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On the fourth-order linear recurrence formula related to classical Gauss sums

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Abstract: Let p be an odd prime with $p \equiv 1 \bmod 4$, k be any positive integer, ψ be any fourth-order character mod p. In this paper, we use the analytic method and the properties of character sums mod p to study the computational problem of $G(k,p) = \tau^k(\psi) + \tau^k\left(\overline{\psi}\right)$, and give an interesting fourth-order linear recurrence formula for it, where $\tau(\psi)$ denotes the classical Gauss sums.

Keywords: The classical Gauss sums, Fourth-order linear recurrence formula, Analytic method, Character sums

MSC: 11L05, 11L07, 11T24

1 Introduction

Let $q \geq 3$ be a positive integer. For any positive integer $k \geq 2$, the k-th Gauss sums $G(m,k;q)$ is defined as

$$A(m,k;q) = \sum_{a=1}^{q} e\left(\frac{ma^k}{q}\right),$$

where $e(y) = e^{2\pi i y}$.

Recently, some scholars have studied the properties of $A(m,k;q)$, and obtained many interesting results. For example, Shen Shimeng and Zhang Wenpeng [1] proved a recurrence formula related to $A(m,4;p)$. Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2, \tag{1}$$

where p is an odd prime with $p \equiv 1 \bmod 4$. They proved the identity

$$\begin{aligned}&\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2 \\ &= \begin{cases} 3p^3 - 3p^2 - 3p + p\left(\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)\right), & \text{if } p \equiv 5 \bmod 8; \\ 3p^3 - 3p^2 - 3p - p\tau^2\left(\overline{\chi}_4\right) - p\tau^2(\chi_4) + 2\tau^5\left(\overline{\chi}_4\right) + 2\tau^5(\chi_4), & \text{if } p \equiv 1 \bmod 8, \end{cases}\end{aligned}$$

where χ_4 denotes any fourth-order character mod p, $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, and $\overline{c}$ denotes the multiplicative inverse of c mod p.

Li Xiaoxue and Hu Jiayuan [2] also computed the exact value of $\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)$ and $\tau^5\left(\overline{\chi}_4\right) + \tau^5(\chi_4)$.

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Other works related to k-th Gauss sums and the generalized k-th Gauss sums can be found in [3-10].

Now let p be an odd prime with $p \equiv 1 \bmod 4$, ψ be any fourth-order character mod p, and we also define $G(k,p)$ as

$$G(k,p) = \tau^k(\psi) + \tau^k(\overline{\psi}).$$

In this paper, we employ a remark of [2] by using the analytic method and the properties of the classical Gauss sums to study the computational problem of $G(k,p)$ for any positive integer k, and give an interesting fourth-order linear recurrence formula for $G(k,p)$. That is, we will prove the following result.

Theorem. *Let p be an odd prime with $p \equiv 1 \bmod 4$. Then for any positive integer k, we have the linear recurrence formulae*

$$G(2k+2,p) = 2\sqrt{p}\cdot\alpha\cdot G(2k,p) - p^2\cdot G(2k-2,p)$$

and

$$G(2k+3,p) = 2\sqrt{p}\cdot\alpha\cdot G(2k+1,p) - p^2\cdot G(2k-1,p),$$

where $G(0,p) = 2$, $G(1,p) = A(1) - \sqrt{p}$, $A(1) = A(1,4;p)$, $G(2,p) = 2\sqrt{p}\alpha$, $G(3,p) = \sqrt{p}\cdot\left(2\alpha - (-1)^{\frac{p-1}{4}}\sqrt{p}\right)\cdot\left(A(1)-\sqrt{p}\right)$, $\alpha = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ *and* $\left(\frac{*}{p}\right)$ *is the Legendre's symbol* mod p.

From these recurrence formulae, we can obtain the exact value of $G(k,p)$ for all positive integer k. Hence, we can also deduce the following corollaries:

Corollary 1.1. *Let p be an odd prime with $p \equiv 5 \bmod 8$. Then we have:*

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2 = 3p^3 - 3p^2 + 2p^{\frac{3}{2}}\alpha - 3p.$$

Corollary 1.2. *For odd prime p with $p \equiv 1 \bmod 8$, we have the identity*

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2$$
$$= 3p^3 - 3p^2 - 2p^{\frac{3}{2}}\alpha - 3p + I(\psi)2\sqrt{2p+2\sqrt{p}\alpha}\left(4p\alpha^2 - p^2 + 2p^{\frac{3}{2}}\alpha\right),$$

where $I(\psi) = 1$ if $G(1,p)$ is positive, and $I(\psi) = -1$ if $G(1,p)$ is negative.

