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# ON THE FUNCTIONAL FORMULATION OF QUANTUM FIELD THEORY

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## ABSTRACT

This paper describes a functional formulation of Euclidean quantum field theory, based on a complete equivalence with classical statistical mechanics. One introduces an extra time variable and sets up a canonical scheme with new Lagrangian and Hamiltonian functions. The generating functional is then defined as the Gibbs average over the ensemble. This allows us, in particular, to control in a simple way the invariance properties of the integration measure. In several cases of physical interest it is seen that the invariance requirement leads to extra determinantal factors in the integration volume and therefore to a set of improved Feynman rules. In particular, enforcing dilatation invariance for the generating functional is shown to lead to a non-zero background, i.e., to spontaneous breaking of the symmetry. The application of the method to constrained systems is discussed in detail and in the case of Yang-Mills theories the Faddeev-Popov prescription for quantization is reproduced with remarkable simplicity. A discussion of the functional quantization of gravity is also offered.

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## 1. - INTRODUCTION

The path integral approach, where quantum dynamics is described as a sum over all field configurations, is, in its Euclidean formulation, at the root of recent progress in quantum field theory[1]. The main reasons for this are the following.

The functional formalism gives the covariant perturbation theory in the most direct way since it leads straight to Feynman rules, in particular in the case of gauge theories. It also represents a very important tool for the investigation of non-perturbative properties in the framework of the recent lattice approach.

Let us recall the most relevant points. At the start, one has a Lagrangian density, a function of the fields and of their derivatives (the metric is Euclidean)

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2} (\partial_\mu \varphi)^2 + \mathcal{U}(\varphi) . \tag{1.1}$$

The fundamental generating functional is then given by

$$Z = \int d\Omega(\varphi) \exp\left\{ -\frac{1}{\hbar} \int d^4x \left( \mathcal{L} + \varphi j_{ext} \right) \right\} , \tag{1.2}$$

$$d\Omega(\varphi) \equiv \Pi_x \, d\varphi(x) ,$$

where $j_{ext}$ is an external source*).

The analogy with classical statistical mechanics is apparent and well known. The field $\varphi$ plays the rôle of a Lagrangian set of co-ordinates defined over a four-dimensional Euclidean manifold. The action is the analogue of the Hamiltonian, $\hbar$ corresponds to the temperature and $Z$ to the partition function.

The Lagrangian formalism embodied in Eqs. (1.1) and (1.2) automatically leads to the so-called "naïve" Feynman rules (i.e., those which follow from inspection of the classical Lagrangian). These rules are valid in the simplest cases, for example for the purely scalar model. However, in more sophisticated theories, which are also more interesting from a physical point of view, the correct perturbative results follow from the use of improved Feynman rules which contain fictitious "ghost" fields, due to the presence in the integration volume $d\Omega$ of

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*) From now on, the source term will be omitted for simplicity.

extra functional determinants. This happens, for instance, for chiral and Yang-Mills gauge theories. The derivation of these rules often has to be done case by case and the correspondence with the canonical formalism is not always straightforward.

The aim of this paper is to describe an improved functional formulation[2] where the similarity with classical statistical mechanics becomes complete equivalence. Such a framework provides an appropriate determination of the functional measure. The prescription coincides with the conventional results in the cases already known of chiral and gauge theories, but leads to new hints when conformal invariant theories are considered. Gravity can also be treated in a simple way.

We now summarize the content of this work. In order to simplify the presentation, the paper is divided into two parts. First of all we shall formulate the method for systems with a finite number of degrees of freedom. This will be done in Sections 2-5.

In particular, Section 2 contains a general description of our procedure, while in Section 3 we apply it to the discussion of constrained systems. Section 4, whose material is fundamental for treating gauge theories later on, deals with the case in which an invariance group is present. This leads to the existence of ignorable variables and we show how to dispose a priori of the relevant integrations in a general manner. In view of the application of the method to fermionic systems, Section 5 offers a discussion of Lagrangians containing only first order "time" derivatives.

The second part, consisting of Sections 6-8, will be devoted to generalizing these results to problems with infinite degrees of freedom, that is Quantum Field Theory.

In Section 6 we examine the more conventional problems related to field theoretical models with chiral and conformal invariance. Sections 7 and 8 contain a detailed application of the formalism developed in Section 4 to non-Abelian gauge theories and to Einstein gravity.

In Section 9 we shall finally discuss some special field theoretical aspects of our approach. For the reader aiming at a quick grasp of the main points of the method, Sections 2, 5 and 6 are suggested. The subtleties of systems endowed with local invariance groups require the longer reading of Sections 3, 4, 7 and 8.

## 2. - GENERAL DESCRIPTION OF THE METHOD

2.1 In order to present our procedure in the simplest and most general way, we introduce a notation where the degrees of freedom are denumerable.

Let us thus consider a system described by $N$ variables, $q_n$, $n = 1,\ldots N$. The index $n$ represents all the discrete labels $\alpha$ of a field (internal degrees of freedom, Lorentz indices, etc.) as well as the variable $x_\mu$, now thought of as an element of a finite four-dimensional lattice. Thus we have the correspondence

$$\varphi_\alpha(x) \longrightarrow q_m \quad , \quad m = 1\ldots N \tag{2.1}$$

This change of notation (essential by the way for the definition of a functional integral) will be extremely convenient. We shall finally assume throughout this paper that the transition from the discrete to the continuum configuration is straightforward and can be performed in an elementary way*).

In this framework, the Lagrangian will be simply considered as a function of the co-ordinates $q_n$, i.e.,

$$\int d^4x \; \mathcal{L}(\varphi, \partial_\mu \varphi) \longrightarrow \mathcal{V}(q_m) \tag{2.2}$$

and the definition of the generating functional will be

$$Z = \int d\Omega(q) \; \exp\{-1/\hbar \, \mathcal{V}(q)\} \quad , \tag{2.3}$$

$d\Omega(q) \equiv \prod_m dq_m$ .

This expression displays a fundamental difference from the Gibbs integral of classical statistical mechanics

$$\int d\Omega(q)\, d\Omega(p)\, \exp\{-\beta H(q,p)\}$$

In the above formula, $H$ not only depends on the co-ordinates $q_n$, it is also a function of the conjugate momenta $p_n$, and the integration measure is the elementary volume in phase space with well-known canonical invariance properties) and not just the volume element of configuration space.

