Nguyen Thanh Hong; Trinh Tuan; Nguyen Xuan Thao

On the Fourier cosine—Kontorovich-Lebedev generalized convolution transforms

Applications of Mathematics, Vol. 58 (2013), No. 4, 473–486

Persistent URL: `http://dml.cz/dmlcz/143341`

ON THE FOURIER COSINE—KONTOROVICH-LEBEDEV GENERALIZED CONVOLUTION TRANSFORMS

Nguyen Thanh Hong, Trinh Tuan, Nguyen Xuan Thao, Hanoi

(Received August 1, 2011)

Abstract. We deal with several classes of integral transformations of the form

$$f(x) \to D \int_{\mathbb{R}_+^2} \frac{1}{u} (\mathrm{e}^{-u \cosh(x+v)} + \mathrm{e}^{-u \cosh(x-v)}) h(u) f(v) \,\mathrm{d}u \,\mathrm{d}v,$$

where D is an operator. In case D is the identity operator, we obtain several operator properties on $L_p(\mathbb{R}_+)$ with weights for a generalized operator related to the Fourier cosine and the Kontorovich-Lebedev integral transforms. For a class of differential operators of infinite order, we prove the unitary property of these transforms on $L_2(\mathbb{R}_+)$ and define the inversion formula. Further, for an other class of differential operators of finite order, we apply these transformations to solve a class of integro-differential problems of generalized convolution type.

Keywords: convolution, Hölder inequality, Young's theorem, Watson's theorem, unitary, Fourier cosine, Kontorovich-Lebedev, transform, integro-differential equation

MSC 2010: 33C10, 44A35, 45E10, 45J05, 47A30, 47B15

1. Introduction

The Fourier cosine integral transform is of the form (see [10], [11])

$$(1.1) \qquad (F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy \,\mathrm{d}x$$

for $f \in L_1(\mathbb{R}_+)$, and

$$(1.2) \qquad (F_c f)(y) = \lim_{N\to\infty} \sqrt{\frac{2}{\pi}} \int_0^N f(x) \cos yx \,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty f(x) \frac{\sin xy}{x} \,\mathrm{d}x$$

This research is funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2011.05.

for $f \in L_2(\mathbb{R}_+)$; here the limit is understood in $L_2(\mathbb{R}_+)$ norm mean. These two definitions are equivalent if $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$.

The Kontorovich-Lebedev integral transform was first investigated by M. J. Kontorovich and N. N. Lebedev in 1938–1939 and has the form (see [5], [6], [14])

$$K[f](y) = \int_0^\infty K_{\mathrm{i}x}(y) f(x)\,\mathrm{d}x, \tag{1.3}$$

which contains as the kernel the Macdonald function $K_\nu(x)$ (see [1]) of the pure imaginary index $\nu = \mathrm{i}y$. The function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + z\frac{\mathrm{d}u}{\mathrm{d}z} - (z^2 + \nu^2)u = 0. \tag{1.4}$$

The Macdonald function has the asymptotic behaviour (see [6])

$$K_\nu(z) = \Big(\frac{\pi}{2z}\Big)^{1/2} \mathrm{e}^{-z}[1 + O(1/z)], \quad z \to \infty, \tag{1.5}$$

and near the origin

$$z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu) + o(1), \quad z \to 0,\ \nu \neq 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.7}$$

The following form for the Macdonald function is very useful (see [1], [6], [14]):

$$K_{\mathrm{i}y}(x) = \int_0^\infty \mathrm{e}^{-x\cosh u} \cos yu\,\mathrm{d}u,\ x > 0. \tag{1.8}$$

The inverse Kontorovich-Lebedev transform (1.3) is of the form (see [5], [6])

$$f(x) = K^{-1}[g](x) = \frac{2}{\pi^2} x\sinh(\pi x) \int_0^\infty \frac{1}{y} K_{\mathrm{i}x}(y) g(y)\,\mathrm{d}y, \tag{1.9}$$

here, $g(y) = K[f](y)$.

Throughout this paper, we are interested in the Kontorovich-Lebedev transform (1.3). However, note that there is another version of the Kontorovich-Lebedev integral transform which is of the form (see [1], [6], [16])

$$g(y) = \widetilde{K}[f](y) = \int_0^\infty K_{\mathrm{i}y}(x) f(x)\,\mathrm{d}x. \tag{1.10}$$

A generalized convolution for the Fourier cosine and the Kontorovich-Lebedev integral transforms has been studied in [12]:

(1.11)

$$(h \overset{\gamma}{*} f)(x) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{1}{u}[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}] h(u) f(v)\,\mathrm{d}u\,\mathrm{d}v,\ x > 0.$$

