ON THE FREQUENCY FUNCTION OF xy

By Cecil C. Craig

Given the distribution function of x and y, what can be said of the distribution of the product xy? The author has had two inquiries during the last two years, one from an investigator in business statistics and the other from a psychologist, concerning the probable error of the product of two quantities, each of known probable error. There seems to be very little in the literature of mathematical statistics on this question.

If x and y are independent and are each distributed according to the same normal frequency law, it is well known that the distribution function of

$$\bar{z} = \frac{x - m_x}{\sigma_x} \cdot \frac{y - m_y}{\sigma_y}$$

is

$$\frac{1}{\pi} K_0(\bar{z}) \text{ ,[1]}$$

in which $K_0(\bar{z})$ is the Bessel function of the second kind of a purely imaginary argument of zero order.[2] If x and y are independent and are each distributed according to a logarithmic normal frequency law, it has been pointed out that the product, $(x - a)$ $(y - b)$, in which a and b are the upper (or lower) limits of the range for x and y respectively, is distributed according to a law of the same type.[3] In both cases the special choice of origins greatly simplifies the problem.

In the present discussion it will be assumed that x and y are distributed normally. It will appear that the distribution of xy is a function of r_{xy}, the coefficient of correlation between x and y, and of the parameters,

$$\rho_1 = \frac{m_1}{\sigma_1} = \frac{m_x}{\sigma_x} \quad \text{and} \quad \rho_2 = \frac{m_2}{\sigma_2} = \frac{m_y}{\sigma_y} ,$$

which are proportional to the reciprocals of the coefficients of variation. The chief difficulty arises when ρ_1 and ρ_2 are small so that zero values of xy occur

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[1] J. Wishart and M. S. Bartlett: The Distribution of Second Order Moment Statistics in a Normal System; Proceedings of the Cambridge Philosophical Society, Vol. XXVIII (1932), pp. 455–459.

[2] G. N. Watson: A Treatise on the Theory of Bessel Functions; Cambridge University Press (1922), p. 78.

[3] P. T. Yuan: On the Logarithmic Frequency Distribution and the Semi-logarithmic Frequency Surface; Annals of Mathematical Statistics, Vol. 4 (1933), pp. 46, 47.

for values of x and y well within their respective ranges of variation. (If ρ_1 and ρ_2 are large, practically one may exclude zero values of x and y from consideration. The author hopes to present an investigation of this case soon.) It is the object of the present paper to study the rather unusual frequency function that arises in this situation. It will first be assumed that x and y are independent ($r_{xy} = r = 0$). Then it will be shown that the distribution function when $r \neq 0$ is readily derived from that arrived at in the special case.

We can find the moment generating function of xy without difficulty. We have,

$$M_{xy}(\vartheta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-m_1)^2}{2\sigma_1^2}-\frac{(y-m_2)^2}{2\sigma_2^2}} e^{xy\vartheta}\,dx\,dy$$

$$= \frac{e^{[(\sigma_1^2 m_2^2+\sigma_2^2 m_1^2)\vartheta^2+2m_1m_2\vartheta]/2(1-\sigma_1^2\sigma_2^2\vartheta^2)}}{(1-\sigma_1^2\sigma_2^2\vartheta^2)^{1/2}}.$$

Setting, for convenience,

$$z = \frac{xy}{\sigma_1\sigma_2},$$

this can be written,

$$M_z(\vartheta) = \frac{e^{[(\rho_1^2+\rho_2^2)\vartheta^2+2\rho_1\rho_2\vartheta]/2(1-\vartheta^2)}}{(1-\vartheta^2)^{1/2}}. \tag{1}$$

This choice of variable and of parameters will be adhered to in the sequel.

On expanding $\log M_z(\vartheta)$ in powers of ϑ, we get for the semi-invariants (of Thiele),

$$\lambda_{2k+1:z} = (2k+1)!\,\rho_1\rho_2, \qquad k = 0, 1, 2, \cdots$$

$$\lambda_{2k:z} = \frac{(2k)!}{2}(\rho_1^2+\rho_2^2) + (2k-1)!, \qquad k = 1, 2, \cdots. \tag{2}$$

These give for the mean and variance of xy,

$$M_{xy} = m_1m_2$$

$$\sigma_{xy}^2 = \sigma_1^2m_2^2 + \sigma_2^2m_1^2 + \sigma_1^2\sigma_2^2.$$

For the standard semi-invariants of z (or of xy), we have,

$$\xi_{2k+1:z} = \frac{\lambda_{2k+1:z}}{\lambda_{2:z}^{\frac{2k+1}{2}}} = \frac{(2k+1)!\,\rho_1\rho_2}{(\rho_1^2+\rho_2^2+1)^{\frac{2k+1}{2}}},$$

$$\xi_{2k:z} = \frac{\lambda_{2k:z}}{\lambda_{2:z}^{k}} = \frac{(2k-1)!\,[k(\rho_1^2+\rho_2^2)+1]}{(\rho_1^2+\rho_2^2+1)^k}.$$

