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ON THE FREE QUADRATIC EXTENSIONS OF A COMMUTATIVE RING

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Let R be a commutative ring with unit element 1. A quadratic extension of R is an R-algebra which is a finitely generated projective R-module of rank 2. Let $Q(R)$ be the set of all R-algebra isomorphism classes of quadratic extensions of R, and $Q_s(R)$ the set of all R-algebra isomorphism classes of separable quadratic extensions of R. In [2], it was shown that the product in $Q_s(R)$, in the sense of [1], [4] and [5], is extended to $Q(R)$, and $Q(R)$ is an abelian semigroup with unit element. In this note, we study the quadratic extensions of R which are free R-modules. We shall call them the *free quadratic extensions* of R. Let $Q_f(R)$ and $Q_{fs}(R)$ be the sets of all classes which are free R-modules in $Q(R)$ and $Q_s(R)$, respectively. We shall show that $Q_f(R)$ is an abelian semigroup with unit element, and $Q_{fs}(R)$ is an abelian group consisting of all invertible elements in $Q_f(R)$. For some special rings, we shall determine the structures of $Q_f(R)$ and $Q_{fs}(R)$. We remark that $Q_{fs}(R)$, $Q_s(R)$ and $Pic(R)_2$; the group of isomorphism classes $[U]$ of R-module U such that $U\otimes_R U\simeq R$, are closely related by the exact sequence $0 \to Q_{fs}(R) \to Q_s(R) \to Pic(R)_2$.

Let R be any commutative ring with unit element 1. For a free quadratic extension S of R, we can write $S=R\oplus Rx$ and $x^2=ax+b$ for some a, b in R, then we denote it by $S=(R, a, b)$, and by $[R, a, b]$ the R-algebra isomorphism class containing (R, a, b).

Lemma 1. *The following two conditions a) and b) are equivalent;*
a) $(R, a, b)\simeq(R, c, d)$ *as R-algebras,*
b) there exist an invertible element α *in R and an element* β *in R such that* $c=\alpha(a-2\beta)$ *and* $d=\alpha^2(\beta a+b-\beta^2)$.
If (R, a, b) *and* (R, c, d) *satisfy a) or b), then we have*
c) $c^2+4d=\alpha^2(a^2+4b)$ *for some invertible element* α *in R.*
Moreover, if 2 is invertible in R, then we have the converse.

Proof. *a)* $\to$ *b)*: Let σ: $(R, a, b)=R\oplus Rx \to (R, c, d)=R\oplus Ry$ be an R-algebra isomorphism, and set $\sigma(x)=\alpha y+\beta$ and $\sigma^{-1}(y)=\alpha' x+\beta'$. Sinec $y=\sigma\cdot\sigma^{-1}(y)=\alpha'\alpha y+\alpha'\beta+\beta'$, we have $\alpha'\alpha=1$, that is, α and α' are invertible. The equalities $(\sigma(x))^2=(\alpha y+\beta)^2=\alpha(\alpha c+2\beta)y+\alpha^2 d+\beta^2$ and $\sigma(x^2)=\sigma(ax+b)=\alpha ay$

$+b+\beta a$ imply that $\alpha c+2\beta=a$ and $\alpha^2 d+\beta^2=b+\beta a$. Then we have $c=\alpha'(a-2\beta)$ and $d=\alpha'^2(\beta a+b-\beta^2)$.

$b) \to a)$: Define a mapping σ: $(R, a, b)=R\oplus Rx \to (R, c, d)=R\oplus Ry$ by $\sigma(x)=\alpha^{-1}y+\beta$, then σ is an R-algebra isomophism.

$b) \to c)$ is obvious. If 2 is invertible, setting $\beta=\frac{1}{2}(a-\alpha^{-1}c)$, we see that $c)$ implies $b)$.

The following lemma is well known.

Lemma 2. *(R, a, b) is R-separable if and only if a^2+4b is invertible in R.*

We shall define a product in $Q_f(R)$ by $[R, a, b]\cdot[R, c, d]=[R, ac, a^2d+bc^2+4bd]$. From the following Lemma 3, it is easily seen that $Q_f(R)$ is an abelian semigroup with unit element $[R, 1, 0]$.