*) We shall not consider in this paper those effects which arise from a more subtle definition of the path integral measure and which lead to anomalous breaking effects, for instance in the case of chiral symmetries[3].

Our functional formulation of quantum systems is based on the idea of substituting to the conventional definition of $Z$ given by (2.2) a new one that generalizes the notion of the Gibbs integral to the present case.

Let us introduce a new parameter $t$ so that our co-ordinates $q_n$ will become $t$ dependent. In the case of field theory, this corresponds to having, besides the four Euclidean co-ordinates $x_\mu$, an extra "time" variable $t$ [4]:

$$q_m(t) \longrightarrow \varphi_\alpha(x_\mu, t) . \tag{2.4}$$

We shall next interpret the usual action $\int d^4 x\, \mathcal{L} \to \mathcal{V}(q)$ as a potential energy. The variation of $q_n(t)$ as a function of the time $t$ will then be determined by "equations of motion" following from the new "Lagrangian"

$$L(q,\dot{q}) \equiv \sum_m L(q_m,\dot{q}_m) = T(q,\dot{q}) - \mathcal{V}(q), \tag{2.5}$$

where a new term $T$, the kinetic energy part, has been introduced. As usual we shall assume that it is at most quadratic in $\dot{q}_n$.

The explicit form of $T$ will be discussed later in this chapter; in the most interesting cases its determination will be quite unique and will follow from general arguments based on locality and invariance. This point is the heart of the matter since we shall see that the expression of $T$ determines the deviation of the Feynman rules from their "naïve" form following from Eqs. (1.2) or (2.3).

Leaving aside for the moment the key question of the determination of $T$, we proceed along pure conventional lines to specify the new definition of $Z$ à la Gibbs.

The co-ordinates $q_n(t)$ obey the Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m} \quad , \quad m = 1 \ldots N . \tag{2.6}$$

Accordingly, their evolution in $t$ is entirely determined giving $q_n(t)$ and $\dot{q}_n(t)$ at a fixed $t_0$. In other words, we have added a parameter $t$, but the dependence of $q_n$ upon $t$ is given in terms of two sets $q_n(t_0)$, $\dot{q}_n(t_0)$ (the initial conditions). Thus we have actually doubled the original co-ordinates.

One can now follow the canonical scheme by introducing the momenta

$$p_n = \frac{\partial L}{\partial \dot{q}_n} \equiv \frac{\partial T}{\partial \dot{q}_n} \tag{2.7}$$

and the "Hamiltonian" function

$$H = \Sigma_n p_n \dot{q}_n - L(q, \dot{q}) . \tag{2.8}$$

We take for $T$ a quadratic form in $\dot{q}_n$ (the case of a linear dependence will be considered separately)

$$T = \frac{1}{2} \Sigma_{nm} T_{nm} \dot{q}_n \dot{q}_m , \tag{2.9}$$

then

$$H(q, p) = T(q, \dot{q}) + V(q) . \tag{2.10}$$

We now define $Z$ as the Gibbs average ($\hbar = 1$)

$$Z = \int d\Omega(q) d\Omega(p) \exp\{-H(q, p)\} . \tag{2.11}$$

This equation represents our general definition of the generating functional.

In the definition of $Z$ we sum over all configurations $q_n(t)$ <u>at fixed $t$</u> (i.e., in terms of fields the functional space is the space of the functions of $x_\mu$ as in the usual formulation). The integral is independent of the particular choice of $t$ because each $q_n(t)$ evolves in "time" according to the equations of motion (2.6), that is according to the canonical transformation generated by $H$. But then the measure $d\Omega(q)d\Omega(p)$ is independent of $t$ by Liouville's theorem (more generally the measure is invariant under canonical transformations) and of course $H$ is independent of $t$; this is the usual situation with a Gibbs integral. We see therefore that the introduction of $t$ has just been a device in order to exploit a definition à la Gibbs that, as we shall see, gives a general way to control the measure and its invariance properties.

In order to keep contact with the usual formulation, we postulate that the matrix $T_{nm}$ in Eq. (2.9) admits an inverse, i.e., $\det T \neq 0$; in this way the definition of $p_n$

$$p_n = T_{nm} \dot{q}_m \tag{2.12}$$

can be inverted to obtain $\dot{q}_n$ and

$$T = \frac{1}{2} \left(T^{-1}\right)_{nm} p_n p_m \; . \tag{2.13}$$

Inserting the expressions (2.10) and (2.13) for $H$ and $T$ in the definition of $Z$, Eq. (2.11), the integration on the momentum variables $p_n$ can be readily performed leading to

$$Z = \int d\Omega(q) \left(\det T\right)^{1/2} \exp\{-\mathcal{V}(q)\} \, . \tag{2.14}$$

This result is fundamental for future developments. We stress the presence of the factor $(\det T)^{1/2}$, which makes all the difference between the Gibbs definition (2.11) and the "naïve" formula (2.3).

The introduction of the quantity

$$\hat{\mathcal{V}} = \mathcal{V} - \frac{1}{2} \ln \det T = \mathcal{V} - \frac{1}{2} \operatorname{tr} \ln T \tag{2.15}$$

allows one to rewrite $Z$ in the form

$$Z = \int d\Omega(q) \exp\{-\hat{\mathcal{V}}(q)\} \, . \tag{2.16}$$

It is fruitful to have a deeper understanding of the general reason which leads to the extra factor $(\det T)^{1/2}$. The key motivation lies in the requirement of invariance of the functional integral under a given group of transformations. One can express this by saying that, while achieving the invariance of the theory at the classical level simply requires the invariance of the action, for the quantum formulation we also need the invariance of the volume element.

The choice of the invariant kinetic term, together with Eq. (2.14) for $Z$, automatically satisfies this requirement.

Let us consider the transformation of co-ordinates

$$q_i \to \alpha_i(q) \,, \quad dq_i \to \alpha_{ij}(q)\, dq_j \tag{2.17}$$

and impose the invariance of the quadratic form*)

$$ds^2 = \Sigma_{nm} T_{nm}\, dq_n\, dq_m \tag{2.18}$$

(the elementary arc element!) This requires that

$$T_{nm} \to (\alpha^{-1} T \alpha^{-1})_{nm} \tag{2.19}$$

As a consequence (as is well known from tensor calculus) one can define an invariant volume

$$d\Omega_I(q) = (\det T)^{1/2} \Pi_m\, dq_m \,. \tag{2.20}$$

One thus realizes that the determination of an invariant volume follows automatically from the choice of an invariant kinetic term.