Taking,

$$\xi_3 = \frac{6\,\rho_1\rho_2}{(\rho_1^2 + \rho_2^2 + 1)^{3/2}},$$

as a measure of skewness, it is easy to verify that

$$|\,\xi_3\,| \leqq \frac{2}{3}\sqrt{3}\,.$$

For either $\rho_1 = 0$ or $\rho_2 = 0$, the distribution is symmetrical about its mean which then falls at the origin.

For the excess or kurtosis, we have,

$$\xi_4 = \frac{6\,[2(\rho_1^2 + \rho_2^2) + 1]}{(\rho_1^2 + \rho_2^2 + 1)^2} \leqq 6\,.$$

Thus the skewness is never great and becomes small with increasing ρ_1 or ρ_2. The excess also becomes small with increasing ρ_1 or ρ_2, but it can be very large for small values of these parameters, attaining its maximum of 6 for $\rho_1 = \rho_2 = 0$. But, as it will appear below, the distribution function always becomes infinite in a logarithmic manner at the origin. (We have already seen, as must obviously be the case, that moments of all orders exist.) It is to be noted, too, that for any given ρ_1 and ρ_2, ξ_{2k} increases without limit with increasing k, and that the same is true of ξ_{2k+1} if neither ρ_1 nor ρ_2 is zero.

Turning now to the derivation of the actual frequency function of z, we set $w = xy$; then for any given x, $y = w/x$, $dy = \dfrac{dw}{x}$ if $x > 0$, and $dy = -\dfrac{dw}{x}$ if $x < 0$. These values are substituted into $\varphi_1(x)\,\varphi_2(y)dx\,dy$, in which $\varphi_1(x)$ and $\varphi_2(y)$ are the frequency functions of x and y respectively, and the resulting expression is integrated over all values of x, giving for the frequency function of w:

$$F(w) = \frac{e^{-\left(\frac{m_1^2}{2\sigma_1^2} + \frac{m_2^2}{2\sigma_2^2}\right)}}{2\pi\sigma_1\sigma_2}\left[\int_0^\infty \Phi(w,\,x)\,\frac{dx}{x} - \int_{-\infty}^0 \Phi(w,\,x)\,\frac{dx}{x}\right]$$

in which,

$$\Phi(w,\,x) = e^{-(\sigma_2^2x^4 - 2m_1\sigma_2^2x^3 - 2m_2\sigma_1^2wx + \sigma_1^2w^2)/2\sigma_1^2\sigma_2^2x^2}\,.$$

Again setting $z = \dfrac{xy}{\sigma_1\sigma_2}$, and introducing the parameters ρ_1 and ρ_2, this reduces to,

$$(4) \qquad F(z) = \frac{e^{-\frac{(\rho_1^2+\rho_2^2)}{2}}}{2\pi}[\psi_1(z) - \psi_2(z)]\,,$$

in which,

$$\psi_1(z) = \int_0^\infty e^{-\left(\frac{x^2}{2} - \rho_1 x - \rho_2 \frac{z}{x} + \frac{z^2}{2x^2}\right)} \frac{dx}{x}, \tag{5}$$

and $\psi_2(z)$ is the integral of the same function over the interval $(-\infty, 0)$.

Now writing,

$$\psi_1(z) = \int_0^\infty e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} \frac{e^{\rho_1 x + \rho_2 \frac{z}{x}}}{x} \, dx, \tag{6}$$

we note that

$$\frac{e^{\rho_1 x + \rho_2 \frac{z}{x}}}{x}$$

can be expanded in a Laurent series in powers of x for all values of x except zero. In this expansion the coefficient of x^{r-1}, $r \geqq 1$, is $\frac{\rho_1^r}{r!} \sum_r(\rho_1\rho_2 z)$, in which

$$\sum_r(\rho_1\rho_2 z) = 1 + \frac{\rho_1\rho_2 z}{r+1} + \frac{(\rho_1\rho_2 z)^2}{(r+2)^{(2)}\, 2!} + \frac{(\rho_1\rho_2 z)^3}{(r+3)^{(3)}\, 3!} + \cdots, \tag{7}$$
$$((r+k)^{(k)} = (r+k)(r+k-1)\cdots(r+1)).$$

We may note parenthetically that

$$\frac{\rho_1^r}{r!} \sum_r(\rho_1\rho_2 z) = \left(\frac{\rho_1}{\rho_2 z}\right)^{\frac{r}{2}} I_r(2\sqrt{\rho_1\rho_2 z}),$$

in which $I_r(x)$ is the Bessel function of the first kind with a purely imaginary argument.[4]

The coefficient of x^{-r-1}, $r \geqq 0$, is $\frac{z^r \rho_2^r}{r!} \sum_r(\rho_1\rho_2 z)$.

