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JUNE 1953

ON THE FREE FLIGHT MOTION OF MISSILES HAVING SLIGHT CONFIGURATIONAL ASYMMETRIES

John D. Nicolaides

Research and Development Project Number TB3-01081

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ON THE FREE FLIGHT MOTION OF MISSILES HAVING SLIGHT CONFIGURATIONAL ASYMMETRIES

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ABSTRACT

The theories for the free flight motion of missiles, as generally considered by the aero-dynamicist and the ballistician, are combined to yield a single theory for the basically symmetrical missile. The force and moment system contains not only the usual aerodynamic forces and moments but also the effects of slight configurational asymmetries and the effects of rolling velocity.

The theory yields the condition for the dynamic stability of both statically stable and statically unstable missiles, and also predicts that the presence of configurational asymmetries together with rolling velocity may result in "resonance instability."

Numerical integrations of the differential equations for the pitching and yawing motion are carried out for three variations in the rolling motion. The results indicate that the rapidity of passage through the resonance region is a significant factor affecting the magnitude of the pitch and yaw of the missile.

Two models of a simple arrow type missile having control surface deflection are gun-launched at supersonic velocity in the Aberdeen Spark Photography Range and the free flight pitching and yawing motion and the transverse displacement are measured. The tricyclic theory is fitted to the experimental data. The results indicate that the theory accurately represents the actual motion of the two models and that the associated static and dynamic aerodynamic derivatives are accurately determined.

SYMBOLS

b fin or wing span

c fin or wing chord

C_{ij} aerodynamic derivatives (See Fig. 2)

$\hat{C}_{ij} = C_{ij} \frac{\rho Sc}{2m}$

d diameter of missile body

I transverse moments of inertia

I_X axial moment of inertia

$k^{-2} = \frac{m c^2}{2 I}$

m mass of missile

ρ air density

V total velocity of missile in space

S reference area for aerodynamic forces and moments, generally $\frac{\pi}{\pi} d^2$

T total kinetic energy of the missile

XYZ orthogonal axis system fixed in missile and rotating with it (See Fig. 2)

X, Y, Z aerodynamic forces along the X, Y, and Z axes respectively

L, M, N aerodynamic moments about the X, Y, and Z axes respectively (See Fig. 2)

u, v, w components of the total linear velocity of the missile in space along the X, Y, and Z axes respectively

$\alpha = \frac{w}{u}$, angle of attack of missile in XYZ system

$\beta = \frac{v}{u}$, angle of sideslip of missile in XYZ system

p, q, r — components of the total angular velocity of the missile in space along the X, Y, and Z axes respectively

δ_A — control surface deflection angle (Aileron) or asymmetry about X axis

δ_E — control surface deflection angle (Elevator) or asymmetry along Z axis

δ_R — control surface deflection angle (Rudder) or asymmetry along Y axis

δ_ϵ — effective asymmetry angle

$\breve{X}\breve{Y}\breve{Z}$ — orthogonal axis system attached to missile ($\breve{X}$ axis lies along axis of mass symmetry of missile, and $\breve{Y}$ axis constrained to lie in xy plane) (See Fig. 1)

$\breve{X}, \breve{Y}, \breve{Z}$ — aerodynamic forces along the $\breve{X}$, $\breve{Y}$, and $\breve{Z}$ axes respectively

$\breve{L}, \breve{M}, \breve{N}$ — aerodynamic moments about the $\breve{X}$, $\breve{Y}$, and $\breve{Z}$ axes respectively (See Fig. 1)

$\breve{u}, \breve{v}, \breve{w}$ — components of the total linear velocity of the missile in space along the $\breve{X}$, $\breve{Y}$, and $\breve{Z}$ axes respectively

$\breve{\alpha} = \frac{\breve{w}}{\breve{u}}$ angle of attack of missile, $\breve{X}\breve{Y}\breve{Z}$ system

$\breve{\beta} = \frac{\breve{v}}{\breve{u}}$ angle of yaw of missile, $\breve{X}\breve{Y}\breve{Z}$ system

$\tilde{p}, \tilde{q}, \tilde{r}$	components of the total angular velocity of the missile in space along the $\breve{X}$, $\breve{Y}$, and $\breve{Z}$ axes respectively (See Fig. 1)
xyz	orthogonal axis system fixed in space (See Fig. 1)
Q_x, Q_y, Q_z	forces along the $\breve{X}$, $\breve{Y}$, and z axes respectively
Q_ϕ, Q_θ, Q_ψ	moments about the X, Y, and z axes respectively
$\dot{x}, \dot{y}, \dot{z}$	linear velocities along the x, y, and z axes respectively
$\dot{\phi}, \dot{\theta}, \dot{\psi}$	angular velocities along the $\breve{X}$, $\breve{Y}$, and z axes respectively
$\dot{\Omega}$	$\dot{\theta} + i\dot{\psi}$, total angular velocity of the missile normal to the axis of mass symmetry
$\vec{\alpha} = \beta + i\alpha$ $\breve{\vec{\alpha}} = \breve{\beta} + i\breve{\alpha}$	complex angle of attack and yaw
$\vec{\varrho} = q + ir$ $\breve{\vec{\varrho}} = \breve{q} + i\breve{r}$	complex angular velocity normal to the missile
$\dot{\vec{\alpha}} = \dot{\beta} + i\dot{\alpha}$ $\dot{\breve{\vec{\alpha}}} = \dot{\breve{\beta}} + i\dot{\breve{\alpha}}$	rate of change of complex angle of attack and yaw
$\dot{\vec{\varrho}} = \dot{q} + i\dot{r}$ $\dot{\breve{\vec{\varrho}}} = \dot{\breve{q}} + i\dot{\breve{r}}$	rate of change of complex angular velocity normal to the missile

INTRODUCTION

Developments of theories for the free flight motion of missiles have, in general, proceeded along two separate paths. The aerodynamicist[1,2] has been primarily concerned with aircraft which, although lacking rotational symmetry and essentially non-rolling, are acted upon by a linear aerodynamic system which includes forces and moments due to control surface deflection and lags in the flow. The ballistician,[3-11] on the other hand, has been primarily concerned with rapidly rolling symmetrical missiles and has included in his treatment of the motion the important gyroscopic and Magnus effects.

In recent years, with the advent of the guided missile, the rocket, supersonic aircraft, and modern finned ordnance weapons, the interests of both groups have merged. Although limited extensions[12-14] of the theories of both groups have been undertaken, the essential differences remain.

It is one of the purposes of this paper to unify these approaches for the class of missiles which are basically symmetrical and are only slightly asymmetrical due to control surface deflection, wing and/or tail incidence, bent body, damaged or malaligned fin, etc.

The necessity for considering this union arises from the failure of the existing theory[1-13] to represent the free flight motion of winged and/or finned missiles and its failure to account for various phenomena which have been generally experienced on statically stable missiles. Some of these phenomena are

(1) that non-rolling statically stable missiles generally have large dispersion and that rolling the missile reduces the dispersion.

(2) that even for generally well performing missiles a few go berserk[15] yielding extremely poor dispersion and sometimes tumbling, and

(3) that peculiarities in the free flight motion seem to occur when the rolling velocity and the pitching velocity approach coincidence.

The general procedure to be followed here will be to develop the differential equations of motion for a missile having basic trigonal or greater configurational symmetry[16] and slight configurational asymmetry. The aerodynamic system will include the forces and moments generally considered by both the aerodynamicist and the ballistician. The important gyroscopic terms resulting from rolling velocity are also included. For simplicity and clarity the differential equations will be solved for the case of constant axial velocity and constant rolling velocity of the missile. The resultant tricyclic theory for the free flight pitching and yawing motion of the missile and the theory for the displacement of the center of gravity of the missile will be applied to the experimental data obtained from the free flight motion of gun-launched models tested in the Aberdeen Spark Photography Range.[17-20] The static and dynamic aerodynamic derivatives associated will be obtained. Finally, the differential equations of motion will be numerically integrated for the case of varying rolling velocity in order to indicate the motions obtained.

THEORY

The difficulties generally encountered in the formulation of a theory of free-flight motion may be separated into two groups, dynamical and aerodynamical. Herein the dynamical problem will be approached by employing a modified Eulerian axis system and by using the Lagrange equation for formulating the basic differential equations of motion. Mathematical simplifications are introduced by limiting the angular displacements and angular velocities (except rolling) to small size. This is the familiar dynamical approach to the linearized motion of a "spinning top" or a "gyroscopic pendulum"[21,22]. The aerodynamical problem will be approached in the manner and nomenclature of the aerodynamicist. A linear force and moment system is assumed and the symmetry arguments of the ballistician employed. In general, then, these two basic problems will be approached separately. Their resolutions will then be combined and the fundamental differential equations of motion will be obtained.

The differential equations of motion will be solved for the case of constant flight velocity and constant rolling velocity, and the resulting expressions for the pitching and yawing motion and for the transverse displacement of a missile will be discussed in detail. In a later section the differential equations of motion will be investigated numerically for the case of constant flight velocity but varying rolling velocity.

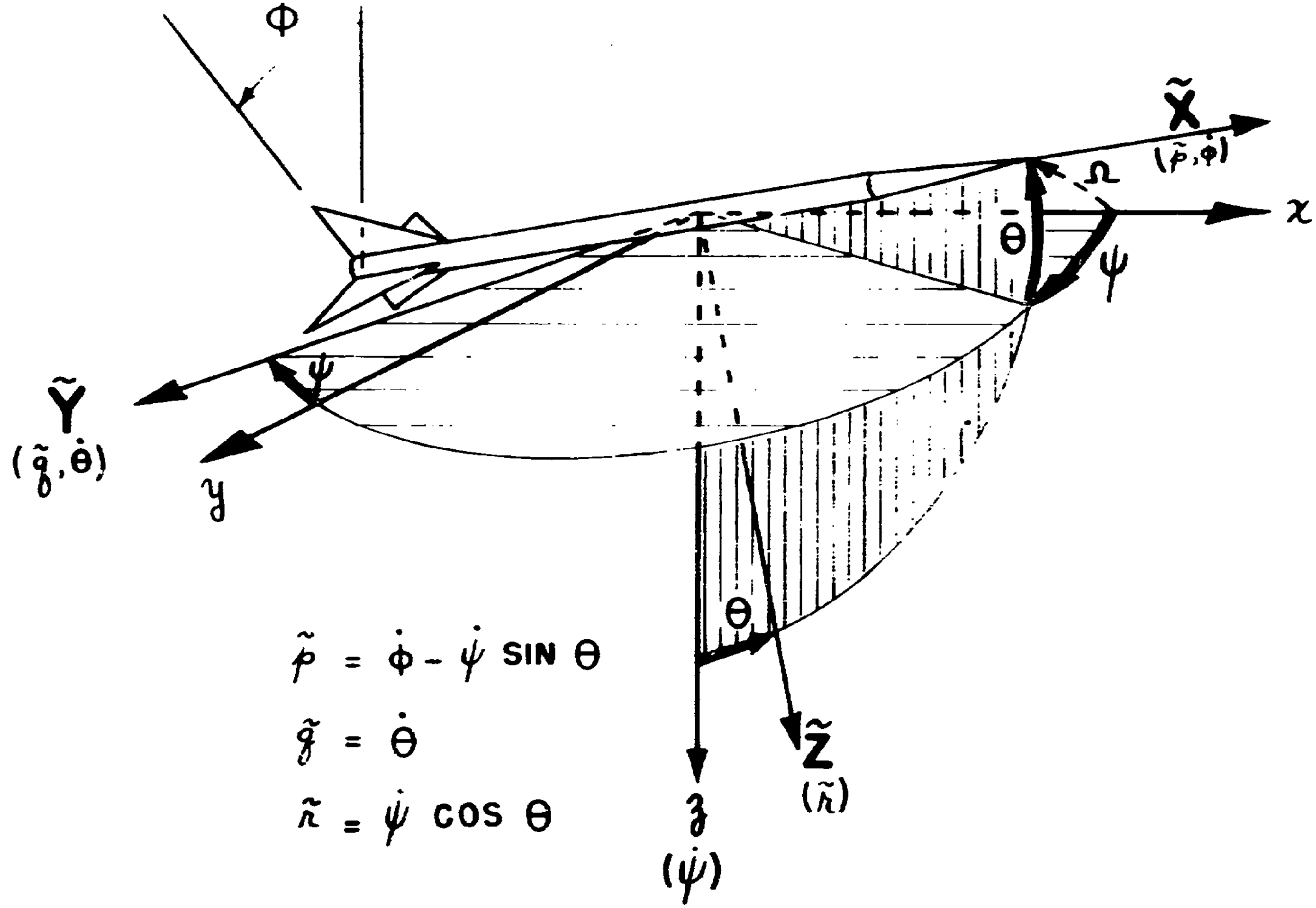
DYNAMICAL SYSTEM

The coordinate systems illustrated in Fig. 1 will be used in considering the free flight motion of a missile. The xyz system is orthogonal and fixed in space. The $\tilde{X}\tilde{Y}\tilde{Z}$ system is orthogonal and is pitching and yawing but not rolling with the missile. (The $\tilde{X}$ axis lies along the axis of mass symmetry of the missile and the $\tilde{Y}$ axis is constrained to lie in the xy plane). The angular velocity of the missile is given by the components $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in the $\tilde{X}$ $\tilde{Y}$ $\tilde{Z}$ system.

The coordinates of the dynamical system are taken as

(1) $\dot{x}$, $\dot{y}$, $\dot{z}$, the components of the linear velocity of the center of gravity of the missile, and

(2) $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$, the components of the angular velocity of the missile about its center of gravity. (It should be noted that these components are in a moving non-orthogonal modified Eulerian axis system.)



$$\tilde{p} = \dot{\phi} - \dot{\psi}\ \mathrm{SIN}\ \Theta$$

$$\tilde{q} = \dot{\Theta}$$

$$\tilde{r} = \dot{\psi}\ \mathrm{COS}\ \Theta$$

FIG. 1 — DYNAMICAL SYSTEM

The total kinetic energy of the missile in free flight is thus given by

$$T = \frac{1}{2} I_{\breve{X}}\breve{p}^{2} + \frac{1}{2} I_{\breve{Y}}\breve{q}^{2} + \frac{1}{2} I_{\breve{Z}}\breve{r}^{2} + \frac{1}{2} m \dot{x}^{2} + \frac{1}{2} m \dot{y}^{2} + \frac{1}{2} m \dot{z}^{2} \tag{1}$$

It is assumed that the missile has symmetrical mass distribution,

$$I_{\breve{Y}} = I_{\breve{Z}} = I \tag{2}$$

and from Fig. 1, it is seen that

$$\breve{p} = \dot{\phi} - \dot{\psi} \sin \theta \tag{3}$$

$$\breve{q} = \dot{\theta} \tag{4}$$

$$\breve{r} = \dot{\psi} \cos \theta \tag{5}$$

Thus the total kinetic energy of the missile may be written as

$$T = \frac{1}{2} I_x \dot{\phi}^2 - I_x \dot{\phi} \dot{\psi} \sin \theta + \frac{1}{2} I_x \dot{\psi}^2 \sin^2 \theta + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} I \dot{\psi}^2 \cos^2 \theta$$

$$+ \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} m \dot{z}^2 \tag{6}$$

Now considering the Lagrange equation[23] for this dynamical system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r \tag{7}$$

where q_r = coordinates of the dynamical system

$(\phi, \theta, \psi, x, y, z)$

Q_r = Generalized Force (i.e., force or moment tending to change the particular coordinate).

Performing the indicated operations, assuming that θ, ψ, $\dot{\theta}$, $\dot{\psi}$ are small quantities and that their products may be neglected, yields

$$I \ddot{\Omega} - i I_X \dot{\phi} \dot{\Omega} = Q_{\Omega} \tag{8}$$

$$\frac{d}{dt} (I_X \dot{\phi} - I_X \dot{\psi} \sin \theta) = Q_{\phi} \tag{9}$$

$$m \ddot{S} = Q_s \tag{10}$$

$$m \ddot{x} = Q_x$$

where complex variables have been introduced in defining (11)

$\dot{\Omega} = \dot{\theta} + i\dot{\psi}$, total angular velocity of the missile normal to the axis of mass symmetry (12)

$S = y + i z$, transverse displacement of the missile (13)

Eqs. (8) - (11) are then the basic differential equations of motion of the missile. They may be completed once the Generalized Forces tending to change the coordinates of the dynamical system are known. These Generalized Forces are derived in the following section from the aerodynamic forces and moments which act on a missile in free flight.

AERODYNAMIC FORCE AND MOMENT SYSTEM

The coordinate system used in considering the aerodynamic forces and moments which act on a missile in free flight will be the standard N.A.C.A. system which is orthogonal and fixed to the missile (i.e., rolling with the missile, principle axis). This system is designated by XYZ and the components of the linear and angular velocity are given by u, v, w and p, q, r respectively (See Fig. 2).

Aerodynamic Force

The total aerodynamic force which acts on a missile in free flight is assumed to depend on the linear velocity of the missile, the angular velocity of the missile, accelerations, the density of the air, the velocity of sound, the size and shape of the missile, and, for the particular case under consideration here, slight asymmetry of the configuration (i.e., control surface deflection, wing and/or tail incidence, bent fins, bent body, etc.).

