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On the Fourier Transform of Rapidly Decreasing Functions of L^p Type on a Symmetric Space

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1. Introduction

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $G=KAN$ be a fixed Iwasawa decomposition and M the centralizer of A in K. In a series of his papers Harish-Chandra introduced the Schwartz space $\mathscr{C}(G)$, in analogy to the space $\mathscr{S}(\boldsymbol{R}^n)$, of rapidly decreasing functions on the real euclidean space $\boldsymbol{R}^n$([10]), and also as one of the family of the whole spaces $\mathscr{C}^p(G)$. It is a problem to know whether one can carry out a Fourier analysis of the member of $\mathscr{C}^p(G)$ and know the image of $\mathscr{C}^p(G)$ by the Fourier transform, when possible.

After Harish-Chandra, Eguchi-Okamoto [3] introduced the Schwartz space $\mathscr{C}(G/K)$ on the symmetric space G/K, which is a subspace of the space $\mathscr{C}(G)$, and characterized the image of it by the Fourier transform. In this paper we consider the Fourier transform of the subspaces $\mathscr{C}^p(G/K)$ $(0<p<2;\ \mathscr{C}^2(G/K)=\mathscr{C}(G/K))$ consisting of functions in $\mathscr{C}^p(G)$ which are invariant under right K action.

Let $0<p<2$. Then the space $\mathscr{C}^p(G/K)$ is contained in $\mathscr{C}(G/K)$ and so, for any $f\in\mathscr{C}^p(G/K)$ its Fourier transform $\tilde{f}$ is defined. For a general element $f\in\mathscr{C}(G/K)$, $\tilde{f}$ is a C^∞ function on $\mathfrak{a}^*\times K/M$ with a growth condition and a property of symmetry; but if f is an element of $\mathscr{C}^p(G/K)$, $\tilde{f}$ extends analytically to the interior of a tubular domain with respect to the first component. We denote the tubular domain by F^p. The main theorem of this paper is that the space $\mathscr{Z}(F^p\times K/M)$ consisting of these functions which have holomorphic extension to $\mathrm{Int}\,F^p$ and such symmetry and growth, is the just image of the Fourier transform of $\mathscr{C}^p(G/K)$ in real rank one case.

A brief sketch of the proof of surjectivity is as follows: Let $\hat{K}^0$ be the set of the equivalence classes of unitary representations of K which are class 1 with respect to M. Let φ be a function in $\mathscr{Z}(F^p\times K/M)$ and f be the Fourier inverse image of φ. Applying the theorem for the Fourier transform of smooth functions on K/M (Sugiura [11]), we obtain a family of functions $\varphi^\delta(\delta\in\hat{K}^0)$ with values in endomorphisms of the representation space of δ. Then φ^δ has a growth with respect to δ. From this and the fact that f is the sum of trace of inverse image f^δ of φ^δ, it follows that $f\in\mathscr{C}^p(G/K)$. In order to show that f^δ satisfies the growth condition, we employ the usual manner which Helgason uses in his papers [9(c), (d)]. For this we need Harish-Chandra's theorem for the asymptotic expansion of Eisenstein integrals ([7(d)], also [14, Chap. IV]), some results about C functions in [9(d)] and an estimate for the coefficients Γ_μ of expansion of Eisenstein integrals by Hashizume [8]. This results in shifting the integral on $\mathfrak{a}^*$ towards the boundary of the tubic domain. This method is similar to the proof of the theorem for $I^1(G)$ by Helgason [9(c)].

The spaces $I^p(G)$, consisting of all functions in $\mathscr{C}^p(G/K)$ which are also invariant under left K-action, were studied by Ehrenpreis-Mautner [4] in the case $G=\boldsymbol{SL}(2, \boldsymbol{R})$, by Helgason [9(c)] for the case when G is either complex or of real rank one and $p=1$. Trombi [12] and Trombi and Varadarajan [13] determined the image of $I^p(G)$ for $0<p<2$, the former for the case of real rank one and the latter for general case respectively. Moreover, in the case $p=2$, Harish-Chandra [7(a)] characterized the spherical Fourier transform of $I(G)$. Arthur [1(a)] and Eguchi [2(a)] obtain the corresponding results for $\mathscr{C}(G)$, the former when G is of real rank one and the latter when G has only one conjugate class of Cartan subgroups. Recently Arthur [1(b)] proved the theorem for the general case and Eguchi [2(b)] characterized the image of Fourier transform of $\mathscr{C}(\boldsymbol{E}_\tau)$, the Schwartz space on the vector bundle on G/K which is associated to a unitary representation τ of K on a finite dimensional vector space.

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2. Notation and Preliminaries

As usual let $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote the ring of integers, the field of real numbers and the field of complex numbers respectively; $\boldsymbol{Z}^+$ denotes the set of non-negative integers. If T is a topological space and S a subset of T, $\mathrm{Int}\, S$ and $Cl(S)$ denote the interior of S and the closure of S in T, respectively. For a vector space V over $\boldsymbol{R}$, V_c denotes the complexification of V.

Let G be a connected semisimple Lie group with finite center, $\mathfrak{g}$ its Lie algebra and $<\ ,\ >$ the Killing form of $\mathfrak{g}_c$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Let K be the analytic subgroup with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its dual and $F=\mathfrak{a}_c^*$. For a root λ of $(\mathfrak{g}, \mathfrak{a})$ let m_λ be the multiplicity of λ. If $\lambda, \mu\in F$ let $H_\lambda\in\mathfrak{a}_c$ be determined by $\lambda(H)=<H_\lambda, H>$ $(H\in\mathfrak{a})$ and put $<\lambda, \mu>=<H_\lambda, H_\mu>$. If $\lambda\in\mathfrak{a}^*$ and $X\in\mathfrak{p}$, put $|\lambda|=<\lambda, \lambda>^{1/2}$, $|X|=<X, X>^{1/2}$. Fix a Weyl chamber $\mathfrak{a}^+\subset\mathfrak{a}$ and let $\mathfrak{a}_+^*$ denote its preimage in A^* under the map $\lambda\to H_\lambda$. Let Σ^+ denote the set of positive roots and put $\rho=(1/2)\sum_{\alpha\in\Sigma^+}m_\alpha\alpha$ and $\mathfrak{n}=$