Setting now,

$$\sum_{n=-\infty}^{\infty} f_n(x) = \frac{e^{\rho_1 x + \rho_2 \frac{z}{x}}}{x},$$

we substitute this series in (6) and seek to justify the expansion it gives for $\psi_1(z)$ obtained by term by term integration. We write,

$$\psi_1(z) = \int_0^1 e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} \sum f_n(x)\, dx + \int_1^\infty e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} \sum f_n(x)\, dx.$$

[4] Watson, loc. cit., p. 77.

For $z > 0$, $\rho_1\rho_2 > 0$, the terms of $\sum f_n(x)$ are all > 0. Then the convergence of

$$\sum \int_0^1 e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} f_n(x)\, dx$$

is sufficient to allow term by term integration in the first integral. In the second integral we observe that $\sum f_n(x)$ converges uniformly in every fixed interval $1 \leqq x \leqq a$. Then term by term integration is permissible here if

$$\sum \int_0^\infty e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} f_n(x)\, dx$$

is convergent.[5] It is evident, then, that it will be sufficient to establish the convergence of

$$\sum \int_0^\infty e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} f_n(x)\, dx\,.$$

If either or both $z < 0$ or $\rho_1\rho_2 < 0$, it will be easily seen that the series involved are still absolutely convergent which is sufficient.

Now using the definition of the Bessel function of a purely imaginary argument of the second kind,

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-\tau - \frac{z^2}{4\tau}} \frac{d\tau}{\tau^{\nu+1}}\,, \text{[6]}$$

it is easy to derive the relation,

$$K_{\frac{n-1}{2}}(z) = z^{\frac{n-1}{2}} \int_0^\infty e^{-\frac{x^2}{2} - \frac{z^2}{2x^2}} \frac{dx}{x^n}\,.$$

Remembering that $K_\nu(z) = K_{-\nu}(z)$, we have for our expansion,

$$\psi_1(z) = \sum\nolimits_0 K_0 + (\rho_1 + \rho_2)\, z^{1/2} \sum\nolimits_1 K_{\frac{1}{2}} + (\rho_1^2 + \rho_2^2)\frac{z}{2!}\sum\nolimits_2 K_1$$

$$+ (\rho_1^3 + \rho_2^3)\frac{z^{3/2}}{3!}\sum\nolimits_3 K_{\frac{3}{2}} + \cdots$$

in which the argument for all the $\sum$-functions is $\rho_1\rho_2 z$, and for all the K-functions is z.

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[5] T. J. I'a Bromwich: An Introduction to the Theory of Infinite Series; Macmillan & Co., London, 2nd edition (1926), pp. 496 and 500.

[6] Watson, loc. cit., pp. 78 and 183.

But we may as well add to this the expansion of $-\psi_2(z)$, which may be written,

$$\int_0^\infty e^{-\frac{x^2}{2}-\frac{z^2}{2x^2}} \frac{e^{-\rho_1 x-\rho_2 \frac{z}{x}}}{x}\, dx\,,$$

and obtain the expansion,

$$F(z) = \frac{e^{-\frac{\rho_1^2+\rho_2^2}{2}}}{\pi}\left[\sum_0 K_0 + (\rho_1^2+\rho_2^2)\frac{z}{2!}\sum_2 K_1 + (\rho_1^4+\rho_2^4)\frac{z^2}{4!}\sum_4 K_2 + \cdots\right],$$

the convergence of which we will examine. But it must be noted that the terms arising from the expansion of

$$\frac{e^{\rho_1 x+\rho_2 \frac{z}{x}}}{x} \quad \text{and} \quad \frac{e^{-\rho_1 x-\rho_2 \frac{z}{x}}}{x}$$

which contribute to the expansion of $F(z)$ as just written are those of the forms,

$$\frac{\rho_1^{2i}}{(2i)!}\sum_{2i} \quad \text{and} \quad \frac{\rho_2^{2i} z^{2i}}{(2i)!}\sum_{2i}\,.$$

Hence the expansion as written is valid in any case only for $z > 0$. For $z \gtrless 0$, we may write however,

$$F(z) = \frac{e^{-\frac{\rho_1^2+\rho_2^2}{2}}}{\pi}\left[\sum_0 K_0 + (\rho_1^2+\rho_2^2)\frac{|z|}{2!}\sum_2 K_1 + (\rho_1^4+\rho_2^4)\frac{z^2}{4!}\sum_4 K_2 + (\rho_1^6+\rho_2^6)\frac{|z|^3}{6!}\sum_6 K_3 + \cdots\right], \tag{8}$$

in which the arguments for the $\sum$ and K-functions are the same as before.

