54. On the Freudenthal's Construction of Exceptional Lie Algebras

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Introduction. In his papers [3], [4], Professor Freudenthal constructed an exceptional simple Lie algebra $\mathfrak{G}$ as follows. Let $\mathfrak{J}$ be an exceptional simple Jordan algebra of all 3×3 Hermitian matrices with coefficients in the algebras of octaves, in which the Jordan product $X\cdot Y$ is defined as $1/2(XY+YX)$. A symmetric cross product $X\times Y$ in $\mathfrak{J}$ is defined by

$$X\times Y=X\cdot Y-\frac{1}{2}(\mathrm{sp}\,(X)Y+\mathrm{sp}\,(Y)X-\mathrm{sp}\,(X)\,\mathrm{sp}\,(Y)I+(X,Y)I),$$

where $\mathrm{sp}\,(X)$ means the spur of X, I is the unit matrix and $(X,Y)=\mathrm{sp}\,(X\cdot Y)$ for $X,Y\in\mathfrak{J}$. Furthermore, for any $X,Y\in\mathfrak{J}$, $\langle X,Y\rangle$ is a linear transformation of $\mathfrak{J}$ defined by

$$\langle X,Y\rangle Z=2Y\times(X\times Z)-\frac{1}{2}(Z,Y)X-\frac{1}{6}(X,Y)Z \qquad \text{for } Z\in\mathfrak{J}.$$

Let $\mathfrak{H}$ be the subspace spanned by $\{\langle X,Y\rangle \mid X,Y\in\mathfrak{J}\}$ in the space of linear transformations on $\mathfrak{J}$. Let $\mathfrak{R}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}$ and $\mathfrak{L}=\mathfrak{H}\oplus\boldsymbol{R}\oplus\mathfrak{J}\oplus\mathfrak{J}$ ($\boldsymbol{R}$ is the field of real numbers) be direct sums as vector spaces, in which elements are denoted as

$$P=\ulcorner X,Y,\xi,\omega\urcorner \quad\text{and}\quad \Theta=\ulcorner \textstyle\sum_i\langle X_i,Y_i\rangle,\rho,A,B\urcorner$$

or

$$P=\begin{bmatrix}X\\Y\\\xi\\\omega\end{bmatrix}\quad\text{and}\quad\Theta=\begin{bmatrix}\sum_i\langle X_i,Y_i\rangle\\\rho\\A\\B\end{bmatrix}.$$

For any elements $P_i=\ulcorner X_i,Y_i,\xi_i,\omega_i\urcorner$ $(i=1,2)$ in $\mathfrak{R}$, an alternating form $\{P_1,P_2\}$ and an element $P_1\times P_2$ of $\mathfrak{L}$ are defined as follows;

$$\{P_1,P_2\}=(X_1,Y_2)-(X_2,Y_1)+\xi_1\omega_2-\xi_2\omega_1,$$

$$P_1\times P_2=\frac{1}{2}\begin{bmatrix}\langle X_1,Y_2\rangle+\langle X_2,Y_1\rangle\\-\frac{1}{4}((X_1,Y_2)+(X_2,Y_1)-3\xi_1\omega_2-3\xi_2\omega_1)\\-Y_1\times Y_2+\frac{1}{2}(\xi_1X_2+\xi_2X_1)\\X_1\times X_2-\frac{1}{2}(\omega_1Y_2+\omega_2Y_1)\end{bmatrix}.$$

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For $\Theta=\ulcorner\sum_i\langle X_i, Y_i\rangle, \rho, A, B\urcorner$ in $\mathfrak{L}$ and $P=\ulcorner X, Y, \xi, \omega\urcorner$ in $\mathfrak{K}$, an element ΘP in $\mathfrak{K}$ is defined by

$$\Theta P=\begin{bmatrix} \left(\sum_i\langle X_i, Y_i\rangle+\frac{1}{3}\rho\right)X+2B\times Y+\omega A \\ -\left(\sum_i\langle Y_i, X_i\rangle+\frac{1}{3}\rho\right)Y+2A\times X+\xi B \\ (A, Y)-\rho\xi \\ (B, X)+\rho\omega \end{bmatrix}.$$