The existence of the generalized convolution (1.11) for two functions in $L_1(\mathbb{R}_+)$ with weight and its application to solving integral equations of generalized convolution type were studied in [12]. Namely, for $h \in L_1(\mathbb{R}_+, 1/x), f \in L_1(\mathbb{R}_+, 1/\sinh x)$, the following factorization equality holds (see [12]):

$$F_c(h \overset{\gamma}{*} f)(y) = \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y), \ \forall y > 0. \tag{1.12}$$

In any convolution $(h * f)$ of two functions h and f, if we fix the function h and let f vary in a certain function space, then one can study convolution transforms of the type $f \mapsto D(f * h)$, where D is an operator. The most famous integral transforms constructed in this way are the Watson transforms that are related to the Mellin convolution and the Mellin transform (see [11])

$$f(x) \longmapsto g(x) = \int_0^\infty k(xy) f(y) \,\mathrm{d}y.$$

Recently, several authors have been interested in the convolution transforms of this type (see [3], [4], [13], [15]). In this paper, we are interested in the transform $f \mapsto D(h \overset{\gamma}{*} f)$, where $(h \overset{\gamma}{*} f)$ is the generalized convolution (1.11). For the case D is the identity operator, in Section 2 we study several further operator properties in the Lebesgue spaces $L_p(\mathbb{R}_+)$ with weight for the generalized convolution (1.11). In particular, Young's theorem and Young's inequality for this generalized convolution are obtained. In Section 3, for a class of differential operators D of infinite order, we obtain the necessary and sufficient condition such that the respective transforms are unitary on $L_2(\mathbb{R}_+)$, and define the inverse transforms. Finally, in Section 4, for an other class of differential operator D of finite order, we obtain the solution in closed form of a class of integro-differential equations.

2. Generalized convolution operator properties

In this section, we will prove several norm properties of the generalized convolution (1.11). Throughout the paper, we are interested in the following two-parametric family of Lebesgue spaces.

Definition 1 (see [16]). For $\alpha \in \mathbb{R}$, $0 < \beta \leqslant 1$, we denote by $L_p^{\alpha,\beta}(\mathbb{R}_+)$ the space of all functions $f(x)$ defined in $\mathbb{R}_+$ such that

$$\int_0^\infty |f(x)|^p K_0(\beta x) x^\alpha \,\mathrm{d}x < \infty. \tag{2.1}$$

The norm of a function in this space is defined by

$$\|f\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \left(\int_0^\infty |f(x)|^p K_0(\beta x) x^\alpha \,\mathrm{d}x\right)^{1/p}.$$

Using the asymptotics of the Macdonald function (1.5), (1.6), (1.7), formula (2.1) can be expressed in an equivalent form

$$\int_0^1 |f(x)|^p |\log x| x^\alpha \,\mathrm{d}x + \int_1^\infty |f(x)|^p x^{\alpha-1/2} \mathrm{e}^{-\beta x} \,\mathrm{d}x < \infty.$$

The boundedness of the generalized convolution (1.11) on the spaces $L_1(\mathbb{R}_+)$ is given by the following theorem; here we consider the function $h \in L_1^{-1,\beta}(\mathbb{R}_+)$.

Theorem 2.1. *Let* $h \in L_1^{-1,\beta}(\mathbb{R}_+)$ *and* $g \in L_1(\mathbb{R}_+)$, $0 < \beta \leqslant 1$. *Then the generalized convolution* (1.11) *exists for almost all* $x > 0$, *belongs to* $L_1(\mathbb{R}_+)$, *and the following estimation holds:*

$$\|h \overset{\gamma}{*} g\|_{L_1(\mathbb{R}_+)} \leqslant \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)} \|f\|_{L_1(\mathbb{R}_+)}. \tag{2.2}$$

Moreover, the factorization property (1.12) *holds true. Furthermore, if* $0 < \beta < 1$, *then the convolution* (1.11) *belongs to* $C_0(\mathbb{R}_+)$, *and the Parseval type equality takes place for all* $x > 0$:

$$(h \overset{\gamma}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \cos xy \,\mathrm{d}y. \tag{2.3}$$

P r o o f. Using formula (1.8) we obtain

$$\frac{1}{2} \int_0^\infty (\mathrm{e}^{-u \cosh(x+v)} + \mathrm{e}^{-u \cosh(x-v)}) \,\mathrm{d}v = K_0(u). \tag{2.4}$$

Then

$$\|h \overset{\gamma}{*} f\|_{L_1(\mathbb{R}_+)} \leqslant \frac{2}{\pi^2} \int_0^\infty \int_0^\infty \frac{|h(u)|}{u} K_0(u) |f(v)| \,\mathrm{d}u \,\mathrm{d}v = \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+, 1/x)} \cdot \|f\|_{L_1(\mathbb{R}_+)}.$$