Let us consider now the question of the convergence of (8), first in the case that $z > 0$. We set

$$c_\nu = \frac{z^\nu K_\nu}{\nu!}\bigg/\frac{z^{\nu-1}K_{\nu-1}}{(\nu-1)!}\,.$$

Then from the relation,

$$K_{\nu-1} - K_{\nu+1} = -\frac{2\nu}{z}K_\nu\,,\text{[7]} \tag{9}$$

we readily derive,

$$\frac{z^2}{(\nu+1)^{(2)}} = c_\nu\left(c_{\nu+1} - \frac{2\nu}{\nu+1}\right)$$

[7] Watson, loc. cit., p. 79.

For $z > 0$, the left hand member and c_ν are both > 0. Thus

$$c_{\nu+1} - \frac{2\nu}{\nu + 1} > 0 .$$

Then let

$$c_{\nu+1} = \frac{2\nu}{\nu + 1} + \delta_{\nu+1} , \qquad \delta_{\nu+1} > 0 ,$$

and we have,

$$\frac{z^2}{(\nu + 1)^{(2)}} > \left(2 - \frac{2}{\nu}\right) \delta_{\nu+1} = 2\,\delta_{\nu+1} - \frac{2\,\delta_{\nu+1}}{\nu} .$$

It is evident from this that for a given $z > 0$, a ν_0 exists such that $c_\nu < 3$ for $\nu \geqq \nu_0$.

Further since

$$\sum\nolimits_r \leqq e^{|\rho_1 \rho_2 z|}$$

the convergence sought follows for $z > 0$. Since K is an even function of z, it is easy to see that (8) is also convergent for $z < 0$. For $z = 0$, the first term possesses a logarithmic discontinuity at the origin.

To calculate ordinates of $F(z)$ there are fairly extensive tables available in Watson's treatise already referred to. These tables may be readily extended by means of the asymptotic formula for $K(z)$ for larger values of z, and by means of (9) for larger values of ν. One can rapidly build up tables of $\sum_r(x)$ by means of the easily obtained recursion formula,

$$\sum\nolimits_r(x) = \sum\nolimits_{r+1}(x) + \frac{x}{(r + 2)^{(2)}} \sum\nolimits_{r+2}(x) .$$

It is unfortunately true that the expansion found for $F(z)$ is very slowly convergent for large values of ρ_1 and ρ_2.

At the end of this paper are shown three charts of $F(z)$ with the tables of ordinates from which they were made by way of illustrating what such curves look like. (On the second for comparison the broken line is the normal curve of error.)

For $\rho_1 = \rho_2 = r = 0$, we have simply the known result,

$$F(z) = \frac{1}{\pi} K_0\,(z) .$$

For $\rho_1 = 1$, $\rho_2 = r = 0$, the curve is symmetrical about its mean (and the origin). Here every $\sum$-function is unity.

For the case in which $\rho_1 = \rho_2 = \frac{1}{2}$, $r = 0$, I first constructed tables of $\sum_i(x)$ for $x = \pm 0.025, \pm 0.05, \pm 0.1$, and by intervals of 0.1 to ± 3.0 for $i = 0, 1, \cdots, 20$. Values of $\sum_0(x)$ and $\sum_2(x)$ for $x = 3.2$ and 3.4 were also used. Not more than five terms of (8) were required to obtain values of $F(z)$ accurate to five places of decimals. This distribution curve is skew with $M_z = 0.25$ and $\xi_{3:z} = \frac{\sqrt{6}}{3}$.

The curves are plotted in standard units with unit total area ($\sigma_z = \sqrt{\rho_1^2 + \rho_2^2 + 1}$). The tables of ordinates are given both in units of $z = \frac{xy}{\sigma_1\sigma_2}$ and of $t = \frac{z - m_z}{\sigma_z}$.

Turning now to the case in which $r \neq 0$, after some computation, we have for the moment generating function,

$$M_z(\vartheta) = \frac{e^{\frac{(\rho_1^2+\rho_2^2-2r\rho_1\rho_2)\vartheta^2+2\rho_1\rho_2\vartheta}{2[1-(1+r)\vartheta][1+(1-r)\vartheta]}}}{\sqrt{[1-(1+r)\vartheta][1+(1-r)\vartheta]}}. \tag{10}$$

As a check on this result, if we set $r = 1$ and $\rho_1 = \rho_2 = \rho$ in it we get,

$$M_{\frac{x^2}{\sigma^2}}(\vartheta) = \frac{e^{\frac{\rho^2\vartheta}{1-2\vartheta}}}{\sqrt{1-2\vartheta}},$$

which may be readily verified to be the moment generating function of $\frac{x^2}{\sigma^2}$ if x is distributed normally with mean m and variance σ^2 $\left(\rho = \frac{m}{\sigma}\right)$.