Assuming that the motion is confined to infinitesimals and that the dependence is linear yields

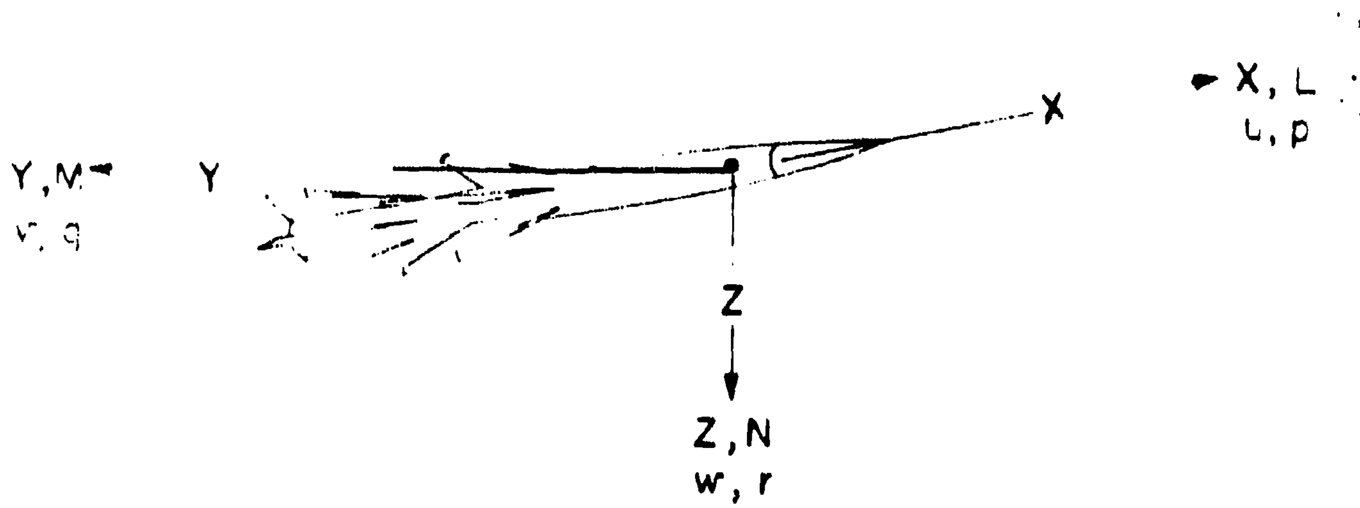
$$Y = Y_{\beta}\beta + Y_r r + Y_{\dot\beta}\dot\beta + Y_{\dot r}\dot r$$

$$+ Y_{p\alpha}p\alpha + Y_{pq}pq + Y_{p\dot\alpha}p\dot\alpha + Y_{p\dot q}p\dot q + Y_{\delta_R}\delta_R \qquad (14)$$

$$Z = Z_{\alpha}\alpha + Z_q q + Z_{\dot\alpha}\dot\alpha + Z_{\dot q}\dot q$$

$$+ Z_{p\beta}p\beta + Z_{pr}pr + Z_{p\dot\beta}p\dot\beta + Z_{p\dot r}p\dot r + Z_{\delta_E}\delta_E \qquad (15)$$

where Y_i and Z_i are the Stability Derivatives.



$$\text{LINEAR VEL.} = \begin{vmatrix} u \\ v \\ w \end{vmatrix} = V \qquad \text{FORCE} = \begin{bmatrix} X = C_X \frac{1}{2} \rho V^2 S \\ Y = C_Y \frac{1}{2} \rho V^2 S \\ Z = C_Z \frac{1}{2} \rho V^2 S \end{bmatrix}$$

$$\text{ANGULAR VEL.} = \begin{vmatrix} p \\ q \\ r \end{vmatrix} \qquad \text{MOMENT} = \begin{bmatrix} L = C_\ell \frac{1}{2} \rho V^2 S b \\ M = C_m \frac{1}{2} \rho V^2 S c \\ N = C_n \frac{1}{2} \rho V^2 S b \end{bmatrix}$$

FIG. 2 - AERODYNAMIC SYSTEM

Multiplying Eq. (15) by i and adding to Eq. (14), and replacing the stability derivatives by the standard nomenclature for the aerodynamic derivatives yields

$$
\begin{aligned}
\frac{Y + iZ}{\frac{1}{2}\rho V^2 S} = & \left[C_{Y_\beta} + i\, C_{Z_{p\beta}} \left(\frac{pb}{2V}\right)\right]\beta + \left[C_{Z_\alpha} - i\, C_{Y_{p\alpha}} \left(\frac{pb}{2V}\right)\right] i\alpha \\
& + \left[C_{Y_{pq}} \left(\frac{pb}{2V}\right) + i\, C_{Z_q}\right]\left(\frac{cq}{2V}\right) + \left[C_{Z_{pr}} \left(\frac{pb}{2V}\right) - i\, C_{Y_r}\right] i \left(\frac{br}{2V}\right) \\
& + \left[C_{Y_{\dot\beta}} + i\, C_{Z_{p\dot\beta}} \left(\frac{pb}{2V}\right)\right]\left(\frac{b\dot\beta}{2V}\right) + \left[C_{Z_{\dot\alpha}} - i\, C_{Y_{p\dot\alpha}} \left(\frac{pb}{2V}\right)\right] i \left(\frac{c\dot\alpha}{2V}\right) \\
& + \left[C_{Y_{p\dot q}} \left(\frac{pb}{2V}\right) + i\, C_{Z_{\dot q}}\right]\left(\frac{c}{2V}\right)^2 \dot q + \left[C_{Z_{p\dot r}} \left(\frac{pb}{2V}\right) - i\, C_{Y_{\dot r}}\right] i \left(\frac{b}{2V}\right)^2 \dot r \\
& + \left[C_{Y_{\delta_R}} \delta_R + i\, C_{Z_{\delta_E}} \delta_E\right] \qquad (16)
\end{aligned}
$$

Here we impose the fundamental assumption that the contribution of the configurational asymmetries is "slight" and thus that not only may it be added linearly but that the basic missile (i.e., the configuration of the missile when $\delta_R = \delta_E = 0$) has trigonal or greater rotation symmetry and mirror symmetry.[10, 11, 16]

As a result of this assumption it follows that

AERODYNAMIC		AEROBALLISTIC**		BALLISTIC[7,10,11]

$$
\begin{array}{lllll}
C_{Y_\beta} & = C_{Z_\alpha} & \equiv \left(\begin{smallmatrix}-\\-\end{smallmatrix}\right) C_N & = -K_N \left(\frac{2d^2}{S}\right) & \\[2ex]
-C_{Y_{p\alpha}} & = C_{Z_{p\beta}} & \equiv \left(\begin{smallmatrix}+\\+\end{smallmatrix}\right) C_{N_p} & = K_F \left(\frac{4d^3}{bS}\right) & \\[2ex]
C_{Y_{pq}} & = C_{Z_{pr}} \left(\frac{b}{c}\right) & \equiv (\pm) C_{N_{pq}} & = K_{XF} \left(\frac{8d^4}{bcS}\right) & \\[2ex]
-C_{Y_r} \left(\frac{b}{c}\right) & = C_{Z_q} & \equiv (\pm) C_{N_q} & = K_S \left(\frac{4d^3}{cS}\right) & (17) \\[2ex]
 & & & \overline{\hphantom{= -K_{LXF} \left(\frac{16}{bc}\frac{d^5}{2S}\right)}}^{\,*} & \\[2ex]
C_{Y_{\dot\beta}} \left(\frac{b}{c}\right) & = C_{Z_{\dot\alpha}} & \equiv \left(\begin{smallmatrix}+\\+\end{smallmatrix}\right) C_{N_{\dot\alpha}} & = K_{LN} \left(\frac{4d^3}{cS}\right) & \\[2ex]
-C_{Y_{p\dot\alpha}} & = C_{Z_{p\dot\beta}} \left(\frac{b}{c}\right) & \equiv \left(\begin{smallmatrix}-\\-\end{smallmatrix}\right) C_{N_p} & = -K_{LF} \left(\frac{8d^4}{bcS}\right) & \\[2ex]
C_{Y_{p\dot q}} & = C_{Z_{p\dot r}} \left(\frac{b}{c}\right)^2 & \equiv (\mp) C_{N_{pq}} & = -K_{LXF} \left(\frac{16}{bc}\frac{d^5}{2S}\right) & \\[2ex]
-C_{Y_{\dot r}} \left(\frac{b}{c}\right)^2 & = C_{Z_{\dot q}} & \equiv (\mp) C_{N_q} & = -K_{LS} \left(\frac{8d^4}{c^2S}\right) &
\end{array}
$$

---

* The introduction of the new Ballistic nomenclature below this line is necessary since forces and moments depending on "Lags" in the flow (i.e., acceleration effects) are not included in the Ballistic Theory.

** (FORCES ACTING FORWARD OF C.G.
FORCES ACTING AFT OF C.G.)

Because of these equalities of the aerodynamic derivatives for the class of configurations having basic symmetry, an Aeroballistic nomenclature[24] is convenient and is introduced as indicated above,

$$\vec{\alpha} = \beta + i\alpha \qquad \vec{q} = q + i\,r \qquad \dot{\vec{\alpha}} = \dot{\beta} + i\dot{\alpha} \qquad \dot{\vec{q}} = \dot{q} + i\,\dot{r}$$

$$\tilde{\alpha} = \tilde{\beta} + i\tilde{\alpha} \qquad \tilde{q} = \tilde{q} + i\,\tilde{r} \qquad \dot{\tilde{\alpha}} = \dot{\tilde{\beta}} + i\dot{\tilde{\alpha}} \qquad \dot{\tilde{q}} = \dot{\tilde{q}} + i\,\dot{\tilde{r}} \tag{18}$$

Substituting Eq. (17) - (18) into Eq. (16) yields

$$\frac{Y + i\,Z}{\frac{1}{2}\rho V^2 S} = \left[C_{N_\alpha} + i\,C_{N_{p\alpha}}\left(\frac{pb}{2V}\right)\right]\vec{\alpha} + \left[C_{N_{pq}}\left(\frac{pb}{2V}\right) + i\,C_{N_q}\right]\left(\frac{c\vec{q}}{2V}\right)$$

$$+ \left[C_{N_{\dot{\alpha}}} + i\,C_{N_{p\dot{\alpha}}}\left(\frac{pb}{2V}\right)\right]\left(\frac{c}{2V}\right)^2 \dot{\vec{\alpha}} + \left[C_{N_{p\dot{q}}}\left(\frac{pb}{2V}\right) + i\,C_{N_{\dot{q}}}\right]\left(\frac{c}{2V}\right)^2 \dot{\vec{q}} + C_{N_{\delta_\ell}}\vec{\delta}_\ell \tag{19}$$

where

$$(=)\ C_{N_{\delta_\ell}}\vec{\delta}_\ell = C_{Y_{\delta_R}}\delta_R + i\,C_{Z_{\delta_E}}\delta_E \quad \text{and } C_{N_{\delta_\ell}} \text{ is real.} \tag{20}$$

Since all the terms, except the last, are independent of the roll orientation of the coordinate system and since the last term represents an asymmetry which is fixed in the missile and thus rolling with the missile, the total normal force may be expressed in terms of the original dynamical XYZ system which was not rolling with the missile as

$$\tilde{Y} + i\,\tilde{Z} = a\tilde{\alpha} + b\,\tilde{q} + c\,\dot{\tilde{\alpha}} + d\,\dot{\tilde{q}} + e\,\delta_{\ell_0}\,e^{i\int p\,dt} \tag{21}$$

$$a = \frac{1}{2}\rho V^2 S\left[C_{N_\alpha} + i\left(\frac{pb}{2V}\right)C_{N_{p\alpha}}\right]$$

$$b = \frac{1}{2}\rho V^2 S\left[C_{N_{pq}}\left(\frac{pb}{2V}\right) + i\,C_{N_q}\right]\left(\frac{c}{2V}\right) \tag{22}$$

$$c = \frac{1}{2}\rho V^2 S \left[C_{N_{\dot{\alpha}}} + i\, C_{N_{p\dot{\alpha}}} \left(\frac{pb}{2V}\right) \right] \left(\frac{c}{2V}\right)$$

$$d = \frac{1}{2}\rho V^2 S \left[C_{N_{p\dot{q}}} \left(\frac{pb}{2V}\right) + i\, C_{N_{\dot{q}}} \right] \left(\frac{c}{2V}\right)^2 \qquad (22)$$

$$e = \frac{1}{2}\rho V^2 S \left[C_{N_{\delta_E}} \right]$$

The Generalized Force, Q_s, which is tending to move the missile in the yz plane is the summation of all the forces acting in this plane. This includes not only the aerodynamic forces but also the gravitational force, thrust malalignment forces, and any others that might be acting. However, in order to emphasize the effects of configurational asymmetries and to keep the treatment as elementary as possible, the Generalized Force will be assumed to include only the aerodynamic forces. (It should be noted, however, that the addition of the gravitational force,[10] the thrust malalignment forces and mass malalignment forces[25] present no fundamental difficulties.)

Since $\breve{Y} + i\,\breve{Z}$ lies in the $\breve{Y}\,\breve{Z}$ plane which has been assumed to make a small angle with the yz plane (i.e., Ω assumed small), then $\breve{Y} + i\,\breve{Z}$ may be taken as equal to Q_s. $Q_s = \breve{Y} + i\,\breve{Z}$ (22a)

Aerodynamic Moment

The specification of the aerodynamic moment acting normal to the missile in free flight proceeds in the same manner as the specification of the normal aerodynamic force. Accordingly M and N are assumed to be linear functions of α, β, q, r, $\dot{\alpha}$, $\dot{\beta}$, $\dot{q}$, $\dot{r}$, δ_R and δ_E as

$$M = M_{\alpha}\alpha + M_q q + M_{\dot{\alpha}}\dot{\alpha} + M_q q$$

$$+ M_{p\beta}\beta p + M_{pr}pr + M_{p\dot{\beta}}p\dot{\beta} + M_{p\dot{r}}p\dot{r} + M_{\delta_E}\delta_E \qquad (23)$$

$$N = N_{\beta}\beta + N_r r + N_{\dot{\beta}}\dot{\beta} + N_{\dot{r}}\dot{r}$$

$$+ N_{p\alpha}p\alpha + N_{pq}pq + N_{p\dot{\alpha}}p\dot{\alpha} + N_{p\dot{q}}p\dot{q} + N_{\delta_R}\delta_R \qquad (24)$$

Multiplying Eq. (24) by i and adding to Eq. (23), and replacing the stability derivatives by the aerodynamic derivatives yields

$$
\begin{aligned}
\frac{M + i\,N}{\frac{1}{2}\rho V^2 S} = {} & \left[C_{m_{p\beta}} \left(\frac{pb}{2V}\right) c + i C_{n_\beta}\, b \right] \beta + \left[C_{n_{p\alpha}} \left(\frac{pb}{2V}\right) b - i\, C_{m_\alpha}\, c \right] i\, \alpha \\
& + \left[C_{m_q}\, c + i\, C_{n_{pq}} \left(\frac{pb}{2V}\right) b \right] \left(\frac{cq}{2V}\right) + \left[C_{n_r}\, b - i\, C_{m_{pr}} \left(\frac{pb}{2V}\right) c \right] i \left(\frac{br}{2V}\right) \\
& + \left[C_{m_{p\dot\beta}} \left(\frac{pb}{2V}\right) c + i\, C_{n_{\dot\beta}}\, b \right] \left(\frac{b\dot\beta}{2V}\right) + \left[C_{n_{p\dot\alpha}} \left(\frac{pb}{2V}\right) b - i\, C_{m_{\dot\alpha}}\, c \right] i \left(\frac{c\dot\alpha}{2V}\right) \\
& + \left[C_{m_{\dot q}}\, c + i\, C_{n_{p\dot q}} \left(\frac{pb}{2V}\right) b \right] \left(\frac{c}{2V}\right)^2 \dot q + \left[C_{n_{\dot r}}\, b - i\, C_{m_{p\dot r}} \left(\frac{pb}{2V}\right) c \right] i \left(\frac{b}{2V}\right)^2 \dot r \\
& + \left[i\, C_{n_{\delta_R}}\, b\, \delta_R - i\, C_{m_{\delta_E}}\, c\, i\, \delta_E \right] \qquad ([illegible])
\end{aligned}
$$

However, from symmetry considerations it follows that

AERODYNAMIC | **AEROBALLISTIC** | **BALLISTIC**[7,10,11]

$$C_{m_{p\beta}} = C_{n_{r}} \left(\frac{b}{c}\right) = (\mp)\, C_{M_{p\alpha}} = -\, K_T \left(\frac{4d^4}{bcS}\right)$$

$$C_{m_\alpha} = -\, C_{n} \left(\frac{b}{c}\right) = (\pm)\, C_{M} = K_M \left(\frac{2d^3}{cS}\right)$$

$$C_{m_q} = C_{n_r} \left(\frac{b}{c}\right)^2 = (\pm)\, C_{M_q} = -\, K_H \left(\frac{4d^4}{c^2S}\right)$$

$$C_{m_{pr}} \left(\frac{b}{c}\right) = -\, C_{n_{pq}} \left(\frac{b}{c}\right) = (\mp)\, C_{M_{pq}} = -\, K_{XT} \left(\frac{8d^5}{bc^2S}\right)$$

$$C_{m_{p\dot\beta}} \left(\frac{b}{c}\right) = C_{n_{p\dot\alpha}} \frac{b}{c} = (\pm)\, C_{M_p} = K_{LT} \left(\frac{8d^5}{c^2bS}\right) \quad (m)$$

$$C_{m_{\dot\alpha}} = -\, C_{n_{\dot\beta}} \left(\frac{b}{c}\right)^2 = (\mp)\, C_{M_{\dot\alpha}} = -\, K_{LH} \left(\frac{4d^4}{c^2S}\right)$$

$$C_{m_{\dot q}} = C_{n_{\dot r}} \left(\frac{b}{c}\right)^3 = (\pm)\, C_{M_{\dot q}} = K_{LH} \left(\frac{8d^5}{c^3S}\right)$$

$$C_{m_{p\dot r}} \left(\frac{b}{c}\right)^2 = -\, C_{n_{p\dot q}} \left(\frac{b}{c}\right) = (\mp)\, C_{M_{p\dot q}} = K_{LXT} \left(\frac{16d^6}{c^3bS}\right)$$

Thus the aerodynamic moment becomes

$$M + i N = A \vec{\alpha} + B \vec{q} + C \vec{\dot{\alpha}} + D \vec{\dot{q}} + E \vec{\delta_\xi} \tag{27}$$

where

$$A = \tfrac{1}{2}\rho V^2 S\, c \left[C_{M_{p\alpha}} \left(\tfrac{pb}{2V}\right) - i\, C_{M_\alpha} \right]$$

$$B = \tfrac{1}{2}\rho V^2 S\, c \left[C_{M_q} - i\, C_{M_{pq}} \left(\tfrac{pb}{2V}\right) \right] \left(\tfrac{c}{2V}\right)$$

$$C = \tfrac{1}{2}\rho V^2 S\, c \left[C_{M_{p\dot{\alpha}}} \left(\tfrac{pb}{2V}\right) - i\, C_{M_{\dot{\alpha}}} \right] \left(\tfrac{c}{2V}\right) \tag{28}$$

$$D = \tfrac{1}{2}\rho V^2 S\, c \left[C_{M_{\dot{q}}} - i\, C_{M_{p\dot{q}}} \left(\tfrac{pb}{2V}\right) \right] \left(\tfrac{c}{2V}\right)^2$$

$$E = \tfrac{1}{2}\rho V^2 S\, c \left[i\, C_{M_{\delta_\xi}} \right]$$

and

$$(+)\ i\, C_{M_{\delta_\xi}}\, c\, \delta_\xi = i\, C_{n_{\delta_R}}\, b\, \delta_R - i\, C_{m_{\delta_E}}\, c\, i\, \delta_E \tag{29}$$

This aerodynamic moment written in terms of the non-rolling system is given by

$$\breve{M} + i \breve{N} = A \vec{\breve{\alpha}} + B \vec{\breve{q}} + C \vec{\breve{\dot{\alpha}}} + D \vec{\breve{\dot{q}}} + E\, \delta_{\xi_0}\, e^{i \int p\,dt} \tag{30}$$

Since the moment normal to the missile's axis, Q_Ω is assumed to contain only the aerodynamic moments and since $\breve{M} + i \breve{N}$ is equal to Q_Ω we have

$$Q_\Omega = \breve{M} + i \breve{N} \tag{31}$$

General Differential Equations of Motion

With the specifications of the aerodynamic forces and moments acting on the missile in free flight in terms of the Generalized Forces, the differential equations of motion may be completed as

$$I\ddot{\Omega} - i I_x \dot{\phi}\,\dot{\Omega} = A\vec{\alpha} + B\vec{\xi} + C\dot{\vec{\alpha}} + D\dot{\vec{\xi}} + E\,\delta_{\varepsilon_o}\,e^{i\int p\,dt} \qquad (32)$$

$$\frac{d}{dt}\left(I_x\dot{\phi} - I_x\dot{\psi}\sin\Theta\right) = Q_{\phi} \qquad (33)$$

$$\mathfrak{M}\ddot{S} = a\vec{\alpha} + b\vec{\xi} + c\dot{\vec{\alpha}} + d\dot{\vec{\xi}} + \ell\,\delta_{\varepsilon_o}\,e^{i\int p\,dt} \qquad (34)$$

$$\mathfrak{M}\ddot{\mathcal{X}} = Q_x \qquad (35)$$

and under the basic assumption of small angles and slowly changing angles it is seen from Eqs. (4), (5), (12), and (18) that $\vec{\xi} = \dot{\Omega}$. (36)

Differential Equations of Motion For Constant Axial Velocity

When Q_x is zero, it follows from Eq. (35) and the assumption of small angles that the axial velocity of the missile is a constant. The sum of the three small angles, namely, (a) the angle between the missile's axis and the total velocity, (b) the angle between the total velocity and the fixed axis, and (c) the angle between the fixed x axis and the missile's axis, is zero.

