# ON THE FUNCTIONAL EQUATION $\sum_{i=0}^{p} a_{i} f_{i}^{n i}=1$ 

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1. Let $a_{0}, \cdots, a_{p}(p \geqslant 1)$ be $p+1$ meromorphic functions in $|z|<R(\leqslant \infty)$ and $n_{0}, \cdots, n_{p}$ positive integers. In this paper we consider whether the functional equation

$$
\sum_{i=0}^{p} a_{i} f_{i}^{n_{t}}=1
$$

has holomorphic solutions $f_{0}, \cdots, f_{p}$ in $|z|<R$.
Recently, Yang [10] has proved the following
Theorem A. The functional equation

$$
\begin{equation*}
a(z) f^{m}(z)+b(z) g^{n}(z)=1 \tag{1}
\end{equation*}
$$

$(a, b, f, g$ meromorphic in $|z|<\infty, m$ and $n$ integers $\geqslant 3$ ) cannot hold, if

$$
\begin{equation*}
T(r, a)=o(T(r, f)), \quad T(r, b)=o(T(r, g)), \tag{2}
\end{equation*}
$$

unless $m=n=3$.
If $f(z)$ and $g(z)$ are entire and (2) holds, then (1) cannot hold, even if $m=n=3$.

This is a generalization of the case $a \equiv b \equiv 1$ treated by Montel [7], Jategaonkar [5] and Gross [2,3]. Further, Iyer [4], Jategaonkar and Gross considered many other cases.

We will generalize the latter half of Theorem A and give some consequences of the generalizations.

It is assumed that the reader is familier with the fundamental concepts of Nevanlinna's theory of meromorphic functions and the symbols $m(r, f), N(r, f)$, $\bar{N}(r, f), T(r, f)$, etc. (see [11]).

[^0]2. First, we will treat the case $R=\infty$. The following lemma is a little sharpened form of Theorem 4 in [9].

Lemma 1. Let $g_{0}, \cdots, g_{p}(p \geqslant 1)$ be $p+1$ non-constant meromorphic functions in $|z|<\infty$ satisfying

$$
\begin{equation*}
\sum_{i=0}^{p} \alpha_{i} g_{i}=1, \quad \alpha_{0} \cdots \alpha_{p} \neq 0 \tag{3}
\end{equation*}
$$

with constant coefficients and $\delta\left(\infty, g_{i}\right)=1(i=0, \cdots, p)$. Then, we have

$$
\sum_{i=0}^{p} \theta_{p}\left(0, g_{i}\right) \leqslant p
$$

Here,

$$
\theta_{p}\left(0, g_{i}\right)=1-\limsup _{r \rightarrow \infty} \frac{N_{p}\left(r, 0, g_{i}\right)}{T\left(r, g_{i}\right)}
$$

and

$$
N_{p}\left(r, 0, g_{i}\right)=\sum_{a_{k} \neq 0} \log ^{+} \frac{r}{\left|a_{k}\right|}+\min \left(\rho_{0}, p\right) \log r
$$

where the summation is taken over the different zeros $a_{k}(\neq 0)$ of $g_{i}$ counted $\min \left(\rho_{k}, p\right)$ times at $a_{k}, \rho_{k}\left(\right.$ resp. $\left.\rho_{0}\right)$ being the order of multiplicity of zero of $g_{i}$ at $a_{k}(r e s p .0)$.

Proof. We proceed the proof in the same way as in the proof of Theorem 4 in [9].

1) The case when $g_{0}, \cdots, g_{p}$ are linearly independent. We note that the Wronskian $\Delta=\left\|g_{0}, \cdots, g_{p}\right\|$ is not identically zero in this case. By differentiating both sides of (3), we have

$$
\begin{equation*}
\sum_{i=0}^{p} \alpha_{i} \frac{g_{i}^{(\mu)}}{g_{i}} g_{i}=0, \mu=1, \cdots, p \tag{4}
\end{equation*}
$$

From (3) and (4), we have

$$
g_{i}=\frac{\widetilde{\Delta}_{i}}{\alpha_{i} \widetilde{\Delta}},
$$

where

$$
\widetilde{\Delta}=\Delta / g_{0} \cdots g_{p}(\not \equiv 0)
$$

and

$$
\widetilde{\Delta}_{i}=g_{i}\left\|g_{0}, \cdots, g_{i-1}, 1, g_{i+1}, \cdots, g_{v}\right\| / g_{0} \cdots g_{p}
$$