To obtain the semi-invariants of z in this case, on expanding $\log M_z(\vartheta)$ in powers of ϑ, setting

$$a = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 r, \quad b = 2\rho_1\rho_2, \quad c = 1 + r, \quad \text{and} \quad d = 1 - r,$$

we have,

$$\begin{aligned} \log M_z(\vartheta) &= \frac{a\vartheta^2 + b\vartheta}{2}(1 - c\vartheta)^{-1}(1 + d\vartheta)^{-1} \\ &\quad - \frac{1}{2}[\log(1 - c\vartheta) + \log(1 + d\vartheta)] \\ &= \frac{a\vartheta^2 + b\vartheta}{4}[2 + (c^2 - d^2)\vartheta + (c^3 + d^3)\vartheta^2 + (c^4 - d^4)\vartheta^3 + \cdots] \\ &\quad + \frac{1}{2}\left[(c - d)\vartheta + (c^2 + d^2)\frac{\vartheta^2}{2} + (c^3 - d^3)\frac{\vartheta^3}{3} + \cdots\right], \end{aligned} \tag{11}$$

from which we derive,

$$\begin{aligned} \lambda_{n:z} &= \frac{n!}{4}[\{c^{n-1} - (-d)^{n-1}\}a + \{c^n - (-d)^n\}b] \\ &\quad + \frac{(n-1)!}{2}\{c^n + (-d)^n\}. \end{aligned} \tag{12}$$

In particular,

$$\lambda_{1:z} = \frac{b}{2} + \frac{c-d}{2} = \rho_1\rho_2 + r$$

$$\lambda_{2.z} = a + \frac{c^2 - d^2}{2} \cdot b + \frac{c^2 + d^2}{2} = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 r + (1 + r^2)$$

$$\begin{aligned}(13) \quad \lambda_{3:z} &= \frac{3}{2}\left[(c^2 - d^2)\, a + (c^3 + d^3)\, b\right] + c^3 - d^3 \\ &= 6\left[(\rho_1^2 + \rho_2^2)\, r + \rho_1\rho_2\,(1 + r^2)\right] + 2\, r\,(3 + r^2) \\ \lambda_{4:z} &= 6\left[(c^3 + d^3)\, a + (c^4 - d^4)\, b\right] + 3\,(c^4 + d^4) \\ &= 12\,(\rho_1^2 + \rho_2^2)\,(1 + 3\, r^2) + 24\,\rho_1\rho_2 r\,(3 + r^2) + 6\,(1 + 6r + r^4)\,.\end{aligned}$$

Noting that

$$\frac{\partial a}{\partial r} = -\,b\,, \qquad \frac{\partial b}{\partial r} = 0\,, \qquad \frac{\partial c}{\partial r} = 1\,, \qquad \frac{\partial d}{\partial r} = -\,1\,,$$

one can easily demonstrate what seems to be a rather striking property of these semi-invariants, viz.,

$$(14) \qquad \frac{\partial \lambda_{n:z}}{\partial r} = n(n-1)\,\lambda_{n-1:z}\,.$$

To gain a notion of the magnitude of the skewness and excess in this case, we form,

$$\xi_{3:z} = \frac{\lambda_{3:z}}{\lambda_{2:z}^{3/2}} \quad \text{and} \quad \xi_{4:z} = \frac{\lambda_{4:z}}{\lambda_{2:z}^{2}}\,.$$

In view of the above property,

$$\frac{\partial \xi_3}{\partial r} = \frac{6\,\lambda_2^2 - 3\,\lambda_3\lambda_1}{\lambda_2^{5/2}}\,.$$

The denominator of this fraction is always > 0. The numerator, after some reduction, can be written,

$$\begin{aligned}(15) \quad & 6\left[\rho_1^4 + \rho_2^4 - \rho_1^2\rho_2^2\,(1 - r^2) + (\rho_1^2 + \rho_2^2)\,(2 - r^2)\right. \\ & \qquad \left. + (\rho_1^2 + \rho_2^2 - 2)\,\rho_1\rho_2 r + 1 - r^2\right].\end{aligned}$$

The first two terms taken together, the third, and the last are all obviously > 0. The term,

$$(\rho_1^2 + \rho_2^2 - 2)\,\rho_1\rho_2 r$$

has its maximum value for $|r| = 1$. But for $r = 1$, (15) becomes,

$$\rho_1^4 + \rho_2^4 + \rho_1\rho_2(\rho_1^2 + \rho_2^2) + (\rho_1 - \rho_2)^2 ,$$

and for $r = -1$, it is,

$$\rho_1^4 + \rho_2^4 - \rho_1\rho_2(\rho_1^2 + \rho_2^2) + (\rho_1 + \rho_2)^2 ,$$

both of which expressions are easily seen to be > 0.