Corollary 1.3. *For any prime p with $p \equiv 5 \bmod 8$, we have the identities*

$$|G(1,p)| = \sqrt{2p - 2\sqrt{p}\alpha} \quad \text{and} \quad |G(5,p)| = \sqrt{2p-2\sqrt{p}\alpha}\cdot\left|4p\alpha^2 - p^2 + 2p^{\frac{3}{2}}\alpha\right|.$$

Corollary 1.4. *Let p be an odd prime with $p \equiv 5 \bmod 8$, then we have*

$$\left|\sum_{a=0}^{p-1} e\left(\frac{a^4}{p}\right)\right| = \sqrt{3p - 2\sqrt{p}\alpha}.$$

It is clear that Corollary 1.1 and Corollary 1.2 improved the results of [2].

2 Several Lemmas

In this section, we need several simple lemmas, which are necessary in the proof of our theorem. Hereinafter, we will use many properties of the classical Gauss sums, all of which can be found in [11]; so they will not be repeated here. First we have the following:

Lemma 2.1. *Let p be an odd prime with $p \equiv 1 \bmod 4$. Then for any quadratic non-residue $r \bmod p$, we have the identity*

$$p = \alpha^2 + \beta^2 \equiv \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p} \right) \right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + r\overline{a}}{p} \right) \right)^2,$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol $\bmod p$, and $\overline{a}$ denotes the multiplicative inverse of $a \bmod p$.

Proof. This is a well known result. See Theorem 4-11 in [12]. □

Lemma 2.2. *Let p be an odd prime with $p \equiv 1 \bmod 4$, ψ be any fourth-order character $\bmod p$. Then we have the identity*

$$G(2,p) = \tau^2(\psi) + \tau^2\left(\overline{\psi}\right) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p} \right) = 2\sqrt{p} \cdot \alpha.$$

Proof. Let ψ be any fourth-order character mod p, it is clear that $\psi^2 = \left(\frac{*}{p}\right) = \chi_2$, the Legendre's symbol mod p. Then from the definition and properties of the classical Gauss sums we have

$$\tau^2(\psi) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \psi(ab) e\left(\frac{a+b}{p} \right) = \sum_{a=1}^{p-1} \psi(a) \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a+1)}{p} \right) = \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \psi(a)\chi_2(a+1). \quad (2)$$

Similarly, we also have

$$\tau^2\left(\overline{\psi}\right) = \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \overline{\psi}(a)\chi_2(a+1). \quad (3)$$

As ψ is a fourth-order character mod p, for any integer m with $(m,p) = 1$, we have

$$1 + \psi(m) + \chi_2(m) + \overline{\psi}(m) = \begin{cases} 4, & \text{if } m \text{ satisfies } m \equiv c^4 \bmod p \text{ for some integer } c \text{ with } (c,p) = 1\,; \\ 0, & \text{if otherwise.} \end{cases}$$

Note that for prime p with $p \equiv 1 \bmod 4$, one has identity $\tau(\chi_2) = \sqrt{p}$. From (2) and (3) we have

$$\begin{aligned}
\tau^2(\psi) + \tau^2\left(\overline{\psi}\right) &= \sqrt{p} \cdot \sum_{a=1}^{p-1} \psi(a)\chi_2(a+1) + \sqrt{p} \cdot \sum_{a=1}^{p-1} \overline{\psi}(a)\chi_2(a+1) \\
&= \sqrt{p} \cdot \sum_{a=1}^{p-1} \left(1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)\right) \chi_2(a) \\
&\quad - \sqrt{p} \sum_{a=1}^{p-1} \chi_2(a+1) - \sqrt{p} \sum_{a=1}^{p-1} \chi_2(a)\chi_2(a+1) \\
&= \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2\left(a^4 + 1\right) - \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2(a) - \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2(1 + \overline{a}) + \sqrt{p} \\
&= \sqrt{p} \cdot \sum_{a=1}^{p-1} (1 + \chi_2(a)) \chi_2\left(a^2 + 1\right) + 2\sqrt{p} \\
&= \sqrt{p} \cdot \sum_{a=1}^{p-1} (1 + \chi_2(a)) \chi_2(a+1) + \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi_2(a + \overline{a}) + 2\sqrt{p} \\
&= \sqrt{p} \cdot \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p} \right) = 2\sqrt{p} \cdot \alpha.
\end{aligned}$$

This proves Lemma 2.2. □

Lemma 2.3. *Let p be an odd prime with $p \equiv 1 \bmod 4$, ψ be any fourth-order character* mod p. *Then we have the identities*

$$\tau(\psi) + \tau\left(\overline{\psi}\right) = A(1) - \sqrt{p} \qquad \text{and} \qquad \tau(\psi)\tau\left(\overline{\psi}\right) = (-1)^{\frac{p-1}{4}} \cdot p,$$

where $A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)$.