Lemma 3. (Lemma 3 in [2]). *If $(R, a, b)\cong(R, a', b')$ and $(R, c, d)\cong(R, c', d')$ are isomorphisms as R-algebras, then so is $(R, ac, a^2d+bc^2+4bd)\cong(R, a'c', a'^2d'+b'c'^2+4b'd')$.*

A separable quadratic extension S of R has a unique automorphism $\sigma=\sigma(S)$ of S such that $S^\sigma=\{x\in S;\ \sigma(x)=x\}=R$. In [1], [4] and [5], the product $S_1\bigstar S_2$ of separable quadratic extension S_1 and S_2 of R was defined as the fixed subalgebra $(S_1\otimes_R S_2)^{\sigma_1\otimes\sigma_2}$, where $\sigma_i=\sigma(S_i)$.

Lemma 4 (Proposition 4 in [2]). *Let (R, a, b) and (R, c, d) be separable quadratic extensions of R. Then we have $[R, a, b]\cdot[R, c, d]=[(R,a, b)\bigstar(R, c, d)]$.*

Theorem 1. *An element $[R, a, b]$ of $Q_f(R)$ is invertible if and only if $[R, a, b]$ is contained in $Q_{fs}(R)$. Therefore, $Q_{fs}(R)$ is the set of all invertible elements in $Q_f(R)$. It is an abelian group of exponent 2.*

Proof. Let $[R, a, b]$ be any element of $Q_{fs}(R)$. By Lemma 2, a^2+4b is invertible in R. Set $\alpha=(a^2+4b)^{-1}$ and $\beta=-2b$, then we have $\alpha(a^2-2\beta)=1$ and $\alpha^2(\beta a^2+(2a^2b+4b^2)-\beta^2)=0$, hence we have $(R, a^2, 2a^2b+4b^2)\cong(R, 1, 0)$ by Lemma 1. Since $[R, a, b]^2=[R, a^2, 2a^2b+4b^2]$, we have $[R, a, b]^2=[R, 1, 0]$, so $[R, a, b]$ is invertible in $Q_f(R)$. Conversely, we assume $[R, a, b]\cdot[R, c, d]=[R, 1, 0]$, then we have $1=\alpha^2\{(ac)^2+4(a^2d+bc^2+4bd)\}=\alpha^2(a^2+4b)(c^2+4d)$ for some invertible element α in R. Thus, a^2+4b is invertible in R, therefore, $[R,a,b]$ is contained in $Q_{fs}(R)$.

Theorem 2. *Let $\{R_\lambda;\ \lambda\in\Lambda\}$ be a family of commutative rings with unit elements, and $R=\prod_{\lambda\in\Lambda}R_\lambda$ a direct product of $\{R_\lambda;\ \lambda\in\Lambda\}$. Then we have isomorphisms $Q_f(R)\cong\prod_{\lambda\in\Lambda}Q_f(R_\lambda)$ and $Q_{fs}(R)\cong\prod_{\lambda\in\Lambda}Q_{fs}(R_\lambda)$ by correspondence $[R, \prod_{\lambda\in\Lambda}a_\lambda, \prod_{\lambda\in\Lambda}b_\lambda] \overset{f}{\mapsto}\prod_{\lambda\in\Lambda}[R_\lambda, a_\lambda, b_\lambda]$.*

Proof. Let $(R, \prod_{\lambda\in\Lambda}a_\lambda, \prod_{\lambda\in\Lambda}b_\lambda)\cong(R, \prod_{\lambda\in\Lambda}c_\lambda, \prod_{\lambda\in\Lambda}d_\lambda)$. Then, there exist $\alpha=\prod_{\lambda\in\Lambda}\alpha_\lambda$

and $\beta=\prod_{\lambda\in\Lambda}\beta_\lambda$ such that α is invertible in R, $\prod c_\lambda=\alpha(\prod a_\lambda-2\beta)$ and $\prod d_\lambda=\alpha^2(\beta\prod a_\lambda+\prod b_\lambda-\beta^2)$. It is equivalent to existence of α_λ and β_λ in R_λ such that α_λ is invertible, $c_\lambda=\alpha_\lambda(a_\lambda-2\beta_\lambda)$ and $d_\lambda=\alpha_\lambda{}^2(\beta_\lambda a_\lambda+b_\lambda-\beta_\lambda{}^2)$ for all $\lambda\in\Lambda$, namely, $\prod_{\lambda\in\Lambda}(R_\lambda, a_\lambda, b_\lambda)\cong\prod_{\lambda\in\Lambda}(R_\lambda, c_\lambda, d_\lambda)$. Thus f is injective. It is clear that f is an epimorphism. Therefore, we have an isomorphism $Q_f(R)\cong\prod_{\lambda\in\Lambda}Q_f(R_\lambda)$ as semigroups, so we have the isomorphism $Q_{fs}(R)\cong\prod_{\lambda\in\Lambda}Q_{fs}(R_\lambda)$ as groups by Theorem 1.