$\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space for $\alpha\in\Sigma^+$. By the usual manner we get an Iwasawa decomposition $G=KAN$; $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$. Let $A^+=\exp\mathfrak{a}^+$. Then $G=KCl(A^+)K$. Any $g\in G$ can be written $g=\kappa(g)\exp H(g)n(g)=k_1ak_2$, where $\kappa(g)\in K$, $n(g)\in N$, $H(g)\in\mathfrak{a}$, $a\in A^+$ are unique. Put $\log a=H(a)$ $(a\in A)$. Let M (resp. M') denote the centralizer (resp. normalizer) of A in K, $W=M'/M$ the Weyl group, which acts as a group of linear transformations on $\mathfrak{a}$ and F. Let ω denote the order of W and put $l=\dim\mathfrak{a}$.

The Killing form induces euclidean measures on A and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-(1/2)l}$ we obtain invariant measures da and $d\lambda$ on A and $\mathfrak{a}^*$ respectively, such that for each $f\in\mathscr{S}(A)$, the following equalities

$$f^*(\lambda)=\int_A f(a)\exp\{-i\lambda(\log a)\}da \qquad (\lambda\in\mathfrak{a}^*), \tag{2.1}$$

$$f(a)=\int_{\mathfrak{a}^*} f^*(\lambda)\exp\{i\lambda(\log a)\}d\lambda \qquad (a\in A), \tag{2.2}$$

hold without any multiplicative constants, where i denotes a square root of -1. We normalize the Haar measures dk and dm on the compact groups K and M respectively so that the total measures are one respectively. The Haar measures of the nilpotent groups N and $\bar{N}=\theta(N)$ are normalized so that

$$\theta(dn)=d\bar{n},\quad \int_{\bar{N}}\exp\{-2\rho(H(\bar{n}))\}d\bar{n}=1.$$

The Haar measure dx on G can be normalized so that
$dx=\exp\{2\rho(\log a)\}dkdadn$ $(x=kan)$ and $dx=\Delta(a)dk_1dadk_2$ $(x=k_1ak_2)$, where the function Δ on A^+ is defined by $\Delta(a)=c\prod_{\alpha\in\Sigma^+}(\sinh\alpha(\log a))^{m_\alpha}$ for a suitable constant c. Let φ_λ $(\lambda\in F)$ be the elementary spherical functions ([7(a)]) and put $\Xi=\varphi_0$. For $x=k\exp X$ $(k\in K, X\in\mathfrak{p})$ put $\sigma(x)=|X|$ $(x\in G)$. Then σ is a spherical function on G. It is known that there exist positive numbers c, d and e such that

$$\Xi(a)\leqq c\exp\{-\rho(\log a)\}(1+\sigma(a))^d \qquad (a\in A^+), \tag{2.3}$$

$$\int_G \Xi(x)^2(1+\sigma(x))^{-e}<\infty. \tag{2.4}$$

(See [7(c), p. 16, 17]).

For any element v of the symmetric algebra $S(\mathfrak{a}_c)$ over $\mathfrak{a}_c$ let $\partial(v)$ denote the corresponding differential operator on $\mathfrak{a}$, then $S(\mathfrak{a}_c)$ (resp. $S(F)$) can be regarded as the algebra of all differential operators with constant coefficients on $\mathfrak{a}$ (resp. F).

Let T be a maximal torus of K and $\mathfrak{t}$ be the corresponding Lie subalgebra of $\mathfrak{k}$. If μ is a pure imaginary valued linear function on $\mathfrak{t}$ we can select a unique element $h_\mu\in\mathfrak{t}$ such that $\mu(H)=-i\langle h_\mu,H\rangle$ for all $H\in\mathfrak{t}$. Let Γ be the set of all $H \in \mathfrak{t}$ with $\exp H = 1$. Let $\hat{\Gamma}$ be the set of all $H \in \mathfrak{t}$ suh that $<H, X> \in 2\pi \boldsymbol{Z}$ for all $X \in \Gamma$, then $\hat{\Gamma}$ is the dual lattice of Γ. Let D be the subset of all $\mu \in \Gamma$ such that $<\mu, \alpha_i> \leqq 0$ $(1 \leqq i \leqq l)$, where $\alpha_1, \dots, \alpha_l$ are the set of all simple roots with respect to a lexicographic order in the set of nonzero roots of $(\mathfrak{k}, \mathfrak{t})$. Then there is a bijective map $\mu \to \sigma(\mu)$ from D onto $\hat{K}$, the set of all unitary equivalence classes of irreducible representations of K. We put

$$|\sigma| \doteq -<\mu, \mu> .$$

3. The Fourier transform of $\mathscr{C}^p\ (G/K)$

Let $0<p<2$ and let $\mathscr{C}^p(G/K)$ denote the set of C^∞ functions f on G which satisfy the following conditions: (i) $f(xk) = f(x)$ for any $x \in G$ and $k \in K$; (ii) For any $r \in \boldsymbol{Z}^+$ and $g, g' \in \mathfrak{G}$

$$\tau^p_{r,g,g'}(f) = \sup_{x \in G} |f(g; x; g')| \Xi(x)^{-2/p}(1+\sigma(x))^r < \infty. \tag{3.1}$$

The seminorms $\tau^p_{r,g;g'}$ convert $\mathscr{C}^p(G/K)$ into a Fréchet space. By definition of $\mathscr{C}^p(G/K)$ and the property of the spherical function Ξ, it is clear that

$$\mathscr{D}(G/K) \subset \mathscr{C}^p(G/K) \subset \mathscr{C}^q(G/K) \subset \mathscr{C}(G/K)$$

if $0<p \leqq q \leqq 2$, where $\mathscr{D}(G/K)$ denotes the space of all C^∞ functions on G with compact support which are invariant under the right K-action. $\mathscr{D}(G/K)$ is dense in $\mathscr{C}^p(G/K)$; this is obtained by a similar proof to the one for the case $p=2$ (cf. [7 (c), §13]). Moreover, since the function Ξ satisfies

$$\int_G \Xi(x)^2(1+\sigma(x))^{-r}dx < \infty$$

for a number $r \geqq 0$, we see easily that $\mathscr{C}^p(G/K) \subset L^p(G/K)$.