Differentiating this sum yields

$$S = V\left(-i\dot{\Omega} + \dot{\vec{\alpha}}\right) \qquad (37)$$

Neglecting products of the aerodynamic derivatives in comparison with the derivatives themselves, Eqs. (32), (34) and (37) may be combined to yield the differential equation for the pitching and yawing motion.

$$\ddot{\tilde{\alpha}} - \frac{V}{c}\left\{\hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right) + i\left(\frac{pb}{2V}\right)\left[k^{-2}\left(\hat{C}_{M_{p\dot\alpha}} - \hat{C}_{M_{pq}}\right) + \hat{C}_{N_{p\alpha}} + \left(\frac{2c}{b}\right)\frac{I_x}{I}\right]\right\}\dot{\tilde{\alpha}}$$

$$+ \frac{V^2}{c^2}\left\{-2k^{-2}\hat{C}_{M_\alpha} + \left(\frac{pb}{2V}\right)\left[-\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\frac{I_x}{I}\hat{C}_{N_{p\alpha}} + i\left(\frac{2c}{b}\right)\frac{I_x}{I}\hat{C}_{N_\alpha} - i\,2k^{-2}\hat{C}_{M_p}\right]\right\}\tilde{\alpha}$$

$$= \frac{V^2}{c^2}\left\{i\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\left(1-\frac{I_x}{I}\right)\hat{C}_{N_{\delta_f}} - 2k^{-2}\hat{C}_{M_{\delta_f}}\right\}\delta_{f_0}\, e^{i\int p\,dt} \qquad (38)$$

$$\text{where} \quad \hat{C}_{i_j} = C_{i_j}\left(\frac{\rho S c}{2m}\right) \qquad k^{-2} = \frac{mc^2}{2I} \qquad (39)$$

Eqs. (33) and (38) will then be considered in the following section. First, the solution will be given for the case of constant rolling velocity (i.e., $Q_\phi = 0$) and, then, in a later section the case of varying rolling velocity will be numerically investigated.

SOLUTION OF PITCHING AND YAWING MOTION FOR CONSTANT ROLLING VELOCITY

When Q_ϕ is zero, it follows from Eq. (33) and the assumption of small angles that the rolling velocity of the missile is a constant. For this case the general solution of the differential equation for the pitching and yawing motion, Eq. (38), is given by

$$\vec{\tilde{\alpha}} = K_1 e^{\phi_1 t} + K_2 e^{\phi_2 t} + K_3 e^{\phi_3 t} \tag{40}$$

where

$$K_1 = \frac{\dot{\vec{\tilde{\alpha}}}_o - \phi_2 \vec{\tilde{\alpha}}_o + K_3 (\phi_2 - \phi_3)}{\phi_1 - \phi_2} \qquad K_2 = \frac{\dot{\vec{\tilde{\alpha}}}_o - \phi_1 \vec{\tilde{\alpha}}_o + K_3 (\phi_1 - \phi_3)}{\phi_2 - \phi_1} . \tag{41)(42}$$

$$K_3 = \frac{\frac{V^2}{c^2}\left\{ i \left(\frac{2c}{b}\right)\left(\frac{pb}{2V}\right)\left(1 - \frac{I_x}{I}\right)\hat{C}_{N_{\delta_r}} - 2k^{-2}\hat{C}_{M_{\delta_r}} \right\}\delta_{r_e}}{(\phi_3 - \phi_1)(\phi_3 - \phi_2)} \tag{43}$$

$$\phi_{1,2} = \frac{V}{2c}\left\{ \hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right) + i\left(\frac{pb}{2V}\right)\left[k^{-2}\left(\hat{C}_{M_{p\dot\alpha}} - \hat{C}_{N_{pq}}\right) + \hat{C}_{N_{p\alpha}} + \left(\frac{2c}{b}\right)\frac{I_x}{I}\right]\right\}$$

$$\pm \frac{V}{2c}\left\{\left[\hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right) + i\left(\frac{pb}{2V}\right)\left[k^{-2}\left(\hat{C}_{M_{p\dot\alpha}} - \hat{C}_{M_{pq}}\right) + \hat{C}_{N_{p\alpha}}\left(\frac{2c}{b}\right)\frac{I_x}{I}\right]\right]^2\right.$$

$$\left. - 4\left\{-2k^{-2}\hat{C}_{M_\alpha} + \left(\frac{pb}{2V}\right)\left[-\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\frac{I_x}{I}\hat{C}_{N_{p\alpha}} + i\left(\frac{2c}{b}\right)\frac{I_x}{I}\hat{C}_{N_\alpha} - i\,2k^{-2}\hat{C}_{M_{p\alpha}}\right]\right\}\right\}^{1/2} \tag{44}$$

$$\phi_3 = ip \tag{45}$$

A physical representation of this solution is given by noting that the motion is "tricylic"; that is to say, the free flight pitching and yawing motion of missiles' having slight configurational asymmetry may be represented by the motion traced out by three rotating vectors. (See Fig. 3). Rewriting Eq. (40) as

$$\tilde{\alpha} = K_1 e^{(\lambda_1 + i\omega_1) t} + K_2 e^{(\lambda_2 + i\omega_2) t} + K_3 e^{ipt} \tag{46}$$

where λ_1 and λ_2 are the real parts and ω_1 and ω_2 are the imaginary parts of $\phi_{1,2}$. It is noted that the real parts, λ_1 and λ_2, cause the K_1 and K_2 vectors to damp or expand and that the imaginary parts, ω_1 and ω_2, cause the vectors to rotate. Since ip is a pure imaginary, the K_3 arm does not change in size but rotates at a constant angular velocity equal to the steady rolling velocity of the missile.

Before considering the effects of configurational asymmetries it is helpful to review the free flight pitching and yawing motions obtained for the case of no configurational asymmetries.

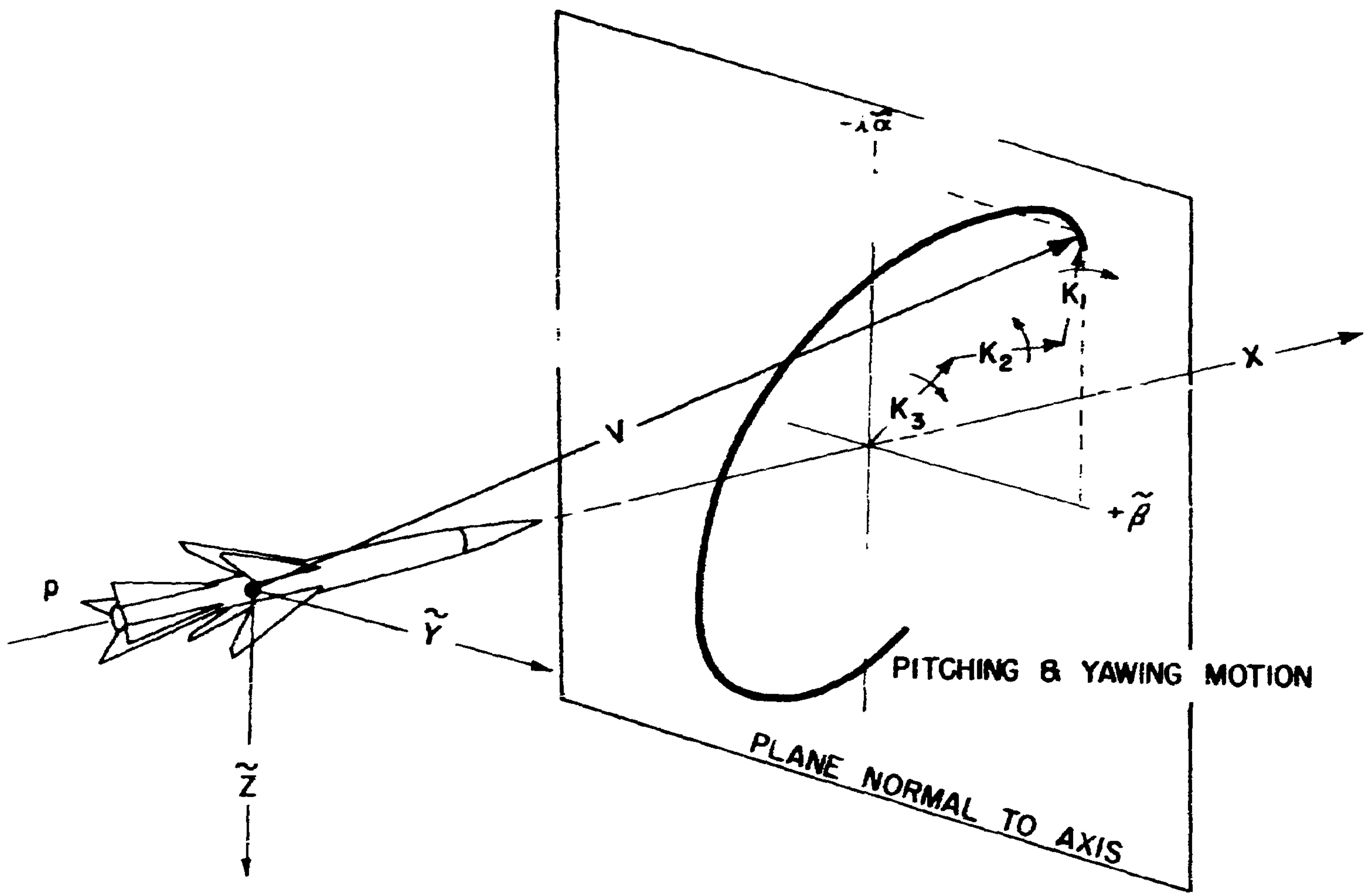


FIG. 3 – TRICYCLIC PITCHING AND YAWING

Epicylic Motion

When the asymmetry is set equal to zero, $\delta_{\varepsilon_0} = 0$, the solution, Eq. (40), reduces to the epicylic form.[3-11]

$$\bar{\tilde{\alpha}} = K_1 e^{\phi_1 t} + K_2 e^{\phi_2 t} \tag{47}$$

$$K_1 = \frac{\dot{\bar{\tilde{\alpha}}}_0 - \phi_2 \bar{\tilde{\alpha}}_0}{\phi_1 - \phi_2} \tag{48}$$

$$K_2 = \frac{\dot{\bar{\tilde{\alpha}}}_0 - \phi_1 \bar{\tilde{\alpha}}_0}{\phi_2 - \phi_1} \tag{49}$$

$$\phi_{1,2} = \text{See Eq. (44)} \tag{44}$$

This solution, like the original, Eq. (40), applies to both statically stable and statically unstable missiles (i.e., to missiles whose center of pressure of the normal force due to angle of attack and yaw is aft of center of gravity of the missile ($-C_{M_\alpha}$) and to missiles whose center of pressure is forward of the center of gravity ($+ C_{M_\alpha}$) .

__Non-Rolling Missiles__. For the case of no rolling velocity ($p = o$) the constants are given by

I. Statically Stable Missiles $(-C_{M_\alpha})$

$$\lambda_{1,2} = \frac{V}{2c}\left\{\hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right)\right\} \tag{50}$$

$$\omega_{1,2} = \pm\frac{V}{2c}\left\{-8\,k^{-2}\,\hat{C}_{M_\alpha}\right\}^{\frac{1}{2}} \tag{51}$$

II. Statically Unstable Missiles $(+C_{M_\alpha})$

$$\lambda_{1,2} = \frac{V}{2C}\left\{\hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right) \pm 8\,k^{-2}\,\hat{C}_{M_\alpha}\right\}^{\frac{1}{2}} \tag{52}$$

$$\omega_{1,2} = 0 \tag{53}$$

For the statically stable missile the vectors K_1 and K_2, rotate in opposite directions with equal velocity, and thus the pitching and yawing motion is given by lines, ellipses, or circles (See Fig. 4).

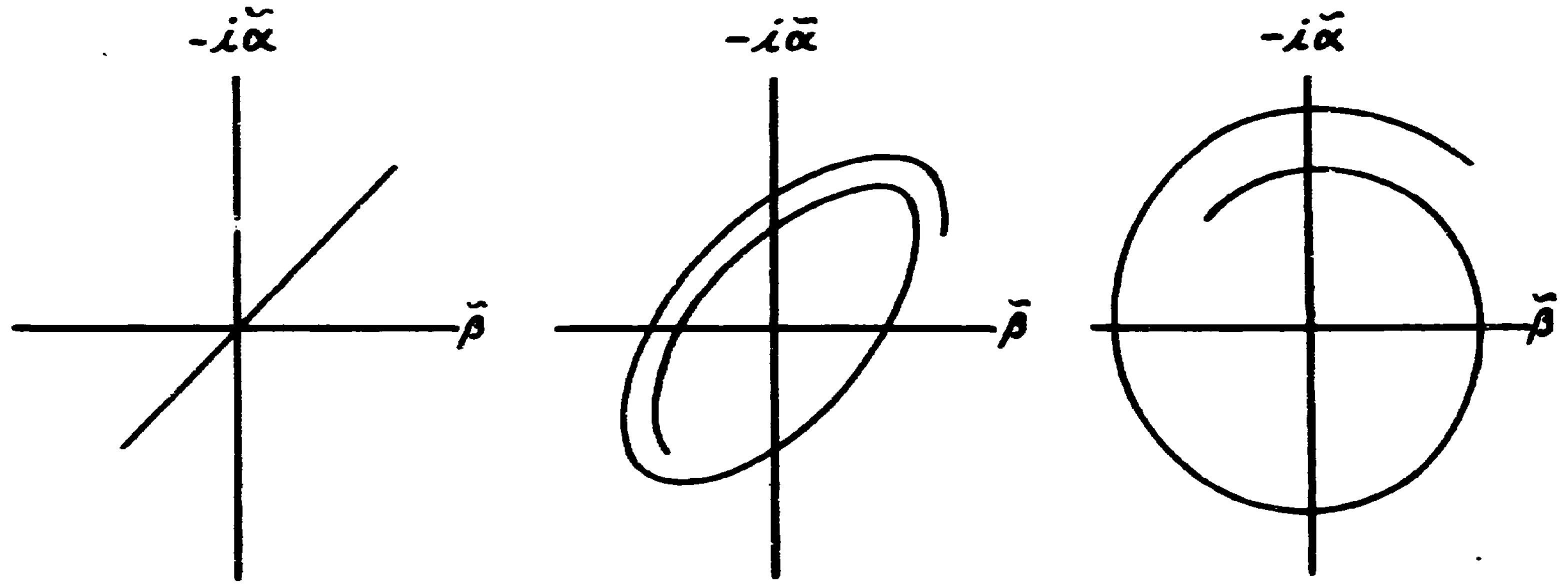


FIG. 4 - TYPICAL FREE FLIGHT PITCHING AND YAWING MOTIONS OF NON-ROLLING STATICALLY STABLE SYMMETRICAL MISSILES.

The general condition for dynamic stability (i.e., the requirement that the motion repeat itself or damp out) is that $\lambda_{1,2} \leq 0$. For the case of the statically stable non-rolling missile this dynamic stability condition reduces to

$$\left| \hat{C}_{N_\alpha} + k^{-2} \hat{C}_{M_q} \right| \geq \left| k^{-2} \hat{C}_{M_{\dot{\alpha}}} \right| \tag{54}$$

since $\hat{C}_{N_\alpha}$ and $\hat{C}_{M_q}$ are negative and $\hat{C}_{M_{\dot{\alpha}}}$ is positive.

For the statically unstable missile the K_1 and K_2 vectors do not rotate and therefore the motion is a line. The motion is, however, generally dynamically unstable since $\left| 8\, k^{-2} \hat{C}_{M_\alpha} \right| > \left| \left[\hat{C}_{N_\alpha} + k^{-2} \left(\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}} \right) \right]^2 \right|$ and thus one of the arms will damp and the other expand.