Let $U(R)$ be the unit group of a ring R, and $U^2(R)$ the set $\{u^2;\ u\in U(R)\}$. We define a relation $\sim$ in R as follows; for a and b in R, $a\sim b$ if there exist c and d in $U^2(R)$ such that $ac=bd$. Then the relation $\sim$ is an equivalence relation and we denote by $R/U^2(R)$ the quotient $R/\sim$. The multiplication in R induces a multiplication in $R/U^2(R)$, and $R/U^2(R)$ is an abelian semigroup with unit element [1], where $[a]$ denotes the class of a in R/U^2(R). It is clear that the set of all invertible elements in $R/U^2(R)$ is $U(R)/U^2(R)$. We define a mapping $D: Q_f(R)\to R/U^2(R)$ by $D([R, a, b])=[a^2+4b]$, and this is a homomorphism, which carries $[R, 1, 0]$ and $[R, 0, 0]$ to [1] and [0], respectively. Indeed, by Lemma 1, D is well defined, and $D([R,a,b]\cdot[R,c,d])=[(ac)^2+4(a^2d+bc^2+4bd)]$ $=[a^2+4b][c^2+4d]$.

Theorem 3. *If 2 is invertible in R, then D is an isomorphism and this induces an isomorphism $Q_{fs}(R)\cong U(R)/U^2(R)$ as groups.* (cf. Proposition 3.3 in [1])

Proof. By Lemma 1, $[R, a, b]=[R, c, d]$ in $Q_f(R)$ if and only if $[a^2+4b]$ $=[c^2+4d]$ in $R/U^2(R)$. Thus D is a monomorphism. For any element a in R, $D\left(\left[R, 0, \frac{a}{4}\right]\right)=[a]$, therefore D is surjective. Thus D is an isomorphism. Furthermore, by Theorem 1, D induces an isomorphism $Q_{fs}(R)\cong U(R)/U^2(R)$ as groups.

In the case where 2 is not invertible in R, we give a sufficient condition such that D ia a monomorphism;

Theorem 4. *If R is a unique factorization domain of charactaristic $\neq 2$, or a ring such that $2R$ is a prime ideal and 2 is a non-zero-divisor, then D is a monomorphism.*

Proof. In the first place, we remark that if $a=a'+2r$ then $(R, a, b)\cong(R, a', ra+b-r^2)$ and $a^2+4b=a'^2+4(ra+b-r^2)$. Let $D([R, a, b])=D([R, c, d])$, that is, $a^2+4b=\alpha^2(c^2+4d)$ for some invertible element α in R. Since $(R, a, b)\cong(R, a/\alpha, b/\alpha^2)$, we may assume that $a^2+4b=c^2+4d$. If $a-c\in 2R$, we may put $a=c$, and so we have $b=d$. Thus, if $a-c\in 2R$, D is a monomorphism. Now, we remain only to show that $a^2+4b=c^2+4d$ implies $a-c\in 2R$. Let R be a unique factorization domain. If $b=d$, the implication is clear. Let $b\neq d$. Put

$2=p_1^{e_1}\cdot p_2^{e_2}\cdots p_n^{e_n}$ the prime factorization of 2. For each i, $(1\leqq i\leqq n)$, let f_i be an integer such that $a+c=p_i^{f_i}\cdot s_i$ and $p_i\not| s_i$. Then from $4|(a+c)(a-c)$, we have $p_i^{2e_i-f_i}|a-c$. If $f_i\leqq e_i$, we have $p_i^{e_i}|a-c$ because of $2e_i-f_i\geqq e_i$. On the other hand, if $f_i>e_i$, we have $p_i^{e_i}|a-c$ because of $a-c=p_i^{f_i}\cdot s_i-2c$. Thus we have $p_i^{e_i}|a-c$ for every i, $(1\leqq i\leqq n)$. Therefore, $a-c\in 2R$. Let R be a ring such that $2R$ is a prime ideal. Since $(a+c)(a-c)=4(d-b)$ is in $2R$, if $a-c\notin 2R$ then $a+c=2r$ for some r in R, and so $a-c=2(r-c)$. It is a contradiction. Thus, $a-c\in 2R$.

