$$

for every $x, y \in \mathbb{T}^n$, for every $\alpha, \beta, \gamma \in \mathbb{N}^n$; such a function a is called an *amplitude* of order $m \in \mathbb{R}$ of type (ρ, δ). Formally we may define

$$(\mathrm{Op}(a)f)(x) := \int_{\mathbb{T}^n} f(y) \sum_{\xi\in\mathbb{Z}^n} a(x,y,\xi)\ \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\ \tilde{\mathrm{d}}y$$

for $f \in \mathcal{D}(\mathbb{T}^n)$.

Remark 3.1. On $\mathbb{T}^n$, Hörmander's usual (ρ,δ)-symbol class of order $m \in \mathbb{R}$ coincides with the class $\mathrm{Op}S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ [9].

Lemma 3.2. *Let $a \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$, and define*

$$a_\alpha(x,y,\xi) := \left(\mathrm{e}^{\mathrm{i}(y-x)} - 1\right)^\alpha a(x,y,\xi),$$

where $\alpha \in \mathbb{N}^n$. Then $\triangle_\xi^\alpha a \in S^{m-\rho|\alpha|}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$ and $Op(a_\alpha) = Op(\triangle_\xi^\alpha a)$.

Proof. Clearly $\triangle_\xi^\alpha a \in S^{m-\rho|\alpha|}_{\rho,\delta}(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$. Now

$$\begin{aligned}
(\mathrm{Op}(a_\alpha)f)(x) &= \int_{\mathbb{T}^n} f(y) \sum_{\xi\in\mathbb{Z}^n} a_\alpha(x,y,\xi)\ \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\ \tilde{\mathrm{d}}y \\
&= \int_{\mathbb{T}^n} f(y) \left(\sum_{\xi\in\mathbb{Z}^n} \left(\mathrm{e}^{\mathrm{i}(y-x)} - 1\right)^\alpha a(x,y,\xi)\ \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\right) \tilde{\mathrm{d}}y \\
&= \int_{\mathbb{T}^n} f(y) \left(\sum_{\xi\in\mathbb{Z}^n} a(x,y,\xi)\ (\triangle_\xi^\alpha)^t \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\right) \tilde{\mathrm{d}}y \\
&= \int_{\mathbb{T}^n} f(y) \left(\sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\ \triangle_\xi^\alpha a(x,y,\xi)\right) \tilde{\mathrm{d}}y.
\end{aligned}$$

Thus $\mathrm{Op}(a_\alpha) = \mathrm{Op}(\triangle_\xi^\alpha a)$.

Theorem 3.3. *For every amplitude $a \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$ there exists a unique symbol $\sigma \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ satisfying $\mathrm{Op}(a) = \mathrm{Op}(\sigma)$, where*

$$\sigma(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \triangle_\xi^\alpha \, \partial_y^{(\alpha)} a(x,y,\xi)|_{y=x}. \tag{3.3}$$

Proof: essentially the same as in [18].

4. Periodisation

Let us define the *periodisation operator* $p : \mathcal{S}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{T}^n)$ by

$$pu(x) := \mathcal{F}_T^{-1}((\mathcal{F}_E u)|_{\mathbb{Z}^n})(x).$$

Let us describe the extension of this periodisation for some nice-enough classes of distributions. From the proof of the Poisson summation formula it follows that

$$pu(x) = \sum_{\xi \in \mathbb{Z}^n} u(x + 2\pi\xi).$$

This formula makes sense almost everywhere for $u \in L^1(\mathbb{R}^n)$. Indeed, formally

$$\begin{aligned} (pu)(x) &= \sum_{k \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot k} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot k} \; u(y) \; \mathrm{d}y = \int_{\mathbb{R}^n} u(y) \left(\sum_{k \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot k} \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^n} u(y) \; \delta_{\mathbb{Z}^n}(2\pi(x-y)) \; \mathrm{d}y = \sum_{\xi \in \mathbb{Z}^n} u(x + 2\pi\xi). \end{aligned}$$