We now prove the Parseval type equality. Using Fubini's theorem and the formula (2.16.48.19) in [9]

$$\int_0^\infty \cos by K_{\mathrm{i}y}(u) \,\mathrm{d}y = \frac{\pi}{2} \mathrm{e}^{-u \cosh b},$$

we have

$$
\begin{aligned}
(h \overset{\gamma}{*} f)(x) &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \frac{1}{u}[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]h(u)f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty \frac{2}{\pi}\frac{1}{u}h(u)f(v)K_{\mathrm{i}y}(u)(\cos(x+v)y + \cos(x-v)y)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y \\
&= \frac{4}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{u}h(u)f(v)K_{\mathrm{i}y}(u)\cos xy\cos vy\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{y\sinh\pi y}\biggl(\frac{2}{\pi^2}y\sinh\pi y\int_0^\infty \frac{1}{u}K_{\mathrm{i}y}(u)h(u)\,\mathrm{d}u\biggr) \\
&\quad\times\biggl(\sqrt{\frac{2}{\pi}}\int_0^\infty f(v)\cos vy\,\mathrm{d}v\biggr)\cos xy\,\mathrm{d}y \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{y\sinh\pi y}K^{-1}[h](y)(F_c f)(y)\cos xy\,\mathrm{d}y.
\end{aligned}
$$

That gives the Parseval identity (2.3), and the proof of the theorem is complete. □

The next theorem draws a parallel with a result studied in [16], namely, the boundedness of the generalized convolution (1.11) on spaces $L_r^{\alpha,\gamma}$, $1 < r < \infty$, $\alpha > -1$, $0 < \gamma \leqslant 1$ is given.

Theorem 2.2. *Let $1 < p < \infty$ be a real number and q its conjugate exponent, i.e. $1/p + 1/q = 1$. Then for any $h \in L_p^{-p,\beta}(\mathbb{R}_+)$ and $f \in L_q(\mathbb{R}_+)$, the generalized convolution $(h \overset{\gamma}{*} f)$ (1.11) is well-defined as a bounded continuous function on $\mathbb{R}_+$. Moreover, $(h \overset{\gamma}{*} f)$ belongs to $L_r^{\alpha,\gamma}(\mathbb{R}_+)$, $1 \leqslant r < \infty$, $\alpha > -1$, $0 < \gamma \leqslant 1$, and*

$$
\|h \overset{\gamma}{*} f\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant C_{\alpha,\gamma}^{1/r}\|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|f\|_{L_q(\mathbb{R}_+)}, \tag{2.5}
$$

where $C_{\alpha,\gamma} = (2^{r+\alpha-1}/\pi^{2r}\gamma^{\alpha+1})\Gamma^2((\alpha+1)/2)$.

Proof. Using the integral representation (2.4) for the function $K_0(u)$, the Hölder inequality, and the fact that $\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)} \leqslant 2\mathrm{e}^{-u}$ for all positive u, x, v, we get

$$
\begin{aligned}
|(h \overset{\gamma}{*} f)(x)| &\leqslant \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \Bigl|\frac{h(u)}{u}\Bigr||f(v)|[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v \\
&\leqslant \frac{1}{\pi^2}\biggl(\int_0^\infty\int_0^\infty \Bigl|\frac{h(u)}{u}\Bigr|^p[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\biggr)^{1/p} \\
&\quad\times\biggl(\int_0^\infty\int_0^\infty |f(v)|^q[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\biggr)^{1/q} \\
&\leqslant \frac{2}{\pi^2}\biggl(\int_0^\infty \Bigl|\frac{h(u)}{u}\Bigr|^p K_0(u)\,\mathrm{d}u\biggr)^{1/p}\|f\|_{L_q(\mathbb{R}_+)}.
\end{aligned} \tag{2.6}
$$

Therefore, the generalized convolution is well-defined as a bounded operator and the estimation (2.6) holds. Moreover, in view of formula (2.16.2.2) in [9] we get

$$\begin{aligned}\|h \overset{\gamma}{*} f\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} &\leqslant \frac{2}{\pi^2}\|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|f\|_{L_q(\mathbb{R}_+)}\Big(\int_0^\infty x^\alpha K_0(\gamma x)\,\mathrm{d}x\Big)^{1/r}\\ &= \frac{2}{\pi^2}(2\gamma)^{-1/r}\Big(\frac{\gamma}{2}\Big)^{-\alpha/r}\Gamma^{2/r}\Big(\frac{\alpha+1}{2}\Big)\|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|f\|_{L_q(\mathbb{R}_+)},\ \alpha > -1.\end{aligned}$$

This yields (2.5) □

For the Fourier convolution (see [10])

$$(2.7)\qquad (h \underset{F}{*} f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x-y)f(y)\,\mathrm{d}y,$$

Young's theorem and its corollary, the so-called Young inequality, are fundamental (see [2]). So, it is useful to study similar topics for convolutions and generalized convolutions for other integral transforms. Next, we will prove Young's type theorem for the generalized convolution (1.11).