<u>Rolling Missiles</u>. The motion and the effects of the aerodynamic derivatives on the stability of the rolling missile are more readily discussed if the radical in Eqs. (44) is approximated by a binomial expansion[10] and if

$$\left| \left(\frac{2\,c}{b} \right) \frac{I_x}{I} \right| >> \left| k^{-2} \left(C_{M_{p\dot{\alpha}}} - \hat{C}_{M_{pq}} \right) + \hat{C}_{N_{p\alpha}} \right| \tag{55}$$

Accordingly, the real and imaginary parts of $\phi_{1,2}$ are given by

$$\lambda_{1,2} = \frac{V}{2c}\left\{\hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right)\right.$$

$$\pm \frac{\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)I_x\left[k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot\alpha}}\right) + \frac{I}{I_x}\left(\frac{2b}{c}\right)k^{-2}\hat{C}_{M_{p\alpha}} - \hat{C}_{N_\alpha}\right]}{I\left[-8k^{-2}\hat{C}_{M_\alpha} + \left(\frac{pb}{2V}\right)^2\left(\frac{2c}{b}\right)^2\left(\frac{I_x}{I}\right)^2\right]^{1/2}} \tag{56}$$

$$\omega_{1,2} = \frac{V}{2c}\left\{\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\left(\frac{I_x}{I}\right) \pm \left[-8k^{-2}\hat{C}_{M_\alpha} + \left(\frac{pb}{2V}\right)^2\left(\frac{2c}{b}\right)^2\left(\frac{I_x}{I}\right)^2\right]^{1/2}\right\} \tag{57}$$

subject, however, to the condition that

$$\left(\frac{pb}{2V}\right)^2\left(\frac{2c}{b}\right)^2\left(\frac{I_x}{I}\right)^2 - 8k^{-2}\hat{C}_{M_\alpha} \geq 0 \tag{58}$$

For the statically stable missile it is seen from Eq. (5) that the two vectors rotate in the opposite direction, as in the non-rolling case, but that the difference in their rotational velocities depends on the rolling velocity of the missile. For small rolling velocity of the missile, the pitching and yawing motion is given by slowly rotating ellipses. As the rolling velocity increases the motion becomes flower-like and finally for large rolling velocity we have motions characteristic of the fast spinning gyroscopic pendulum.[21] (See Fig. 5)

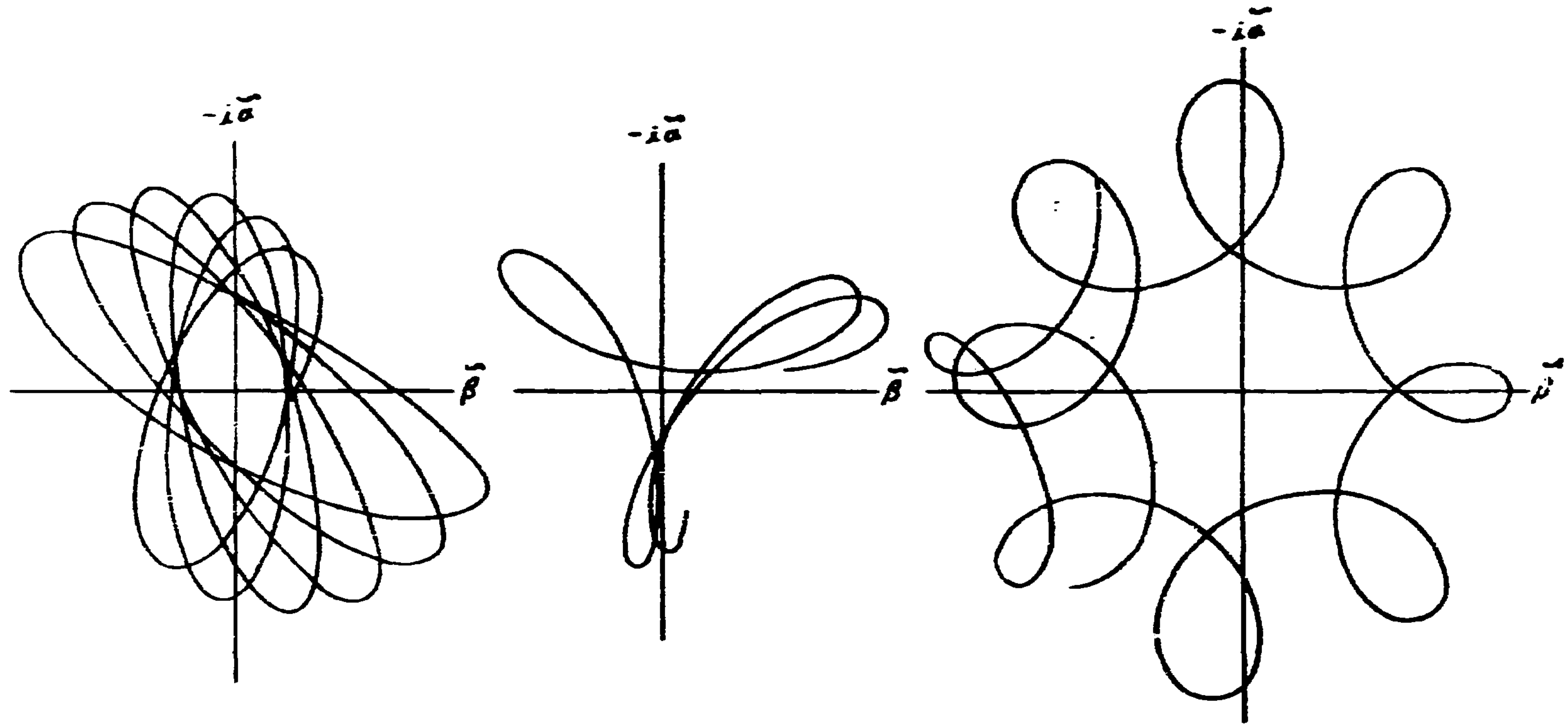


FIG. 5 - TYPICAL FREE FLIGHT PITCHING AND YAWING MOTIONS OF ROLLING STATICALLY STABLE SYMMETRICAL MISSILES.

For the statically unstable missile it is seen from Eq. (57) that the vectors rotate in the same direction (which is also the same direction as the rolling) and thus give the familiar motions characteristic of a "top".[21] (See Fig. 6).

The effects of the various aerodynamic derivatives on the dynamic stability of a missile are more readily seen by writing Eq. (56) in the form*

$$\lambda_{1,2} = \frac{V}{2c}\left\{ \hat{C}_{N_\alpha} (1 \mp \tau) + k^{-2}\hat{C}_{M_q} (1 \pm \tau) + k^{-2} C_{M_{\dot\alpha}} (1 \pm \tau) \pm \frac{I\, k^{-2}\left(\frac{2b}{\delta}\right)\tau}{I_x} \hat{C}_{M_{p\alpha}} \right. \tag{59}$$

where $t = \dfrac{1}{\sqrt{1 - \frac{1}{S}}}$, $S = \dfrac{\left(\frac{pb}{2V}\right)^2 \left(\frac{2c}{b}\right)^2 \left(\frac{I_x}{I}\right)^2}{8\, k^{-2}\hat{C}_{M_\alpha}}$ = (stability factor of the ballistician)[7]. (60) (61)

Ranges of values for the stability factor for the various types of missiles are given in Table I.

Statically Stable Missiles	$S < 0$	$\tau < 1$
Statically Unstable Missiles	$S > 0$	$\tau > 1$
Non-Rolling Missiles	$S = 0$	$\tau = 0$
Neutrally Stable Rolling Missiles	$S = \infty$	$\tau = 1$

TABLE I

Values of S in the range $0 \angle S < 1$ may not be considered when using Eq. (59) due to the condition imposed by Eq. (58).

* ~~Eq. (57) in~~ similar form is given by

$$\lambda_{1,2} = \frac{V}{2c}\left\{\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\left(1 \pm \frac{1}{\tau}\right)\right\}$$

For statically stable rolling missiles it is seen from Eq. (59) that $\hat{C}_{N_\alpha}$ and $\hat{C}_{M_q}$ tend to make the motion dynamically stable, whereas $\hat{C}_{M_{\dot\alpha}}$ and $\hat{C}_{M_{p\alpha}}$ (the Magnus moment coeffifient is generally taken as positive for the statically stable missile) have the opposite effect. $\hat{C}_{M_{p\alpha}}$ tends to undamp the fast rotating vector (nutation) and damp the slower rotating vector (precession). Since the size of the $\hat{C}_{M_{p\alpha}}$ term slowly increases with rolling velocity, there might be limiting rolling velocity beyond which the missile cannot be dynamically stable.[15]

For the statically unstable missile the effects of the aerodynamic derivatives on the dynamic stability are not as simple. From Eq. (59) it is seen that $\hat{C}_{N_\alpha}$ tends to undamp the nutation and damp the precession, whereas $\hat{C}_{M_q}$, $\hat{C}_{M_{\dot\alpha}}$, and $\hat{C}_{M_{p\alpha}}$ (here the sign of $\hat{C}_{M_{p\alpha}}$ is taken as negative, however, positive values are not uncommon) tend to damp the nutation and undamp the precession.

Dynamic Stability Criterion

The criterion for dynamic stability is that

$$\lambda_{1,2} \leq 0 \tag{62}$$

or that

$$\hat{C}_{N_\alpha} + k^{-2} (\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}) \geq \text{REAL} \left\{ \left[\hat{C}_{N_\alpha} + k^{-2} (\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}) + i \left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\left(\frac{I_x}{I}\right) \right]^2 \right.$$

$$\left. + \left[8\, k^{-2} \hat{C}_{M_\alpha} - 4\, i \left(\frac{2c}{b}\right)\left(\frac{I_x}{I}\right) \hat{C}_{N_\alpha} + i\, 8\, k^{-2} C_{M_{p\alpha}} \right] \right\}^{1/2} \tag{63}$$

Extracting the real part of the radical, this criterion may be replaced by two conditions:

(1) when[10,26,27]

$$-\hat{C}_{N_\alpha} - k^{-2} (\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}) > 0 \tag{64}$$

then

$$\left(\frac{pb}{2V}\right)^2 \left(\frac{2c}{b}\right)^2 \left(\frac{I_x}{I}\right)^2 - 8k^{-2}\hat{C}_{M_\alpha} \geq \frac{\left(\frac{pb}{2V}\right)^2 \left(\frac{2c}{b}\right)^2 \left(\frac{I_x}{I}\right)^2 \left\{ 2\left[-\hat{C}_{N_\alpha} + k^{-2}\left(\frac{b}{c}\right) \frac{I}{I_x} \hat{C}_{M_{pq}} \right] + k^{-2}(\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}) + \hat{C}_{N_\alpha} \right\}^2}{\left[\hat{C}_{N_\alpha} + k^{-2} (C_{M_q} + C_{M_{\dot{\alpha}}}) \right]^2} \tag{65}$$

(2) when [27]

$$-\hat{C}_{N_\alpha} - k^{-2} (\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}) = 0 \tag{66}$$

then

$$\left(\frac{pb}{2V}\right)^2 \left(\frac{2c}{b}\right)^2 \left(\frac{I_x}{I}\right)^2 - 8\, k^{-2} \hat{C}_{M_\alpha} \geq 0 \tag{67}$$

and

$$\left(\frac{pb}{2V}\right)\left(\frac{2c}{b}\right)\left(\frac{I_x}{I}\right)\left\{2\left[-\hat{C}_{N_\alpha} + k^{-2}\left(\frac{b}{c}\right)\frac{I}{I_x}\hat{C}_{M_{p\alpha}}\right] + \hat{C}_{N_\alpha} + k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}\right)\right\} = 0 \tag{68}$$

For the case when

$$-\hat{C}_{N_\alpha} - k^{-2}\left(\hat{C}_{M_q} + \hat{C}_{M_{\dot{\alpha}}}\right) < 0 \tag{69}$$

no dynamic stability is possible.

It should be noted that although for this case of constant flight velocity and constant rolling velocity these conditions must be satisfied, for the case of varying rolling velocity and flight velocity dynamic instability may be and is tolerated in some designs for short durations.

Tricyclic Motion

The presence of slight configurational asymmetries has been shown to produce tricyclic motion which differs from the epicyclic motion only in the addition of a third term and a modification in the initial size of the nutational and precessional vectors. Therefore, the remarks of the previous paragraph on the dynamic stability and the contribution of the aerodynamic coefficients to the motion still apply. It remains, however, to consider the effect of this additional third vector.

For the dynamically stable missile after the nutational and precessional arms have damped the third vector is seen to represent the steady state pitching and yawing of the missile.

Non-Rolling Missiles. For the statically and dynamically stable non-rolling missile, the size of the third vector is given by

$$K_{3_0} = + \left(\frac{\hat{C}_{M_{\delta_\varepsilon}}}{\hat{C}_{M_\alpha}} \right) \delta_{\varepsilon_0} \tag{70}$$

which is the familiar expression for the steady state "trim" of an aircraft due to elevator deflection.[1,2] This non-rolling trim angle and the angle of effective control surface deflection lie in the same plane (i.e., K_3 and δ_{ε_0} are parallel). The transient pitching and yawing motion is therefore given by ellipses, lines, or circles whose centers are displaced from the origin by this angle of trim. (See Fig. 7) The case of the statically unstable non-rolling missile is handled by the theory but is hardly worth discussion since the motion is generally dynamically unstable. (See Eq. (52).)

Rolling Missiles. The addition of rolling velocity produces profound changes in the nature and size of the pitching and yawing motion. The size of the third vector as effected by the rolling velocity is best seen by considering the ratio of the rolling trim to the non-rolling trim.

$$\frac{K_3}{K_{3_0}} = \frac{\frac{V^2}{c^2}\left[i \left(\frac{2c}{b}\right)\left(\frac{pb}{2V}\right)\left(1 - \frac{I_x}{I}\right) \hat{C}_{N_{\delta_\varepsilon}} - 2\, k^{-2} \hat{C}_{M_{\delta_\varepsilon}} \right]}{\left[\lambda_1 + i\,(\omega_1 - p)\right]\left[\lambda_2 + i\,(\omega_2 - p)\right]} \left(\frac{\hat{C}_{M_\alpha}}{\hat{C}_{M_{\delta_\varepsilon}}} \right) \tag{71}$$

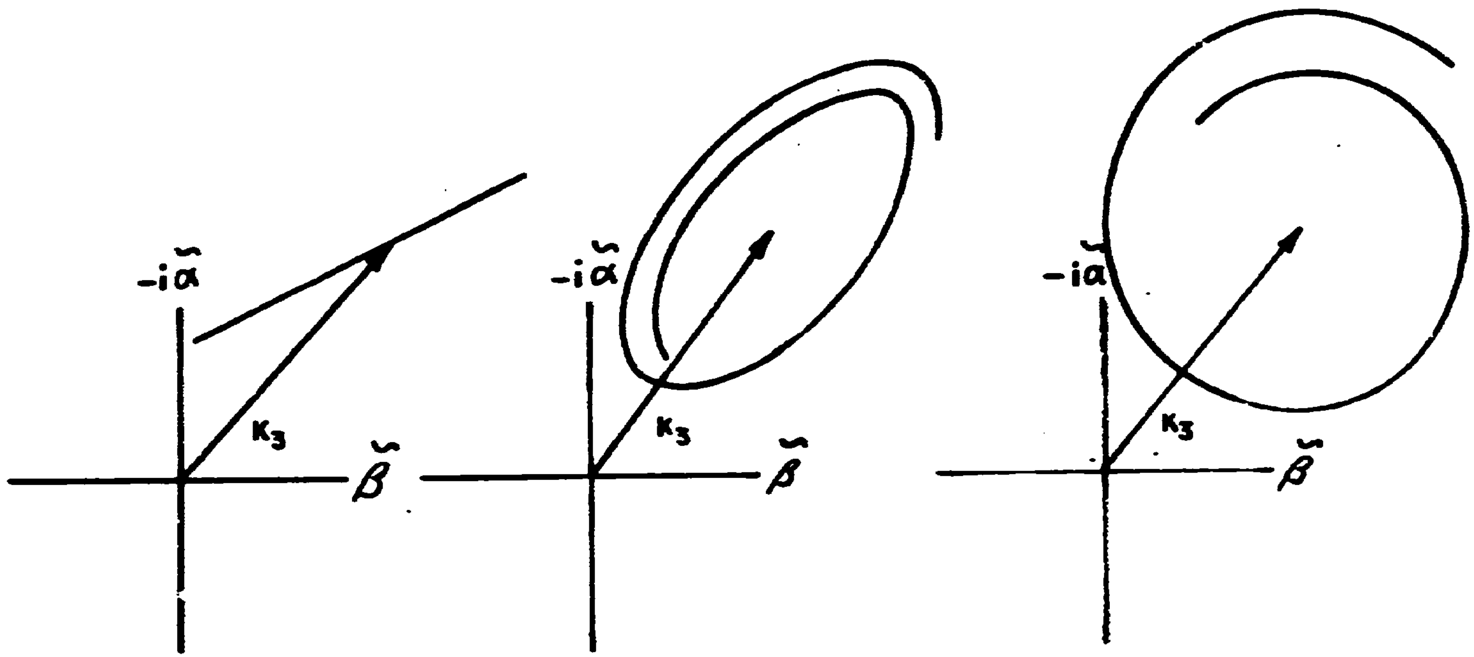


FIG. 7 - TYPICAL FREE FLIGHT PITCHING AND YAWING MOTIONS OF NON-ROLLING STATISTICALLY STABLE ASYMMETRICAL MISSILES.

This ratio is called the "magnification factor" and typical values as a function of rolling velocity are given in Fig. 8 which were calculated for the statically and dynamically stable missile used in the experimental tests to be reported in a later section. From Eq. (71) and from Fig. 8 it is apparent that a resonance phenomenon appears possible if the rolling velocity approaches the rotational velocity of either the nutation vector or the precession vector. However, for statically stable missiles it has been shown that the precessional vector and the nutational vector rotate in different directions and thus only an equality of the angular velocity of the nutation vector and the rolling velocity may be considered. Since for the statically unstable missile both the nutational vector and the precessional vector rotate in the same direction as the roll, resonance with both should be considered.

<u>Resonance Criterion</u>. Substituting $\omega_1 = p$ or $\omega_2 = p$ into Eq. (5) and solving for $\hat{C}_{M_\alpha}$ yields

$$\hat{C}_{M_\alpha} = \frac{I_p^2}{V^2 M} \left(\frac{I_x}{I} - 1 \right) \tag{72}$$

For the statically stable missile (i.e., $\hat{C}_{M_\alpha} < 0$), a necessary condition for resonance is given by

$$\frac{I_x}{I} < 1 \tag{73}$$

which is generally satisfied for all types of missiles. However, for the statically unstable missile (i.e., $\hat{C}_{M_\alpha} > 0$), the necessary condition for resonance is given by

$$\frac{I_x}{I} > 1 \tag{74}$$

which is virtually impossible to satisfy in a practical design. As a

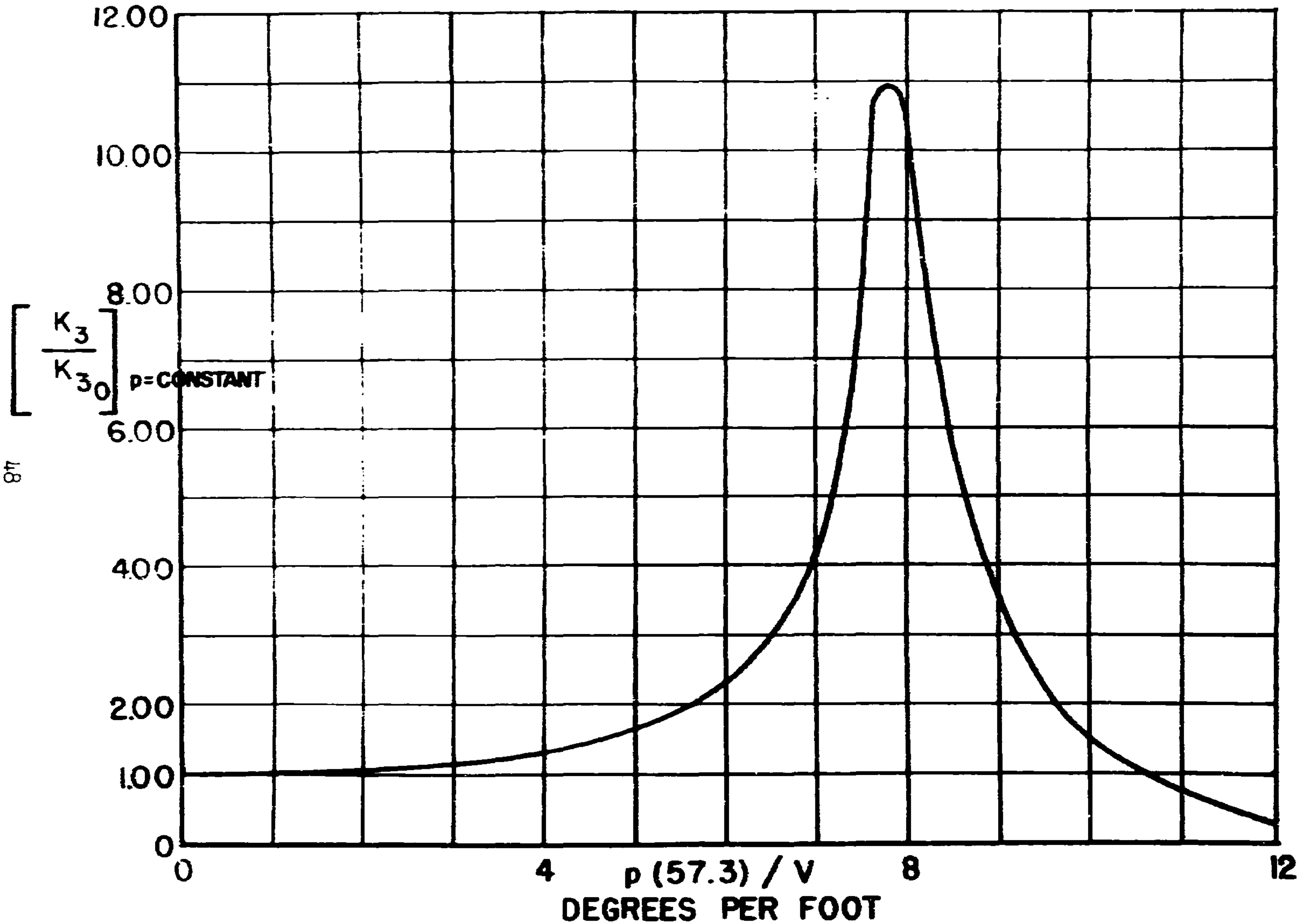


FIG. 8 - MAGNIFICATION FACTOR FOR TEST MISSILE.

result of these conditions it is seen that, in general, resonance can only occur for the case of the statically stable missile and, then, only when the angular velocity of the nutational vector equals the rolling velocity of the missile.