For each p let F^p be the set of all linear functionals λ on $\mathfrak{a}_c$ such that $|\mathrm{Im}\, s\lambda(H)| \leqq \varepsilon\rho(H)$ for any $H \in \mathfrak{a}^+$ and $s \in W$, where $\varepsilon = 2/p - 1$ and Im denotes the imaginary part. For any continuous function φ on $\mathrm{Int}\, F^p \times K/M$ we define a function $\check{\varphi}$ on $\mathrm{Int}\, F^p \times G$ by

$$\check{\varphi}(\lambda : x) = \int_K \varphi(\lambda : \kappa(xk)M) \exp\{(i\lambda - \rho)(H(xk))\} dk. \tag{3.2}$$

Now let $\mathscr{Z}(F^p \times K/M)$ denote the space consisting of all C^∞ functions φ on $\mathfrak{a}^* \times K/M$ which satisfy the following conditions: (i) For fixed $k \in K$ the function $\lambda \to \varphi(\lambda : kM)$ extends to $\mathrm{Int}\, F^p$ as a holomorphic function; (ii) $\check{\varphi}(s\lambda : x) = \check{\varphi}(\lambda : x)$ for any $\lambda \in \mathrm{Int}\, F^p$, $s \in W$ and $x \in G$; (iii) For any $q, r \in \boldsymbol{Z}^+$ and $u \in S(F)$

$$\zeta^p_{q,r,u}(\varphi) = \sup_{\mathrm{Int}\, F^p \times K/M} |\varphi(\lambda; \partial(u) : kM; \omega^r_k)| (1+|\lambda|)^q < \infty, \tag{3.3}$$

where ω_k denotes the Casimir operator for K. The seminorms $\zeta^p_{q,r,u}$ convert $\mathscr{Z}^p(F\times K/M)$ into a Fréchet space.

For any function f in $\mathscr{C}^p(G/K)$ its Fourier transform is defined by

$$(3.4)\quad \tilde{f}(\lambda\colon kM)=(\mathscr{F}f)(\lambda\colon kM)=\int_{AN}f(kan)\exp\{(-i\lambda+\rho)(\log a)\}dadn.$$

By formula (2.3) it is easy to check that the above expression is equal to

$$\int_G f(x)\exp\{(i\lambda-\rho)(H(x^{-1}k))dx.$$

THEOREM 3.1. *The Fourier transform* $\mathscr{F}$ *is a continuous mapping of* $\mathscr{C}^p(G/K)$ *into* $\mathscr{Z}(F^p\times K/M)$. *In the special case, when the real rank of* G *equals one,* $\mathscr{F}$ *is a linear topological isomorphism of* $\mathscr{C}^p(G/K)$ *onto* $\mathscr{Z}^p(F^p\times K/M)$.

In order to prove this theorem we need some lemmas.

4. The proof of injectivity

LEMMA 4.1. *Let* $f\in C^p(G/K)$. *For each* $\lambda\in\operatorname{Int}F^p$ *and* $k\in K$ *the integral*

$$(4.1)\qquad \tilde{f}(\lambda\colon kM)=\int_{AN}f(kan)\exp\{(-i\lambda+\rho)(\log a)\}dadn$$

is uniformly convergent for $\lambda\in\operatorname{Int}F^p$, *and for any fixed* $k\in K$ *the function* $\lambda\to f(\lambda\colon kM)$ *is holomorphic on* $\operatorname{Int}F^p$.

PROOF. Let $\alpha_1,\dots,\alpha_l$ be all simple restricted roots and $\varepsilon_1,\dots,\varepsilon_l$ be the elements in F such that $<\alpha_i,\varepsilon_j>=\delta_{ij}$. Then $\{\varepsilon_j\}_{1\le j\le l}$ is a basis for F. We introduce a global coordinate on F by $\lambda=\Sigma_{1\le j\le l}\lambda_j\varepsilon_j$. Then we have for any j $(1\le j\le l)$

$$(4.2)\qquad \left|f(kan)\frac{\partial}{\partial\lambda_j}\exp\{(-i\lambda+\rho)(\log a)\}\right|$$
$$\le|f(kan)|\,|\varepsilon_j(\log a)\,|\exp\{(\eta+\rho)(\log a)\},$$

where $\lambda=\xi+i\eta$ ($\xi,\eta\in\mathfrak{a}^*$). Since we can find a constant $c\ge 1$ such that

$$(4.3)\qquad 1+\sigma(a)\le c(1+\sigma(an))\qquad (a\in A,\ n\in N)$$

(see [7(c), p. 106]), we have

$$(4.4)\qquad |\varepsilon_j(\log a)|\le c|\varepsilon_j|(1+\sigma(an))\qquad (a\in A,\ n\in N).$$

Let d be the constant in (2.3). Then from (2.4) we can choose $r>0$ such that

$$\int_G \Xi(x)^2(1+\sigma(x))^{2(1+d)/p+1-d-r}dx < \infty. \tag{4.5}$$

Since $f\in \mathscr{C}^p(G/K)$, for this r we can choose a constant $c>0$ such that

$$|f(kan)| \leqq c(1+\sigma(an))^{(2/p)-r}\Xi(an)^{2/p}$$

for all $k\in K$, $a\in A$ and $n\in N$. Therefore, the expression (4.2) is bounded by

$$c(1+\sigma(an))^{(2/p)+1-r}\Xi(an)^{2/p}\exp\{(\eta+\rho)(\log a)\},$$

where c is a positive constant. If this expression is integrable on AN, then

$$\begin{aligned}(4.6)\quad &\int_{AN}(1+\sigma(an))^{(2/p)+1-r}\Xi(an)^{2/p}\exp\{(\eta+\rho)(\log a)\}dadn\\ &=\int_G(1+\sigma(x))^{(2/p)+1-r}\Xi(x)^{2/p}\exp\{(\eta-\rho)(H(x))\}dx\\ &=\int_{A^+K}(1+\sigma(a))^{(2/p)+1-r}\Xi(a)^{2/p}\exp\{(\eta-\rho)(H(ak)\}\Delta(a)dadk.\end{aligned}$$