We shall see that in several cases of physical interest, invariance arguments like conformal invariance or chiral invariance will imply $T_{nm} \neq \delta_{nm}$. As a consequence, this will lead to modified rules, to be derived from the correct form of the functional integral.

We conclude this section by proving an important property of the formula (2.14) for the partition function: its invariance in form with respect to any change of co-ordinates.

Let us introduce a new set of variables $\bar{q}_n$ defined by the transformation

$$q_\ell = q_\ell(\bar{q}_1 \cdots \bar{q}_N) \,, \quad \ell = 1 \ldots N \,. \tag{2.21}$$

The Lagrangian $L$ expressed in terms of the new co-ordinates will take the form

$$L(\bar{q}, \dot{\bar{q}}) = \frac{1}{2} \Sigma_{ij} \bar{T}_{ij}\, \dot{\bar{q}}_i \dot{\bar{q}}_j + \bar{\mathcal{V}}(\bar{q}) \tag{2.22}$$

*) Since the transformation law (2.17) of the $q_n$ is considered at fixed $t$, this is, of course, equivalent to the invariance of the kinetic term $\Sigma_n T_{nm} \dot{q}_n \dot{q}_m$.

where

$$\bar{V}(\bar{q}) = V[q(\bar{q})] \tag{2.23}$$

and

$$\bar{T}_{ij}(\bar{q}) = \frac{\partial q_m}{\partial \bar{q}_i} T_{nm}[q(\bar{q})] \frac{\partial q_m}{\partial \bar{q}_j} \tag{2.24}$$

As a consequence of (2.24), one obtains

$$(\det \bar{T})^{1/2} = (\det T)^{1/2} \det\left(\frac{\partial q_n}{\partial \bar{q}_i}\right). \tag{2.25}$$

Let us now write the expression of Z in the new co-ordinates

$$Z = \int d\Omega(\bar{q}) (\det \bar{T})^{1/2} \exp\{-\bar{V}(\bar{q})\} \tag{2.26}$$

The change of variables between q and $\bar{q}$

$$d\Omega(q) = d\Omega(\bar{q}) \det\left(\frac{\partial q_n}{\partial \bar{q}_i}\right) \tag{2.27}$$

introduces a Jacobian which compensates the similar factor implied by Eq. (2.25). Further, taking into account (2.23), one ascertains at once that Z as given by Eq. (2.26) coincides with the formula (2.14) in terms of the variables $q_n$.

## 3. - THE TREATMENT OF CONSTRAINED SYSTEMS

We now treat the case of constrained systems that will also be instrumental to the discussion of gauge theories. Let us assume that the system is described by the N co-ordinates $q_n$, $n = 1,\ldots,N$ with a Lagrangian

$$L = T - V$$

and

$$T = \frac{1}{2} \Sigma_{nm} T_{nm} \dot{q}_n \dot{q}_m . \tag{3.1}$$

The constraint conditions will, in general, be given through R equations in the implicit form

$$f_a(q) = 0 \qquad a = 1\ldots R \tag{3.2}$$

The number of independent variables is thus $I = N - R$.

The treatment of this case is somewhat more cumbersome than the corresponding non-constrained one discussed in Section 2. We shall proceed as follows. We first state the fundamental formula for Z which will then be proved to be the correct one on the basis of its transformation properties.

The proposed expression for Z is

$$Z = \int d\Omega(q) \prod_{a=1}^{R} \delta[f_a(q)][\Delta(f)]^{1/2} \exp\{-\mathcal{V}\} \tag{3.3}$$

where $\Delta(f)$ is defined as follows. We first introduce the quantities

$$F_{\mu\mu} = \sum_a \frac{\partial f_a}{\partial q_m} \frac{\partial f_a}{\partial q_\mu} \tag{3.4}$$

and the differential operator

$$D = \sum_{\mu m} F_{\mu m} \frac{\partial}{\partial T_{\mu m}} \ . \tag{3.5}$$

$\Delta$ then has the form

$$\Delta(f) = \frac{1}{R!} D^R (\det T) \ . \tag{3.6}$$

In order to establish the validity of this result, it is convenient to proceed in two steps. First of all it will be shown that the expression (3.3) for Z is invariant in form with respect to any change of co-ordinates. One can then choose a set of variables for which the system to be studied reduces trivially to an unconstrained problem in such a way that the formula (3.3) becomes (2.14) of the last section.

Let us thus consider the change of variables $q \to \bar{q}$.

$$q_m = q_m(\bar{q}_1 \cdots \bar{q}_N) \tag{3.7}$$

The kinetic and potential parts of L transform according to the simple rules

$$\mathcal{V}(q) \to \bar{\mathcal{V}}(\bar{q}) = \mathcal{V}[q(\bar{q})] \tag{3.8a}$$

$$T_{\mu m}(q) \to \bar{T}_{ij}(\bar{q}) = \sum_{\mu m} \frac{\partial q_\mu}{\partial \bar{q}_i} T_{\mu m}[q(\bar{q})] \frac{\partial q_m}{\partial \bar{q}_j} \tag{3.8b}$$

In addition

$$f_a(q) \to \bar{f}_a(\bar{q}) = f_a[q(\bar{q})] \tag{3.8c}$$

and as a consequence

$$F_{nm}(q) \to \bar{F}_{ij}(\bar{q}) = \sum_{nm} \frac{\partial q_n}{\partial \bar{q}_i} F_{nm}[q(\bar{q})] \frac{\partial q_m}{\partial \bar{q}_j} \tag{3.8d}$$

Notice that, as made explicit by (3.8d), $F_{nm}(q)$ transforms in the same way as the kinetic energy matrix $T_{nm}(q)$. We then immediately see that

$$[\Delta(f)]^{1/2} \to [\bar{\Delta}(\bar{f})]^{1/2} = [\Delta(f)]^{1/2} \det\left(\frac{\partial q_\ell}{\partial \bar{q}_i}\right) \tag{3.9}$$

This fundamental property, together with the well-known transformation law for the volume element

$$d\Omega(q) = d\Omega(\bar{q}) \det\left(\frac{\partial q_n}{\partial \bar{q}_i}\right) \tag{3.10}$$

ensures, as in Section 2, that the definition (3.3) of $Z$ is invariant in form for any change of variables.