Then the following relations hold (cf. [2], [4])

$$(1) \qquad (P\times Q)R-(P\times R)Q+\frac{1}{8}\{P, Q\}R-\frac{1}{8}\{P, R\}Q-\frac{1}{4}\{Q, R\}P=0,$$

$$(2) \qquad \{(P\times Q)R, S\}+\{R, (P\times Q)S\}=0,$$

$$(3) \qquad [P\times Q, R\times S]=(P\times Q)R\times S+R\times(P\times Q)S.$$

Put $\mathfrak{M}=\{P\in\mathfrak{K}\mid P\times P=0\}$ and let Inv $(\mathfrak{M})$ be a Lie algebra of the group of projective transformations of $\mathfrak{K}$ which leave the manifold $\mathfrak{M}$ invariant. Freudenthal introduced a Lie product [,] in the vector space direct sum $\mathfrak{G}=\text{Inv}(\mathfrak{M})\oplus\mathfrak{A}_1\oplus\mathfrak{K}\oplus\mathfrak{K}$, where $\mathfrak{A}_1$ was a three dimensional simple Lie algebra, and he proved that $\mathfrak{G}$ became a simple Lie algebra of type E_8.

Then the vector space $\mathfrak{T}=\mathfrak{K}\oplus\mathfrak{K}$ becomes a Lie triple system relative to the ternary product $[t_1t_2t_3]=[[t_1, t_2], t_3]$ $(t_i\in\mathfrak{T})$, since $[\mathfrak{K}\oplus\mathfrak{K}, \mathfrak{K}\oplus\mathfrak{K}]\subset\text{Inv}(\mathfrak{M})\oplus\mathfrak{A}_1$, $[\text{Inv}(\mathfrak{M})\oplus\mathfrak{A}_1, \mathfrak{K}\oplus\mathfrak{K}]\subset\mathfrak{K}\oplus\mathfrak{K}$ and $\text{Inv}(\mathfrak{M})\oplus\mathfrak{A}_1$ is a subalgebra of $\mathfrak{G}$. Therefore it follows from the simplicity of $\mathfrak{G}$ that $\mathfrak{T}$ is simple as Lie triple system (cf. [5], [6]).

In this paper, we shall give a direct proof of this result without using of simplicity of $\mathfrak{G}$ (see Theorem 1), and a reformation of the Freudenthal's construction by means of a kind of triple systems and a criterion for simplicity of $\mathfrak{G}$ (see Theorem 4).

§ 1. We denote an element of the vector space $\mathfrak{T}=\mathfrak{K}\oplus\mathfrak{K}$ in matrix form as $\begin{pmatrix}P\\Q\end{pmatrix}$ for $P, Q\in\mathfrak{K}$. Following Freudenthal [3], a ternary product in $\mathfrak{T}$ is defined abstractly as follows:

$$\left[\begin{pmatrix}P_1\\P_1'\end{pmatrix}\begin{pmatrix}P_2\\P_2'\end{pmatrix}\begin{pmatrix}P_3\\P_3'\end{pmatrix}\right]$$

$$(4) \qquad =\begin{bmatrix}(P_1\times P_2'-P_2\times P_1')P_3-\frac{1}{8}(\{P_1, P_2'\}-\{P_2, P_1'\})P_3+\frac{1}{4}\{P_1, P_2\}P_3' \\ (P_1\times P_2'-P_2\times P_1')P_3'+\frac{1}{8}(\{P_1, P_2'\}-\{P_2, P_1'\})P_3'-\frac{1}{4}\{P_1', P_2'\}P_3\end{bmatrix}.$$

Then it is easily seen that $\mathfrak{T}$ has a structure of Lie triple system relative to this product, that is, the following identities are satisfied for any $t, u, v, x, y\in\mathfrak{T}$;

$$(5) \qquad [ttu]=0,$$

$$(6) \qquad [tuv]+[uvt]+[vtu]=0,$$

(7) $$[xy[tuv]]=[[xyt]uv]+[t[xyu]v]+[tu[xyv]].$$

A subspace $\mathfrak{S}$ is called an ideal of $\mathfrak{T}$ if $[\mathfrak{S}\mathfrak{T}\mathfrak{T}]\subset\mathfrak{S}$. And $\mathfrak{T}$ is said to be simple if $\mathfrak{T}$ has no proper ideals and $\dim\mathfrak{T}>1$.

Now we put $\boldsymbol{P}_1=\ulcorner 0,0,1,0\urcorner$ and $\boldsymbol{P}_2=\ulcorner 0,0,0,1\urcorner$.