This calculation can be justified in the standard way. Now $pu \in L^1(\mathbb{T}^n)$ with $\|pu\|_{L^1(\mathbb{T}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)}$. Moreover, if $\xi \in \mathbb{Z}^n$ then

$$\widehat{pu}_T(\xi) = \widehat{u}_E(\xi).$$

Clearly $p : L^1(\mathbb{R}^n) \to L^1(\mathbb{T}^n)$ is a surjection. We will also use that

$$(pu)(x) = \sum_{\xi \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \widehat{pu}_T(\xi) = \sum_{\xi \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} \widehat{u}_E(\xi).$$

Let us establish basic properties of pseudodifferential operators with respect to periodisation. Let us call symbol $a(x,\xi)$ periodic if the function $x \mapsto a(x,\xi)$ is 2π-periodic. We will use tildes to denote corresponding restricted operators.

Proposition 4.1. *Let $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ be a periodic symbol. Let $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$. Then $p \circ a(X,D) = \tilde{a}(X,D) \circ p$.*

Notice that a does not have to be a symbol, as the same property holds when we define $a(X,D)$ in the usual sense, by even quite irregular amplitude $a(x,\xi)$.

Proof. Notice that $\tilde{a} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$. Let $f \in L^1(\mathbb{R}^n)$. Then we have

$$\begin{aligned}
p(a(X,D)f)(x) &= \sum_{k\in\mathbb{Z}^n} a(x+2\pi k, D) f(x+2\pi k) \\
&= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x+2\pi k)\cdot\xi}\ a(x+2\pi k,\xi)\ \widehat{f}_E(\xi)\ \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}2\pi k\cdot\xi}\right) \mathrm{e}^{\mathrm{i}x\cdot\xi}\ a(x,\xi)\ \widehat{f}_E(\xi)\ \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ a(x,\xi)\ \widehat{f}_E(\xi)\ \delta_{\mathbb{Z}^n}(\xi)\ \mathrm{d}\xi \\
&= \sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ a(x,\xi)\ \widehat{f}_E(\xi) \\
&= \sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ a(x,\xi)\ \widehat{pf}_T(\xi) \\
&= \tilde{a}(X,D)(pf)(x);
\end{aligned}$$

these calculations are justified in the sense of distributions. The proof is complete.

Since we will not always work with periodic symbols it may be convenient to periodize them. If $a(X,D)$ is a pseudodifferential operator with symbol $a(x,\xi)$, by $(pa)(X,D)$ we will denote a pseudodifferential operator with symbol $(pa)(x,\xi) = \sum_{k\in\mathbb{Z}^n} a(x+2\pi k,\xi)$. This makes sense if, for example, a in integrable in x.

Proposition 4.2. *Let $a \in S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ satisfy $a(x,\xi)=0$ for all $x\in\mathbb{R}^n\setminus[-\pi,\pi]^n$. Then we have*

$$a(X,D)f = (pa)(X,D)f + Rf,$$

for all f supported in $[-\pi,\pi]^n$. Here $R : \mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is a smoothing pseudodifferential operator.

Proof. By our definition we can write

$$(pa)(X,D)f(x) = \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ a(x+2\pi k,\xi)\ \widehat{f}_E(\xi)\ \mathrm{d}\xi,$$

and let $Rf = a(X,D)f-(pa)(X,D)f$. The assumption on the support of a implies that for every x there is only one $k\in\mathbb{Z}^n$ for which $a(x+2\pi k,\xi)\neq 0$, so the sum consists of only one term. It follows that $Rf(x)=0$ for $x\in[-\pi,\pi]^n$. Let now $x\in\mathbb{R}^n\setminus[-\pi,\pi]^n$. Since

$$Rf(x) = -\sum_{k\in\mathbb{Z}^n,k\neq 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\ a(x+2\pi k,\xi)\ f(y)\ \mathrm{d}y\ \mathrm{d}\xi$$

is just a single term and $|x-y|>0$, we can integrate by parts with respect to ξ any number of times. This implies that $R\in\Psi^{-\infty}$ and that Rf decays at infinity faster than any power. The proof is complete since the same argument can be applied to the derivatives of Rf.