We can now take advantage of this property and choose a particularly simple set of co-ordinates such that the evaluation in this frame of $Z$ of Eq. (3.3) leads back to Eq. (2.14).

Let us thus perform the following change of variables

$$\begin{aligned} \bar{q}_a &= q_i & i &= 1 \ldots I \,, \\ \bar{q}_{I+a} &= f_a(q) & a &= 1 \ldots R \,. \end{aligned} \tag{3.11}$$

For this basis, the constraints are therefore of the elementary form

$$f_a \equiv \bar{q}_{I+a} = 0 \,.$$

It then follows that

$$F_{nm} = F_n \delta_{nm} \tag{3.12}$$

$$F_\mu = 0 \qquad m = 1 \dots I\,,$$
$$F_\mu = 1 \qquad m = I+1 \dots N\,. \tag{3.13}$$

As a consequence

$$\Delta(\beta) = \det \hat{T} \tag{3.14}$$

where $\hat{T}_{ij}$ is the minor of $T_{ij}$ in which all rows $i = I + 1,\dots,N$ and all columns $j = I + 1,\dots,N$ have been eliminated. The partition function can thus be written as

$$Z = \int \prod_{i=1}^{I} dq_i \, (\det \hat{T})^{1/2} \exp\{-\hat{\mathcal{V}}\} \tag{3.15}$$

where we have set

$$\bar{q}_{I+a} = 0$$

both in $\hat{T}$ and $\hat{\mathcal{V}}$.

This result is obviously equivalent to the expression (2.14) for Z. Indeed, one could have treated the complete system by simply eliminating from the start all the vanishing $\bar{q}_{I+a}$ and by considering the reduced Lagrangian

$$\hat{L}(\bar{q}_1 \cdots \bar{q}_I) = \hat{T}(\bar{q}_1 \cdots \bar{q}_I, \dot{\bar{q}}_1 \cdots \dot{\bar{q}}_I) + \hat{\mathcal{V}}(\bar{q}_1 \cdots \bar{q}_I) \tag{3.16}$$

in which $\bar{q}_i$, $i = 1,\dots,I$ are unconstrained variables. We are thus allowed to use the definition (2.14) of Z which leads us directly to the formula (3.15).

## 4. - THE CASE OF AN INVARIANCE GROUP

In order to achieve a satisfactory formulation of systems with local invariance properties, it is fruitful to supplement the previous treatment of the constrained case with some remarks which take into account the symmetry properties of the theory. This is interesting on its own, but it also provides the correct hint on how to treat gauge invariant theories like QCD and gravitation.

We start from a Lagrangian of the form

$$T = \frac{1}{2} t_{nm} \dot{q}_n \dot{q}_m - \mathcal{V}(q) \tag{4.1}$$

and suppose it is invariant in form under the transformations $q_n \to q'_n$ of an $R$ parameter group

$$q'_n = f_n(q_1 \dots q_N ; \theta_1 \dots \theta_R), \quad n = 1 \dots M. \tag{4.2}$$

$\theta_1, \dots, \theta_R$ are the parameters of the group that define the transformation. Notice that the transformations (4.2) involve the $q_n$ but not the "time" $t$.

The general formulation of Section 2 applies perfectly to the present case and one can express the partition function $Z$ by means of Eq. (2.14)*)

$$Z = \int d\Omega(q) \left(\det T\right)^{1/2} \exp\left\{-\mathcal{V}(q)\right\}$$

On the other hand, the existence of an $R$ parameter symmetry group tells us that there are $R$ ignorable co-ordinates. This means that $R$ of the functional integrations (those involving the ignorable variables) can be performed on general grounds, leading to a reduced form of $Z$ in which the effective integration is performed over the remaining $M = N - R$ physical co-ordinates.

In order to achieve this result we perform a change of variables which is at the root of our treatment and consists of the explicit introduction of the group parameters as dynamical variables. Let us write

$$\begin{gathered} q_n(t) = f_n\left[Q_r(t), \phi_\alpha(t)\right] \\ n, r = 1 \dots M ; \quad \alpha = 1 \dots R . \end{gathered} \tag{4.3}$$

Namely, through (4.3), we consider quantities $Q_1(t), \dots, Q_M(t)$ <u>and</u> $R$ variables $\phi_1(t), \dots, \phi_R(t)$ as new dynamical variables $M$. The physical reason for this choice is that the group variables $\phi_1(t), \dots, \phi_R(t)$ are ignorable and do not appear in the "potential" because of the invariance. Only their momenta will be present in the "kinetic" term. Of course, now we have more variables than

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*) Of course, we tacitly assumed that the source term is invariant under the same group of transformations.

needed since there are $M + R = N$ co-ordinates. We dispose of this redundancy by imposing $R$ constraint equations among the $Q$'s (gauge fixing procedure):

$$F_\alpha(Q) = 0 \qquad \alpha = 1 \ldots R \, . \tag{4.4}$$

This implies that $R < M$.

Equation (4.3) can also be viewed as a group transformation with parameters $\phi_\alpha(t)$ from the variables $Q_r(t)$, fixed, to the variables $q_n(t)$. It will be convenient to consider the inverse transformation $g_r$ where the $Q_r(t)$ are expressed as functions of the $q_n$ and $\phi_\alpha$:

$$Q_r(t) = g_r[q_n, \phi_\alpha] \tag{4.5}$$

Differentiation of both (4.3) and (4.5) gives

$$dq_n = \frac{\partial q_n}{\partial Q_r} dQ_r + \frac{\partial q_n}{\partial \phi_\alpha} d\phi_\alpha$$

$$dQ_r = \frac{\partial Q_r}{\partial q_n} dq_n + \frac{\partial Q_r}{\partial \phi_\alpha} d\phi_\alpha \tag{4.6}$$

Consistency between these two relations leads to the important identity

$$\frac{\partial q_n}{\partial \phi_\alpha} = - \frac{\partial Q_r}{\partial \phi_\alpha} \frac{\partial q_n}{\partial Q_r} \tag{4.7}$$

where the derivative $\partial Q_r / \partial \phi_\alpha$ is understood to be evaluated at fixed $q$ and so on.