Since it is known that

$$\int_K \exp\{(\eta-\rho)(H(ak))\}dk \leqq e^{\eta(\log a)}\Xi(a) \qquad (a\in A^+)$$

([12, p. 282]), from (2.3) it follows that (4.6) is bounded by

$$c\int_{A^+}\Xi(a)^2(1+\sigma(a))^{-q}\Delta(a)\exp\{(\eta-\varepsilon\rho)(\log a)\}da, \tag{4.7}$$

where c is a positive constant and $q=r+d-1-2(1+d)p$. If $\lambda\in \operatorname{Int}F^p$, $|s\eta(H)|\leqq\varepsilon\rho(H)$ $(H\in\mathfrak{a}^+,\ s\in W)$. So the above expression is bounded by

$$c\int_{A^+}\Xi(a)^2(1+\sigma(a))^{-q}\Delta(a)da = c\int_G\Xi(x)^2(1+\sigma(x))^{-q}dx.$$

This proves that (4.6) is absolutely convergent. Hence the integral

$$\int_{AN} f(kan)\frac{\partial}{\partial\lambda_j}\exp\{(-i\lambda+\rho)(\log a)\}dadn$$

converges uniformly for $\lambda\in\operatorname{Int}F^p$. More generally, iterating the above discussion we see that for each polynomial P in l variables the integral

$$\int_{AN} f(kan)P\left(\frac{\partial}{\partial\lambda_1},\dots,\frac{\partial}{\partial\lambda_l}\right)\exp\{(-i\lambda+\rho)(\log a)\}dadn$$

converges uniformly for $\lambda \in \operatorname{Int} F^p$. Therefore formula (4.1) can be differentiated under the integral. So, the function $\lambda \to f(\lambda : kM)$ is holomorphic on $\operatorname{Int} F^p$ for any fixed $k \in K$. This completes the proof of the lemma.

LEMMA 4.2. *For any $p, r \in \boldsymbol{Z}^+$ and $u \in S(F)$ we can select $q \in \boldsymbol{Z}^+$, finite elements $g_0, g_1, \dots, g_s \in \mathfrak{G}$ and a positive number c such that*

$$\sup_{\operatorname{Int} F^p \times K/M} |\tilde{f}(\lambda;\, \partial(u) : kM;\, \omega_k^r)|(1+|\lambda|)^p$$

$$\leqq c \sum_{1 \leqq i \leqq s} \sup_{x \in G} |f(g_0;\, x;\, g_i)|\, \Xi(x)^{-2/p}(1+\sigma(x))^q.$$

PROOF. Let $\{H_j\}_{1 \leqq j \leqq l}$ be an orthonormal basis of $\mathfrak{a}$ and consider an element of $\mathfrak{A}$ (the subalgebra of $\mathfrak{G}$ generated by 1 and $\mathfrak{a}_c$) defined by

$$h = -\sum_{1 \leqq j \leqq l} H_j^2 + 2H_\rho.$$

Put

$$\psi_\lambda(a) = \exp\{(-i\lambda + \rho)(\log a)\} \qquad (a \in A).$$

Then, by simple calculation we have

$$\psi_\lambda(a;\, h) = (|\lambda|^2 + |\rho|^2)\psi_\lambda(a).$$

Let $n \in \boldsymbol{Z}^+$ and $u \in S(F)$. Then we see that

$$(4.8) \qquad (|\lambda|^2 + |\rho|^2)^n u_\lambda(\omega_k^r)_k \int_{AN} f(kan) \exp\{(-i\lambda + \rho)(\log a)\} da\, dn$$

$$= \int_{AN} f(\omega_k^r;\, kan) P_u(a) \psi_\lambda(a;\, h^n) da\, dn,$$

where P_u is a polynomial which is determined by u;

$$P_u(a) = \sum_{0 \leqq r \leqq d} \sum_{i_1 + \cdots + i_l = r} a_{i_1 \cdots i_l} \varepsilon_1^{i_1}(-i \log a) \dots \varepsilon_l^{i_l}(-i \log a) \qquad (a \in A),$$

$$u_\lambda = \sum_{0 \leqq r \leqq d} \sum_{i_1 + \cdots + i_l = r} a_{i_1 \cdots i_l} \left(\frac{\partial}{\partial \lambda_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial \lambda_l}\right)^{i_l},$$

here $a_{i_1 \cdots i_l}$ are constants. We put $f(k : a : n) = f(kan)$ $(k \in K,\, a \in A,\, n \in N)$. If $H \in \mathfrak{a}$,

$$\int_{AN} f(k;\, \omega_k^r : a : n) P_u(a) \psi_\lambda(a;\, H) da\, dn$$

$$= \int_{AN} f(\omega_k^r;\, k : a;\, -H : n) P_u(a) \psi_\lambda(a) da\, dn$$

$$+ \int_{AN} f(\omega_k^r;\, kan) P_u(a;\, -H) \psi_\lambda(a) da\, dn.$$

Since for a good function ϕ on AN

$$\int_N \phi(na)dn = \exp\{2\rho(\log a)\}\int_N \phi(an)dn \qquad (a \in A),$$

the first term is equal to

$$\int_{AN} f(\omega_k^r;\, kan)P_u(a)\psi_\lambda(a)dadn - 2\rho(-H)\int_{AN} f(\omega_k^r;\, kan)P_u(a)\psi_\lambda(a)dadn.$$