Thus (15) is always positive and the maximum value of $\xi_{3:z}$ is attained for $r = 1$, the minimum value for $r = -1$. These values are respectively,

$$\frac{6\,(\rho_1 + \rho_2)^2 + 8}{[(\rho_1 + \rho_2)^2 + 2]^{\frac{3}{2}}} \quad \text{and} \quad \frac{-\,6\,(\rho_1 - \rho_2)^2 - 8}{[(\rho_1 - \rho_2)^2 + 2]^{\frac{3}{2}}} ,$$

the absolute value of either being $\leqq 2\sqrt{2}$, which is attained in the first case for $\rho_1 = -\rho_2$ and in the second for $\rho_1 = \rho_2$. It is seen that for high correlation between x and y the skewness of xy can be quite large.

For the excess, we see that

$$\xi_{4:z} = \frac{\lambda_{4:z}}{\lambda_{2:z}^2}$$

attains a value of 12 when $\rho_1 = -\rho_2$, $r = 1$ or when $\rho_1 = \rho_2$, $r = -1$. Since this is such an extraordinary value it does not seem worth while to carry out the extended computation that seems to be required to verify one's surmise that this is the maximum of the absolute value.

Now, to derive the frequency function we proceed as before. We set $z = \dfrac{xy}{\sigma_1\sigma_2}$ and then

$$F(z) = I_1(z) - I_2(z) ,$$

in which,

$$I_1(z) = \frac{1}{2\pi\sqrt{1 - r^2}} \int_0^\infty e^{-\frac{1}{2(1-r^2)}\left[(x-\rho_1)^2 - 2r(x-\rho_1)\left(\frac{z}{x}-\rho_2\right) + \left(\frac{z}{x}-\rho_2\right)^2\right]} \frac{dx}{x} ,$$

and $I_2(z)$ is the integral of the same function over the interval $(-\infty, 0)$.

We can write $I_1(z)$:

$$\frac{e^{-\frac{\rho_1^2 - 2r\rho_1\rho_2 + \rho_2^2}{2(1-r^2)} + \frac{rz}{1-r^2}}}{2\pi\sqrt{1 - r^2}} \int_0^\infty e^{-\frac{1}{2(1-r^2)}\left(x^2 + \frac{z^2}{x^2}\right) + \frac{1}{1-r^2}\left[(\rho_1 - r\rho_2)x + (\rho_2 - r\rho_1)\frac{z}{x}\right]} \frac{dx}{x} .$$

Setting,

$$\frac{x}{\sqrt{1 - r^2}} = u \quad \text{and} \quad \frac{z}{1 - r^2} = \zeta ,$$

this becomes,

$$(16)\qquad \frac{\sqrt{1-r^2}\, e^{-\frac{\rho_1^2-2r\rho_1\rho_2+\rho_2^2}{2(1-r^2)}+\frac{r\zeta}{(1-r^2)^2}}}{2\pi} \times \int_0^\infty e^{-\frac{1}{2}\left(u^2+\frac{\zeta^2}{u^2}\right)} e^{\frac{\rho_1-r\rho_2}{\sqrt{1-r^2}}u+\frac{\rho_2-r\rho_1}{\sqrt{1-r^2}}\frac{\zeta}{u}} \frac{du}{u}.$$

But on writing,

$$\frac{\rho_1-r\rho_2}{\sqrt{1-r^2}} = R_1 \quad \text{and} \quad \frac{\rho_2-r\rho_1}{\sqrt{1-r^2}} = R_2,$$

the integral in the last expression is of the same form as the $\psi_1(z)$ in the uncorrelated case. It is evident, then, that the distribution function of ζ can be written,

$$(17)\qquad \frac{\sqrt{1-r^2}}{\pi} e^{-\frac{\rho_1^2-2r\rho_1\rho_2+\rho_2^2}{2(1-r^2)}} e^{\frac{r\zeta}{(1-r^2)^2}} \Big[\textstyle\sum_0 (R_1R_2\zeta)K_0(\zeta) + (R_1^2+R_2^2)\frac{|\zeta|}{2!}\sum_2 (R_1R_2\zeta)K_1(\zeta) + (R_1^4+R_2^4)\frac{\zeta^2}{4!}\sum_4 (R_1R_2\zeta)K_2(\zeta) + (R_1^6+R_2^6)\frac{|\zeta|^3}{6!}\sum_6 (R_1R_2\zeta)K_3(\zeta) + \cdots \Big],$$

and is essentially of the form of $F(z)$, reached when $r = 0$, multiplied by an exponential function.