Corollary 1. *Let $\boldsymbol{Z}$ be the ring of rational integers. $Q(\boldsymbol{Z})$ is isomorphic to a multiplicative subsemigroup $\{n;\ n=4r$ or $n=4r+1,\ r\in\boldsymbol{Z}\}$ of $\boldsymbol{Z}$. Therefore, $Q_s(\boldsymbol{Z})$ is trivial.* (cf. Proposition 4 in [3]).

Corollary 2. *Let $R=\boldsymbol{Z}[i]$ be the ring of Gaussian integers. $Q(R)=Q_f(R)$ is isomorphic to the subsemigroup $\{[\alpha]\in R/\{1, -1\};\ \alpha=4b,\ 4b+1,\ 4b+2i$ for all $b\in R\}$ of $R/U^2(R)=\boldsymbol{Z}[i]/\{1, -1\}$. And $Q_s(R)$ is trivial.*

Proof. Sinec $R/2R=\{\bar{0}, \bar{1}, \bar{i}, \overline{1+i}\}$, we get $Q(R)=\{[R, 0, b], [R, 1, b], [R, i, b], [R, 1+i, b];\ b\in R\}$. Therefore, we have $Q(R)\simeq Im\ D=\{[\alpha]\in R/\{1, -1\};\ \alpha=4b,\ 4b+1,\ 4b+2i$ for all b in $R\}$, hence $Q_s(R)$ is trivial..

REMARK 1. In Theorem 4, we can not omit the condition that 2 is a non-zero-divisor. For example, let $R=\boldsymbol{Z}/(4)$, then we have $Q(R)=\{[R, \bar{0}, \bar{0}], [R, \bar{0}, \bar{1}], [R, \bar{0}, \bar{2}], [R, \bar{0}, \bar{3}], [R, \bar{1}, \bar{0}], [R, \bar{1}, \bar{1}]\}$, $Q_s(R)=\{[R, \bar{1}, \bar{0}], [R, \bar{1}, \bar{1}]\}$, $D(Q(R))=\{\bar{0}, \bar{1}\}\subset\boldsymbol{Z}/(4)$ and $D(Q_s(R))=\{\bar{1}\}\subset\boldsymbol{Z}/(4)$. Then D is neither monomorphic nor epimorphic.

REMARK 2. In the case where R is not a unique factorization domain, we can not omit the condition in Theorem 4 that $2R$ is a prime ideal. For example, let $R=\boldsymbol{Z}[\sqrt{5}]$. Then we have $[R, \sqrt{5}, -1]\neq[R, 1, 0]$ but $D([R, \sqrt{5}, -1])=D([D, 1, 0])=[1]$. D is not a monomorphism.

Theorem 5. *Let $K=GF(p^n)$ be finite field, then $Q(K)$ is isomorphic to the multiplicative semigroup $\boldsymbol{Z}/(3)$. Further, the isomorphism induces an isomorphism $Q_s(K)\simeq\{\bar{1}, -\bar{1}\}=U(\boldsymbol{Z}/(3))$.*

Proof. The case $p\neq 2$. In the first place, we note that $(R, a, b)\simeq(R, 0, a^2+4b)$ and $U(K)=K^*=K-\{0\}$. From Theorem 3 and $(K^*: K^{*2})=2$, we have $Q(K)=\{[K, 0, 0], [K, 0, 1], [K, 0, \alpha]\}$, where α is an element K^* which is not contained in K^{*2}. By the correspondence $[K, 0, 0]\mapsto\bar{0}$, $[K, 0, 1]\mapsto\bar{1}$ and $[K, 0, \alpha]\mapsto -\bar{1}$, we have an isomorphism $Q(K)\simeq\boldsymbol{Z}/(3)$ as multiplicative semigroups, and it induces $Q_s(K)\simeq\{\bar{1}, -\bar{1}\}=U(\boldsymbol{Z}/(3))$ as groups.