The size of the rolling trim at resonance depends on the rate of damping ($\lambda_{1,2}$). If the damping is zero, the size of the rolling trim is infinite. However, since most missiles are dynamically stable the size of the rolling trim is finite. For missiles with good dynamic stability magnification factors of 10 are not unusual. For missiles of marginal dynamic stability, values from 50 to 100 are quite possible.

The effect of this resonance phenomenon is to produce large angles of pitch and yaw with the result that the basic assumptions of the theory may no longer be valid and instability may result from causes not considered. For this reason stability considerations must include "resonance instability"[25] as well as static and dynamic stability.

Orientation of Trim and Asymmetry. The addition of rolling velocity not only affects the size of the rolling trim but also the angular orientation between the plane containing the effective control surface deflection and the plane containing the trim (i.e., the angle between K_3 and δ_{c_0}). This angle is given by

$$\gamma = \tan^{-1}\left(\frac{\text{Imag. } K_3}{\text{Real } K_3}\right) - \tan^{-1}\left(\frac{\text{Imag. } \delta_{c_0}}{\text{Real } \delta_{c_0}}\right) \tag{75}$$

and typical values are given in Fig. 9 for the particular missile previously mentioned.

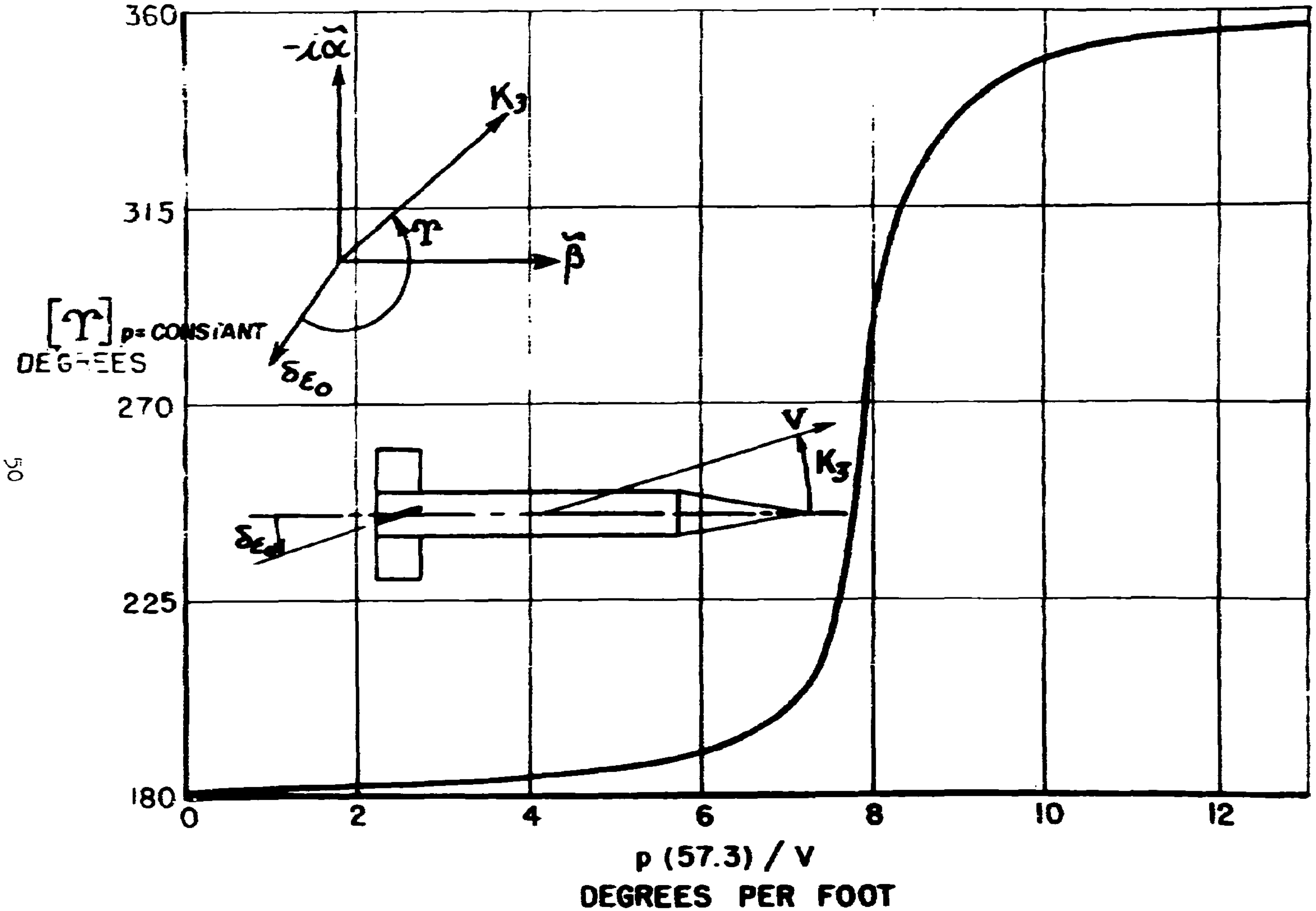


FIG. 9 - ANGULAR ORIENTATION OF TRIM AND ASYMMETRY.

SOLUTION FOR THE TRANSVERSE DISPLACEMENT FOR CONSTANT ROLLING VELOCITY

The motion of the center of gravity of the missile may now be considered. The displacement of the missile in the x direction has already been assumed to be one of constant velocity. Thus, the problem reduces to solving the differential equation of motion for the transverse displacement of the missile which is given by Eq. (34). The function, $\bar{\tilde{\alpha}}$ (t) is known from Eq. (40), and the function, $\bar{\tilde{\varphi}}$ (t), may be obtained from Eq. (36) and Eq. (37) as

$$\bar{\tilde{\varphi}}(t) = \frac{i\,s}{V} - i\,\dot{\bar{\tilde{\alpha}}}(t) \qquad (76)$$

where $\dot{\bar{\tilde{\alpha}}}$ (t) is obtained by differentiating Eq. (40). Substituting these functions into Eq. (34) and solving gives the solution for the transverse displacement of the missile,

$$S = \left[\frac{a + (c - i\,b - i d \phi_1)\phi_1}{m - b/v}\right]\left[\frac{K_1}{\phi_1^2}\right] e^{\phi_1 t} + \left[\frac{a + (c - ib - id\phi_2)\phi_2}{m - i\,b/v}\right]\left[\frac{K_2}{\phi_2^2}\right] e^{\phi_2 t}$$

$$+ \left[\frac{a + (c - ib - id\,\phi_3)\phi_1 + \ell\left(\delta_o/K_3\right)}{m - i\,b/v}\right]\left[\frac{K_3}{\phi_3^2}\right] e^{\ 3t} + k_4 t + k_5 \qquad (77)$$

where k_4 and k_5 are functions of the initial conditions of flight.

This solution for the transverse displacement is seen to be a tricyclic motion plus a linear term and a constant. The constant term depends upon the selection of the coordinate system. For the case where the origin is the muzzle of the gun or launcher and the x axis lies along the center line of the gun or launcher, the constant of the linear term, k_4, is called the "jump angle" [28] and the term itself represents the line about which the oscillatory motion (i.e., swerving motion) takes place.

For the case of the non-rolling missile the solution for the transverse displacement reduces to

$$S = \left[\frac{a + (c - i\,b - i d\phi_1)\phi_1}{m - i\,b/v}\right]\left[\frac{K_1}{\phi_1^{\,2}}\right] e^{\phi_1 t} + \left[\frac{a + (c - ib - id\phi_2)\phi_2}{m - i\,b/v}\right]\left[\frac{K_2}{\phi_2^{\,2}}\right] e^{\phi_2 t}$$

$$+ \left[\frac{a + e\,(\delta_{L\,o}/K_3)}{m - i\,b/v}\right] K_3\, t^2 + k_{4_o}\, t + k_{5_o} \tag{78}$$

Pitching and Yawing Motion for Varying Rolling Velocity

In the previous sections the solution of the differential equations of motion for a missile having slight configurational asymmetry and flying at constant axial velocity and constant rolling velocity was given. The purpose of this section is to investigate* the free flight motion for the case of varying rolling velocity and constant axial velocity.

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*The author is indebted to Dr. M. Lotkin for the numerical integrations hand computed by the Numerical Analysis Section, BRL. A promising new technique is discussed in Ref. 29.

In the design and construction of most fin-stabilized missiles, particularly ordnance weapons (i.e., bombs, mortars, bazookas, rockets, finned artillery shells, etc.), configurational asymmetries are present due to various causes, some of which are manufacturing inaccuracies, damage in handling, damage in launching and intentional design[*]. These asymmetries, besides producing angle of incidence (δ_R , δ_E), have an equal probability of producing differential angle of incidence (δ_A). This differential angle of incidence causes a varying rolling velocity which for most designs starts at zero and approaches a steady state value which is determined by an equality of the roll moment due to differential angle of incidence and the roll moment due to rolling velocity.[19,20]

$$Q_\phi = L_{\delta_A} \delta_A + L_p p \tag{79}$$

The differential equation of pitching and yawing motion, Eq. (38) has been integrated numerically for the following rolling motions:

(1) zero rolling velocity to steady state rolling velocity equal to the nutation rate

(2) zero rolling velocity to steady state rolling velocity equal to 1.5 nutation rate

(3) zero rolling velocity to steady state rolling velocity equal to 5 nutation rate

The resulting pitching and yawing motions are plotted in Figs. 10, 11, and 12 for the missile used in the experimental tests. It is seen from the figures that the speed of passage through the resonance region has a profound influence on the magnitude of the pitching and yawing motion.

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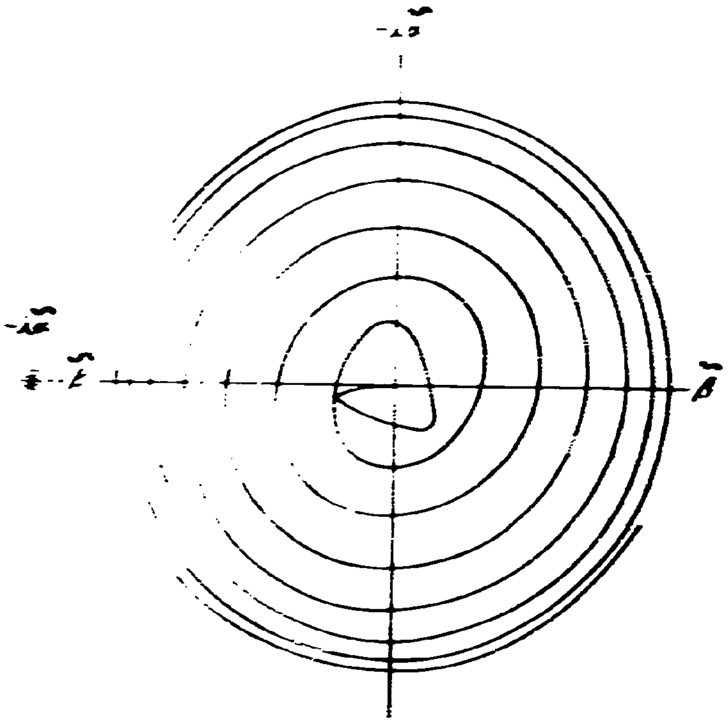
[*]Canting the fins, for example, is a standard practice in many ordnance designs.



$p = 0$
$\delta_\epsilon = 0$

FIG. 10
RESONANCE
$0 \leq p \leq \omega_1$
$\delta_\epsilon \neq 0$

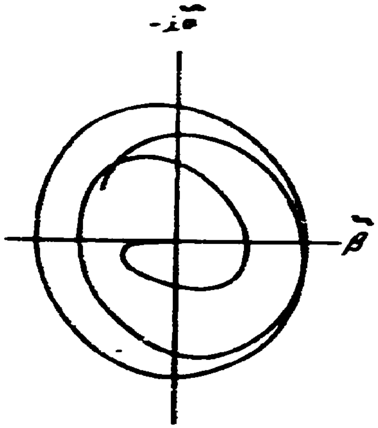


FIG. 11
$0 \leq p \leq 1.5\, \omega_1$
$\delta_\epsilon \neq 0$

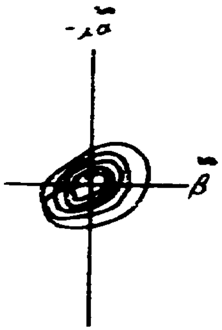


FIG. 12
$0 \leq p \leq 5\, \omega_1$
$\delta_\epsilon \neq 0$

COMMENT ON THEORY

In summary, the solutions for the free flight motions of a missile having slight configurational asymmetries and flying at constant velocity, axial and rolling, indicates that:

(1) The pitching and yawing motion is Tricyclic.

(2) The transverse displacement is Tricyclic, plus a linear term and a constant. (For the case of zero rolling velocity the third arm is replaced by a quadratic term.)

(3) The precession and nutation vectors rotate and change size as in the Epicyclic case.

(4) The rotation and change in size of the new third vector, trim, arising from the asymmetry depends primarily on the rolling velocity of the missile.

(5) The steady state solution for the pitching and yawing motion is given by the third vector which is rotating at the rolling velocity of the missile. As a result, the steady state roll orientation of the missile to the instantaneous plane of pitch and yaw is a constant.

(6) For statically stable missiles the free flight motions resonate when the rolling velocity of the missile and the nutation rate approach coincidence.

(7) The size of the motions at resonance are limited only by the degree of dynamic stability of the missile.

The above characteristics of the solutions for the free flight motions appear to form a basis for a satisfactory explanation for the three phenomena mentioned in the INTRODUCTION. The effect of the rolling velocity on the dispersion of the missile is apparent from consideration of Eq. (77) and Eq. (78). When the rolling velocity of the missile is zero, large dispersion may result from the quadratic term; however, when roll is introduced, the quadratic term disappears and the dispersion may be reduced, provided, of course, that the resonance region is avoided. It is this resonance condition that appears to provide an explanation for the second and third phenomena mentioned, since the large angles of attack and yaw and the resultant transverse displacement, predicted by the theory when the roll rate and nutation rate are coincident, tend to bear out the observed phenomena.

Although the Tricyclic theory appears to contain the seeds for a satisfactory explanation of the observed physical phenomena, no general acceptance may be expected until the ability of the theory to represent accurate experimental data of free flight motion has been thoroughly investigated.

Accordingly, a program has been initiated in the Spark Photography[17-20,33] Ranges to obtain the required accurate free flight data. The general program consists of an investigation of motions over a large range of rolling velocities and includes a detailed investigation of the resonance region. However, only the results for the small rolling velocity case are now available and are given in the following section.

EXPERIMENTAL PROGRAM

The purposes of the present preliminary experimental program are (1) to investigate the ability of the tricylic theory to represent the actual free flight motion of missiles having slight configurational asymmetry, and

(2) to determine the static and dynamic aerodynamic force and moment derivatives which are associated with the motion. In order to obtain the required free flight data two models were tested in the Aberdeen Spark Photography Range.[17-20] (See Fig. 13)

The models employed in the tests had a simple arrow configuration with a cone-cylinder body of fineness ratio of 10 and with cruciformed single wedge rectangular fins of aspect ratio 3 and 8% thickness. One set of fins had an angle of incidence of $.3^{\circ}$ and the other set of fins had no incidence. (See Fig. 14). The models were launched from a special railed gun which enables launching of winged and/or finned missiles. (See Fig. 15)

RESULTS

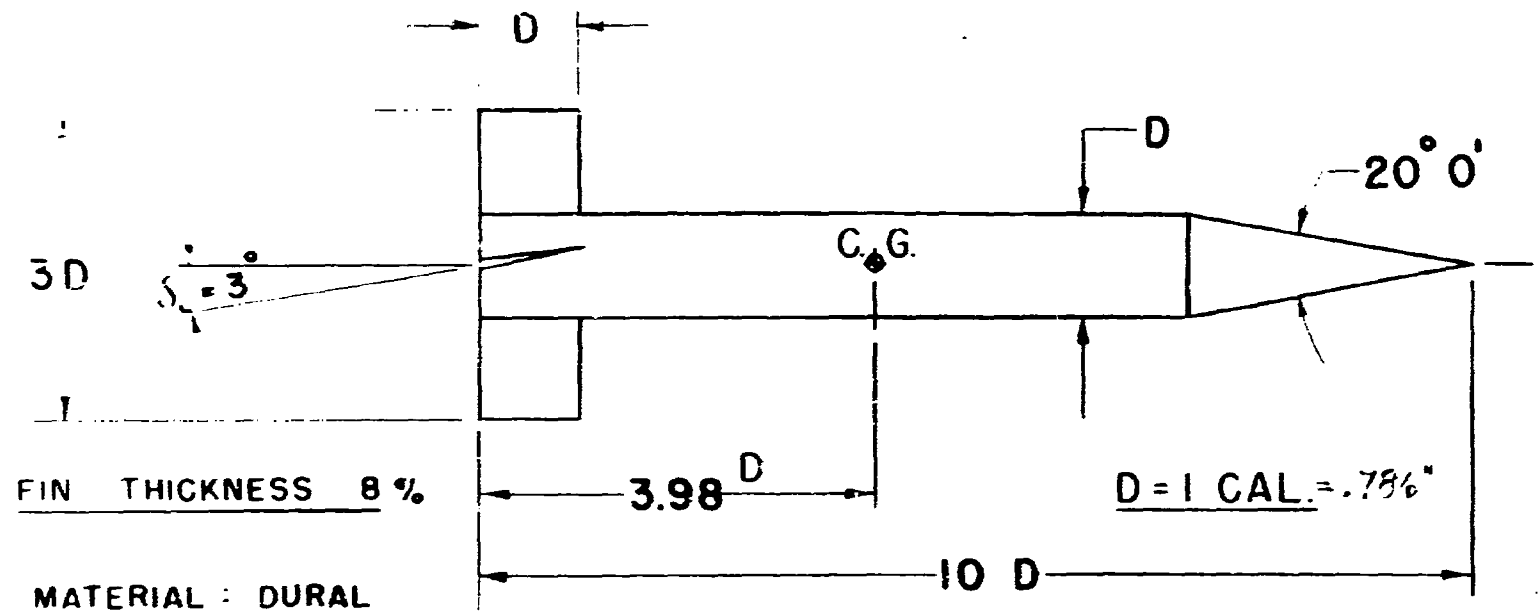
Motion

The experimental data for the pitching and yawing motion and also for the transverse displacement of the models as obtained from the shadowgraphs (Fig. 16) taken in the Range are given by the points in Figs. 17, 18, 19, and 20. In these same figures are plotted the theoretical curves of the tricyclic motion (Eqs. (40) and (77) which were "fitted" to the experimental data by the reduction technique.**[30,32] The "fit" of the theory to the experimental data* is given by the Probable Error in Table II

* The Method of Differential Corrections is used to "fit" Eqs. (40) and (77) to the experimental data. The author is indebted to Mr. C. H. Murphy, Mathematician, B.R.L., for programming the reduction technique for automatic computation by the Bell Relay Computer, and also for suggesting the inclusion of the $\tilde{\bar{\dot{q}}}$ terms in Eqs. (21) and (27).