Then,

$$
\begin{aligned}
m\left(r, g_{i}\right) & \leqslant m\left(r, \widetilde{\Delta}_{i}\right)+m(r, 1 / \widetilde{\Delta})+O(1) \\
& \leqslant m\left(r, \widetilde{\Delta}_{i}\right)+m(r, \widetilde{\Delta})+N(r, \widetilde{\Delta})+O(1) \\
& \leqslant \sum_{j=0}^{p} N_{p}\left(r, 0, g_{j}\right)+p \sum_{j=0}^{p} N\left(r, g_{j}\right)+S(r),
\end{aligned}
$$

where $S(r)$ is independent of $i$ and

$$
S(r)= \begin{cases}O(1),(r \rightarrow \infty), & \text { when all } g_{i} \text { are rational } \\
o\left(\sum_{j=1}^{p} T\left(r, g_{j}\right)\right), & \begin{array}{l}
\text { with a possible exceptional set of } r \\
\text { of finite linear measure, in the other } \\
\text { cases. }
\end{array}\end{cases}
$$

Hence,

$$
\begin{aligned}
T(r) & \equiv \max _{0 \leqslant i \leqslant p}\left\{m\left(r, g_{i}\right)+N\left(r, g_{i}\right)\right\} \\
& \leqslant \sum_{j=0}^{p} N_{p}\left(r, 0, g_{j}\right)+(p+1) \sum_{j=0}^{p} N\left(r, g_{j}\right)+S(r)
\end{aligned}
$$

By the definition of $\theta_{p}\left(0, g_{j}\right)$ and $\delta\left(\infty, g_{j}\right)$, for an arbitrary given $\varepsilon>0$,

$$
N_{p}\left(r, 0, g_{j}\right)<\left(1-\theta_{p}\left(0, g_{j}\right)+\varepsilon\right) T\left(r, g_{j}\right)
$$

and

$$
N\left(r, g_{j}\right)<\left(1-\delta\left(\infty, g_{j}\right)+\varepsilon\right) T\left(r, g_{j}\right)
$$

for $r \geqslant r_{0}$ and $j=0, \cdots, p$. Using these inequalities and by the hypothesis, we have

$$
T(r) \leqslant \sum_{j=0}^{p}\left(1-\theta_{p}\left(0, g_{j}\right)+\varepsilon\right) T(r)+(p+1)^{2} \varepsilon T(r)+S(r)
$$

which implies

$$
\sum_{j=0}^{p} \theta_{p}\left(0, g_{j}\right) \leqslant p
$$

2) The case when $g_{0}, \cdots, g_{p}$ are linearly dependent. Porceeding as in [9] and using the inequality

$$
\theta_{p}\left(0, g_{j}\right) \leqslant \theta_{s}\left(0, g_{j}\right) \leqslant 1 \quad \text { for } \quad 1 \leqslant s \leqslant p,
$$

we have the desired

$$
\sum_{j=0}^{p} \theta_{p}\left(0, g_{j}\right) \leqslant p .
$$

Applying this lemma, we have the following
ThEOREM 1. Let $f_{0}, \cdots, f_{p}(p \geqslant 1)$ be $p+1$ non-constant entire functions and let $a_{0}, \cdots, a_{p}$ be $p+1$ meromorphic functions $(\not \equiv 0)$ in $|z|<\infty$ such that

$$
\begin{equation*}
T\left(r, a_{i}\right)=o\left(T\left(r, f_{i}\right)\right),(i=0, \cdots, p) \tag{5}
\end{equation*}
$$

Then, if the following functional equation

$$
\begin{equation*}
\sum_{i=0}^{p} a_{i}(z) \cdot f_{i}^{n_{i}}(z)=1 \tag{6}
\end{equation*}
$$

holds for some integers $n_{0}, \cdots, n_{p}(\geqslant 1)$, it must be

$$
\sum_{i=0}^{p} \frac{1}{n_{i}} \geqslant \frac{1}{p} .
$$

Proof. We have for any meromorphic function $f$ in $|z|<\infty$ and a positive integer $n$

$$
T\left(r, f^{n}\right) \sim n T(r, f),(r \rightarrow \infty)
$$

Therefore, by (5),

$$
\begin{gathered}
\text { ON THE FUNCTIONAL EQUATION } \\
T\left(r, a_{i} f_{i}^{n_{i}}\right) \sim n_{i} T\left(r, f_{i}\right),(r \rightarrow \infty), i=0, \cdots, p .
\end{gathered}
$$