Proof. As ψ is a fourth-order character mod p, from the definition of the classical Gauss sums, we have:

$$\begin{aligned} A(1) &= \sum_{a=0}^{p-1} e\left(\frac{a^4}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)\right) e\left(\frac{a}{p}\right) \\ &= \sum_{a=0}^{p-1} e\left(\frac{a}{p}\right) + \sum_{a=1}^{p-1} \psi(a) e\left(\frac{a}{p}\right) + \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{a}{p}\right) + \sum_{a=1}^{p-1} \overline{\psi}(a) e\left(\frac{a}{p}\right) \\ &= \tau(\psi) + \sqrt{p} + \tau\left(\overline{\psi}\right). \end{aligned} \tag{4}$$

Note the identities:

$$\overline{\tau(\psi)} = \sum_{a=1}^{p-1} \overline{\psi}(a) e\left(\frac{-a}{p}\right) = \overline{\psi}(-1) \cdot \sum_{a=1}^{p-1} \overline{\psi}(a) e\left(\frac{a}{p}\right) = \overline{\psi}(-1) \cdot \tau\left(\overline{\psi}\right),$$

$\tau(\psi) \cdot \overline{\tau(\psi)} = p$ and $\psi(-1) = (-1)^{\frac{p-1}{4}}$. Hence,

$$\tau(\psi) \cdot \tau\left(\overline{\psi}\right) = (-1)^{\frac{p-1}{4}} \cdot \tau(\psi) \cdot \overline{\tau(\psi)} = (-1)^{\frac{p-1}{4}} \cdot p. \tag{5}$$

Now Lemma 2.3 follows from equations (4) and (5). □

3 Proof of the main results

In this section, we shall complete the proof of our theorem. Let $G(k, p) = \tau^k(\psi) + \tau^k\left(\overline{\psi}\right)$, then for any integer $k \geq 1$, note that $\tau^2(\psi)\tau^2\left(\overline{\psi}\right) = p^2$. From Lemmas 2.1 and 2.2 we have

$$\begin{aligned} 2\sqrt{p} \cdot \alpha \cdot G(2k, p) &= G(2, p) \cdot G(2k, p) \\ &= \left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right)\left(\tau^{2k}(\psi) + \tau^{2k}\left(\overline{\psi}\right)\right) \\ &= \tau^{2k+2}(\psi) + \tau^{2k+2}\left(\overline{\psi}\right) + p^2\tau^{2k-2}(\psi) + p^2\tau^{2k-2}\left(\overline{\psi}\right) \\ &= G(2k+2, p) + p^2 \cdot G(2k-2, p). \end{aligned}$$

So we have

$$G(2k+2, p) = 2\sqrt{p} \cdot \alpha \cdot G(2k, p) - p^2 \cdot G(2k-2, p). \tag{6}$$

Similarly, from Lemma 2.2 and Lemma 2.3 we also have

$$\begin{aligned} & 2\sqrt{p} \cdot \alpha \cdot G(2k+1, p) = G(2, p) \cdot G(2k+1, p) \\ &= \left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right)\left(\tau^{2k+1}(\psi) + \tau^{2k+1}\left(\overline{\psi}\right)\right) \\ &= \tau^{2k+3}(\psi) + \tau^{2k+3}\left(\overline{\psi}\right) + p^2\tau^{2k-1}(\psi) + p^2\tau^{2k-1}\left(\overline{\psi}\right) \\ &= G(2k+3, p) + p^2 \cdot G(2k-1, p), \end{aligned}$$

The last equation implies:

$$G(2k+3, p) = 2\sqrt{p} \cdot \alpha \cdot G(2k+1, p) - p^2 \cdot G(2k-1, p) \tag{7}$$

for all integer $k \geq 1$.

Now our Theorem follows from equations (6) and (7).

Proof of Corollary 1.3. For any prime p with $p \equiv 5 \bmod p$, note that $\psi(-1) = -1$ and $\overline{G(1,p)} = -G(1,p)$; so $G(1,p)$ is pure imaginary. Thus, from Lemma 2.2 we have

$$|G(1,p)| = \sqrt{\left|\tau^2(\psi) + \tau^2\left(\overline{\psi}\right) - 2p\right|} = \sqrt{2p - 2\sqrt{p}\alpha}. \tag{8}$$

Applying the main Theorem proved and Lemma 2.1 together with $\tau(\psi)\tau\left(\overline{\psi}\right) = -p$, we have $G(4,p) = G^2(2,p) - 2p^2 = 2p\left(\alpha^2 - \beta^2\right)$ and

$$G(1,p)G(4,p) = G(5,p) - pG(3,p) = G(5) - pG(1,p)\left(G(2,p) + p\right).$$

Hence

$$G(5,p) = G(1,p)\left(2p\left(\alpha^2 - \beta^2\right) - 2p^{\frac{3}{2}}\alpha + p^2\right) = G(1,p)\left(4p\alpha^2 - p^2 - 2p^{\frac{3}{2}}\alpha\right). \tag{9}$$

Now Corollary 1.3 follows from equations (8) and (9). □

Proof of Corollary 1.4. we note that $A(1) = G(1,p) + \sqrt{p}$ and $G(1,p)$ is pure imaginary; so from (8) we have

$$|A(1)| = \left|\sum_{a=0}^{p-1} e\left(\frac{a^4}{p}\right)\right| = \sqrt{|G(1,p)|^2 + p} = \sqrt{3p - 2\sqrt{p}\alpha}.$$

This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

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