Frequency curves for xy (in standard units) are given in Fig. 1, Fig. 2 and Fig. 3.

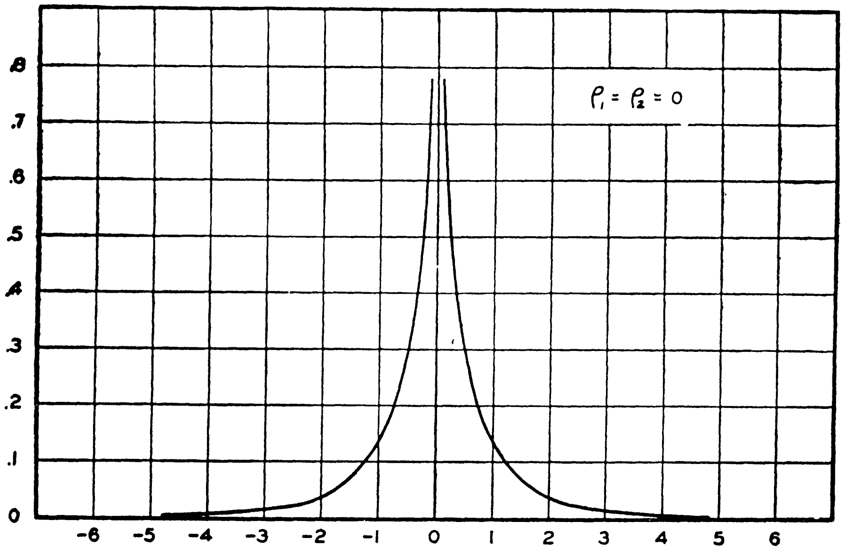

FIG. 1

TABLES OF ORDINATES OF THE DISTRIBUTION FUNCTIONS, $F(z)$ AND $F(t)$

For $\rho_1 = \rho_2 = 0, r = 0$

(Curve is symmetrical with respect to origin)

$M_z = 0, \sigma_z = 1$

$z = t$	$F(z) \equiv F(t)$
0.1	0.77256
0.2	.55790
0.3	.43887
0.4	.35477
0.5	.29425
0.6	0.24749
0.7	.21025
0.8	.17996
0.9	.15493
1.0	.13402
1.2	0.10138
1.4	.07756
1.6	.05983
1.8	.04645
2.0	.03625
2.4	0.02235
2.8	.01395
3.2	.00878
3.6	.00557
4.0	.00355
4.8	0.00146
5.6	.00061
6.4	.00026
7.2	.00011
8.0	.00005
9.0	0.00002
10.0	.00001

$\rho_1 = 1, \rho_2 = 0, r = 0$

(Curve is symmetrical with respect to origin)

$M_z = 0, \sigma_z = \sqrt{2}$

z	$F(z)$	t	$F(t)$
0.1	0.58215	0.07	0.82328
0.2	.44891	.14	.63485
0.3	.37159	.21	.52551
0.4	.31736	.28	.44882
0.5	.27593	.35	.39023
0.6	0.24270	0.42	0.34323
0.7	.21519	.49	.30432
0.8	.19193	.57	.27143
0.9	.17195	.64	.24318
1.0	.15460	.71	.21863
1.2	0.12595	0.85	0.17812
1.4	.10340	0.99	.14623
1.6	.08533	1.13	.12068
1.8	.07069	1.27	.09997
2.0	.05873	1.41	.08306
2.4	0.04078	1.70	0.05767
2.8	.02846	1.98	.04025
3.2	.01992	2.26	.02818
3.6	.01397	2.55	.01976
4.0	.00981	2.83	.01388
4.8	0.00485	3.39	0.00685
5.6	.00239	3.96	.00338
6.4	.00118	4.53	.00167
7.2	.00058	5.09	.00082
8.0	.00029	5.66	.00040
9.0	0.00012	6.36	0.00017
10.0	.00005	7.07	.00007
11.0	.00002	7.78	.00003
12.0	.00001	8.49	.00001