By the definition of $N_{p}\left(r, 0, g_{i}\right)$, we have

$$
N_{v}\left(r, 0, a_{i} f_{i}^{n_{i}}\right) \leqslant N_{p}\left(r, 0, a_{i}\right)+p \bar{N}\left(r, 0, f_{i}\right),
$$

so that

$$
\limsup _{r \rightarrow \infty} \frac{N_{p}\left(r, 0, a_{i} f_{i}^{n_{i}}\right)}{T\left(r, a_{i} f_{i}^{n_{i}}\right)} \leqslant \frac{p}{n_{i}} \limsup _{r \rightarrow \infty} \frac{\bar{N}\left(r, 0, f_{i}\right)}{T\left(r, f_{i}\right)} \leqslant \frac{p}{n_{i}},
$$

that is, by the definition of $\theta_{p}$,

$$
\theta_{p}\left(0, a_{i} f_{i}^{n_{i}}\right) \geqslant 1-\frac{p}{n_{i}} \quad(i=0, \cdots, p) .
$$

Summing over these inequalities and using Lemma 1 and (5), we have

$$
\sum_{i=0}^{p}\left(1-\frac{p}{n_{i}}\right) \leqslant \sum_{i=0}^{p} \theta_{p}\left(0, a_{i} f_{i}^{n_{i}}\right) \leqslant p,
$$

that is,

$$
\begin{equation*}
\sum_{i=0}^{p} \frac{1}{n_{i}} \geqslant \frac{1}{p} \tag{7}
\end{equation*}
$$

which is the desired.
As consequences of this theorem, we have generalizations of the latter half of Theorem A.

Corollary 1. If

$$
\sum_{i=0}^{p} \frac{1}{n_{i}}<\frac{1}{p},
$$

then the equation (6) cannot hold.
Corollary 2. Let $p$ be equal to 1. Then, (6) cannot hold for $n_{0} \geqslant 2$, $n_{1} \geqslant 3$ or $n_{0} \geqslant 3, n_{1} \geqslant 2$.

This is an improvement of the latter half of Theorem A and of Lemma 2 in [5].

In [3], Gross considered the functional equation

$$
\sum_{i=0}^{p} f_{i}^{n}(z)=1
$$

for non-constant entire functions $f_{i}(i=0, \cdots, p)$ and a positive integer $n$. When $n=3, p=2([6])$ or $n=4, p=2$ ([3]), there is a solution Of course, these cases satisfy the inequality (7).

As an application of Theorem 1, we give some generalizations of Theorem 2 in [10] when $f$ and $g$ are entire.

THEOREM 2. Let $f_{0}, \cdots, f_{p}(p \geqslant 1)$ be $p+1$ non-constant entire functions and let $n_{0}, \cdots, n_{p}$ be $p+1$ integers not less than one such that at least one of $f_{i}^{n_{1}} / f_{j}^{n_{j},}(i \neq j)$ is transcendental and

$$
\begin{equation*}
\sum_{i=0}^{p} \frac{1}{n_{i}}<\frac{1}{p} . \tag{8}
\end{equation*}
$$

Then, for rational funcctions $R_{i}(z)(\not \equiv 0)(i=0, \cdots, p)$

$$
F(z) \equiv \sum_{i=0}^{p} R_{i}(z) f_{i}^{n_{i}}(z)
$$

has infinitely many zeros or some of partial sums (one of which may be F) are equal to zero identically.

Proof. Assume that the statement is false. Then $F(z)$ can be written as

$$
F(z)=R(z) e^{f(z)}
$$

where $R(z)$ is rational $(\not \equiv 0)$ and $f(z)$ is entire.
Let

$$
-\frac{f(z)}{n_{i}}=g_{i}(z) \quad(i=0, \cdots, p) .
$$

Then $g_{i}(z)$ is entire $(i=0, \cdots, P)$ and

$$
\sum_{i=0}^{p} R_{i} R^{-1}\left(f_{i} e^{a_{i}}\right)^{n_{i}}=1
$$

Since $R_{i} R^{-1}$ is rational, it holds that

$$
T\left(r, R_{i} R^{-1}\right)=O(\log r),(r \rightarrow \infty), i=0, \cdots, p
$$