** See Appendix C.

FIG. 13.



$I_x = 0.61 \times 10^{-5}$ LB.-SEC.^{2}-FT.

I[illegible] $= 25.78 \times 10^{-5}$ LB.-SEC.^{2}-FT.

$$M = .010 \ \frac{\text{LB. SEC.}^2}{\text{FT.}}$$

FIG. 14 - MODEL DESIGN.

FIG. 15. Railed Gun

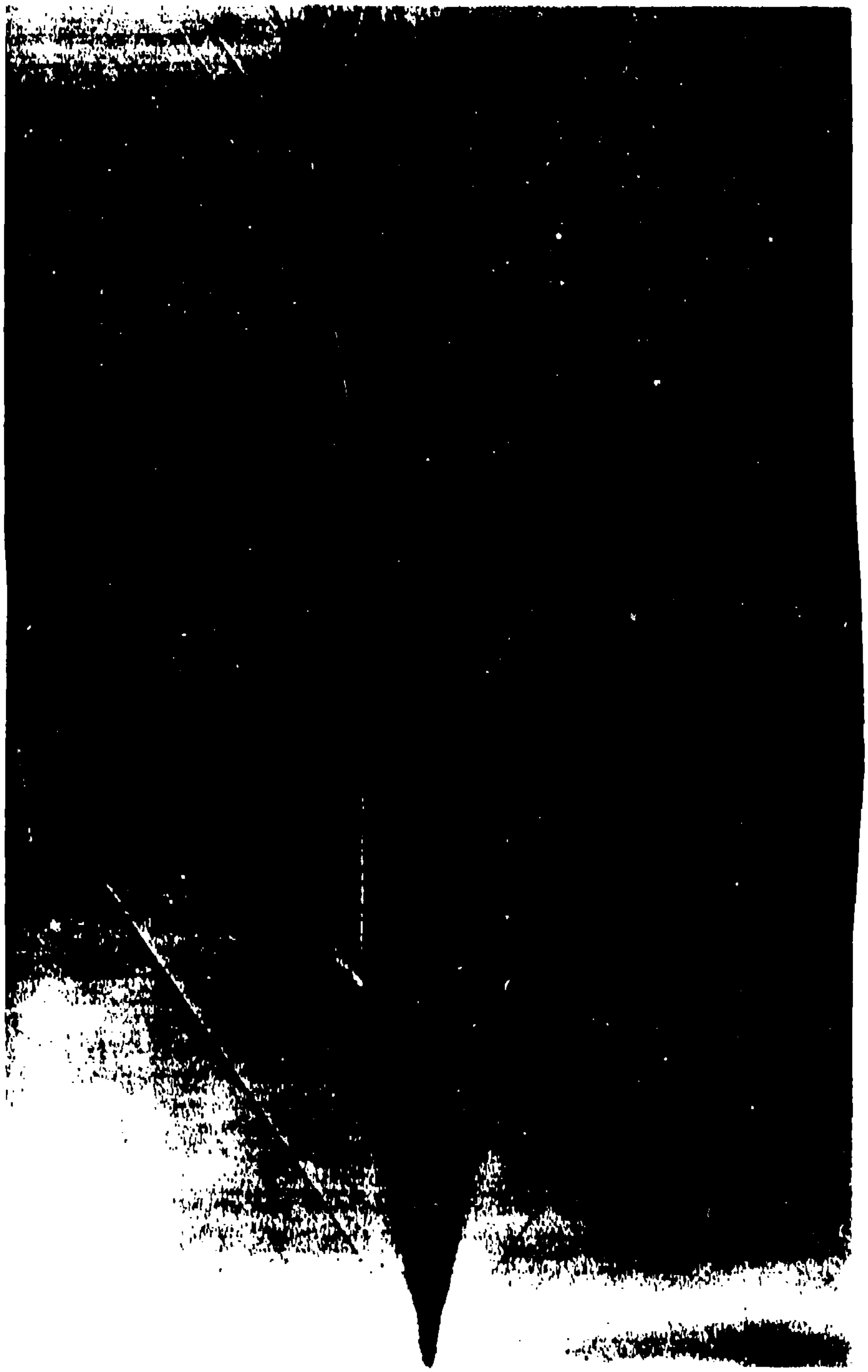


FIG. 16. Shadowgraph of Missile in Flight.

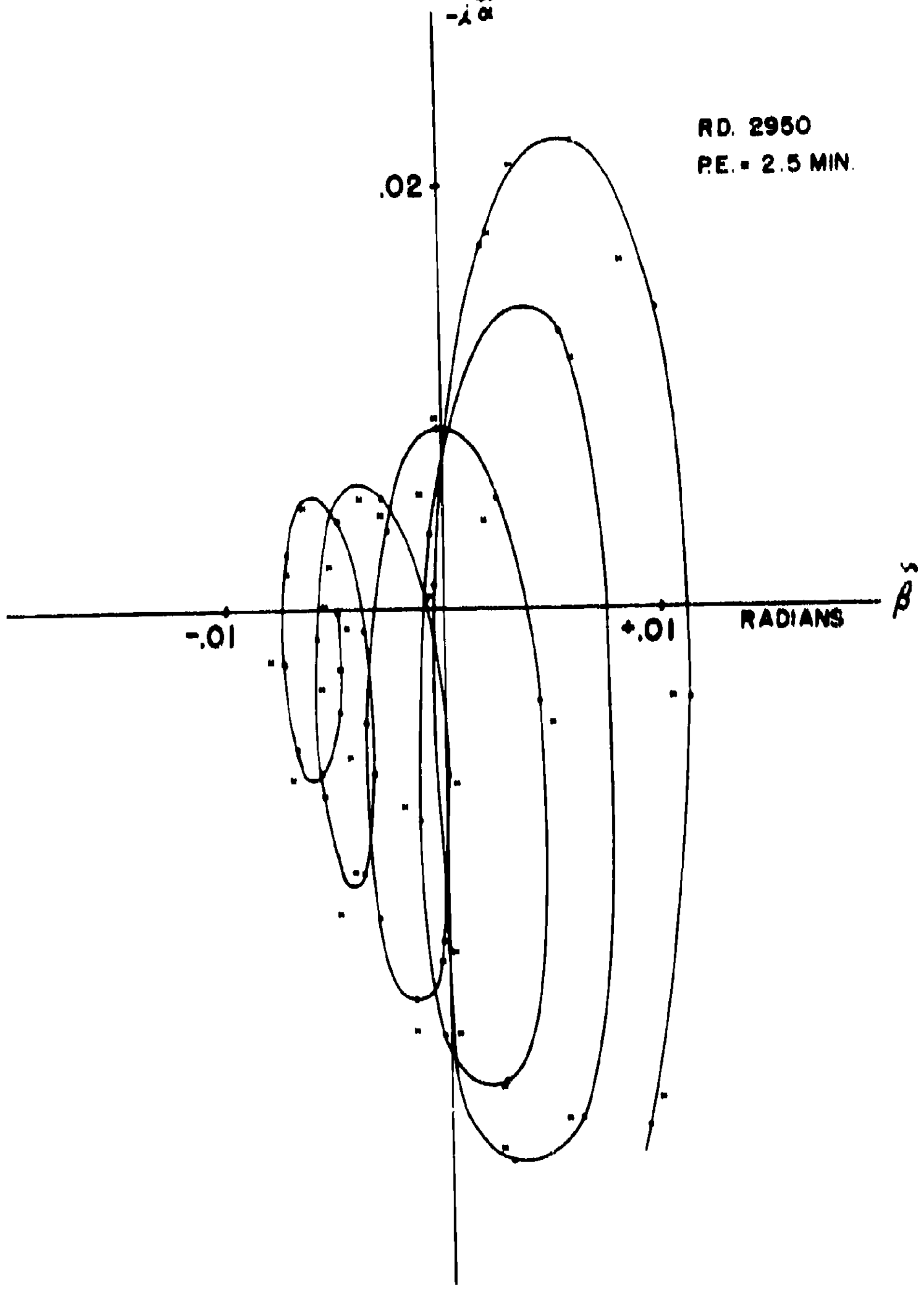


FIG. 17 - EXPERIMENTAL AND THEORETICAL FREE FLIGHT PITCHING AND YAWING MOTION. (RD. 2950)

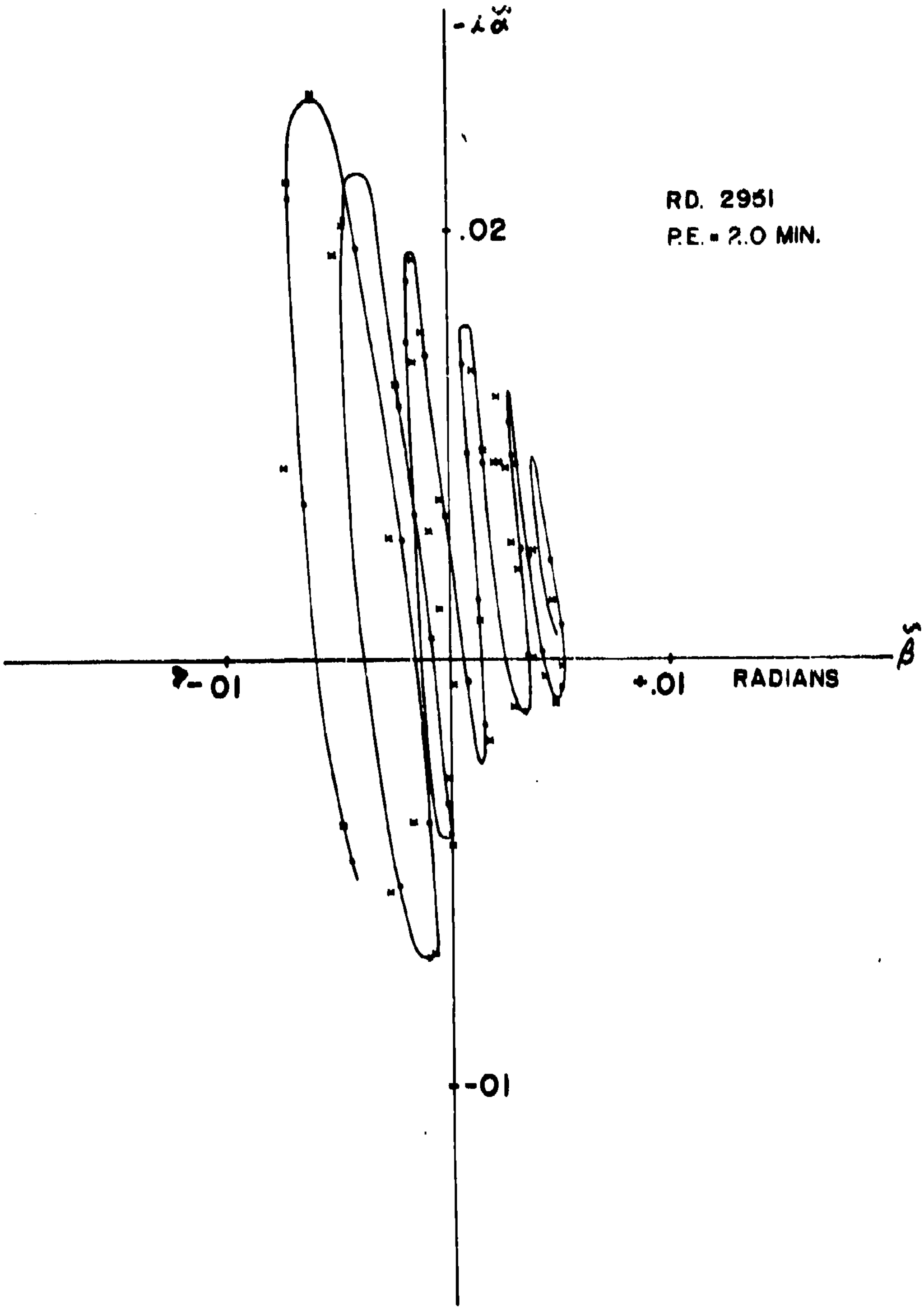


FIG. 18 – EXPERIMENTAL AND THEORETICAL FREE FLIGHT PITCHING AND YAWING MOTION. (RD. 2951)

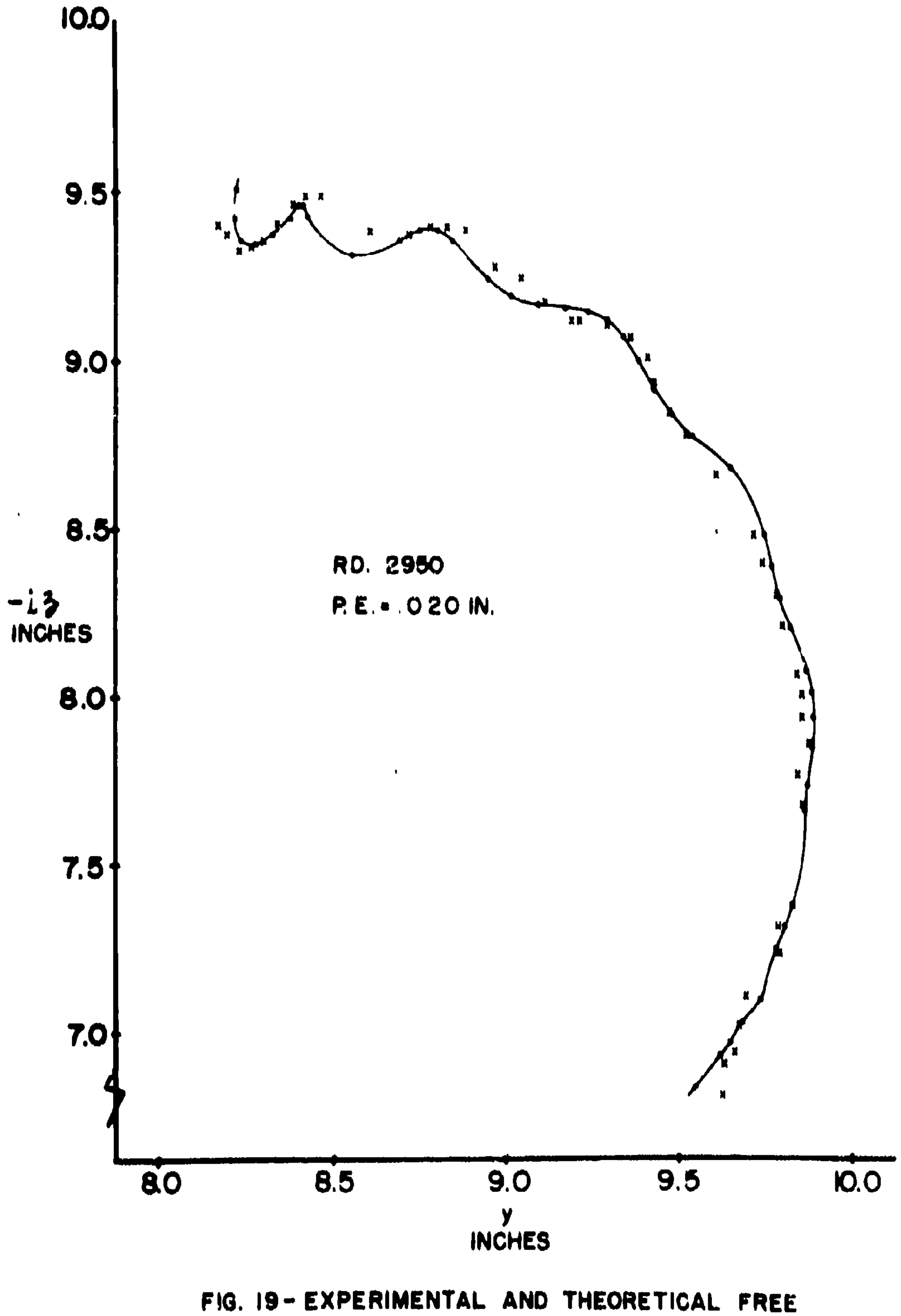


FIG. 19 - EXPERIMENTAL AND THEORETICAL FREE FLIGHT TRANSVERSE DISPLACEMENT. (RD. 2950)