Lemma 1. *Let $\mathfrak{S}$ be an ideal of the Lie triple system $\mathfrak{T}=\mathfrak{K}\oplus\mathfrak{K}$, then $\begin{pmatrix}P\\0\end{pmatrix}\in\mathfrak{S}$ is equivalent to $\begin{pmatrix}0\\P\end{pmatrix}\in\mathfrak{S}$.*

Proof. If $\begin{pmatrix}P\\0\end{pmatrix}\in\mathfrak{S}$, *then* $\left[\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_2\end{pmatrix}\begin{pmatrix}P\\0\end{pmatrix}\right]=-\frac{1}{4}\begin{pmatrix}0\\P\end{pmatrix}\in\mathfrak{S}$. Conversely, $\begin{pmatrix}0\\P\end{pmatrix}\in\mathfrak{S}$ implies $\left[\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\begin{pmatrix}\boldsymbol{P}_2\\0\end{pmatrix}\begin{pmatrix}0\\P\end{pmatrix}\right]=\frac{1}{4}\begin{pmatrix}P\\0\end{pmatrix}\in\mathfrak{S}$.

Theorem 1. *The Lie triple system $\mathfrak{T}=\mathfrak{K}\oplus\mathfrak{K}$ with the ternary product* (4) *is simple.*

Proof. Let $\mathfrak{S}$ be a non-zero ideal of $\mathfrak{T}$. If $\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\in\mathfrak{S}$, then $\left[\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\begin{pmatrix}\boldsymbol{P}_2\\0\end{pmatrix}\begin{pmatrix}0\\P\end{pmatrix}\right]\in\mathfrak{S}$ for any $P\in\mathfrak{K}$, hence $\frac{1}{4}\begin{pmatrix}P\\0\end{pmatrix}\in\mathfrak{S}$ and $\mathfrak{K}\oplus\{0\}\subset\mathfrak{S}$. Using Lemma 1, we have $\{0\}\oplus\mathfrak{K}\subset\mathfrak{S}$, hence $\mathfrak{K}\oplus\mathfrak{K}\subset\mathfrak{S}$, which implies that $\mathfrak{T}$ is simple. So we shall show that $\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\in\mathfrak{S}$. Let $\begin{pmatrix}P\\P'\end{pmatrix}$ be a non-zero element in $\mathfrak{S}$. Then, from (4) we see $\left[\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\begin{pmatrix}\boldsymbol{P}_2\\0\end{pmatrix}\begin{pmatrix}P\\P'\end{pmatrix}\right]=\frac{1}{4}\begin{pmatrix}P'\\0\end{pmatrix}\in\mathfrak{S}$. Hence we may assume without loss of generality that $P\neq 0$. Put $P=\ulcorner X,Y,\xi,\omega\urcorner$.

Case 1. For $\xi\neq 0$: $\left[\left[\begin{pmatrix}P\\P'\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_2\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_2\end{pmatrix}\right]\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\right]\in\mathfrak{S}$, hence $\frac{1}{4}\xi\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\in\mathfrak{S}$.

Case 2. For $\omega\neq 0$: $\left[\begin{pmatrix}P\\P'\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\right]=-\frac{1}{2}\omega\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\in\mathfrak{S}$. From Lemma 1, it follows that $\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\in\mathfrak{S}$.

Case 3. For $\xi=\omega=0$ and $X\neq 0$: Choose $Z\in\mathfrak{J}$ such that $(X,Z)\neq 0$, then $\left[\begin{pmatrix}P\\P'\end{pmatrix}\begin{pmatrix}0\\Q\end{pmatrix}\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\right]=\frac{1}{4}(X,Z)\begin{pmatrix}0\\\boldsymbol{P}_1\end{pmatrix}\in\mathfrak{S}$, where $Q=\ulcorner 0,Z,0,0\urcorner$, hence $\begin{pmatrix}\boldsymbol{P}_1\\0\end{pmatrix}\in\mathfrak{S}$.

Case 4. For $\xi=\omega=0$ and $Y\neq 0$: Choose $Z\in\mathfrak{J}$ such that $(Y,Z)\neq 0$, then a proof is similar to Case 3.

§ 2. We assume that any vector space considered in this section is a finite dimensional vector space over a field F of characteristic 0. A triple system $\mathfrak{A}$ is a vector space with a trilinear map $\mathfrak{A}\times\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$: $(x,y,z)\mapsto[xyz]$. A subspace $\mathfrak{B}$ of $\mathfrak{A}$ is called an ideal if $[\mathfrak{B}\mathfrak{A}\mathfrak{A}]+[\mathfrak{A}\mathfrak{B}\mathfrak{A}]+[\mathfrak{A}\mathfrak{A}\mathfrak{B}]\subset\mathfrak{B}$. We call a triple system $\mathfrak{A}$ simple when it has only trivial ideals and $\dim \mathfrak{A}>1$. A Lie triple system is a triple system satisfying (5), (6), (7). We define a *symplectic triple system* $\mathfrak{K}$ as a triple system with a non-zero alternating bilinear form $\mathfrak{K}\times\mathfrak{K}\to F: (x,y)\mapsto\{x,y\}$ satisfying the following identities