Iterating the above discussion, we can choose finite elements $g_0 = \omega_k^r, g_1, \dots, g_s \in \mathfrak{G}$, $b_1, \dots, b_s \in \mathfrak{A}$ and $c_1, \dots, c_s \in \boldsymbol{R}$ so that formula (4.8) equals

$$\sum_{1 \leq j \leq s} c_j \int_{AN} f(g_0;\, kan;\, g_j)P_u(a;\, b_j)\exp\{(-i\lambda+\rho)(\log a)\}dadn. \tag{4.9}$$

Now for each j $(1 \leqq j \leqq r)$ we can choose $d_j \geqq 0$ and $s_j \in \boldsymbol{Z}^+$ such that

$$|P_u(a;\, b_j)| \leqq d_j(1+|\log a|)^{s_j} = d_j(1+\sigma(a))^{s_j} \qquad (a \in A).$$

The absolute value of the integral in (4.9) is bounded by

$$cd_j \cdot \sup_{x \in G}\{|f(g_0;\, x;\, g_j)|\Xi(x)^{-2/p}(1+\sigma(x))^{s_j+t}\} \cdot$$

$$\cdot \int_{AN} \Xi(an)^{2/p}(1+\sigma(an))^{-t}\exp\{(\eta+\rho)(\log a)\}dadn,$$

here we use the relation (4.3). By the same discussion as in the proof of Lemma 4.1, for a sufficiently large $t > 0$ the last integral is finite if $\lambda \in \operatorname{Int} F^p$. This proves our lemma.

LEMMA 4.3. *Let $f \in \mathscr{C}^p(G/K)$. Then $\tilde{f}$ satisfies the following functional equation with respect to the Weyl group W;*

$$(\tilde{f})^\vee_{s\lambda} = (\tilde{f})^\vee_\lambda \qquad (\lambda \in \operatorname{Int} F^p,\ s \in W).$$

PROOF. By definition of the Fourier transform $\mathscr{F}$ and the dual Radon transform $\vee$ we have

$$(\tilde{f})^\vee_\lambda(x) = \int_K \tilde{f}(\lambda\colon \kappa(xk))\{\exp(i\lambda-\rho)(H(xk))\}dk$$

$$= \int_{K \times G} f(g)\exp\{(i\lambda-\rho)(H(g^{-1}\kappa(xk))+H(xk))\}dgdk.$$

Since $H(g^{-1}xk) = H(g^{-1}\kappa(xk)) + H(xk)$, the last integral equals

$$\int_{K \times G} f(g)\exp\{(i\lambda-\rho)(H(g^{-1}xk))\}dgdk = f \times \varphi_\lambda(x),$$

where φ_λ is the elementary spherical function and $\times$ denotes the convolution. So $\varphi_\lambda=\varphi_{s\lambda}$ implies that $(\tilde{f})^\vee_\lambda=(\tilde{f})^\vee_{s\lambda}$ ($\lambda\in \mathrm{Int}\, F^p$, $s\in W$). This proves our lemma.

Since $\mathscr{C}^p(G/K)\subset L^2(G/K)$, now Plancherel's theorem ([9(c), p. 15]), Lemmas 4.1, 4.2 and 4.3 complete the proof of the injectivity and the continuity of the Fourier transform $\mathscr{F}: \mathscr{C}^p(G/K)\to\mathscr{Z}(F^p\times K/M)$.

5. The proof of surjectivity

In this section we assume the real rank of G to be one.

Let $\psi\in\mathscr{Z}(F^p\times K/M)$. Then its Fourier inversion is given by

$$f(x) = \omega^{-1}\int_{\mathfrak{a}^*}\check{\psi}(\lambda: x)|c(\lambda)|^{-2}d\lambda, \tag{5.1}$$

where c is Harish-Chandra's c-function. (See [3] and [9(c)]). In order to prove that $f\in\mathscr{C}^p(G/K)$, we use a theorem of Fourier analysis on the compact group K.

Let $\hat{K}^0$ denote the set of equivalence classes of irreducible unitary representations of K of class 1 with respect to M. Let δ be such a representation of K and V_δ be the representation space of dimension $d(\delta)$. For $F\in C^\infty(G/K)$ we put

$$F^\delta(x) = d(\delta)\int_K F(kx)\delta(k^{-1})dk. \tag{5.2}$$

Then F^δ is a C^∞ function on G with values in $\mathrm{Hom}(V_\delta, V_\delta)$, the space of endomorphism of V_δ, and satisfies

$$F^\delta(kx) = \delta(k)F^\delta(x). \tag{5.3}$$

For $\delta\in\hat{K}^0$ we derive from (5.1)

$$f^\delta(x) = \omega^{-1}\int_{\mathfrak{a}^*}\left(\int_K \exp\{-(i\lambda+\rho)(H(x^{-1}k))\delta(k)\}dk\right)\psi^\delta(\lambda)|c(\lambda)|^{-2}d\lambda, \tag{5.4}$$

where

$$\psi^\delta(\lambda: kM) = d(\delta)\int_K \psi(\lambda: k_1kM)\delta(k_1^{-1})dk_1 = \delta(k)\psi^\delta(\lambda: eM), \tag{5.5}$$

$$\psi^\delta(\lambda) = \psi^\delta(\lambda: eM).$$

From the theorem of the Fourier transform of smooth functions on the compact group K ([11]) it follows that for each r, $s\in\mathbf{Z}^+$ and $u\in S(F)$

$$\sup_{\mathrm{Int}\,F^p\times\hat{K}^0}\|\psi^\delta(\lambda;\partial(u))\|(1+|\delta|)^r(1+|\lambda|)^s<\infty, \tag{5.6}$$

where $\|A\|$ denotes the Hilbert-Schmidt norm of the endomorphism A. We also denote the trace of A by $Tr\,A$.