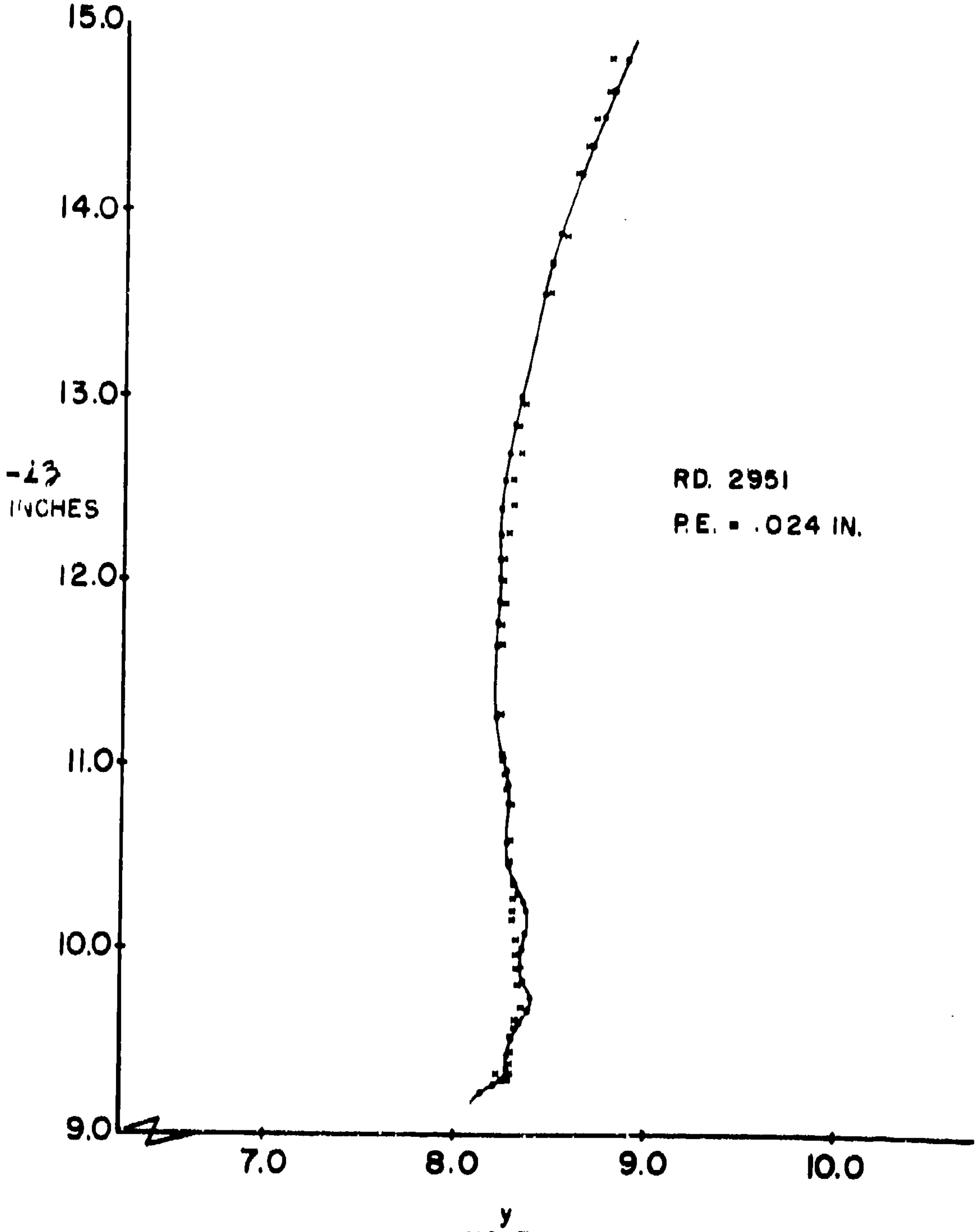


FIG. 20 - EXPERIMENTAL AND THEORETICAL FREE FLIGHT TRANSVERSE DISPLACEMENT. (RD. 2951)

ROUND	2950	2951
PITCH AND YAW P.E.	2.5 MIN.	2.0 MIN.
TRANSVERSE DISP. P.E.	.020 IN.	.024 IN.

TABLE II

Thus, since the Probable Errors of the residuals is of the same order of magnitude as the estimated error in measurement in the Range (Fig. 13) the tricyclic theory of pitching and yawing motion and transverse displacement may be considered to accurately represent the actual free flight pitching and yawing motions of these models.

Aerodynamic Derivatives

The values for the aerodynamic derivatives as obtained from the constants of the motion are given in Table III together with their probable error as obtained from the least squares fit of the theory to the experimental data.

ROUND	2950	2951
C_{M_α}	- 22.6	-22.0
P.E.	0.22%	0.21%
$C_{M_q} + C_{M_{\dot\alpha}}$	-304	-298
P. E.	3.4%	3.2%
$C_{M_{\delta_\ell}}$	19.8	19.9
P.E.	1.9%	1.6%

TABLE III

It is noted that the values for the moment derivatives due to angle of attack and yaw, C_{M_α}, as obtained from the motions of each of the identical missiles differ by 2.6%, that the values for the sum of the moment derivative due to pitching and yawing velocity and the moment derivative due to rate of change of the angle of attack and yaw, $C_{M_q} + C_{M_{\dot\alpha}}$, for the missiles differ by 2.0%, and that the values for the moment derivative due to control surface deflection (asymmetry), $C_{M_{\delta_\ell}}$, differ by 0.5%. Although the sample is small, the results suggest a good reproducibility of the static and dynamic aerodynamic derivatives as obtained from the Spark Photography Range technique.

ACKNOWLEDGMENT

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REFERENCES

[1]Durand, W. F., Aerodynamic Theory, Vol. V, Durant Reprinting Committee, 1943

[2]Perkins, C. D., and Hage, R.E., Airplane Performance Stability and Control, John Wiley & Sons, Inc., 1949

[3]Fowler, R. H., Gallop, E. G., Lock, C. N. H., and Richmond, H. W., The Aerodynamics of a Spinning Shell, Phil. Trans. Roy. Soc., London, 1920.

[4]Fowler, R. H., and Lock, C. N. H., The Aerodynamics of a Spinning Shell, Part II, Phil. Trans. Roy. Soc., London, 1921.

[5]Cranz, C., Lehrbuch der Ballistik, J. Springer, Berlin, 1925, p. 358

[6]Moulton, F. R., New Methods in Exterior Ballistics, University of Chicago Press, Chicago, 1926, Chap. 6.

[7]Kent, R. H., An Elementary Treatment of the Motion of a Spinning Projectile About Its Center of Gravity, B.R.L. No. 85, 1937, and revision with McShane, E. J., B.R.L. No. 459, 1944.

[8]Kent, R. H., Notes for a Course on Exterior Ballistics given at John Hopkins University, 1948.

[9]Nielsen, K. L., and Synge, J. L., On The Motion of a Spinning Shell, Quarterly of Applied Mathematics, Vol. IV, No. 3, October, 1946.

[10]Kelly, J. L., and McShane, E. J., On The Motion of a Projectile with Small or Slowly Changing Yaw, B.R.L. No. 446, 1944.

[11]Kelly, J. L., McShane, E.J., and Reno, F., A Forthcoming Text on Exterior Ballistics, Denver Press, 1953

[12]Phillips, W. H., Effect of Steady Rolling on Longitudinal and Directional Stability, N.A.C.A. T.N. 1627, June, 1948.

[13]Rosser, J. B., Newton, R. R., and Gross, G. L., Mathematical Theory of Rocket Flight, McGraw-Hill Book Company, Inc., 1947.

[14] Prendergast, B., and Kelly, J. L., Second Report of Contract No. W 18-001 ORD. 590, Unpublished.

[15] Zaroodny, S. S., On the Mechanism of Dispersion and Short Ranges of Mortar Fire (Restricted), B.R.L. No. 668, 1948.

[16] Maple, C. G., and Synge, J. L., Aerodynamic Symmetry of Projectiles, Quarterly of Applied Mathematics, Vol. VI, No. 4, January, 1949.

[17] Charters, A.C., and Thomas, R. N., The Aerodynamic Performance of Small Spheres from Subsonic to High Supersonic Velocities, Journal of the Aeronautical Sciences, Vol. 12, No. 4, October, 1945.

[18] Charters, A.C., Some Ballistic Contributions to Aerodynamics, Journal of the Aeronautical Sciences, Vol. 14, No. 3, March, 1947.

[19] Bolz, R. E., and Nicolaides, J. D., A Method of Determining Some Aerodynamic Coefficients from Supersonic Free-Flight Tests of a Rolling Missile, Journal of the Aeronautical Sciences, Vol. 17, No. 10, October 1950, and B.R.L. No. 711, 1949.

[20] Nicolaides, J. D., and Bolz, R. E., On the Pure Rolling Motion of Winged and/or Finned Missiles in Varying Supersonic Flight, B.R.L. Report No. 799, 1952, and J.A.S., Vol. 20, No. 3, March, 1953.

[21] Webster, A. G., The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies, Leipzig: B. G. Teubner, 1904. (New York: Stechert-Hafner, 1920).

[22] Timoshenko, S., and Young, D. H., Advanced Dynamics, McGraw-Hill Book Company, Inc., 1948.

[23] Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Dover Publications, New York, 1944.

[24] Nicolaides, J. D., Variation of the Aerodynamic Force and Moment Coefficients with Reference Position, B.R.L. TN No. 746, 1952.

[25] Rankin, R. A., The Mathematical Theory of the Motion of Rotated and Unrotated Rockets, Phil. Trans. of the Royal Society of London, No. 837, Vol. 241, March 1949.

[26] Bolz, R. E., Dynamic Stability of a Missile in Rolling Flight, Journal of the Aeronautical Sciences, Vol. 19, No. 6, June 1952.

[27] Murphy, C. H., On Dynamic Stability Criteria of the Kelly-McShane Linearized Theory of Motion, B.R.L. No. 853, 1953.

[28] Zaroodny, S. S., On Jump Due to Muzzle Disturbances, B.R.L. Report No. 703, June, 1949.

[29] Murphy, C. H., Analogue Computer Determination of Certain Aerodynamic Coefficients, B.R.L. Report No. 807, 1952.

[30] Turetsky, R. A., Reduction of Spark Data, B.R.L. No. 684, 1948.

[31] Scarborough, J. B., Numerical Mathematical Analysis, The Johns Hopkins Press, 1930.

[32] Kopal, Z., Kavanagh, K. E., and Rodier, N. K., A Manual of Reduction of Spinner Rocket Shadowgrams, M.I.T. TN No. 4, Cambridge, 1949.

[33] Rogers, W. K., The Transonic Free Flight Range, B.R.L. No. 849, 1953.

APPENDIX A

Derivation of Basic Differential Equations of Motion

The derivation of the basic differential equations of motion by employing the total kinetic energy of the missile (Eq. 6) and the Lagrange equation (Eq. 7) on each of the coordinates of the dynamical system (ϕ, θ, ψ, $\dot{x}$, $\dot{y}$ and $\dot{z}$) is given below:

(1) For θ : The Lagrange equation for θ is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \tag{1A}$$

Substituting the expression for the total kinetic energy of the missile (Eq. 6) and performing the indicated operations yields.

$$\frac{d}{dt}\left[I \dot{\theta}\right] - \left[- I_x \dot{\phi} \dot{\psi} \cos\theta + I_x \dot{\psi}^2 \sin\theta \cos\theta - I \dot{\psi}^2 \cos\theta \sin\theta\right] = Q_\theta \tag{2A}$$

Now assuming that θ, ψ, $\dot{\theta}$, and $\dot{\psi}$ are small and that their squares and products may be neglected yields,

$$I \ddot{\theta} + I_x \dot{\phi} \dot{\psi} = Q_\theta \tag{3A}$$

(2) For ψ: The Lagrange equation for ψ is given by

$$\frac{d}{dt}\left(\frac{\partial I}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_\psi \tag{4A}$$

Substituting Eq. 6 and differentiating yields

$$\frac{d}{dt}\left[- I_x \dot{\phi} \sin\theta + I_x \sin^2 \theta \dot{\psi} + I \cos^2\theta \dot{\psi}\right] - \left[0\right] = Q_\psi \tag{5A}$$

or

$$-I_x\dot{\phi} \cos\theta \dot{\theta} - I_x \sin\theta \ddot{\phi} + I_x \sin^2\theta \ddot{\psi} + I_x \dot{\psi} 2 \sin\theta \cos\theta \dot{\theta}$$

$$+ I \cos^2\theta \ddot{\psi} - I \dot{\psi} 2 \cos\theta \sin\theta \dot{\theta} = Q_\psi \tag{6A}$$

assuming that θ, ψ, $\dot{\theta}$, $\dot{\psi}$, and $\ddot{\phi}$ are small quantities yields

$$I\ddot{\psi} - I_x \dot{\phi}\dot{\theta} = Q_\psi \qquad (7A)$$

(3) For ϕ : The Lagrange equation for ϕ is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = Q_\phi \qquad (8A)$$

Substituting Eq. 6 and differentiating yields

$$\boxed{\frac{d}{dt}\left[I_x \ddot{\phi} - I_x \dot{\psi} \sin\theta\right] = Q_\phi} \qquad (9A)$$

(4) For x : The Lagrange equation for x is given by

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = Q_x \qquad (10A)$$

Substituting Eq. 6 and differentiating yields

$$\frac{d}{dt}\left[m\dot{x}\right] - \left[0\right] = Q_x \qquad (11A)$$

or

$$\boxed{m\ddot{x} = Q_x} \qquad (12A)$$

(5) For y : The Lagrange Equation for y is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = Q_y \qquad (13A)$$

Substituting Eq. 6 and differentiating yields

$$\frac{d}{dt}\left[m\dot{y}\right] - \left[0\right] = Q_y \qquad (14A)$$

or

$$m\ddot{y} = Q_y \qquad (15A)$$

(6). For z : The Lagrange equation for z is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = Q_z \qquad (16A)$$

Substituting Eq. 6 and differentiating yields

$$\frac{d}{dt}\left[m\,\dot{z}\right] - \left[0\right] = Q_z \qquad (17A)$$

or

$$m\,\ddot{z} = Q_z \qquad (18A)$$

Now multiplying Eq. (7A) by i and adding to Eq. (3A) yields

$$I\,(\ddot{\theta} + i\,\ddot{\psi}) - i\,I_x\,\dot{\phi}\,(\dot{\theta} + i\,\dot{\psi}) = Q_\theta + i\,Q_\psi \qquad (19A)$$

Introducing complex variables by defining

$$\dot{\Omega} = \dot{\theta} + i\,\dot{\psi} \qquad (20A)$$

$$\ddot{\Omega} = \ddot{\theta} + i\,\ddot{\psi} \qquad (21A)$$

and substituting into Eq. (19A) yields

$$\boxed{I\,\ddot{\Omega} + i\,I_x\,\dot{\phi}\,\dot{\Omega} = Q_\Omega} \qquad (22A)$$

Also multiplying Eq. (18A) by i and adding to Eq. (15A) yields

$$m\,(\ddot{y} + i\,\ddot{z}) = Q_y + i\,Q_z \qquad (23A)$$

defining

$$S = y + i\,z \qquad (24A)$$

$$\dot{S} = \dot{y} + i\,\dot{z} \qquad (25A)$$

$$\ddot{S} = \ddot{y} + i\,\ddot{z} \qquad (26A)$$

APPENDIX B

Aerodynamic and Ballistic Nomenclature

The aerodynamic forces and moments in both the Aerodynamic nomenclature* and the Ballistic nomenclature, and also in the Aero-Ballistic nomenclature (for missiles possessing trigonal or greater rotational symmetry and mirror symmetry) are given in Table I.

The relation between the Ballistic J's and the Aero-Ballistic $\hat{C}$'s are given in Table II.

From the definitions and relations of Tables' I and II, some of the basic equations of the report may be converted to Ballistic form.

The constants of the Aerodynamic force, Eq. (21), and the aerodynamic moment, Eq. (27), in Ballistic nomenclature are given by:

FORCE CONSTANTS

$$a = -K_N \rho u^2 d^2 + i K_F \rho u \omega d^3$$

$$b = K_{XF} \rho \omega d^4 + i K_s \rho u d^3$$

$$c = K_{LN} \rho u d^3 - i K_{LF} \rho \omega d^4$$

$$d = K_{LXF} \rho \omega/u\, d^5 - i K_{LS} \rho d^4$$

$$e = K_{N_\epsilon} \rho u^2 d^2$$

*Hopgood, R. C., "A Proposed Revision of American Standard Letter Symbols for Aeronautical Sciences", Aeronautical Engineering Review, January, 1953.

TABLE I

AERODYNAMIC FORCES

AERODYNAMIC

$$Y_\beta\ \beta = C_{Y_\beta}\ \beta\ \bar{q}\ S$$

$$Y_{p\alpha}\ p\alpha = C_{Y_{p\alpha}} \left(\frac{pb}{2V}\right) \alpha\ \bar{q}\ S$$

$$Y_r\ r = C_{Y_r} \left(\frac{rb}{2V}\right) \bar{q}\ S$$

$$Y_{pq}\ pq = C_{Y_{pq}} \left(\frac{pb}{2V}\right)\left(\frac{qc}{2V}\right) \bar{q}\ S$$

$$Y_{\dot\beta}\ \dot\beta = C_{Y_{\dot\beta}} \left(\frac{\dot\beta b}{2V}\right) \bar{q}\ S$$

$$Y_{p\dot\alpha}\ p\dot\alpha = C_{Y_{p\dot\alpha}} \left(\frac{pb}{2V}\right)\left(\frac{\dot\alpha c}{2V}\right) \bar{q}\ S$$

$$Y_{\dot r}\ \dot r = C_{Y_{\dot r}} \left(\frac{\dot r b}{2V}\right)\left(\frac{b}{2V}\right) \bar{q}\ S$$

$$Y_{p\dot q}\ p\dot q = C_{Y_{p\dot q}} \left(\frac{pb}{2V}\right)\left(\frac{\dot q c}{2V}\right)\left(\frac{c}{2V}\right) \bar{q}\ S$$

$$Y_{\delta_R}\ \delta_R = C_{Y_{\delta_R}}\ \delta_R\ \bar{q}\ S$$

$$Z_\alpha\ \alpha = C_{Z_\alpha}\ \alpha\ \bar{q}\ S$$

$$Z_{p\beta}\ p\beta = C_{Z_{p\beta}} \left(\frac{pb}{2V}\right) \beta\ \bar{q}\ S$$

$$Z_q\ q = C_{Z_q} \left(\frac{qc}{2V}\right) \bar{q}\ S$$

$$Z_{pr}\ pr = C_{Z_{pr}} \left(\frac{pb}{2V}\right)\left(\frac{rb}{2V}\right) \bar{q}\ S$$

$$Z_{\dot\alpha}\ \dot\alpha = C_{Z_{\dot\alpha}} \left(\frac{\dot\alpha c}{2V}\right) \bar{q}\ S$$

$$Z_{p\dot\beta}\ p\dot\beta = C_{Z_{p\dot\beta}} \left(\frac{pb}{2V}\right)\left(\frac{\dot\beta b}{2V}\right) \bar{q}\ S$$

$$Z_{\dot q}\ \dot q = C_{Z_{\dot q}} \left(\frac{\dot q c}{2V}\right)\left(\frac{c}{2V}\right) \bar{q}\ S$$

$$Z_{p\dot r}\ p\dot r = C_{Z_{p\dot r}} \left(\frac{pb}{2V}\right)\left(\frac{\dot r b}{2V}\right)\left(\frac{b}{2V}\right) \bar{q}\ S$$

$$Z_{\delta_E}\ \delta_E = C_{Z_{\delta_E}}\ \delta_E\ \bar{q}\ S$$

TABLE I

AERODYNAMIC FORCES

AERO - BALLISTIC	BALLISTICS
$N_{\zeta} \zeta = C_{N_{\alpha}} \zeta \; \bar{q} \; S$	$N = - K_N \rho u^2 d^2 \zeta$
$N_{p\zeta} \, p \zeta = i \, C_{N_{p\alpha}} \left(\frac{pb}{2V}\right) \zeta \; \bar{q} \; S$	$F = i \, K_F \, \rho \, u \, \omega \, d^3 \zeta$
$N_{\eta} \eta = i \, C_{N_q} \left(\frac{\eta c}{2V}\right) \bar{q} \; S$	$S = i \, K_S \, \rho \, u \, d^3 \eta$
$N_{p\eta} \, p \eta = C_{N_{p_q}} \left(\frac{pb}{2V}\right)\left(\frac{\eta c}{2V}\right) \bar{q} \; S$	$XF = K_{XF} \, \rho \, \omega \, d^4 \eta$
$N_{\dot{\zeta}} \dot{\zeta} = C_{N_{\dot{\alpha}}} \left(\frac{\dot{\zeta} c}{2V}\right) \bar{q} \; S$	$LN = K_{LN} \, \rho \, u \, d^3 \dot{\zeta}$
$N_{p\dot{\zeta}} \, p \dot{\zeta} = i \, C_{N_{p\dot{\alpha}}} \left(\frac{pb}{2V}\right)\left(\frac{\dot{\zeta} c}{2V}\right) \bar{q} \; S$	$LF = - i \, K_{LF} \, \rho \, \omega \, d^4 \dot{\zeta}$
$N_{\dot{\eta}} \dot{\eta} = i \, C_{N_{\dot{q}}} \left(\frac{\dot{\eta} c}{2V}\right)\left(\frac{c}{2V}\right) \bar{q} \; S$	$LS = - i \, K_{LS} \, \rho \, d^4 \dot{\eta}$
$N_{p\dot{\eta}} \, p \dot{\eta} = C_{N_{p\dot{q}}} \left(\frac{pb}{2V}\right)\left(\frac{\dot{\eta} c}{2V}\right)\left(\frac{c}{2V}\right) \bar{q} \; S$	$LXF = - K_{LXF} \, \rho \, \frac{\omega}{u} \, d^5 \dot{\eta}$
$N_{\delta_{\varepsilon}} \delta_{\varepsilon} = C_{N_{\delta_{\varepsilon}}} \delta_{\varepsilon} \; \bar{q} \; S$	$N_{\varepsilon} = K_{N_{\varepsilon}} \, \rho \, u^2 \, d^2 \, \delta_{\varepsilon}$