Let us now perform the time derivative of $q_n(t)$ as defined by (4.3),

$$\dot{q}_n(t) = \frac{\partial q_n}{\partial Q_r} \dot{Q}_r + \frac{\partial q_n}{\partial \phi_\alpha} \dot{\phi}_\alpha = \tag{4.8}$$

$$= \frac{\partial q_n}{\partial Q_r} \left( \dot{Q}_r - \dot{\phi}_\alpha \frac{\partial Q_r}{\partial \phi_\alpha} \right) \equiv \frac{\partial q_n}{\partial Q_r} \Pi_r$$

where we have introduced the important quantity (not necessarily a conjugate momentum)

$$\Pi_r = \dot{Q}_r - \left( \frac{\partial Q_r}{\partial \phi_\alpha} \right)_{\text{fixed } q} \dot{\phi}_\alpha \quad . \tag{4.9}$$

Thus equipped we can now move on to exploit the parametrization (4.3) of the co-ordinates. The first step is to re-express the kinetic and potential parts in terms of $Q_r$ and $\phi_\alpha$. Owing to the invariance of the Lagrangian under the transformation (4.2), one has [compare with Eqs. (3.7)]

$$t_{um}(Q) = \frac{\partial q_u}{\partial Q_r} t_{rs}[q(Q,\phi)] \frac{\partial q_m}{\partial Q_s} \tag{4.10a}$$

$$\mathcal{V}(Q) = \mathcal{V}[q(Q,\phi)] \tag{4.10b}$$

We thus obtain, using (4.8),

$$T = \frac{1}{2} t_{rs}(Q) \Pi_r \Pi_s \ , \tag{4.11a}$$

$$\mathcal{V} \equiv \mathcal{V}(Q) \ . \tag{4.11b}$$

The result (4.11) is fundamental in the sense that all the important cases will be given this form.

We see that the problem is now reduced to a constrained system as discussed in the previous section. Our $N = M + R$ Lagrange co-ordinates are $Q_r$ and $\phi_\alpha$, subjected to the $R$ constraints represented by Eq. (4.4). Finally, the expression of the Lagrangian in terms of these variables follows from Eqs. (4.11) and (4.9).

One can thus apply the result (3.3) for the generating functional $Z$. After some cumbersome calculations in the evaluation of $\Delta(f)$, this leads to the formula

$$Z = \int d\Omega(Q)\, d\Omega(\phi) \, \exp\{-\mathcal{V}(Q)\} \cdot \prod_{1}^{R}{}_\alpha \, \delta(F_\alpha) \, (\det t)^{1/2} \, \det\left( \frac{\partial F_\alpha}{\partial \phi_\beta} \right) \tag{4.12}$$

The validity of this result can be checked explicitly but it is amusing to present a simplified proof which follows from the invariance of $Z$ under any transformation of the co-ordinates $Q_r$ We can then perform a choice of co-ordinates such that the problem is reduced to an unconstrained one: this allows the fundamental formula (2.14) for $Z$ to be directly applied and its equivalence to (4.12) established.

Let us proceed through the steps of the proof. It is first of all quite easy to ascertain the invariance in the form of Eq. (4.12) for any change of variables

$$Q_r = Q_r(\bar{Q}) \tag{4.13}$$

which leaves the $\phi$'s unchanged.

This follows (as usual!) from a compensation between the factor arising from the transformation of $(\det t)^{1/2}$ and of the integration volume $d\Omega(Q)$. The subsequent transformation is then natural (cf. the analogous procedure in the previous section)

$$\begin{aligned} \bar{Q}_i &= Q_i \qquad i = 1 \ldots I = M-R\,, \\ \bar{Q}_{I+\alpha} &= F_\alpha \qquad \alpha = 1 \ldots R\,. \end{aligned} \tag{4.14}$$

In that "reference frame" the constraint conditions are simply

$$\bar{Q}_{I+\alpha} = 0$$

These variables can thus be eliminated and one is finally led to the unconstrained system whose $M$ co-ordinates $\xi_m$, $m = 1,\ldots,M$ are defined as follows:

$$\begin{aligned} \xi_i &= \bar{Q}_i \equiv Q_i \qquad i = 1 \ldots I\,, \\ \xi_{I+\alpha} &= \phi_\alpha \qquad \alpha = 1 \ldots R\,. \end{aligned} \tag{4.15}$$

We can now write the kinetic term in the form

$$T = \frac{1}{2} T_{nm} \dot{\xi}_n \dot{\xi}_m \tag{4.16}$$

where

$$T_{uu} = S_{mr}\, t_{rs}\, S_{su} \qquad (4.17)$$
$$u, m, r, s = 1 \dots M ,$$

and S is the M × M square matrix of elements (the others are vanishing)

$$S_{mr} = \delta_{mr} \qquad m, r = 1 \dots I ,$$
$$S_{mr} = -\frac{\partial \bar{Q}_u}{\partial \phi_\alpha} \qquad m = 1 \dots M ; \; r = I + \alpha . \qquad (4.18)$$

From these equations we learn that

$$(\det T)^{1/2} = (\det t)^{1/2} \, | \det S | \qquad (4.19)$$

and that

$$| \det S | = \left| \det \left( \frac{\partial \bar{Q}_{I+\alpha}}{\partial \phi_\beta} \right) \right| \qquad (4.20)$$

The final part of the argument consists of applying to the present case the general formula (2.14) for the generating functional (for an unconstrained system), namely

$$Z = \int d\Omega(\xi) (\det T)^{1/2} \exp\{-\mathcal{V}\} =$$
$$= \int d\Omega(\xi) (\det t)^{1/2} \left| \det \left( \frac{\partial \bar{Q}_{I+\alpha}}{\partial \phi_\beta} \right) \right|_{\bar{Q}=0} \exp\{-\mathcal{V}\} . \qquad (4.21)$$

It is now immediate to realize that the expression (4.12) for the partition function reduces to the above one for our special choice of variables, confirming the consistency of the general result.

At this point the formula (4.12) can be further simplified by introducing the invariant group measure

$$d\Omega_I = \frac{d\phi_1 \dots d\phi_R}{\det(\delta_\alpha \phi_\beta)} \qquad (4.22)$$

(this well-known result is, for convenience, quickly rederived in Appendix A). $\delta_\alpha \phi_\beta$ is the infinitesimal variation along the "direction" $\alpha$ of the group variable $\phi_\beta$.