Remark 4.3. Note that if f is compactly supported, but not necessarily in the cube $[-\pi,\pi]^n$, sums in the proof may consist of finite number of terms. This means that on $\mathcal{E}'(\mathbb{R}^n)$, modulo a smoothing operator, we can write $a(X,D)$ as a finite sum of operators with periodic symbols. Moreover, the same argument applies if $a(x,\xi)$ is compactly supported in x, but not necessarily in $[-\pi,\pi]^n$.

This proposition allows us to extend formula of Proposition 4.1 to perturbations of periodic symbols. We will use it when $a(x,D)$ is a sum of a constant coefficient operator and an operator with symbol having compact x-support.

Corollary 4.4. *Let $a(X,D)$ be an operator with symbol*

$$a(x,\xi) = a_1(x,\xi) + a_0(x,\xi),$$

where $a_1 \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is periodic in x and $a_0 \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ has compact x-support. Then there is a symbol $\tilde{b} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ such that

$$p(a(X,D)f) = \tilde{b}(X,D)(pf) + p(Rf),\ f \in \mathcal{E}'(\mathbb{R}^n),$$

where $R : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. In particular, if $\operatorname{supp}(a_0(\cdot,\xi)), \operatorname{supp}(f) \subset [-\pi,\pi]^n$, we can take $\tilde{b}(X,D) = \widetilde{a_1}(X,D) + \widetilde{pa_0}(X,D)$.

We assumed that symbols are smooth, but the requirement of the smoothness of $a_1(x,\xi)$ is not necessary similar to Proposition 4.1.

Proof. By Proposition 4.2, $a(X,D) = a_1(X,D) + (pa_0)(X,D) + R$. Since operator $b(X,D) = a_1(X,D) + (pa_0)(X,D)$ has periodic symbol, by Proposition 4.1 we have $p \circ b(X,D) = \tilde{b}(X,D) \circ p = \widetilde{a_1}(X,D) \circ p + \widetilde{pa_0}(X,D) \circ p$. Since $R : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, we also have $p \circ R : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}(\mathbb{T}^n)$. The proof is complete.

5. Conditions for L^2-boundedness

Next we study conditions on a toroidal symbol σ_A that guarantee L^2-boundedness for the corresponding operator $A : \mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$. Notice that $x \mapsto \sigma_A(x,\xi) \in C^\infty(\mathbb{T}^n)$ for every $\xi \in \mathbb{Z}^n$.

Proposition 5.1. *If*

$$\left|\partial_x^\beta \sigma_A(x,\xi)\right| \le C$$

when $|\beta| \le n/2 + 1$ then $A \in \mathcal{L}(L^2(\mathbb{T}^n))$.

Proof. Now

$$\begin{aligned} Af(x) &= \sum_{\xi \in \mathbb{Z}^n} \sigma_A(x,\xi)\ \widehat{f}_T(\xi)\ \mathrm{e}^{\mathrm{i}x\cdot\xi} \\ &= \sum_{\xi,\eta \in \mathbb{Z}^n} \widehat{\sigma_A}_T(\eta,\xi)\ \widehat{f}_T(\xi)\ \mathrm{e}^{\mathrm{i}x\cdot(\xi+\eta)} \\ &= \sum_{\omega \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\omega} \sum_{\xi \in \mathbb{Z}^n} \widehat{\sigma_A}_T(\omega-\xi,\xi)\ \widehat{f}_T(\xi). \end{aligned}$$