TABLE I

AERODYNAMIC MOMENTS

$M_\alpha \alpha = C_{m_\alpha} \alpha \bar{q} S c$	$N_\beta \beta = C_{n_\beta} \beta \bar{q} S b$
$M_{p\beta} p\beta = C_{m_{p\beta}} (\frac{pb}{2V}) \beta \bar{q} S c$	$N_{p\alpha} p\alpha = C_{n_{p\alpha}} (\frac{pb}{2V}) \alpha \bar{q} S b$
$M_q q = C_{m_q} (\frac{qc}{2V}) \bar{q} S c$	$N_r r = C_{n_r} (\frac{rb}{2V}) \bar{q} S b$
$M_{pr} pr = C_{m_{pr}} (\frac{pb}{2V})(\frac{rb}{2V}) \bar{q} S c$	$N_{pq} pq = C_{n_{pq}} (\frac{pb}{2V})(\frac{qc}{2V}) \bar{q} S b$
$M_{\dot\alpha} \dot\alpha = C_{m_{\dot\alpha}} (\frac{\dot\alpha c}{2V}) \bar{q} S c$	$N_{\dot\beta} \dot\beta = C_{n_{\dot\beta}} (\frac{\dot\beta b}{2V}) \bar{q} S b$
$M_{p\dot\beta} p\dot\beta = C_{m_{p\dot\beta}} (\frac{pb}{2V})(\frac{\dot\beta b}{2V}) \bar{q} S c$	$N_{p\dot\alpha} p\dot\alpha = C_{n_{p\dot\alpha}} (\frac{pb}{2V})(\frac{\dot\alpha c}{2V}) \bar{q} S b$
$M_{\dot q} \dot q = C_{m_{\dot q}} (\frac{\dot q c}{2V})(\frac{c}{2V}) \bar{q} S c$	$N_{\dot r} \dot r = C_{n_{\dot r}} (\frac{\dot r b}{2V})(\frac{b}{2V}) \bar{q} S b$
$M_{p\dot r} p\dot r = C_{m_{p\dot r}} (\frac{pb}{2V})(\frac{\dot r b}{2V})(\frac{b}{2V}) \bar{q} S c$	$N_{p\dot q} p\dot q = C_{n_{p\dot q}} (\frac{pb}{2V})(\frac{\dot q c}{2V})(\frac{c}{2V}) \bar{q} S b$
$M_{\delta_E} \delta_E = C_{m_{\delta_E}} \delta_E \bar{q} S c$	$N_{\delta_R} \delta_R = C_{n_{\delta_R}} \delta_R \bar{q} S b$

WHERE $\xi = \beta + i\,\alpha$, $\dot\xi = \dot\beta + i\,\dot\alpha$,

TABLE I

AERODYNAMIC MOMENTS

AERO-BALLISTIC	BALLISTIC
$M_{\xi}\xi = -i\, C_{M_\alpha}\xi\, \bar{q}\, S\, c$	$M = -i\, K_M\, \rho\, u^2\, d^3\, \xi$
$M_{p\xi}\, p\xi = C_{M_{p\alpha}}(\frac{pb}{2V})\xi\, \bar{q}\, S\, c$	$T = -K_T\, \rho\, u\, \omega\, d^4\, \xi$
$M_{\eta}\eta = C_{M_q}(\frac{\eta c}{2V})\, \bar{q}\, S\, c$	$H = -K_H\, \rho\, u\, d^4\, \eta$
$M_{p\eta}\, p\eta = -iC_{M_{pq}}(\frac{pb}{2V})(\frac{\eta c}{2V})\, \bar{q}\, S\, c$	$XT = -iK_{XT}\, \rho\, \omega\, d^5\, \eta$
$M_{\dot\xi}\dot\xi = -iC_{M_{\dot\alpha}}(\frac{\dot\xi c}{2V})\, \bar{q}\, S\, c$	$LM = iK_{LM}\, \rho\, u\, d^4\, \dot\xi$
$M_{p\dot\xi}\, p\dot\xi = C_{M_{p\dot\alpha}}(\frac{pb}{2V})(\frac{\dot\xi c}{2V})\, \bar{q}\, S\, c$	$LT = K_{LT}\, \rho\, \omega\, d^5\, \dot\xi$
$M_{\dot\eta}\dot\eta = C_{M_{\dot q}}(\frac{\dot\eta c}{2V})(\frac{c}{2V})\, \bar{q}\, S\, c$	$LH = K_{LH}\, \rho\, d^5\, \dot\eta$
$M_{p\dot\eta}\, p\dot\eta = -iC_{M_{p\dot q}}(\frac{pb}{2V})(\frac{\dot\eta c}{2V})(\frac{c}{2V})\, \bar{q}\, S\, c$	$LXT = -iK_{LXT}\, \rho\, \frac{\omega}{u}\, d^6\, \dot\eta$
$M_{\delta_\epsilon}\delta_\epsilon = i\, C_{M_{\delta_\epsilon}}\delta_\epsilon\, \bar{q}\, S\, c$	$M_\epsilon = iK_{M_\epsilon}\, \rho\, u^2\, d^3\, \delta_\epsilon$

$\eta = q + i\, r$, $\dot\eta = \dot q + i\, \dot r$, and $\bar{q} = \frac{1}{2}\rho V^2$

TABLE II

AERO-BALLISTIC		BALLISTIC
$\hat{C}_{N\alpha}$	=	$-J_N \; c/d$
$\hat{C}_{N_{p\alpha}}$	=	$J_F \; 2c/b$
$\hat{C}_{N_{\delta_\varepsilon}}$	=	$J_{N_\varepsilon} \; c/d$
$\hat{C}_{M_q}$	=	$-J_H \; 2d/c$
$\hat{C}_{M_{\dot{\alpha}}}$	=	$-J_{LM} \; 2d/c$
$\hat{C}_{M_{p\dot{\alpha}}}$	=	$J_{LT} \; 4d^2/cb$
$\hat{C}_{M_{pq}}$	=	$-J_{XT} \; 4d^2/cb$
$\hat{C}_{M_\alpha}$	=	J_M
$\hat{C}_{M_{p\alpha}}$	=	$-J_T \; 2d/b$
$\hat{C}_{M_{\delta_\varepsilon}}$	=	J_{M_ε}

$$K_{ij} = J_{ij} \; \frac{m}{\rho} d^{-3}$$

$$p = \frac{\nu u}{d}$$

MOMENT CONSTANTS

$$A = -K_T \rho u \omega d^4 - i K_M' \rho u^2 d^3$$

$$B = -K_H \rho u d^4 + i K_{XT} \rho \omega d^5$$

$$C = K_{LT} \rho \omega d^5 + i K_{LM} \rho u d^4 \tag{2B}$$

$$D = K_{LH} \rho d^5 - i K_{LXT} \rho \omega'_{n} d^6$$

$$E = i K_{M_\varepsilon} \rho u^2 d^3$$

Substituting these expressions to Eqs. (32), and (34), and combining yields the differential equation for the pitching and yawing motion

$$\ddot{\tilde{\xi}} - \frac{V}{d}\left\{ - J_N - k^{-2}\left(J_H + J_{LM}\right) + i\nu\left[k^{-2}\left(J_{LT} + J_{XT}\right) + J_F + A/B\right]\right\}\dot{\tilde{\xi}}$$
$$+ \frac{V^2}{d^2}\left\{ - k^{-2} J_M - \nu^2 A/B\, J_F - i\nu A/B\, J_N + i\nu k^{-2} J_T\right\}\tilde{\xi}$$
$$= V^2\left\{ i \frac{2\nu}{d^2}\left(1 - A/B\right) J_{N_\varepsilon} - \frac{m}{B} J_{M_\varepsilon}\right\}\delta_{\varepsilon_0}\, e^{i\int\left(\frac{\nu u}{d}\right) dt} \tag{3B}$$

WHERE

$$k^{-2} = \frac{md^2}{B}$$

The general solution in Ballistic form is given by

$$\tilde{\xi} = K_1 e^{\Phi_1 t} + K_2 e^{\Phi_2 t} + K_3 e^{\Phi_3 t} \tag{4B}$$

WHERE

$$K_1 = \frac{\dot{\tilde{\xi}}_o - \Phi_2 \tilde{\xi}_o + K_3\left(\Phi_2 - \Phi_3\right)}{\Phi_1 - \Phi_2} \tag{5B}$$

$$K_2 = \frac{\dot{\tilde{\xi}}_o - \Phi_1 \tilde{\xi}_o + K_3\left(\Phi_1 - \Phi_3\right)}{\Phi_2 - \Phi_1} \tag{6B}$$

$$K_3 = \frac{V^2/d^2\left\{i\, \nu\left(1 - A/B\right) J_{N_\epsilon} - k^{-2} J_{M_\epsilon}\right\} \delta\varepsilon_o}{\left(\Phi_3 - \Phi_1\right)\left(\Phi_3 - \Phi_2\right)} \tag{7B}$$

$$\begin{aligned}\Phi_{1,2} = {} & V/_{2d}\left\{- J_N - k^{-2}\left(J_H + J_{LM}\right) + i\,\nu\left[k^{-2}\left(J_{LT} + J_{XT}\right) + J_F\, A/_B\right]\right\} \\ & \pm V/_{2d}\left\{\left\{- J_N - k^{-2}\left(J_H + J_{LM}\right) + i\,\nu\left[k^{-2}\left(J_{LT} + J_{XT}\right) + J_F + A/_B\right]\right\}^2\right. \\ & \left. - 4\left\{- k^{-2} J_M - \nu^2\, A/_B\, J_F - i\,\nu\, A/_B\, J_N + i\,\nu\, k^{-2} J_T\right\}\right\}^{\frac{1}{2}}\end{aligned} \tag{8B}$$

$$\Phi_3 = i\,\nu u/_d \tag{9B}$$

APPENDIX C

Reduction Technique

It is the purpose of the reduction technique to "fit" the Tricyclic Theory to the experimental data obtained from the Aerodynamics Range and to determine the various constants associated with the motion. The present procedure is to first consider the pitching and yawing motion and then, using the values of the constants obtained from the pitching and yawing motion, reduce the transverse displacement. Once these constants have been obtained the representation of the experimental data by the theory (i.e., "fit") may be determined and the aerodynamic derivatives may be calculated.

Yawing Motion

For the purpose of Range reduction the equation of yawing motion (Eq. 40) is modified in two ways:

(1) the independent variable is changed from time to distance along the trajectory of the missile, s.

(2) the pitch and yaw are measured from the instantaneous velocity vector of the center of gravity to the missile and are taken as positive in the first quadrant looking at the approaching missile*, and is written as

$$\tilde{\beta} + i\tilde{\alpha} = e^{2.302\left(a_{K_1} + b_{K_1} s\right)} e^{i\left(a_1 + b_1 s\right)} + e^{2.302\left(a_{K_2} + b_{K_2} s\right)} e^{i\left(a_2 + b_2 s\right)} + e^{2.302\left(a_{K_3}\right)} e^{i\left(a_3 + b_3 s\right)} \qquad (1C)$$

where the new constants are related to those in Eq. (46) by

$$a_{K_1} = \log_{10} |K_1| \quad b_{K_1} = \frac{\lambda_1}{2.302\, V} \quad b_1 = \frac{\omega_1}{V} \qquad (2C)(5C)(7C)$$

$$a_{K_2} = \log_{10} |K_2| \quad b_{K_2} = \frac{\lambda_2}{2.302\, V} \quad b_2 = \frac{\omega_2}{V} \qquad (3C)(6C)(8C)$$

$$a_{K_3} = \log_{10} |K_3| \qquad b_3 = \frac{p}{V} \qquad (4C)(9C)$$

* In the standard system the yaw is measured from the missile to the instantaneous velocity vector of the center of gravity and is taken as positive in the third quadrant looking at the rear of the missile (See Fig. 3).

Substituting the Range data for $\tilde{\beta}$, $\tilde{\alpha}$ and z at each station into Eq. (1C) yields a set of 50 equations which are non-linear in the constants to be determined thus the Method of Least Squares[31] cannot be applied.

Differential Corrections[31]

Eq. (1C) is therefore expanded in a Taylor's Series in which the higher order terms are neglected as

$$
\begin{aligned}
\Delta \xi = \xi_M - \xi_o = {} & e^{2.302\left(a_{K_{1_o}} + b_{K_{1_o}} z\right)} e^{i\left(a_{1_o} + b_{1_o} z\right)} \Big[2.302 \Delta a_{K_1} + 2.302 \Delta b_{K_1} \\
& \qquad + i \Delta a_1 + i\, z \Delta b_1 \Big] \\
& + e^{2.302\left(a_{K_{2_o}} + b_{K_{2_o}} z\right)} e^{i\left(a_{2_o} + b_{2_o} z\right)} \Big[2.302 \Delta a_{K_2} + 2.302 \Delta b_{K_2} \\
& \qquad + i \Delta a_2 + i\, z \Delta b_2 \Big] \\
& + e^{2.302\left(a_{K_{3_o}}\right)} e^{i\left(a_{3_o} + b_{3_o} z\right)} \Big[2.302 \quad a_{K_3} + i \quad a_3 + i\, z \quad b_3 \Big] \qquad (10C)
\end{aligned}
$$

where ξ_M = measured value of angle attack and yaw in Range

ξ_o = value computed from Eq. (1C) using initial values of constants

Since these equations are linear in the differential corrections the Method of Least Squares may be applied for their determination, provided that estimates of initial values of the constants may be made. Once values of the differential corrections are determined, they may be added to the original initial values of the constants and this process of Differential it is noted that initial estimates of these constants may be readily determined for

(1) b_3, the rolling velocity of the missile as a function of z may be measured directly from the observed motion in the Range[19,20]

(2) a_3, the angular position of K_3 at z may be obtained from a knowledge of the roll orientation of the missile at z (see above (1) and a knowledge of the orientation of K_3 with respect to the asymmetry* which is fixed in the missile (Eq.75)

(3) $e^{2.302\,[a_{K_3}]}$, the size of K_3 at z_o may be estimated from Eq. (43) provided that the asymmetry is known from the physical measurements of the missile and that estimated values of the aerodynamic coefficients are available. In many cases, inspection of the experimental pitching and yawing motion yields a good indication of the size of K_3.

Thus initial values for a_{K_3}, a_{3_o} and b_{3_o} may be obtained. Subtracting the third term from the experimental data and applying the standard technique[32,30] yields estimates of the remaining initial values of the constants.

Transverse Displacement

The reduction of the transverse displacement of the missile follows directly from the reduction of the pitching and yawing motion. Writing Eq. (77) as

$$S = k_1 e^{(\phi_1/V)Z} + k_2 e^{(\phi_2/V)Z} + k_3 e^{(\phi_3/V)Z} + (k_4/V)Z + k_5 \qquad (130)$$

and substituting the Range data for S and Z and the pitch and yaw contents previously determined yields a set of 25 equations which are linear in the unknowns, k_1, k_2, k_3, k_4 and k_5; and thus the Method of Least Squares may be applied for their determinations.

---

*The angular orientation of the asymmetry with respect to the missile may be determined by measurement prior to firing.

Corrections successively repeated until the sum of the squares of the residuals is a minimum. The values of the final constants so obtained may be considered as the best the data can yield employing the theory of motion. If the probable Error from the residuals,

$$P\,E = .6745\sqrt{\frac{\Sigma\Delta\xi^2}{n - 10}} \qquad (11C)$$

is of the same order of magnitude as the estimated error in measurement of the experimental data, then the theory may be considered to represent accurately the data.

Initial Values

The determination of the original initial values of the constants is critical for employing the Method of Differential Corrections; for if their determination is poor then the process may not be convergent.*

The technique used in obtaining these initial values is to remove the effect of the third term from the experimental data; then the resultant data will be epicyclic and the standard technique for determining the initial values may be used.[30,32]

The effect of the third arm on the experimental data may be subtracted once the constants of this term are estimated.

Considering this term

$$e^{2.302(a_{K_3})}\, e^{i(a_{3_o} + b_{3_o} z)}\left[2.302\,\Delta a_{K_3} + \Delta a_3 + i\,z\,\Delta b_3\right] \qquad (12C)$$

where $e^{2.302(a_{K_3})}$ = size of arm $|K_3|$

a_3 = angular position of arm at z_o

b_3 = rolling velocity of the missile as a function of z

---

* Of course if the theory is not correct, the Process may also be divergent.

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