Theorem 2.3 (Young's Type Theorem). *Let p, q, r be real numbers in $(1;\infty)$ such that $1/p + 1/q + 1/r = 2$ and let $f \in L_p^{-p,\beta}(\mathbb{R}_+)$, $0 < \beta \leqslant 1$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R}_+)$. Then*

$$(2.8)\qquad \Big|\int_0^\infty (f \overset{\gamma}{*} g)(x)\cdot h(x)\,\mathrm{d}x\Big| \leqslant \frac{2^{(p-1)/p}}{\pi^2}\|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R}_+)}.$$

Proof. Let p_1, q_1, r_1 be the conjugate exponentials of p, q, r, respectively, it means

$$\frac{1}{p}+\frac{1}{p_1} = \frac{1}{q}+\frac{1}{q_1} = \frac{1}{r}+\frac{1}{r_1} = 1.$$

Then it is obvious that $1/p_1 + 1/q_1 + 1/r_1 = 1$. Put

$$\begin{aligned}F(x,u,v) &= |g(v)|^{q/p_1}|h(x)|^{r/p_1}[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]^{1/p_1},\\ G(x,u,v) &= \Big|\frac{f(u)}{u}\Big|^{p/q_1}|h(x)|^{r/q_1}[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]^{1/q_1},\\ H(x,u,v) &= \Big|\frac{f(u)}{u}\Big|^{p/r_1}|g(v)|^{q/r_1}[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}]^{1/r_1}.\end{aligned}$$

We have

$$(2.9)\qquad (F\cdot G\cdot H)(x,u,v) = \Big|\frac{f(u)}{u}\Big||g(v)||h(x)|[\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}].$$

On the other hand, in the space $L_{p_1}(\mathbb{R}^3_+)$ we have

(2.10)

$$\begin{aligned}\|F\|^{p_1}_{L_{p_1}(\mathbb{R}^3_+)} &= \int_0^\infty\int_0^\infty\int_0^\infty |g(v)|^q|h(x)|^r[\mathrm{e}^{-u\cosh(x+v)}+\mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &\leqslant 2\int_0^\infty\int_0^\infty\int_0^\infty |g(v)|^q|h(x)|^r\mathrm{e}^{-u}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &= 2\|g\|^q_{L_q(\mathbb{R}_+)}\|h\|^r_{L_r(\mathbb{R}_+)}.\end{aligned}$$

Further, the fact that $K_0(u)\leqslant K_0(\beta u)$ for $0<\beta\leqslant 1$ (see [16]) yields

(2.11)

$$\begin{aligned}\|G\|^{p_1}_{L_{q_1}(\mathbb{R}^3_+)} &= \int_0^\infty\int_0^\infty\int_0^\infty \Big|\frac{f(u)}{u}\Big|^p|h(x)|^r[\mathrm{e}^{-u\cosh(x+v)}+\mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &\leqslant \int_0^\infty\int_0^\infty\int_0^\infty \Big|\frac{f(u)}{u}\Big|^p K_0(\beta u)|h(x)|^r\,\mathrm{d}u\,\mathrm{d}x\\ &= \|f\|^p_{L_p^{-p,\beta}(\mathbb{R}_+)}\|h\|^r_{L_r(\mathbb{R}_+)},\end{aligned}$$

and similarly,

(2.12)

$$\begin{aligned}\|H\|^{r_1}_{L_{r_1}(\mathbb{R}^3_+)} &= \int_0^\infty\int_0^\infty\int_0^\infty \Big|\frac{f(u)}{u}\Big|^p|g(v)|^q[\mathrm{e}^{-u\cosh(x+v)}+\mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &\leqslant \int_0^\infty\int_0^\infty\int_0^\infty \Big|\frac{f(u)}{u}\Big|^p K_0(\beta u)|g(v)|^r\,\mathrm{d}u\,\mathrm{d}v\\ &= \|f\|^p_{L_p^{-p,\beta}(\mathbb{R}_+)}\|g\|^q_{L_q(\mathbb{R}_+)}.\end{aligned}$$

From (2.10), (2.11) and (2.12) we have

$$\begin{aligned}(2.13)\qquad &\|F\|_{L_{p_1}(\mathbb{R}^3_+)}\|G\|_{L_{q_1}(\mathbb{R}^3_+)}\|H\|_{L_{r_1}(\mathbb{R}^3_+)}\\ &\qquad\leqslant 2^{(p-1)/p}\|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R}_+)}.\end{aligned}$$