$$[xyz]=[yxz], \tag{8}$$

$$[xyz]-[xzy]+\{x,y\}z-\{x,z\}y-2\{y,z\}x=0, \tag{9}$$

$$[xy[uvw]]=[[xyu]vw]+[u[xyv]w]+[uv[xyw]]. \tag{10}$$

A linear mapping D of $\mathfrak{K}$ is called a derivation of $\mathfrak{K}$ if $D[xyz]=[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$. The identity (10) implies that a linear mapping $L(x,y): z\mapsto[xyz]$ is a derivation of $\mathfrak{K}$. Using (9) and (10), we can prove that the bilinear form $\{\ ,\ \}$ is invariant under any derivation of $\mathfrak{K}$. Especially we have

$$\{[xyz],w\}+\{z,[xyw]\}=0. \tag{11}$$

The vector space $\mathfrak{K}$ considered in Introduction and § 1 becomes a symplectic triple system by putting $[P_1P_2P_3]=(P_1\times P_2)P_3$ and $\{P_1,P_2\}=1/8\{P_1,P_2\}$ from (1) and (3). Another example is a vector space with a non-zero alternating bilinear form $\{\ ,\ \}$ and the trilinear map $(x,y,z)\mapsto[xyz]=\{x,z\}y+\{y,z\}x$.

Remark. The symplectic triple systems are variations on the Freudenthal triple systems (see [7]) or the balanced symplectic ternary algebras (see [1]).

Lemma 2. *Let $\mathfrak{N}$ be an ideal of a symplectic triple system $\mathfrak{K}$. Then*

(i) $\{\mathfrak{N},\mathfrak{K}\}\mathfrak{K}\subset\mathfrak{N}$,

(ii) $\mathfrak{N}^\perp=\{x\in\mathfrak{K}\mid\{x,\mathfrak{N}\}=0\}$ *is an ideal of* $\mathfrak{K}$,

(iii) $\mathfrak{K}^\perp$ *is the maximal ideal of* $\mathfrak{K}$.

Using this lemma, we have

Theorem 2. *Let $\mathfrak{K}$ be a symplectic triple system with an alternating bilinear form $\{\ ,\ \}$. Then $\mathfrak{K}$ is simple if and only if the form $\{\ ,\ \}$ is non-degenerate.*

To construct a Lie triple system $\mathfrak{T}$ from a symplectic triple system $\mathfrak{K}$, put $\mathfrak{T}=\mathfrak{K}\oplus\mathfrak{K}$. We denote an element $t=x\oplus y$ in matrix form as $t=\begin{pmatrix}x\\y\end{pmatrix}$ and define a triple product in $\mathfrak{T}$ by

$$[t_1t_2t_3]=\begin{pmatrix}[x_1y_2x_3]-[x_2y_1x_3]-\{x_1,y_2\}x_3+\{x_2,y_1\}x_3+2\{x_1,x_2\}y_3\\ [x_1y_2y_3]-[x_2y_1y_3]+\{x_1,y_2\}y_3-\{x_2,y_1\}y_3-2\{y_1,y_2\}x_3\end{pmatrix} \tag{12}$$

for $t_i=\begin{pmatrix}x_i\\y_i\end{pmatrix}$ with $x_i,y_i\in\mathfrak{K}$.

Then it is easily shown that $\mathfrak{T}$ is a Lie triple system with respect to this product. We call $\mathfrak{T}$ the Lie triple system associated with $\mathfrak{K}$. By modification of the proofs of Lemma 1 and Theorem 1, we have the following

Lemma 3. *Let $\mathfrak{T}$ be the Lie triple system associated with a sym-*

plectic triple system $\mathfrak{R}$. Then,

(i) *if $\mathfrak{N}$ is an ideal of $\mathfrak{R}$, then $\mathfrak{N}\oplus\mathfrak{N}$ is an ideal of $\mathfrak{T}$,*

(ii) *if $\mathfrak{S}$ is an ideal of $\mathfrak{T}$, then $\mathfrak{S}=(\mathfrak{S}\cap\mathfrak{R})\oplus(\mathfrak{S}\cap\mathfrak{R})$ and $\mathfrak{S}\cap\mathfrak{R}$ is an ideal of $\mathfrak{R}$.*

Using this Lemma 3 and Theorem 2, we obtain a generalization of Theorem 1 as follows.