Further, we obtain

$$\det\left(\frac{\partial F_\alpha(Q)}{\partial \phi_\gamma}\right) \cdot \det\left(\delta_\beta \phi_\gamma\right) = \det\left(\delta_\beta F_\alpha\right) \tag{4.23}$$

so that there is the identity

$$d\Omega(\phi)\ \det\left(\frac{\partial F_\alpha}{\partial \phi_\beta}\right) = d\Omega_I\ \det\left(\delta_\alpha F_\beta\right) \tag{4.24}$$

Remark that $\delta_\alpha F_\beta$ is a function of the Q's only. Therefore, when inserting (4.24) into the representation (4.12) for Z, the integration over the (ignorable) group variables can be performed and gives a trivial numerical factor. Apart from it, the final formula reads

$$\begin{aligned} Z = \int d\Omega(Q)(\det t)^{1/2} \det(\delta_\alpha F_\beta)\ \Pi_\alpha\, \delta(F_\alpha) \times \\ \times \exp\{-V(Q)\} = \int d\Omega(Q) \det(\delta_\alpha F_\beta)\ \Pi_\alpha \delta(F_\alpha) \exp\{-\hat{V}(Q)\} \end{aligned} \tag{4.25}$$

This is the result we wanted to establish. In a later chapter we shall see how this formula applies naturally to the case of gauge theories.

Let us conclude this section with a comment about the usefulness of the above approach. It is clear that the method, which is based on the elimination of the ignorable co-ordinates, is completely general and can be applied to any kind of problems, even the most elementary ones with familiar groups of invariance like O(3) and the like. In the discrete cases, with compact invariance groups, the method clearly leads to strong simplifications, because several integrations are automatic, but it is not indispensable. One can, in principle, perform the integration over all the co-ordinates, which is a well-defined operation, even if the knowledge of the symmetry properties of the system would allow a quicker handling.

The situation is different for the case of continuum quantum field theory and our procedure has to be considered as indispensable when one has to deal with infinite parameter groups or with non-compact groups. In such a case, the integration over the group variables gives infinities, which are eliminated a priori in our framework.

## 5. - THE LINEAR FORMALISM

Some specific applications of the previous formalism, in particular to the case where fermionic fields are involved, require a discussion of systems whose Lagrangian exhibits a linear dependence on the time derivative of the co-ordinates. Accepting the discrete formulation, we shall examine the case where

$$L = \Sigma_i C_i(q)\dot{q}_i - \mathcal{V}(q) . \tag{5.1}$$

This immediately leads to the equations of motion

$$A_{nm}(q)\dot{q}_m = \frac{\partial \mathcal{V}}{\partial q_n} = F_m \tag{5.2}$$

with the antisymmetric matrix $A_{nm}$ given by

$$A_{nm}(q) = \frac{\partial C_n}{\partial q_m} - \frac{\partial C_m}{\partial q_n} . \tag{5.3}$$

Of course, the Lagrangian $L$ is defined apart from a total $t$ derivative $d\phi/dt$. This amounts to saying that the physical content of the theory is unchanged for a transformation

$$C_i(q) \rightarrow C_i(q) + \frac{\partial \phi}{\partial q_i} \tag{5.4}$$

Indeed, the antisymmetric tensor $A_{nm}$ is invariant and the equations of motion remain unchanged.

The Hamiltonian takes on the particularly simple form (peculiar to this class of systems)

$$H = \frac{\partial L}{\partial \dot{q}_m}\dot{q}_m - L = \mathcal{V}(q) . \tag{5.5}$$

Its time independence is guaranteed by the antisymmetry of $A_{nm}$ (i.e., by the equations of motion)

$$\frac{dH}{dt} = \frac{\partial \mathcal{V}}{\partial q_n}\dot{q}_m = A_{nm}\dot{q}_n\dot{q}_m = 0 .$$

Let us now consider the volume element in the configuration space, $d\Omega(q) = \Pi_n dq_n$. In general, $d\Omega$ is not time independent since

$$\frac{d}{dt}[d\Omega(q)] = d\Omega(q)\Sigma_n \frac{\partial \dot{q}_n}{\partial q_n} \tag{5.6}$$

but a suitable density function can be introduced to compensate for such a lack of invariance. Following the general lines of the ensuing discussion it is not hard to be convinced that the volume $d\Omega_I$, invariant under time translations, is

$$d\Omega_I = d\Omega(q)(\det A)^{1/2} \tag{5.7}$$

The importance of this quantity lies in the fact that it represents in this case, where no canonical momenta appear, the analogue of the invariant phase space volume.

It is thus quite natural to assume in this case as the Gibbs definition of the generating functional, the expression

$$Z = \int d\Omega(q)(\det A)^{1/2} \exp\{-\mathcal{V}(q)\} . \tag{5.8}$$

The result for $Z$ given by Eq. (5.8) has quite a general meaning which can easily be understood by recalling the discussion at the end of Section 2. If there exists a symmetry group of point transformations for the equation of motion, the invariant volume is given by (5.7) and the invariance of $Z$ is ensured.

In order to check this property, let us assume $L$ to be invariant under the transformations

$$q_i \rightarrow \alpha_i(q) , \quad dq_i \rightarrow \alpha_{ij}(q)\, dq_j \tag{5.9}$$

The $A_{nm}$ then transforms as

$$A_{nm} \longrightarrow (\alpha^{-1} A\, \alpha^{-1})_{nm} \tag{5.10}$$

and again the invariant volume is given by the previous expression, i.e.,

$$d\Omega_I = d\Omega(q)\,(\det A)^{1/2} .$$

In particular one can choose a transformation such that the new $A_{nm}$ does not depend on $q$, which indeed guarantees the time independence of the volume [from the equations of motion $\Sigma_n \partial\dot{q}_n/\partial q_n = \partial/\partial q_s (A^{-1})_{rs} F_r$].

Since $A_{nm}$ is antisymmetric, its determinant vanishes when the number of degrees of freedom is odd. For an even number of variables, $\det A$ is non-vanishing and is the square of the Pfaffian of $A$, namely

$$\det A = (\mathrm{Pf}\,A)^2 \tag{5.11}$$

This remark is interesting since the case of an even number of variables can be considered formally similar to the usual Hamiltonian version of the equations of motion if one does not distinguish between $q_n$ and $p_n$. This analogy actually becomes an identity because it is always possible to reduce $A_{nm}$, through a suitable change of variables, to the "canonical" structure

$$A_{nm} \longrightarrow \begin{vmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} \tag{5.12}$$

and $\mathcal{V} = H$ *).