From (2.9) and (2.13), by three-function form of the Hölder inequality [2] we have

$$\begin{aligned}&\Big|\int_0^\infty (f\overset{\gamma}{*}g)(x)\cdot h(x)\,\mathrm{d}x\Big|\\ &\quad\leqslant \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty \Big|\frac{f(u)}{u}\Big||g(v)||h(x)|[\mathrm{e}^{-u\cosh(x+v)}+\mathrm{e}^{-u\cosh(x-v)}]\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &\quad= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty F(x,u,v)G(x,u,v)H(x,u,v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\\ &\quad\leqslant \frac{1}{\pi^2}\|F\|_{L_{p_1}(\mathbb{R}^3_+)}\|G\|_{L_{q_1}(\mathbb{R}^3_+)}\|H\|_{L_{r_1}(\mathbb{R}^3_+)}\\ &\quad\leqslant \frac{2^{(p-1)/p}}{\pi^2}\|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R}_+)}.\end{aligned}$$

The proof is complete. □

The following Young's type inequality is the direct corollary of the above theorem

Corollary 2.1 (A Young's Type Inequality). *Let* $1 < p < \infty$, $1 < q < \infty$, $1 < r < \infty$ *be such that* $1/p + 1/q = 1 + 1/r$ *and let* $f \in L_p^{-p,\beta}(\mathbb{R}_+)$, $0 < \beta \leqslant 1$, $g \in L_q(\mathbb{R}_+)$. *Then the generalized convolution* (1.11) *is well-defined in* $L_r(\mathbb{R}_+)$, *moreover, the following inequality holds:*

$$\|f \overset{\gamma}{*} g\|_{L_r(\mathbb{R}_+)} \leqslant \frac{2^{(p-1)/p}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}. \tag{2.14}$$

3. A Watson type theorem

An important part of the integral transforms theory is to study unitary transforms. In this section, for a class of differential operators of infinite order, we give a condition on the kernel h such that the convolution transformation (3.3) defines a unitary operator in $L_2(\mathbb{R}_+)$, and calculate the inverse transformation.

By an argument similar to that in the proof of Theorem 2.1, one can easily prove the following lemma.

Lemma 3.1. *Let* $h \in L_2^{-2,\beta}(\mathbb{R}_+)$, $0 < \beta \leqslant 1$, *and* $f \in L_2(\mathbb{R}_+)$. *Then the generalized convolution* (1.11) *satisfies the factorization equality* (1.12). *Furthermore, the following generalized Parseval identity holds:*

$$(h \overset{\gamma}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \cos xy \, \mathrm{d}y, \tag{3.1}$$

where the integral is understood in the $L_2(\mathbb{R}_+)$ *norm, if necessary.*

Theorem 3.1. *Let* $h \in L_2^{-2,\beta}(\mathbb{R}_+)$, $0 < \beta \leqslant 1$. *Then the condition*

$$|K^{-1}[h](\tau)| = \frac{1}{\cosh(\pi\tau)} \tag{3.2}$$

is necessary and sufficient for the transformation $f \mapsto g$ *given by formula*

$$g(x) = \frac{\mathrm{d}^2}{\pi^2 \, \mathrm{d}x^2} \prod_{k=0}^{\infty} \Big(1 - \frac{4 \, \mathrm{d}^2}{k^2 \, \mathrm{d}x^2}\Big) \int_0^\infty \int_0^\infty \frac{1}{u} (\mathrm{e}^{-u \cosh(x+v)} + \mathrm{e}^{-u \cosh(x-v)}) h(u) f(v) \, \mathrm{d}u \, \mathrm{d}v, \tag{3.3}$$

to be unitary on $L_2(\mathbb{R}_+)$. Moreover, the inverse transformation can be written in the symmetric form

(3.4)

$$f(x) = \lim_{N\to\infty} \frac{\mathrm{d}^2}{\pi^2\,\mathrm{d}x^2} \prod_{k=0}^{N} \Big(1 - \frac{4\,\mathrm{d}^2}{k^2\,\mathrm{d}x^2}\Big) \int_0^\infty\!\!\int_0^\infty \frac{1}{u}(\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)})\bar{h}(u)f(v)\,\mathrm{d}u\,\mathrm{d}v.$$

Here, the limit is understood in the $L_2(\mathbb{R}_+)$ norm.