LEMMA 5.1. *Let* $\{\psi^\delta\}_{\delta\in\hat{K}^0}$ *be a family of* C^∞ *functions* ψ^δ *from* $\mathfrak{a}^*\times K/M$ *to* $\mathrm{Hom}(V_\delta, V_\delta)$ *which satisfy the following conditions*: (i) *For each* $k\in K$ *the function* $\lambda\mapsto\psi^\delta(\lambda: kM)$ *extends to a holomorphic function on* $\mathrm{Int}\,F^p$; (ii) $(\psi^\delta)^\vee_{s\lambda}=(\psi^\delta)^\vee_\lambda$ *for any* $\lambda\in\mathrm{Int}\,F^p$ *and* $s\in W$; (iii) *For each* r, $s\in\boldsymbol{Z}^+$ *and* $u\in S(F)$, ψ^δ *satisfies the relation* (5.6); (iv) $\psi^\delta(\lambda: kM)=\delta(k)\psi^\delta(\lambda: eM)$. *Then the functions* $F^\delta(x)$ $(\delta\in\hat{K}^0)$ *from* G/K *to* $\mathrm{Hom}(V_\delta, V_\delta)$ *defined by*

$$F^\delta(x) = \omega^{-1}\int_{\mathfrak{a}^*}(\psi^\delta)^\vee(\lambda: x)|c(\lambda)|^{-2}d\lambda$$

are infinitely differentiable and satisfy that for each q, $r\in\boldsymbol{Z}^+$ *and* g, $g'\in\mathfrak{G}$

$$\sup_{G\times\hat{K}^0}|\mathrm{Tr}\,F^\delta(g;\, x;\, g')|\Xi(x)^{-2/p}(1+\sigma(x))^q(1+|\delta|)^r < \infty. \tag{5.7}$$

We shall prove the lemma in following sections. Now we assume this lemma. By the lemma it is clear that the sum

$$\sum_{\delta\in\hat{K}^0}\mathrm{Tr}\,f^\delta(g;\, x;\, g')$$

is absolutely convergent for each $g\in\mathfrak{G}$. So we have

$$f(g;\, x;\, g') = \sum_{\delta\in\hat{K}^0}\mathrm{Tr}\,f^\delta(g;\, x;\, g') \tag{5.8}$$

Take a sufficiently large $r\in\boldsymbol{Z}^+$ so that

$$\sum_{\delta\in\hat{K}^0}(1+|\delta|)^{-r}$$

is convergent. Then for each $q\in\boldsymbol{Z}^+$ and g, $g'\in\mathfrak{G}$, we have obviously

$$\sup_{x\in G}f(g;\, x;\, g')|\Xi(x)^{-2/p}(1+\sigma(x))^q < \infty.$$

This shows that $f\in\mathscr{C}^p(G/K)$. As is well-known, since a continuous and bijective mapping from a Fréchet space onto a Fréche space is a topological isomorphism, we obtain Theorem 3.1.

It is left only to prove Lemma 5.1.

6. Harish-Chandra's C function and an estimate for Γ_μ

Let $\sigma=(\sigma_1, \sigma_2)$ be a double unitary representation on a finite dimensional Hilbert space V, σ_1 and σ_2 acting on the left and right respectively. Let $\lambda\in F$ and consider the function

$$\varphi(x)v = \int_K\sigma_1(\kappa(xk))v\sigma_2(k^{-1})\exp\{(i\lambda-\rho)(H(xk))\}dk$$

($x \in G$, $v \in V$), then the function φ is a σ-spherical function. Let

$$V_\sigma^M = \{v \in V | \sigma_1(m)v = v\sigma_2(m) \text{ for all } m \in M\}.$$

Harish-Chandra gives the following series expansion.

Let $\alpha_1, \ldots, \alpha_l$ be the simple restricted roots, L the set of integral linear combinations $n_1\alpha_1 + \cdots + n_l\alpha_l$ ($n_i \in \boldsymbol{Z}^+$) and $L' = L - \{0\}$.

LEMMA 6.1. *There exist certain meromorphic functions C_s ($s \in W$) on F and rational functions Γ_μ ($\mu \in L$) on F all with values in* $\mathrm{Hom}(V_\sigma^M, V_\sigma^M)$ *such that for $a \in A^+$, $v \in V_\sigma^M$*

$$\exp\{\rho(\log a)\}\int_K \sigma_1(\kappa(ak))v\sigma_2(k^{-1})\exp\{(i\lambda - \rho)(H(ak)\}dk = \textstyle\sum_{s\in W}\Phi(s\lambda: a)C_s(\lambda)v,$$

where

$$\Phi(\lambda: a) = \exp\{i\lambda(\log a)\}\textstyle\sum_{\mu\in L}\Gamma_\mu(\lambda)\exp\{-\mu(\log a)\}.$$

Here λ varies in a certain open dense subset $^*F'$ of F, the functions Γ_μ are given by certain explicit recursion formulas, depending on σ (see [14, Chap. IX]).

Just for the case σ_2 = identity representation we shall need this theorem and an estimate of Γ_μ, which Hashizume [8] obtained by a generalization of Gangolli's method [5]. Let

$$R = \{\lambda \in F \,|\, \mathrm{Im}\,\lambda \in Cl(\mathfrak{a}_+^*)\}.$$

If $\mu \in L$, $\mu = \sum_{1\leq i\leq l} m_i\alpha_i$ ($m_i \geqq 0$), then the number $m(\mu) = \sum_{1\leq i\leq l} m_i$ is called the level of μ.

LEMMA 6.2 ([8]). *We can choose positive numbers a, b such that*

$$\|\Gamma_\mu(\lambda)\| \leqq a(1 + m(\mu)^b)$$

for all $\lambda \in R$.

Recall the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}_c$. Let λ be the canonical symmetrization from the symmetric algebra $S(\mathfrak{g}_c)$ ober $\mathfrak{g}_c$ onto $\mathfrak{G}$. Let $\mathfrak{q}$ be the orthogonal complement of (the Lie subalgebra corresponding to M) in $\mathfrak{k}$. Put $\lambda(S(\mathfrak{q}_c)) = \mathfrak{Q}$. Let $\mathfrak{A}$, $\mathfrak{K}$ be the subalgebras of $\mathfrak{G}$ generated by 1 and $\mathfrak{a}$, 1 and $\mathfrak{k}$, respectively. For $\alpha \in \Sigma^+$, let us write

$$f_\alpha^\pm(a) = (\exp\alpha(\log a) \pm 1)^{-1} \qquad (a \in A'),$$

where A' denotes the set of all $a \in A$ such that $\alpha(\log a) \neq 0$ for all $\alpha \in \Sigma^+$. Let F_0 denote the algebra generated over $\boldsymbol{C}$ by $f_\alpha^\pm$ ($\alpha \in \Sigma^+$). Then for any $g \in \mathfrak{G}$ there exist finite sets $\{f_i\} \subset F_0$, $\{q_i\} \subset \mathfrak{Q}$, $\{h_i\} \subset \mathfrak{A}$ and $\{d_i\} \subset \mathfrak{K}$ ($1 \leqq i \leqq l$) such that

$$D = \sum_i f_i(a) q_i^{q-1} h_i d_i \qquad (a \in A').$$

(See [7(a)], also [14, Chap. IX].) We use this fact in the following section.