*) The present treatment of the first order formalism might be given a more familiar aspect by defining Poisson brackets, canonical transformations and the like. Consider, for instance, the time derivative of a dynamical variable, $f(q,t)$. After using the equations of motion one obtains

$$\frac{df}{dt} = \frac{\partial f}{\partial q_m} (A^{-1})_{mn} \frac{\partial H}{\partial q_n} \equiv \{H, f\} . \tag{x}$$

If $A_{nm}$ is reduced to the semidiagonal form (2.30) and the $2M$ variables are grouped in two sets to be called $q_n$ and $p_n$, one can recognize in Eq. (x) the standard definition.

Similarly, a canonical transformation, generated by $G$, will be defined through the relation

$$\delta q_m = \varepsilon\,(A^{-1})_{mn}\,\partial G/\partial q_n \equiv \varepsilon\,\{G, q_m\} .$$

One could check that $Z$ is canonical invariant and, in general, proceed along the familiar lines of analytical mechanics, but we do not pursue this matter.

The main aim of this discussion has been to establish a formalism suitable for the further generalization to the case of spin 1/2 fields. The only new ingredient is to accept that the variables $q_n$ are anticommuting. For our purposes the only important change will be represented by the form of the invariant volumes.

Taking into account the peculiar rules for the integration of anticommuting variables, one now finds

$$d\Omega_I = d\Omega(q)(\det A)^{-1/2} = d\Omega(q)(\mathrm{Pf}\,A)^{-1} \tag{5.13}$$

and

$$Z_F = \int d\Omega(q)\,(\det A)^{-1/2}\, \exp\{-\mathcal{V}(q)\}\,. \tag{5.14}$$

## 6. - APPLICATIONS TO FIELD THEORY

6.1 We are now ready to apply the above-developed formalism to several cases of immediate physical interest. This will enable us to test the soundness of the approach, on the one hand by rederiving already-known results and, on the other, by obtaining some new consequences of this unified framework.

Let us recall that the main point of the method lies in the addition to the usual action (to be considered as a potential part)

$$\int d^4x\, \mathcal{L} \longrightarrow \mathcal{V} \tag{6.1}$$

of a kinetic term T. Starting from Eq. (2.11), this enables us to obtain Z, as given by (2.14) and (5.8) for quadratic and linear forms, respectively.

The main problem is therefore to select the expression for T. To this aim, and also in view of the forthcoming applications, it is fruitful to use the field theoretical point of view which implies the transition to the continuum limit, i.e.,

$$q_n(t) \longrightarrow \varphi_\alpha(x,t) \tag{6.2}$$

$$d\Omega(q) = \Pi_n\, dq_n \longrightarrow d\Omega(\varphi) = \Pi_x \Pi_\alpha\, d\varphi_\alpha(x,t)\,.$$

The main arguments determining the choice of $T$ are substantially the same as those which are exploited in order to limit the freedom in the form of the Lagrangian (or of the action) for the conventional theories. These are locality, implying a finite number of derivatives (for simplicity, two at most) and invariance requirements. We shall take into account the Poincaré group, internal symmetries of the global type like isospin or chirality, or local like gauge invariance. Conformal and dilatation invariance, as well as general co-ordinate transformations, will also provide some interesting hints. Notice that, as usual in these considerations, "t" has to be considered as a fixed parameter of a nature different from the $x_\mu$ labels. As such, no transformations will be considered where they can get intermixed.

The locality (and simplicity) requirement will be expressed in our formulation by accepting the following form of the kinetic term

$$T = \frac{1}{2}\int d^4x\, d^4y\, T_{\alpha\beta}(x,y)\,\dot{\varphi}_\alpha(x,t)\,\dot{\varphi}_\beta(y,t) = \tag{6.3}$$
$$= \frac{1}{2}\int d^4x\, t_{\alpha\beta}(x)\,\dot{\varphi}_\alpha(x,t)\,\dot{\varphi}_\beta(y,t)\,,$$

i.e., to a "matrix" $T_{\alpha\beta}(x,y)$ which is diagonal in the $x$ space

$$T_{\alpha\beta}(x,y) = t_{\alpha\beta}(x)\,\delta^{(4)}(x-y) \tag{6.4}$$

As a consequence, the generalized functional determinant $\mathrm{Det}\,T_{\alpha\beta}$ can be formally written in the <u>factorized</u> form

$$\mathrm{Det}\,T_{\alpha\beta}(x,y) = \exp \mathrm{Tr}\ln T_{\alpha\beta}(x,y) =$$
$$= \prod_x \det t_{\alpha\beta}(x) = \exp \delta^{(4)}(0)\int d^4x \ln\det t_{\alpha\beta}(x) \tag{6.5}$$

where $\det t_{\alpha\beta}(x)$ is now understood to be evaluated for the internal degrees of freedom only.

The fundamental expression of the partition function $Z$, Eq. (2.14),

$$Z = \int d\Omega(\varphi)\,(\mathrm{Det}\,T)^{1/2} \exp\{-\mathcal{V}\}\,,$$

can then be written in the equivalent form

$$Z = \int d\hat{\Omega}(\varphi)\, \exp\left\{-\int d^4x\, \mathcal{L}\right\}, \tag{6.6}$$

$$d\hat{\Omega}(\varphi) = d\Omega(\varphi)(\mathrm{Det}\,T)^{1/2} = \prod_{x,\alpha} \left[\det t_{\alpha\beta}(x)\right]^{1/2} d\varphi_\alpha(x,t) \tag{6.7}$$

or

$$Z = \int d\Omega(\varphi)\, \exp\left\{-\int d^4x\, \hat{\mathcal{L}}\right\}, \tag{6.8}$$

$$\hat{\mathcal{L}} = \mathcal{L} - \frac{1}{2}\,\delta^{(4)}(0)\, \ln \det t_{\alpha\beta}(x), \tag{6.9}$$

depending on whether one wants to emphasize the invariance properties of the path integral measure or the correct set of rules for perturbation theory.

In this Section we shall analyze in several cases of interest the possible forms of $T$ which follow from the above criteria. Given their interest and the non-elementary character of the discussion, Yang-Mills theories and gravitation will be considered separately.