Proof. *Sufficiency.* Suppose that the function h satisfies condition (3.2). Applying Lemma 3.1, it is easy to see that the generalized convolution transform (3.3) can be written in the form

(3.5)

$$g(x) = \sqrt{\frac{2}{\pi}} \lim_{N\to\infty} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \prod_{k=0}^{N} \Big(1 - \frac{4\,\mathrm{d}^2}{k^2\,\mathrm{d}x^2}\Big) \int_0^\infty \frac{1}{y\sinh \pi y}(K_{\mathrm{i}y}[h])(F_c f)(y)\cos xy\,\mathrm{d}y,$$

or equivalently, $g(x) = \lim_{N\to\infty} g_N(x)$, where

$$g_N(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \prod_{k=0}^{N} \Big(1 - \frac{4\,\mathrm{d}^2}{k^2\,\mathrm{d}x^2}\Big) F_c\Big(\frac{1}{y\sinh\pi y}(K_{\mathrm{i}y}[h])(F_c f)(y)\Big)(x).$$

It is well-known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R}_+)$ if and only if $(Fh)(x)$, $(\mathrm{d}(Fh)(x)/\mathrm{d}x)$, $(\mathrm{d}^2(Fh)(x)/\mathrm{d}x^2) \in L_2(\mathbb{R}_+)$ (Theorem 68, page 92, [11]). Therefore, $h(y)$, $yh(y)$, $y^2h(y), \ldots$, $y^n h(y) \in L_2(\mathbb{R}_+)$ if and only if $(Fh)(x)$, $(\mathrm{d}(Fh)(x)/\mathrm{d}x)$, $(\mathrm{d}^2(Fh)(x)/\mathrm{d}x^2), \ldots, (\mathrm{d}^n(Fh)(x)/\mathrm{d}x^n) \in L_2(\mathbb{R}_+)$. Moreover, for each positive integer n we have

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(Fh)(x) = (-1)^n F(y^{2n}h(y))(x).$$

Therefore, if $y^2 \prod_{k=0}^{N} (1 + 4y^2/k^2)h(y) \in L_2(\mathbb{R}_+)$ then the following formula holds:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \prod_{k=0}^{N} \Big(1 - \frac{4\,\mathrm{d}^2}{k^2\,\mathrm{d}x^2}\Big)(F_c h)(x) = -F_c\Big[y^2 \prod_{k=0}^{N}\Big(1 + \frac{4y^2}{k^2}\Big)h(y)\Big](x). \tag{3.6}$$

From condition (3.2) and the infinite product form of $\sinh z$ (see formula (4.5.68) in [1]) we have

$$\Big|y^2 \prod_{k=0}^{N}(1 + (4y^2/k^2))(1/y\sinh\pi y)K^{-1}[h](y)\Big| = 1\Big/ \prod_{k=N+1}^{\infty}(1 + (4y^2/k^2)) < 1,$$

and hence it is bounded. Therefore

$$y^2 \prod_{k=0}^{N} \Big(1 + \frac{4y^2}{k^2}\Big) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \in L_2(\mathbb{R}_+),$$

and formula (3.6) yields

$$g_N(x) = F_c \Big[y^2 \prod_{k=0}^{N} \Big(1 + \frac{4y^2}{k^2}\Big) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \Big](x) \in L_2(\mathbb{R}_+).$$

This shows that g_N belongs to $L_2(\mathbb{R}_+)$. Applying the Fourier cosine transform to both sides of the above relation, we have

$$(F_c g_N)(y) = y^2 \prod_{k=0}^{N} \Big(1 + \frac{4y^2}{k^2}\Big) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y).$$

Besides, from the Parseval equality for the Fourier cosine transform $\|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$, it follows that

$$\|F_c g_N - F_c g\|_{L_2(\mathbb{R}_+)} = \|g_N - g\|_{L_2(\mathbb{R}_+)} \to 0, \ N \to \infty.$$

Therefore, using formula (4.5.68) in [1] we conclude that

$$\begin{aligned} (F_c g)(y) &= y^2 \prod_{k=0}^{\infty} \Big(1 + \frac{4y^2}{k^2}\Big) \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \\ &= y \sinh 2\pi y \frac{1}{2y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) \\ &= \cosh \pi y K^{-1}[h](y)(F_c f)(y). \end{aligned}$$

From condition (3.2), it is easy to see that $|(F_c g)(y)| \equiv |(F_c f)(y)|$, then $\|f\|_{L_2(\mathbb{R}_+)} = \|g\|_{L_2(\mathbb{R}_+)}$, which implies that the transform (3.3) is unitary. Again from condition (3.2) we obtain

$$\cosh \pi y K^{-1}[\bar{h}](y)(F_c g)(y) = (F_c f)(y).$$

Thus, in the same manner as above it corresponds to (3.4) and the inversion formula of the transform (3.3) follows.