By the hypothesis (8) and by using Theorem 1, at least one of $f_{i} e^{g_{4}}$ is polynomial. The number $s$ of $i$ such that $f_{i} e^{g_{i}}$ is polynomial is at most $p-1$. In fact, if $s \geqslant p$ we may suppose that $f_{0} e^{\sigma_{0}}, \cdots, f_{p-1} e^{\rho_{p-1}}$ are polynomial, so that

$$
\sum_{i=0}^{p-1} R_{i} R^{-1}\left(f_{i} e^{e_{i}}\right)^{n_{i}} \equiv A
$$

is rational. This implies that

$$
\left(f_{p} e^{g_{p}}\right)^{n_{p}}=(1-A) R R_{p}^{-1}
$$

is polynomial. Hence $f_{\nu} e^{\sigma_{p}}$ is polynomial. This shows that $s=p+1$. In this case, any ratio

$$
\frac{\left(f_{i} e^{\sigma_{i}}\right)^{n_{i}}}{\left(f_{j} e^{\sigma_{j}}\right)^{n_{j}}}=\frac{f_{i}^{n_{i}}}{f_{j}^{n_{j}}}
$$

is rational, which is a contradiction. Therefore

$$
1 \leqslant s \leqslant p-1
$$

We may suppose that $f_{0} e^{\sigma_{0}}, \cdots, f_{s-1} e^{\sigma_{s-1}}$ are polynomial, so that

$$
\sum_{i=0}^{s-1} R_{i} R^{-1}\left(f_{i} e^{g_{i}}\right)^{n_{i}}
$$

is rational and is not equal to 1 identically by the assumption. We have

$$
\sum_{i=s}^{p} P_{i}\left(f_{i} e^{\sigma_{i}}\right)^{n_{i}}=1,
$$

where $f_{i} e^{\sigma_{i}}(i=s, \cdots, p)$ are transcendental and $P_{i}(\equiv 0)(i=s, \cdots, p)$ are rational. Therefore, using

$$
T\left(r, P_{i}\right)=o\left(T\left(r, f_{i} e^{\theta_{i}}\right)\right) \quad(i=s, \cdots p),
$$

we have by Theorem 1

$$
\sum_{i=0}^{p} \frac{1}{n_{i}}>\sum_{i=s}^{p} \frac{1}{n_{i}} \geqslant \frac{1}{p-s}>\frac{1}{p}
$$

which contradicts (8). Thus we have our theorem.
3. In [10], it is remarked that the analogous result to Theorem A for some meromorphic functions in the unit disc can be proved as in Theorem A, but it is not noted whether analogous results to the type of Theorem 2 are valid or not. As we can prove some results concerning this type, we shall state these briefly in the following.

Lemma 2. Let $g(z)$ be meromorphic in $|z|<1$. Then for $1>r>r_{0}>0$ and $\lambda>0$

$$
\int_{r_{0}}^{r} m\left(t, g^{\prime} / g\right)(1-t)^{\lambda-1} d t=O\left(\int_{r_{0}}^{r} \log ^{+} T(t, g)(1-t)^{\lambda-1} d t\right)
$$

([8]).
Using this lemma, we can give a lemma analogous to Lemma 1.
Lemma 3. Let $g_{0}, \cdots, g_{p}(p \geqslant 1)$ be $p+1$ functions meromorphic in $|z|<1$ such that

$$
\limsup _{r \rightarrow 1} \frac{T\left(r, g_{i}\right)}{\log \frac{1}{1-r}}=\infty,(i=0, \cdots, p)
$$

and $\delta\left(\infty, g_{i}\right)=1(i=0, \cdots, p)$.
If, for non-zero constants $\alpha_{i}(i=0, \cdots, p)$,

$$
\sum_{i=0}^{p} \alpha_{i} g_{i}=1
$$

then we have

$$
\sum_{i=0}^{p} \theta_{p}\left(0, g_{i}\right) \leqslant p
$$

where

$$
\theta_{p}\left(0, g_{i}\right)=1-\lim _{r \rightarrow 1} \frac{N_{p}\left(r, 0, g_{i}\right)}{T\left(r, g_{i}\right)}
$$

We can prove this in the same way as in the proof of Lemma 1 by using Lemma 2 only when at least one of $g_{i}$ is of order infinite.