Here $|\widehat{\sigma_A}_T(\eta,\xi)| \leq C\ \langle\eta\rangle^{-k}$, so that

$$\begin{aligned}
\|Af\|^2_{L^2(\mathbb{T}^n)} &= \int_{\mathbb{T}^n} |Af(x)|^2\ \tilde{\mathrm{d}}x \\
&= \sum_{\omega\in\mathbb{Z}^n} |\widehat{Af}_T(\omega)|^2 \\
&= \sum_{\omega\in\mathbb{Z}^n} \left| \sum_{\xi\in\mathbb{Z}^n} \widehat{\sigma_A}_T(\omega-\xi,\xi)\ \widehat{f}_T(\xi) \right|^2 \\
&\leq \left(\sup_{\omega\in\mathbb{Z}^n} \sum_{\xi\in\mathbb{Z}^n} |\widehat{\sigma_A}_T(\omega-\xi,\xi)|^2 \right) \\
&\quad \left(\sup_{\xi\in\mathbb{Z}^n} \sum_{\omega\in\mathbb{Z}^n} |\widehat{\sigma_A}_T(\omega-\xi,\xi)|^2 \right) \sum_{\xi\in\mathbb{Z}^n} \left| \widehat{f}_T(\xi) \right|^2 \\
&\leq C'\ \|f\|^2_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

Note that no difference conditions for the ξ-variable were needed. In fact, this is related to the following more general result.

Theorem 5.2. *Let*

$$Tu(x) = \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\phi(x,k)}\ a(x,k)\ \hat{u}(k).$$

Assume that for every α for which $|\alpha| \leq 2n+1$,

$$|\partial_x^\alpha a(x,k)| \leq C, \qquad |\partial_x^\alpha \Delta_k \phi(x,k)| \leq C. \tag{5.1}$$

Assume also that for every $x \in \mathbb{T}^n$ and for every $k, l \in \mathbb{Z}^n$,

$$|\nabla_x \phi(x,k) - \nabla_x \phi(x,l)| \geq C|k-l|. \tag{5.2}$$

Then $T \in \mathcal{L}(L^2(\mathbb{T}^n))$.

Note that condition (5.2) is a discrete version of the usual local graph condition for Fourier integral operators, necessary for the local L^2-boundedness. We also note that these conditions roughly correspond to C^1 properties of the phase in ξ. Finally, if ϕ and a are not 2π–periodic in x, operator T is bounded from $L^2(\mathbb{T}^n)$ to $L^2_{loc}(\mathbb{R}^n)$. Theorem 5.2 is the discrete version of the global boundedness theorem in [12].

6. Extending symbols

It is often useful to extend toroidal symbols from $\mathbb{T}^n \times \mathbb{Z}^n$ to $\mathbb{T}^n \times \mathbb{R}^n$. This can be done with a suitable convolution that respects the symbol inequalities. The idea of the following lemma goes probably back to Y. Meyer.

Lemma 6.1. *There exist $\theta, \phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\theta}_E|_{\mathbb{Z}^n}(\xi) = \delta_{0,\xi}$ and $\partial_\xi^\alpha \widehat{\theta}_E(\xi) = (\triangle_\xi^\alpha)^t \phi_\alpha(\xi)$ for every multi-index α.*

Proof. Let us first consider the case $n = 1$. Let $\theta = \theta_1 \in C^\infty(\mathbb{R}^1)$ such that

$$\mathrm{supp}(\theta_1) \subset]-2\pi, 2\pi[, \quad \theta_1(-x) = \theta_1(x), \quad \theta_1(\pi - y) + \theta_1(\pi + y) = 1$$

for $x \in \mathbb{R}$ and for $0 \leq y \leq \pi$. These assumptions for θ are enough for us, and of course the choice is not unique. In any case, $\widehat{\theta_1}_E \in \mathcal{S}(\mathbb{R}^1)$. If $\xi \in \mathbb{Z}^n$ then

$$\begin{aligned}
\widehat{\theta_1}_E(\xi) &= \int_{\mathbb{R}^1} \theta_1(x)\ \mathrm{e}^{-\mathrm{i}x\cdot\xi}\ \tilde{\mathrm{d}}x \\
&= \int_0^{2\pi} (\theta_1(x - 2\pi) + \theta_1(x))\ \mathrm{e}^{-\mathrm{i}x\cdot\xi}\ \tilde{\mathrm{d}}x \\
&= \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}x\cdot\xi}\ \tilde{\mathrm{d}}x \\
&= \delta_{0,\xi}.
\end{aligned}$$