$$\rho_1 = \rho_2 = \tfrac{1}{2},\ r = 0$$

$$M_z = 0.25,\ \sigma_z = \frac{\sqrt{6}}{2}.$$

z	$F(z)$	t	$F(t)$
−9.6	0.00001	−8.04	0.00001
−8.8	.00002	−7.39	.00002
−8.0	0.00004	−6.74	0.00005
−7.2	.00010	−6.08	.00012
−6.4	.00023	−5.43	.00028
−5.6	.00054	−4.78	.00066
−4.8	.00128	−4.12	.00157
−4.0	0.00311	−3.47	0.00381
−3.6	.00488	−3.14	.00598
−3.2	.00769	−2.82	.00942
−2.8	.01221	−2.49	.01495
−2.4	.01954	−2.16	.02393
−2.0	0.03165	−1.84	0.03876
−1.6	.05213	−1.51	.06384
−1.2	.08809	−1.18	.10788
−0.8	.15568	−0.86	.19066
−0.4	.30423	−0.53	.37259
−0.2	0.47388	−0.37	0.58036
−0.1	.64994	−0.28	.79598
0.1	0.68106	−0.12	0.83409
0.2	.51947	−0.04	.63619
0.4	0.36322	0.12	0.44484
0.8	.21768	.45	.26659
1.2	.14230	.78	.17427
1.6	.09621	1.10	.11783
2.0	.06614	1.43	.08100
2.4	0.04589	1.76	0.05620
2.8	.03201	2.08	.03920
3.2	.02241	2.41	.02745
3.6	.01571	2.74	.01924
4.0	.01103	3.06	.01351

$$\rho_1 = \rho_2 = \tfrac{1}{2},\ r = 0$$

$$M_z = 0.25,\ \sigma_z = \frac{\sqrt{6}}{2}.$$

z	$F(z)$	t	$F(t)$
4.8	0.00545	3.72	0.00667
5.6	.00269	4.36	.00329
6.4	.00133	5.02	.00163
7.2	.00065	5.67	.00080
8.0	.00032	6.33	.00039
8.8	0.00016	6.98	0.00020
9.6	.00008	7.63	.00010
10.4	.00004	8.29	.00005
11.2	.00002	8.94	.00002
12.0	.00001	9.59	.00001

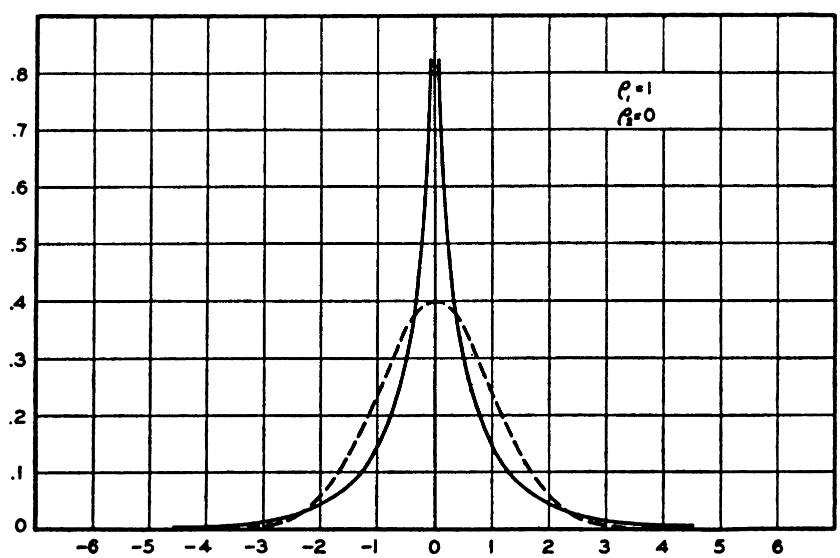

FIG. 2

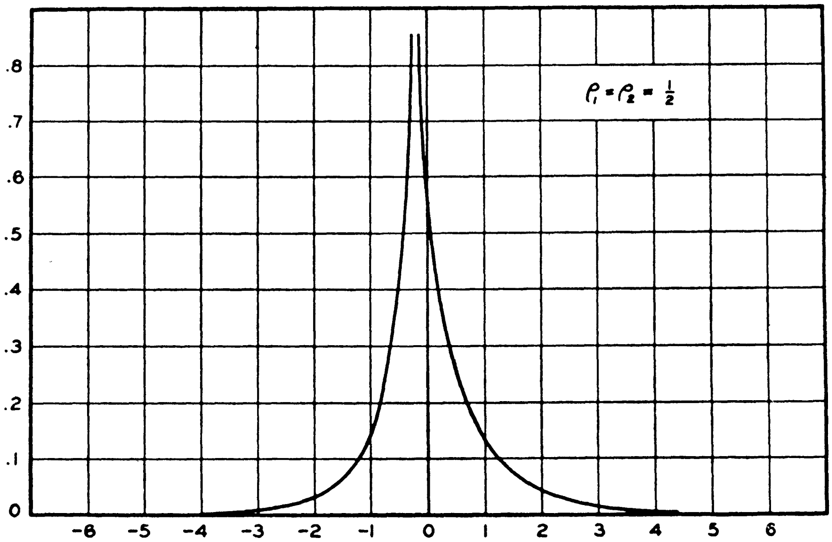

FIG. 3

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