The case $p=2$. Since $a^2+a=a(a+1)$ for a in K, we have $\#\{a^2+a;\ a\in K\}=2^{n-1}<\#(K)$, where $\#$(K) denotes the number of elements in K. Then, there exists α in K such that $\alpha\notin\{a^2+a;\ a\in K\}$, and the quadratic equation $x^2+x+\alpha=0$ has no roots in K. Then, we can see the equalities $\#\{a^2+a;\ a\in K\}=\#\{a^2+a+\alpha;\ a\in K\}=2^{n-1}$ and $\{a^2+a;\ a\in K\}\cap\{a^2+a+\alpha;\ a\in K\}=\phi$. For, if $c=a^2+a$ and $c=b^2+b+\alpha$ for some a, b in K, then $(a+b)^2+(a+b)+\alpha=0$. It is a contradiction. Therefore, we have $K=\{a^2+a;\ a\in K\}\cup\{a^2+a+\alpha;\ a\in K\}$, (disjoint sum), namely, any element a in K verifies either $\beta^2+\beta+a=0$ or $\beta^2+\beta+a+\alpha=0$ for some β in K. On the other hand, by Lemma 1, $(K, 1, 0)\simeq(K, 1, a)$ if and only if there exists β in K such that $\beta^2+\beta+a=0$. And $(K, 1, \alpha)\simeq(K, 1, a)$ if and only if there exists β in K such that $\beta^2+\beta+a+\alpha=0$. Accordingly, we have $Q_s(K)=\{[K, 1, 0], [K, 1, \alpha]\}$. Furthermore, since $U^2(K)=U(K)$, $(K, 0, 0)\simeq(K, 0, a)$ for all a in K, hence $Q(K)=\{[K, 0, 0], [K, 1, 0], [K, 1, \alpha]\}$. By the correspondence $[K, 0, 0]\mapsto\bar{0}$, $[K, 1, 0]\mapsto\bar{1}$ and $[K, 1, \alpha]\mapsto-\bar{1}$ we have the isomorphism $Q(K)\simeq \boldsymbol{Z}/(3)$, and it induces $Q_s(K)\simeq\{\bar{1}, -\bar{1}\}=U(\boldsymbol{Z}/(3))$.

REMARK 3. Let $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ be the fields of rational numbers, real numbers and complex numbers, respectively. By the same argument as the proof of Theorem 5 (in case $p\neq 2$), we can see that $Q(\boldsymbol{R})=\{[\boldsymbol{R}, 0, 0], [\boldsymbol{R}, 0, 1], [\boldsymbol{R}, 0, -1]\}$, $Q(\boldsymbol{C})=\{[\boldsymbol{C}, 0, 0], [\boldsymbol{C}. 1, 0]\}$. Further, $Q_s(\boldsymbol{Q})$ is an infinite ableian group of exponent 2, $Q_s(\boldsymbol{R})$ is a group of order 2 and $Q_s(\boldsymbol{C})$ is trivial.

REMARK 4. In the case $R=\mathrm{GF}(2^n)$, the homomorphism D is not a monomorphism but an epimorphism.

Theorem 6. *Let $R=\boldsymbol{Z}/(n)$, and let $n=p_1^{e_1}\cdot p_2^{e_2}\cdots p_r^{e_r}$ be the prime factorization of n. Then $Q_{fs}(R)$ is the abelian group of type $(2, 2,\cdots, 2)$, r-times.*