7. The proof of the lemma

In this section we assume the real rank of G to be one. Let $\{\psi^\delta\}_{\delta \in \hat{K}^0}$ be a family of C^∞ functions ψ^δ from $F^p \times K/M$ to $\mathrm{End}(V_\delta, V_\delta)$ which satisfy the conditions (i), (ii), (iii) and (iv) in Lemma 5.1, that is; (i) For each $\delta \in \hat{K}^0$ and $k \in K$ the function $\lambda \to \psi^\delta(\lambda: kM)$ extends to a holomorphic function in $\mathrm{Int}\, F^p$; (ii) $(\psi^\delta)^\vee_{s\lambda} = (\psi^\delta)^\vee_\lambda$ for any $\lambda \in \mathrm{Int}\, F^p$ and $s \in W$; (iii) For each $r, s \in \boldsymbol{Z}^+$ and $u \in S(F)$

$$\sup_{\mathrm{Int}\, F^p \times \hat{K}^0} \|\psi^\delta(\lambda; \partial(u))\| (1+|\delta|)^r (1+|\lambda|)^s < \infty;$$

(iv) $\psi^\delta(\lambda: kM) = \delta(k)\psi^\delta(\lambda: eM)$.

For simplicity we write $\psi^\delta(\lambda) = \psi^\delta(\lambda: eM)$. Put

$$\varphi^\delta(x) = \omega^{-1} \int_{\mathfrak{a}^*} \left(\int_K \psi^\delta(\lambda: \kappa(xk)) \exp\{(i\lambda - \rho)(H(xk))\} dk \right) |c(\lambda)|^{-2} d\lambda, \tag{7.1}$$

which is equal to the expression (5.4) and

$$\omega^{-1} \int_{\mathfrak{a}^*} \left(\int_K \delta(\kappa(xk)) \exp\{(i\lambda - \rho)(H(xk))\} dk \right) \psi^\delta(\lambda) |c(\lambda)|^{-2} d\lambda.$$

Using Harish-Chandra's asymptotic expansion theorem for the Eisenstein integral

$$\int_K \delta(\kappa(xk)) \exp\{(i\lambda - \rho)(H(xk))\} dk,$$

we have for $x = k_1 a k_2$ ($k_1, k_2 \in K$, $a \in A^+$)

$$\begin{aligned} \varphi^\delta(k_1 a k_2) &= \varphi^\delta(k_1 a) \\ &= \omega^{-1} \exp\{-\rho(\log a)\}\, \delta(k_1) \int_{\mathfrak{a}^*} \psi^\delta(\lambda) \sum_{s \in W} \exp\{is\lambda(\log a)\} \\ &\quad \cdot \sum_{\mu \in L} \Gamma_\mu(s\lambda) \exp\{-\mu(\log a)\} C_s(\lambda) |c(\lambda)|^{-2} d\lambda. \end{aligned}$$

Transforming λ as $-s^{-1}\lambda$, we see that the last expression equals

$$\begin{aligned} &\omega^{-1} \exp\{-\rho(\log a)\} \delta(k_1) \sum_{s \in W} \int_{\mathfrak{a}^*} \exp\{-i\lambda(\log a)\} \psi^\delta(-s^{-1}\lambda) \\ &\quad \cdot \sum_{\mu \in L} \Gamma_\mu(-\lambda) \exp\{-\mu(\log a)\} C_s(-s^{-1}\lambda) |c(\lambda)|^{-2} d\lambda, \end{aligned}$$

here we use the relation $|c(\lambda)|^2 = c(s\lambda)c(-s\lambda)$, $(\lambda \in \mathfrak{a}^*, s \in W)$. By means of the relation ([9(d), p. 465])

$$\psi^\delta(-s^{-1}\lambda) = c(\lambda)^{-1}C_s(-s^{-1}\bar{\lambda})^*\psi^\delta(-\lambda),$$

we obtain that the last expression equals

$$(7.2)\qquad \omega^{-1}\exp\{-\rho(\log a)\}\delta(k_1)\sum_{s\in W}\int_{\mathfrak{a}^*}\exp\{-i\lambda(\log a)\}$$
$$\cdot\sum_{\mu\in L}\exp\{-\mu(\log a)\}\Gamma_\mu(-\lambda)c(\lambda)^{-1}\left\{\frac{C_s(-s^{-1}\lambda)C_s(-s^{-1}\bar{\lambda})^*}{c(-\lambda)c(\lambda)}\right\}$$
$$\cdot\psi^\delta(-\lambda)d\lambda.$$

We know then that the braces are equal to one ([9(d), p. 465]). By Cauchy's theorem to shift the integration from $\mathfrak{a}^*$ to $\mathfrak{a}^*-i\varepsilon\rho$, we claim that the last expression equals

$$\exp\{-(\varepsilon+1)\rho(\log a)\}\delta(k_1)\int_{\mathfrak{a}^*}\exp\{-i\lambda(\log a)\}\sum_{\mu\in L}\exp\{-\mu(\log a)\}$$
$$\cdot\Gamma_\mu(i\varepsilon\rho-\lambda)c(\lambda-i\varepsilon\rho)^{-1}\psi^\delta(-\lambda+i\varepsilon\rho)d\lambda.$$

This shift is permissible because if $0<\varepsilon'<\varepsilon$, the integral is a holomorphic function of λ on the closed strip bounded by $\mathfrak{a}^*$ and $\mathfrak{a}^*-i\varepsilon'\rho$ and the integral behaves suitably at ∞ because of the rapid decrease of ψ^δ and the mentioned estimates in the previous section for C-function and Γ_μ. Let $\varepsilon'\to\varepsilon$, the claimed relation follows.