If desired $\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$ exists, it must satisfy

$$\int_{\mathbb{R}^1} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ \phi^{(\alpha)}(\xi)\ \mathrm{d}\xi = \int_{\mathbb{R}^1} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ (\triangle_\xi^\alpha)^t \phi_\alpha(\xi)\ \mathrm{d}\xi$$

because $\mathcal{F}_E : \mathcal{S}(\mathbb{R}^1) \to \mathcal{S}(\mathbb{R}^1)$ is bijective. Integration by parts yields

$$(-\mathrm{i}x)^\alpha \theta(x) = (1 - \mathrm{e}^{\mathrm{i}x})^\alpha (\mathcal{F}_E^{-1}\phi_\alpha)(x).$$

Thus

$$(\mathcal{F}_E^{-1}\phi_\alpha)(x) = \left(\frac{-\mathrm{i}x}{1 - \mathrm{e}^{\mathrm{i}x}}\right)^\alpha \theta(x).$$

The general n-dimensional case is reduced to the 1-dimensional case, since $\theta = (x \mapsto \theta_1(x_1)\theta_1(x_2)\cdots\theta_1(x_n)) \in \mathcal{S}(\mathbb{R}^n)$ has desired properties. The following two results can be easily obtained from the discrete Taylor's theorem.

Lemma 6.2. *Let $a : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ belong to $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the restriction $\sigma = a|_{\mathbb{T}^n \times \mathbb{Z}^n} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$.*

Proposition 6.3. *Let $a, b : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ such that $a, b \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $a|_{\mathbb{T}^n \times \mathbb{Z}^n} = b|_{\mathbb{T}^n \times \mathbb{Z}^n}$. Then $a - b$ is smoothing, $a - b \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$.*

The main theorem of this paragraph is that we can extend toroidal symbols in a unique smooth way.

Theorem 6.4. *Let $\sigma \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$. Then there exists $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\sigma = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$; this extended symbol is unique up to smoothing.*

Proof. Uniqueness up to smoothing follows from Proposition 6.3, so the existence is the main issue here. Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ be as in Lemma 6.1. Define $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$a(x,\xi) := \sum_{\eta\in\mathbb{Z}^n} \widehat{\theta}_E(\xi-\eta)\ \sigma(x,\eta).$$

It is easy to see that $\sigma = a|_{\mathbb{T}^n\times\mathbb{Z}^n}$. Furthermore,

$$\begin{aligned}
\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| &= \left|\sum_{\eta\in\mathbb{Z}^n} \partial_\xi^\alpha \widehat{\theta}_E(\xi-\eta)\ \partial_x^\beta \sigma(x,\eta)\right| \\
&= \left|\sum_{\eta\in\mathbb{Z}^n} (\triangle_\xi^\alpha)^t \phi_\alpha(\xi-\eta)\ \partial_x^\beta \sigma(x,\eta)\right| \\
&= \left|\sum_{\eta\in\mathbb{Z}^n} \phi_\alpha(\xi-\eta)\ \triangle_\eta^\alpha \partial_x^\beta \sigma(x,\eta)\ (-1)^{|\alpha|}\right| \\
&\le \sum_{\eta\in\mathbb{Z}^n} |\phi_\alpha(\xi-\eta)|\ C_{\alpha\beta m}\ \langle\eta\rangle^{m-\rho|\alpha|+\delta|\beta|} \\
&\le C'_{\alpha\beta m}\ \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \sum_{\eta\in\mathbb{Z}^n} |\phi_\alpha(\eta)|\ \langle\eta\rangle^{|m-\rho|\alpha|+\delta|\beta||} \\
&\le C''_{\alpha\beta m}\ \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}.
\end{aligned}$$

Thus $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

7. Fourier series operator calculus

In this section we will describe composition formulae of Fourier series operators with pseudo-differential operators. They are similar to the global composition formulae in [11] and [13] in $\mathbb{R}^n$. However, the situation on the torus is technically much simpler since it does not require the global in space analysis of the corresponding remainders.