Proof. It is enough to prove that $Q_s(\boldsymbol{Z}/(p^e))$ is the group of order 2 for any prime integer p. In the case $p\neq 2$, by Theorem 3, $Q_s(\boldsymbol{Z}/(p^e))$ is isomorphic to the group $U(\boldsymbol{Z}/(p^e))/U^2(\boldsymbol{Z}/(p^e))$. The index $(U(\boldsymbol{Z}/(p^e)):\ U^2(\boldsymbol{Z}/(p^e)))$ is 2, since $U(\boldsymbol{Z}/(p^e))$ is a cyclic group of order $\varphi(p^e)=(p-1)p^{e-1}$. Thus, $Q_s(\boldsymbol{Z}/(p^e))$ is the group of order 2. In the case $p=2$, put $\boldsymbol{Z}/(2^e)=R$. We shall remark that $\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}=2R$. In fact, let f: $2R\rightarrow\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}$ be a mapping defined by $f(\bar{a})=\bar{a}^2-\bar{a}$. If $f(\bar{a})=f(\bar{b})$, we have $(a-b)(a+b-1)\equiv 0\ mod\ 2^e$. Since $2\not| a+b-1$, we have $2^e|a-b$, hence $\bar{a}=\bar{b}$, Furthermore, $\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}$ and $2R$ are finite sets and $\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}\subseteqq 2\mathrm{R}$. Hence, $\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}=2R$. Now, we shall show that $(R, \bar{1}, \overline{a+2})\simeq(R, \bar{1}, \bar{a})$ for all integer a. $(R, \bar{1}, \overline{a+2})\simeq(R, \bar{1}, \bar{a})$ if and only if there exist an odd integer α and an integer β such that $1\equiv\alpha(1-2\beta)$ and $a\equiv\alpha^2(\beta+a+2-\beta^2)\ mod\ 2^e$, namely, there exists an integer β such that $(4a+1)\beta^2-(4a+1)\beta-2\equiv 0\ mod\ 2^e$. Since $\{\bar{a}^2-\bar{a};\ \bar{a}\in R\}=2R$, we can take an integer β such that $\bar{\beta}^2-\bar{\beta}=2(\overline{4a+1})^{-1}$, and we have $(4a+1)\beta^2-(4a+1)\beta-2\equiv 0\ mod\ 2^e$. Hence, we have $(R, \bar{1}, \overline{a+2})\simeq(R, \bar{1}, \bar{a})$ for all integer a. Accordingly we have $(R, \bar{1}, \overline{2a})\simeq(R, \bar{1}, \bar{0})$ and $(R, \bar{1}, \overline{2a+1})\simeq(R, \bar{1}, \bar{1})$ for all integer a, But $[R, \bar{1}, \bar{0}] \neq [R, \bar{1}, \bar{1}]$. Therefore, $Q_s(R)$ is the group of order 2.

REMARK 5. Let $R = \boldsymbol{Z}/(2^e)$. Then we have following;
i) if $e=1$, $Q(R) = \{[R, \bar{0}, \bar{0}], [R, \bar{1}, \bar{0}], [R, \bar{1}, \bar{1}]\}$.
ii) if $e \geqq 2$, $Q(R) = \{[R, \bar{0}, \bar{a}_i]; i=1, 2, \cdots, r\} \cup \{[R, \bar{1}, \bar{0}], [R, \bar{1}, \bar{1}]\}$, (disjoint sum), where $\{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_r\}$ is the representatives of $R/U^2(R)$.

Proof. i) is a special case of Theorem 5.

ii) $(R, \bar{0}, \bar{a}) \simeq (R, \bar{0}, \bar{b})$ if and only if there exist an odd integer α and an integer β such that $2\beta \equiv 0$ and $b \equiv \alpha^2(a - \beta^2)$ *mod* 2^e. Put $\beta \equiv 2^{e-1}n$ mod 2^e and $2 \not| n$, then we have $\beta^2 \equiv 0$ *mod* 2^e. Therefore, $(R, \bar{0}, \bar{a}) \simeq (R, 0, \bar{b})$ if and only if $\bar{b} = \bar{\alpha}^2 \bar{a}$ for some $\bar{\alpha}$ in $U(R)$, namely, $[\bar{a}] = [\bar{b}]$ in $R/U^2(R)$.

REMARK 6. There is a commutative ring R with the homomorphism D: $Q_f(R) \to R/U^2(R)$ which is not a monomoprhism but the restriction $D \mid Q_{fs}(R)$ is a monomorphism. For example, if $R = \boldsymbol{Z}/(2^e)$, $(e \geqq 3)$, then we have $D([R, \bar{1}, \bar{0}]) = [\bar{1}]$, $D([R, \bar{1}, \bar{1}]) = [\bar{5}]$ and $[\bar{1}] \neq [\bar{5}]$ in $U(R)/U^2(R)$. Thus, the restriction $D \mid Q_{fs}(R)$ is a monomorphism. But, we have $[R, \bar{0}, \bar{0}] \neq [R, \bar{0}, \overline{2^{e-2}}]$ and $D([R, \bar{0}, \bar{0}]) = D([R, \bar{0}, \overline{2^{e-2}}]) = [\bar{0}]$, Then D is not a monomorphism.

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