Necessity. Suppose that transform (3.3) is unitary on $L_2(\mathbb{R}_+)$ and the inversion formula is defined by (3.4). Then using the Parseval type identity (3.1), the Parseval identity for the Fourier cosine transform, and formula (4.5.68) in [1] we obtain

$$\|g\|_{L_2(\mathbb{R}_+)} = \| \cosh \pi y K^{-1}[h](y)(F_c f)(y)\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

The middle equality holds for all $f \in L_2(\mathbb{R}_+)$ if and only if h satisfies the condition (3.2). This completes the proof of the theorem. □

4. A class of integro-differential problems

In spite of having many useful applications (see [7]), not many integro-differential equations can be solved in closed form. No application of convolution type transforms of solving integro-differential was presented in recent investigations [3], [4], [13], [15]. In this section, we apply a general class of Fourier cosine and Kontorovich-Lebedev generalized convolution transforms to solve a class of integro-differential problems, which seems to be difficult to solve in closed form by using other techniques. Namely, in case $D = \big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big) \prod\limits_{k=1}^{n-1} \big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\big)$, the transform $f \mapsto K_h(f) := D(h \overset{\gamma}{*} f)$ is of the form

(4.1)

$$(K_h f)(x) = \frac{1}{\pi^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \prod_{k=1}^{n-1} \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\Big) \int_0^\infty \int_0^\infty \frac{1}{u} [\mathrm{e}^{-u\cosh(x+v)} + \mathrm{e}^{-u\cosh(x-v)}] h(u) f(v) \,\mathrm{d}u \,\mathrm{d}v.$$

We consider the integro-differential problem

$$\begin{gathered} f(x) + (K_h f)(x) = g(x), \\ \frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}} f(0) = 0, \ k = \overline{1, n}, \\ \lim_{x\to\infty} f^{(k)}(x) = 0, \ k = \overline{0, 2n-1}. \end{gathered} \tag{4.2}$$

Here, h, g are given functions in $L_1(\mathbb{R}_+)$, and f is the unknown function.

In order to give a solution of the above problem, note that, for $h \in L_1(\mathbb{R}_+)$ such that $h(0) = 0$, $\lim\limits_{x\to\infty} h'(x) = 0$, the Fourier sine and Fourier cosine transforms of h, h' exist. Furthermore,

$$\begin{aligned} (F_s h')(y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty h'(x) \sin xy \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \Big\{ h(x) \sin xy \Big|_0^\infty - y \int_0^\infty h(x) \cos xy \,\mathrm{d}x \Big\} \\ &= -y(F_c h)(y), \end{aligned} \tag{4.3}$$

and

$$(F_c h')(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty h'(x) \cos xy \,\mathrm{d}x = y(F_s h)(y). \tag{4.4}$$

Theorem 4.1. *Suppose the following condition holds:*

$$1 - \frac{(2n-1)!}{\sqrt{2\pi}\cdot 2^{2n-1}} F_c\Big(h \overset{\gamma}{*} \frac{1}{\cosh^{2n}\tau/2}\Big)(y) \neq 0, \quad \forall y > 0. \tag{4.5}$$

Then problem (4.2) *has a unique solution in* $L_1(\mathbb{R}_+)$ *whose closed form is*

$$f(x) = g(x) + (g \underset{F_c}{*} l)(x), \tag{4.6}$$

where $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}{1-((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}.$$

Proof. The equation (4.2) can be rewritten in the form

$$f(x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\prod_{k=1}^{n-1}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\Big)\{(h \overset{\gamma}{*} f)(x)\} = g(x). \tag{4.7}$$

Applying the Fourier cosine transform to both sides of (4.7), by original conditions (4.2) and by virtue of the factorization equality (1.12) and formulas (4.3), (4.4) we obtain

$$(F_c f)(y) - y^2 \prod_{k=1}^{n-1}(y^2+k^2)\cdot \frac{1}{y \sinh \pi y} K^{-1}[h](y)(F_c f)(y) = (F_c g)(y). \tag{4.8}$$

Using formula (see relation (1.9.3) in [5])

$$F_c\Big(\frac{1}{\cosh^{2n}\tau/2}\Big)(y) = \frac{\sqrt{2\pi}\cdot 2^{2n-1} y}{(2n-1)!\sinh \pi y}\prod_{k=1}^{n-1}(y^2+k^2),$$

we have

$$(F_c f)(y) - \frac{(2n-1)!}{\sqrt{2\pi}\cdot 2^{2n-1}} F_c\Big(\frac{1}{\cosh^{2n}\tau/2}\Big)(y) K^{-1}[h](y)(F_c f)(y) = (F_c g)(y),$$

or equivalently,

$$(F_c f)(y)\Big[1 - \frac{(2n-1)!}{\sqrt{2\pi}\cdot 2^{2n-1}} F_c\Big(h(\tau) \overset{\gamma}{*} \frac{1}{\cosh^{2n}\tau/2}\Big)(y)\Big] = (F_c g)(y).$$

From condition (4.5) we get

(4.9)

$$(F_c f)(y) = \left(1 + \frac{((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}{1 - ((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}\right)(F_c g)(y).$$

Recall that the Wiener-Levy theorem [8] states that if f is the Fourier transform of an $L_1(\mathbb{R})$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier transform of an $L_1(\mathbb{R})$ function. For the Fourier cosine transform it means that if f is the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}_+\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function.