Theorem 7.1 (composition TP). *Let $T : \mathcal{D}(\mathbb{T}^n) \to \mathcal{D}'(\mathbb{T}^n)$ be defined by*

$$Tu(x) := \sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathrm{e}^{\mathrm{i}(\phi(x,\xi)-y\cdot\xi)}\ a(x,y,\xi)\ u(y)\ \tilde{\mathrm{d}}y,$$

where the amplitude $a \in C^\infty(\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{Z}^n)$ satisfies

$$\left|\partial_x^\alpha \partial_y^\beta a(x,y,\xi)\right| \le C_{\alpha\beta m}\ \langle\xi\rangle^m$$

for every $x, y \in \mathbb{T}^n$, $\xi \in \mathbb{Z}^n$ and $\alpha, \beta \in \mathbb{N}^n$; no restrictions for ϕ here. Let $p \in S^t(\mathbb{T}^n \times \mathbb{Z}^n)$. Then

$$TPu(x) = \sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathrm{e}^{\mathrm{i}(\phi(x,\xi)-z\cdot\xi)}\ c(x,z,\xi)\ u(z)\ \tilde{\mathrm{d}}z,$$

where

$$c(x,z,\xi) = \sum_{\eta\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathrm{e}^{\mathrm{i}(y-z)\cdot(\eta-\xi)}\ a(x,y,\xi)\ p(y,\eta)\ \tilde{\mathrm{d}}y$$

satisfying

$$\left|\partial_x^\alpha \partial_z^\beta c(x,z,\xi)\right| \le C_{\alpha\beta mt}\ \langle\xi\rangle^{m+t}$$

for every $x,z \in \mathbb{T}^n$, $\xi\in\mathbb{Z}^n$ *and* $\alpha,\beta\in\mathbb{N}^n$. *Moreover,*

$$c(x,z,\xi) \sim \sum_{\alpha\in\ge 0} \frac{1}{\alpha!}\ \partial_y^{(\alpha)} \left[a(x,y,\xi)\ \triangle_\xi^\alpha p(y,\xi)\right]|_{y=z}.$$

Composition in the other direction is given by the following theorem.

Theorem 7.2 (composition PT)**.** *Let* $T:\mathcal{D}(\mathbb{T}^n)\to\mathcal{D}'(\mathbb{T}^n)$ *such that*

$$Tu(x) := \sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathrm{e}^{\mathrm{i}(\phi(x,\xi)-y\cdot\xi)}\ a(x,y,\xi)\ u(y)\ \tilde{\mathrm{d}}y,$$

where $a\in C^\infty(\mathbb{T}^n\times\mathbb{T}^n\times\mathbb{Z}^n)$ *satisfying*

$$\left|\partial_x^\alpha \partial_y^\beta a(x,y,\xi)\right| \le C_{\alpha\beta m}\ \langle\xi\rangle^{m}$$

for every $x,y\in\mathbb{T}^n$, $\xi\in\mathbb{Z}^n$ *and* $\alpha,\beta\in\mathbb{N}^n$*; we assume that* $\phi\in C^\infty(\mathbb{T}^n\times\mathbb{Z}^n)$ *satisfies*

$$C^{-1}\ \langle\xi\rangle \le \langle\nabla_x\phi(x,\xi)\rangle \le C\ \langle\xi\rangle$$

for some C *for every* $x\in\mathbb{T}^n$, $\xi\in\mathbb{Z}^n$, *and that*

$$|\partial_x^\alpha\phi(x,\xi)| \le C_\alpha\ \langle\xi\rangle, \quad \left|\partial_x^\alpha\triangle_\xi^\beta\phi(x,\xi)\right| \le C_{\alpha\beta}$$

for every $x\in\mathbb{T}^n$, $\xi\in\mathbb{Z}^n$ *and* $\alpha,\beta\in\mathbb{N}^n\setminus\{0\}$. *Let* $p\in S^t(\mathbb{T}^n\times\mathbb{Z}^n)$. *Then*

$$p(x,D)Tu(x) = \sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\phi(x,\xi)-z\cdot\xi)}\ c(x,z,\xi)\ u(z)\ \tilde{\mathrm{d}}z,$$

where

$$\left|\partial_x^\alpha \partial_z^\beta c(x,z,\xi)\right| \le C_{\alpha\beta}\ \langle\xi\rangle^{m+t}$$