THEOREM 3. Let $f_{0}, \cdots, f_{p}(p \geqslant 1)$ be $p+1$ holomorphic functions in $|z|<1$ such that

$$
\limsup _{r \rightarrow 1} \frac{T\left(r, f_{i}\right)}{\log \frac{1}{1-r}}=\infty,(i=0, \cdots, p)
$$

and let $a_{0}, \cdots, a_{p}$ be $p+1$ meromorphic functions $(\not \equiv 0)$ in $|z|<1$ such that

$$
T\left(r, a_{i}\right)=o\left(T\left(r, f_{i}\right)\right),(r \rightarrow 1), i=0, \cdots, p
$$

If the functional equation

$$
\sum_{i=0}^{p} a_{i} f_{i}^{n_{i}}=1
$$

holds for integers $n_{i}(i=0, \cdots, p)$ not less than one, then the following inequality must be valid:

$$
\sum_{i=0}^{p} \frac{1}{n_{i}} \geqslant \frac{1}{p}
$$

We can prove this as in the proof of Theorem 1 by using Lemma 3 in place of Lemma 1.

REMARK. This is an improvement of Lemma $1^{\prime}$ in [5].
THEOREM 4. Let $f_{0}, \cdots, f_{p}(p \geqslant 1)$ be $p+1$ holomorphic functions in $|z|<1$ and let $n_{0}, \cdots, n_{p}$ be $p+1$ integers not less than one such that at least one of $f_{i}^{n_{1}} / f_{j}^{n j}(i \neq j)$ satisfies

$$
\limsup _{r \rightarrow 1} \frac{T\left(r, f_{i}^{n_{1}} / f_{j}^{n_{j}}\right)}{\log \frac{1}{1-r}}=\infty
$$

and

$$
\sum_{i=0}^{p} \frac{1}{n_{i}}<\frac{1}{p} .
$$

Then, for meromorphic functions $a_{i}(\not \equiv 0)(i=0, \cdots, p)$ of bounded type in $|z|<1$, the zeros $\left\{z_{k}\right\}$ of

$$
\begin{equation*}
F(z) \equiv \sum_{i=0}^{p} a_{i}(z) f_{i}^{n_{i}}(z) \tag{9}
\end{equation*}
$$

satisfy

$$
\sum_{k}\left(1-\left|z_{k}\right|\right)=\infty
$$

or some of partial sums of $a_{i} f_{i}^{n_{4}}$ (one of which may be $F$ ) are equal to zero identically.

Proof. Assume that the statement is false. Let $P(z)$ be a canonical product of $\left\{z_{k}\right\}$. Then $P(z)$ is holomorphic, $|P(z)|<1$ in $|z|<1$ and $P\left(z_{k}\right)=0$. Let $Q(z)$ be a canonical product of the poles of $F(z)$. Then by the hypothesis, $Q(z)$ is holomorphic, $|Q(z)|<1$ in $|z|<1$. Put

$$
F(z)=P(z) Q^{-1}(z) H(z)
$$

Then $H(z)$ is holomorphic in $|z|<1$ and has no zeros in $|z|<1$. Let $g_{i}=H^{-1 / n_{i}}$ be a branch in $|z|<1$. Then it is holomorphic in $|z|<1$ and has no zeros. We have from (9)

$$
\sum_{i=0}^{p} a_{i} Q P^{-1}\left(f_{i} g_{i}\right)_{i^{n_{i}}}=1,
$$

where $a_{i} Q P^{-1}$ is of bounded type ( $i=0, \cdots, p$ ).
We can prove the rest as in the proof of Theorem 2 by using Theorem 3 in place of Theorem 1.

## References

[1] W. K. Hayman, Meromorphic functions, OMM, Clarendon Press, Oxford 1964.
[2] F. Gross, On the equation $f^{n}+g^{n}=1$, Bull. Amer. Math. Soc., 72(1966), 86-88.
[3] F. Gross, On the functional equation $f^{n}+g^{n}=h^{n}$, Amer. Math. Monthly, 73(1966), 1093-1096.
[ 4 ] V. G. IYER, On certain functional equations, J. Indian Math. Soc., 3(1938-39), 312-315.
[5] A. V. JATEGAONKAR, Elementary. proof of a theorem of P. Montel on entire functions, J. London Math. Soc., 40(1965), 166-170.
[6] D. H. Lefmer, On the Diophantine equation $x^{3}+y^{3}+z^{3}=1$, J. London Math. Soc., 31 (1965), 275-280.
[7] P. MONTEL, Leçons sur les familles normales de fonctions analytiques et leurs applications, Gauthier-Villars, Paris 1927.
[8] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris 1929.
[9] K. Nitno and M. Ozawa, Deficiencies of an entire algebroid function, Kōdai Math. Sem. Rep., 22(1970), 98-113.
[10] C. -C. Yang, A generalization of a theorem of P. Montel on entire functions, Proc. Amer. Math. Soc., 26-2 (1970), 332-334.

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