Theorem 3. *Let $\mathfrak{T}$ be the Lie triple system associated with a symplectic triple system $\mathfrak{R}$. Then $\mathfrak{T}$ is simple if and only if $\mathfrak{R}$ is simple.*

For $r, s \in \mathfrak{T}$, a linear mapping $L(r, s): t \mapsto [rst]$ on $\mathfrak{T}$ is also a derivation by (7). By $\mathcal{L}(\mathfrak{T}, \mathfrak{T})$ (resp. $\mathcal{L}(\mathfrak{R}, \mathfrak{R})$), we denote the Lie algebra generated by $\{L(r, s) \mid r, s \in \mathfrak{T}\}$ (resp. $\{L(x, y) \mid x, y \in \mathfrak{R}\}$), of which elements are called inner derivations.

For any $D \in \mathcal{L}(\mathfrak{R}, \mathfrak{R})$, a linear mapping $\bar{D}$ of $\mathfrak{T}$ is defined by $\bar{D}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Dx \\ Dy \end{pmatrix}$, and three special linear mappings U, V, W are defined by

$$U\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ -y \end{pmatrix}, \quad V\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} y \\ 0 \end{pmatrix}, \quad W\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ x \end{pmatrix}.$$

From (12), we have

$$L\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)=\overline{L(x_1, y_2)}-\overline{L(x_2, y_1)}-(\{x_1, y_2\}-\{x_2, y_1\})U$$
$$+2\{x_1, x_2\}V-2\{y_1, y_2\}W.$$

Conversely,

$$2\overline{L(x_1, y_2)}=L\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2 \end{pmatrix}\right)+L\left(\begin{pmatrix} y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix}\right),$$
$$2\{x_1, y_2\}U=L\left(\begin{pmatrix} y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix}\right)-L\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2 \end{pmatrix}\right),$$
$$2\{x_1, x_2\}V=L\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix}\right), \qquad 2\{y_1, y_2\}W=-L\left(\begin{pmatrix} 0 \\ y_1 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2 \end{pmatrix}\right).$$

These identities mean that endomorphisms $\overline{L(x, y)}, U, V, W$ are inner derivations of $\mathfrak{T}$. Hence, we see that

$$\mathcal{L}(\mathfrak{T}, \mathfrak{T})=\overline{\mathcal{L}(\mathfrak{R}, \mathfrak{R})}\oplus FU\oplus FV\oplus FW.$$

The Lie products among these endomorphisms are

$$[\overline{L(x, y)}, \overline{L(u, v)}]=\overline{L([xyu], v)}+\overline{L(u, [xyv])},$$
$$[\overline{L(x, y)}, U]=[\overline{L(x, y)}, V]=[\overline{L(x, y)}, W]=0,$$
$$[U, V]=2V, \quad [U, W]=-2W, \quad [V, W]=U.$$

Let $\mathfrak{G}$ be the standard enveloping Lie algebra of $\mathfrak{T}$ (cf. [5]), that is, $\mathfrak{G}=\mathfrak{T}\oplus\mathcal{L}(\mathfrak{T}, \mathfrak{T})=\mathfrak{R}\oplus\mathfrak{R}\oplus\overline{\mathcal{L}(\mathfrak{R}, \mathfrak{R})}\oplus FU\oplus FV\oplus FW$. Then, we have the following

Theorem 4. *Let $\mathfrak{R}$ be a symplectic triple system and $\mathfrak{G}$ the standard enveloping Lie algebra of the Lie triple system associated with $\mathfrak{R}$. Then $\mathfrak{G}$ is simple if and only if $\mathfrak{R}$ is simple.*

Remark. The Lie algebra $\mathfrak{G}$ considered in Introduction is isomorphic to one obtained in this method from a symplectic triple system. Furthermore, by Theorem 2 and Theorem 4, it is easily shown that $\mathfrak{S}$ is simple. Linear mappings $\sum_i \overline{L(x_i, y_i)}, U, V, W$ considered in § 2 correspond respectively to operators $\Theta, \Gamma, \underline{\Delta}, \bar{\Delta}$ in the Freudenthal's construction (cf. [3]). In case that $\mathfrak{J}$ is the Jordan algebra of all 3×3 complex Hermitian matrices, the associated Lie triple system $\mathfrak{T}$ (see Introduction) is T_4 in the Lister's classification of simple Lie triple systems (cf. [5], 240–241).

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