By the given condition (4.5) the function $\varphi(z) = z/(1+z)$ satisfies the conditions of the Wiener-Levy theorem, and therefore, there exists a unique function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}{1 - ((2n-1)!/\sqrt{2\pi}\cdot 2^{2n-1})F_c(h \overset{\gamma}{*} \cosh^{-2n}\tau/2)(y)}.$$

Therefore the equation (4.9) becomes

$$(F_c f)(y) = (1 + (F_c l)(y))(F_c g) = F_c(g + g \underset{F_c}{*} l)(y),$$

which implies $f(x) = g(x) + (g \underset{F_c}{*} l)(x)$. The proof is complete. $\square$

References

[1] *M. Abramowitz, I. A. Stegun*: Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables. U.S. Department of Commerce, Washington, 1964.

[2] *R. A. Adams, J. J. F. Fournier*: Sobolev Spaces, 2nd ed. Pure and Applied Mathematics 140. Academic Press, New York, 2003.

[3] *F. Al-Musallam, V. K. Tuan*: Integral transforms related to a generalized convolution. Result. Math. *38* (2000), 197–208.

[4] *L. E. Britvina*: A class of integral transforms related to the Fourier cosine convolution. Integral Transforms Spec. Funct. *16* (2005), 379–389.

[5] *A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi*: Tables of Integral Transforms, Vol. I. Bateman Manuscript Project. California Institute of Technology. McGraw-Hill Book Co., New York, 1954.

[6] *H.-J. Glaeske, A. P. Prudnikov, K. A. Skórnik*: Operational Calculus and Related Topics. Analytical Methods and Special Functions 10. Chapman & Hall/CRC, Boca Raton, 2006.

[7] *Y. N. Grigoriev, N. H. Ibragimov, V. F. Kovalev, S. V. Meleshko*: Symmetries of Integro-Differential Equations. With Applications in Mechanics and Plasma Physics. Lecture Notes in Physics 806. Springer, Dordrecht, 2010.

[8] *M. A. Najmark*: Normed Algebras. Translated from the Second Russian Edition by Leo F. Boron. 3rd Completely Revised American Ed. Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics. Wolters-Noordhoff Publishing, Groningen, 1972.

[9] *A. P. Prudnikov, Y. A. Brychkov, O. I. Marichev*: Integrals and Series Vol. 2: Special Functions. Transl. from the Russian by N. M. Queen. Gordon & Breach Science Publishers, New York, 1986.

[10] *I. N. Sneddon*: Fourier Transforms. McGray-Hill Book Company, New York, 1950.

[11] *E. C. Titchmarsh*: Introduction to the Theory of Fourier Integrals. Third edition. Chelsea Publishing Co., New York, 1986.

[12] *T. Tuan*: On the generalized convolution with a weight function for the Fourier cosine and the inverse Kontorovich-Lebedev integral transformations. Nonlinear Funct. Anal. Appl. *12* (2007), 325–341.

[13] *V. K. Tuan*: Integral transforms of Fourier cosine convolution type. J. Math. Anal. Appl. *229* (1999), 519–529.

[14] *J. Wimp*: A class of integral transforms. Proc. Edinb. Math. Soc., II. Ser. *14* (1964), 33–40.

[15] *S. B. Yakubovich*: Integral transforms of the Kontorovich-Lebedev convolution type. Collect. Math. *54* (2003), 99–110.

[16] *S. B. Yakubovich, L. E. Britvina*: Convolutions related to the Fourier and Kontorovich-Lebedev transforms revisited. Integral Transforms Spec. Funct. *21* (2010), 259–276.

Authors' addresses: *Nguyen Thanh Hong*, High School for Gifted Students, Hanoi National University of Education, 136 Xuan Thuy, Hanoi, Vietnam, e-mail: hongdhsp1@yahoo.com; *Trinh Tuan*, Department of Mathematics, Electric Power University, 235 Hoang Quoc Viet, Hanoi, Vietnam, e-mail: tuantrinhpsac@yahoo.com; *Nguyen Xuan Thao*, Faculty of Applied Mathematics and Informatics, Hanoi University of Technology, No. 1, Dai Co Viet, Hanoi, Vietnam, e-mail: thaonxbmai@yahoo.com.