for every $x,z \in \mathbb{T}^n$, $\xi\in\mathbb{Z}^n$ *and* $\alpha,\beta\in\mathbb{N}^n$. *Moreover,*

$$c(x,z,\xi) \sim \sum_{\alpha\ge 0} \frac{\mathrm{i}^{-|\alpha|}}{\alpha!}\ \partial_\eta^\alpha p(x,\eta)|_{\eta=\nabla_x\phi(x,\xi)}\ \partial_y^\alpha\left[\mathrm{e}^{\mathrm{i}\Psi(x,y,\xi)}a(y,z,\xi)\right]|_{y=x},$$

(here we use a smooth extension for the symbol $p(x,\eta)$*) where*

$$\Psi(x,y,\xi) := \phi(y,\xi)-\phi(x,\xi)+(x-y)\cdot\nabla_x\phi(x,\xi), \tag{7.1}$$

when $x\approx y$.

8. Applications to hyperbolic equations

Let $a(x,D) \in \Psi^m(\mathbb{R}^n)$ (with some properties to be specified). If u depends on x and t, we write

$$\begin{aligned} a(x,D)u(x,t) &= \int_{\mathbb{R}^n} a(x,\xi)\ \widehat{u}_E(\xi,t)\ \mathrm{e}^{\mathrm{i}x\cdot\xi}\ \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\ a(x,\xi)\ u(y,t)\ \tilde{\mathrm{d}}y\ \mathrm{d}\xi. \end{aligned}$$

Let $u(\cdot,t) \in L^1(\mathbb{R}^n)$ $(0 < t < t_0)$ be a solution to the hyperbolic problem

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}u(x,t) = a(x,D)u(x,t), \\ u(x,0) = f(x), \end{cases} \tag{8.1}$$

where $f \in L^1(\mathbb{R}^n)$ is compactly supported.

Assume now that $a(X,D) = a_1(X,D) + a_0(X,D)$ where $a_1(x,\xi)$ is periodic and $a_0(x,\xi)$ is compactly supported in x (assume even that $\mathrm{supp}(a_0(\cdot,\xi)) \subset [-\pi,\pi]^n$). Typically, we will want to have $a_1(x,\xi) = a_1(\xi)$ a constant coefficient operator, not necessarily smooth in ξ. Let us also assume that $\mathrm{supp}(f) \subset [-\pi,\pi]^n$.

We will now describe a way to periodise problem (8.1). According to Proposition 4.2, we can replace (8.1) by

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}u(x,t) = (a_1(x,D) + (pa_0)(X,D))u(x,t) + Ru(x,t), \\ u(x,0) = f(x), \end{cases} \tag{8.2}$$

where the symbol $a_1 + pa_0$ is periodic and R is a smoothing operator. To study singularities of (8.1), it is sufficient to analyse the Cauchy problem

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}v(x,t) = (a_1(x,D) + (pa_0)(X,D))v(x,t), \\ v(x,0) = f(x) \end{cases} \tag{8.3}$$

since by Duhamel's formula $\mathrm{WF}(u-v) = \emptyset$. This problem can be transfered to the torus. Let $w(x,t) = pv(\cdot,t)(x)$. In view of Proposition 4.1 it will solve Cauchy problem

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}w(x,t) = (\widetilde{a_1}(x,D) + \widetilde{pa_0}(X,D))w(x,t), \\ w(x,0) = pf(x). \end{cases} \tag{8.4}$$

Calculus constructed in previous sections provides the solution in the form

$$w(x,t) = \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\phi(t,x,k)} c(t,x,k) \widehat{f}_E(k).$$

Here we note that $\widehat{pf}_T(k) = \widehat{f}_E(k)$. Also, if the symbol $a_1(x,\xi) = a_1(\xi)$ has constant coefficients and a_0 is of order zero, we have $\phi(t,x,k) = x\cdot k + ta_1(k)$. In particular, $\nabla_x\phi(x,k) = k$, so composition formulas for $b(x,D)w$ in previous sections can be applied. Details of this analysis and investigation of the corresponding properties will appear in [14].

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