By the results of the previous section, there exist positive numbers c, d such that for $\mu\in L$ and $-\lambda\in R$

$$(7.3)\qquad \|\Gamma_\mu(-\lambda)\| \leqq c(1+m(\mu)^d).$$

In particular, this inequality remains valid for $\lambda=\xi-i\eta$ (ξ, $\eta\in\mathfrak{a}^*$) in a strip around the line $\eta=\varepsilon\rho$. So we can use Cauchy's formula to estimate the derivatives of the function $\lambda\to\Gamma_\mu(-\lambda)$ for points on the line; for each $n\in\mathbf{Z}^+$ there exists a number c_n such that

$$(7.4)\qquad \left\|\frac{d^n}{d\xi^n}\Gamma_\mu(i\varepsilon\rho-\xi)\right\| \leqq c_n(1+m(\mu)^d).$$

The functions $c(\lambda)^{-1}$ and $c(\lambda-i\varepsilon\rho)^{-1}$ are products of Gamma factors $\Gamma(a+i\lambda)/\Gamma(b+i\lambda)$ where a, $b>0$ ([7(a)] or [6]), so by [9(b), p. 574] $c(\lambda)^{-1}$ and $c(\lambda-i\varepsilon\rho)^{-1}$ have each derivative bounded by a polynomial in $|\lambda|$. Hence, for each $\mu\in L$ the function

$$\psi^\delta(-\lambda)c(\lambda)^{-1}\Gamma_\mu(-i\lambda)\exp\{-i\lambda(\log a)\}$$

is integrable and since

$$\sum_{\mu\in L}\exp\{-\mu(H)\}(1+m(\mu)^d) < \infty,$$

the interchange of summation and integration in formula (7.2) is legitimate. We have

$$\begin{aligned}(7.5)\qquad &\exp\{(\varepsilon+1)\rho(\log a)\}\varphi^\delta(k_1a)\\ &= \delta(k_1)\sum_{\mu\in L}\exp\{-\mu(\log a)\}\int_{\mathfrak{a}^*}\exp\{-i\lambda(\log a)\}\\ &\cdot\Gamma_\mu(i\varepsilon\rho-\lambda)c(\lambda-i\varepsilon\rho)^{-1}\psi^\delta(i\varepsilon\rho-\lambda)d\lambda.\end{aligned}$$

For any positive integer q we can choose a differential operator $u\in S(F)$ and a polynomial P_u, depending on u, such that

$$u_\lambda(\exp\{-i\lambda(\log a)\}) = P_u(\log a)\exp\{-i\lambda(\log a)\}$$

and

$$\sup_{a\in A^+}(1+|\log a|)^q/|P_u(\log a)| < \infty.$$

Since the last integral is the euclidean Fourier transform, by means of integration by parts and (2.3) and the estimates which we state above, we know that for any q, $r\in \boldsymbol{Z}^+$ and $H\in\mathfrak{A}$ there exist a positive constant c and $n'\in\boldsymbol{Z}^+$ and a finite number of differential operators $u_1,\dots, u_\alpha$ in $S(F)$ such that

$$\begin{aligned}(7.6)\qquad &|\mathrm{Tr}\,\varphi^\delta(k_1a;H)\Xi(x)^{-2/p}(1+\sigma(a))^q(1+|\delta|)^r|\\ &\leqq c\sum_{1\leqq i\leqq\alpha}\|\delta(k_1)\|(1+|\delta|)^r\|u_{i,\lambda}\psi^\delta(i\varepsilon\rho-\lambda)\|(1+|\lambda|)^{n'}\end{aligned}$$

for $k_1\in K$, $a\in A^+$. Since $\delta(k_1)$ is a unitary matrix of order $d(\delta)$ the Hilbert-Schmidt norm of $\delta(k_1)$ is equal to $d(\delta)^{1/2}$. From Weyl's dimension formula it follows that we can choose $r'\in\boldsymbol{Z}^+$ and a positive constant c', independent of δ, such that

$$\|\delta(k_1)\| \leqq c'(1+|\delta|)^{r'}, \qquad (k_1\in K).$$

Therefore the expression (7.6) is bounded by

$$cc'\sum_{1\leqq i\leqq\alpha}\sup_{\mathrm{Int}\,F^p\times\hat{K}^0}\|\psi^\delta(\lambda;u_i)\|(1+|\delta|)^s(1+|\lambda|)^n,$$

where s and n are sufficiently large positive integers. Now any $g\in\mathfrak{G}$ can be written in the form

$$g \equiv \sum_j f_j(a)Q_j^{a^{-1}}H_j \qquad (\mathrm{mod}\,\mathfrak{G}\mathfrak{k})\ (a\in A'),$$

where $f_j\in F_0$, $Q_j\in\mathfrak{Q}$, $H_j\in\mathfrak{A}$ and the sum is finite, so we have

$$\varphi^\delta(k_1 a; g) = \delta(k_1)\sum_j f_j(a)\delta(Q_j)\varphi^\delta(a; H_j).$$

Since we are in the real rank one case and F_0 is generated by the function $H \to (\exp\{2\alpha(H)\}\pm 1)^{-1}$, each f_j is bounded except near the origin. From (7.6), the fact that $1 \leqq \Xi(a)\exp\{\rho(\log a)\}$ ([7(c), p. 17]) and [7(c), Lemma 17] it follows that for any $q, r \in \boldsymbol{Z}^+$ and $g, g' \in \mathfrak{G}$, we can choose $t \in \boldsymbol{Z}^+$ and a finite number of elements $u_1,\dots, u_l$ of $S(F)$ such that the inequality (5.7) holds. This completes the proof of the Lemma 5.1.

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