The first example to be discussed is, of course, represented by the model of a single scalar field $\phi$. The general form of the "potential" is

$$\mathcal{V} \longrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + P(\phi) \tag{6.10}$$

where $P(\phi)$ is arbitrary. Similarly, locality and simplicity lead to the following form of the (local) kinetic term, $T = \int d^4x\ \mathcal{T}$

$$\mathcal{T} = \frac{1}{2} F(\phi)\, \dot{\phi}^2 \tag{6.11}$$

with $F(\phi)$ a Lorentz invariant function of the field $\phi$. The

$$Z = \int d\Omega(\phi)\, \exp\left\{-\int d^4x \left[\mathcal{L} - \frac{1}{2}\delta^{(4)}(0)\ln F\right]\right.. \tag{6.12}$$

This is the most one can say on the basis of just Poincaré invariance. We realize once again that the freedom in the measure of the functional integral is replaced in the present method by the arbitrariness in the form of the kinetic term. Of course, the simplest choice $F(\phi) = 1$ leads back to the usual expression of $Z$ and to the conventional Feynman rules.

It is amusing to rephrase the boson problem in the framework of the first order formalism. The introduction of an auxiliary field $B(x,t)$ is particularly convenient. We start from

$$\mathcal{V} \to \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\lambda^2\phi^4 + \frac{1}{2}\mu^2\phi^2 + + \frac{1}{2}\left(B + i\mu^2/2\lambda + i\lambda\phi^2\right)^2 \equiv \mathcal{L}_\phi + \frac{1}{2}\left[B + i\left(\mu^2/2\lambda + \lambda\phi^2\right)\right]^2. \tag{6.13}$$

We dispose of the freedom in the choice of the kinetic term by writing it as

$$\mathcal{T} = F(\phi,B)\dot{\phi} + G(\phi,B)\dot{B} \tag{6.14}$$

Notice that from the point of view of the Lagrange equations (2.6) describing the evolution in the parameter $t$, one can add to $\mathcal{T}$ any total $t$ derivative and the two terms in Eq. (4.6) can be considered equivalent when

$$G\dot{B} = F\dot{\phi} + dT/dt$$

Since this would imply that $G = \partial\Gamma/\partial B$ and $F = -\partial\Gamma/\partial\phi$, one has also

$$\frac{\partial G}{\partial \phi} = -\frac{\partial F}{\partial B} \tag{6.15}$$

and the simplest solution is $F = B$, $G = -\phi$.

We shall therefore choose

$$\mathcal{T} = B\dot{\phi} - \dot{B}\phi \tag{6.16}$$

Referring back to the previous formulae of Section 5, we find that

$$A_{ij} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{6.17}$$

so that the generating functional becomes simply

$$Z = \int d\Omega(\phi)\, d\Omega(B) \exp\left\{-\int d^4x\, \mathcal{L}\right\} = = \int d\Omega(\phi) \exp\left\{-\int d^4x\, \mathcal{L}_\phi\right\} \tag{6.18}$$

after performing the elementary integration over B *).

This discussion of the first order formalism leads us naturally to the case of spinor fields. One has

$$\mathcal{L} = \bar{\psi}(\gamma\cdot\partial+m)\psi + \mathcal{L}'(\psi,\chi) \rightarrow \mathcal{V} \tag{6.19}$$

($\chi$ represents the set of other fields). Following the previous discussion, one can write for $\mathcal{T}$

$$\mathcal{T} = \bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi \,. \tag{6.20}$$

Since $H \equiv \mathcal{V}$ and $\det A = 1$ we recover at once the usual expression for

$$Z = \int d\Omega(\chi)\, d\Omega(\psi)\, d\Omega(\bar{\psi}) \exp\left\{-\int d^4x\, \mathcal{L}\right\} \tag{6.21}$$

As appears from these examples, we have so far reproduced well-known results so that little advantage seems to derive from the use of our approach. This is, however, related to the particularly simple form of the kinetic term, on which very little has been imposed from the point of view of invariance requirements. If more specific symmetry properties are enforced on $\mathcal{T}$, our method provides an unambiguous determination of the invariant volume to be used in the functional approach. The case of conformal invariance we are going to discuss represents a good illustration of this point.

6.2 Let us determine the integration measure to be used for a conformal invariant formulation of the functional integral. For the sake of simplicity, the analysis will be limited to dilatation, but the generalization to the other conformal transformations is straightforward.

We first examine the scalar model where the "potential" $\mathcal{V} = \int \mathcal{L}\, d^4x$ is, as usual, determined to be

$$\mathcal{V} \rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\lambda}{4}\phi^4 \tag{6.22}$$

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*) As already mentioned, given the form (6.17) of $A_{ij}$, one might think of B as the conjugate momentum to $\phi$, $B = \partial\tau/\partial\dot{\phi}$ (or analogously $\phi$ to B).

and is invariant under dilatations (at the classical level). As far as the kinetic term is concerned, the simple choice we used in the previous section, namely $\tau = 1/2\ (\dot{\phi})^2$, is not satisfactory from the point of view of the transformation properties under dilatations.

Since dilatations concern $x_\mu$ and the field $\phi$ and <u>not</u> the parameter $t$ (remember that it is part of our general philosophy to define symmetry operations to work at fixed $t$), the dimensionality of $\phi$ and $\dot{\phi}$ is the same, namely $-1$ (in units of length). Thus $\mathcal{T} = 1/2\dot{\phi}^2$ has dimensionality $-2$ rather than $-4$ as it is required to make $T$ (and therefore $Z$) invariant.

The above discussion leads to the unique choice

$$\mathcal{T} = \frac{1}{2}\phi^2\dot{\phi}^2 \quad \text{i.e.} \quad F(\phi) = \phi^2. \tag{6.23}$$

This result has quite an interesting consequence. We obtain

$$\begin{aligned} Z &= \int d\Omega(\phi)\,(\mathrm{Det}\,\phi)\,\exp\{-\mathcal{V}\} = \int d\hat{\Omega}(\phi)\,\exp\{-\mathcal{V}\} \\ &= \int d\Omega(\phi)\,\exp\{-\int d^4x\,\hat{\mathcal{L}}\} \end{aligned} \tag{6.24}$$

where

$$\hat{\mathcal{L}} = \mathcal{L} - \delta^{(4)}(0)\ \ln\phi\,. \tag{6.25}$$

Should we have chosen $\mathcal{T} = 1/2\ \dot{\phi}^2$, the generating functional would have taken the usual form

$$Z = \int d\Omega(\phi)\,\exp\{-\int d^4x\,\mathcal{L}\}\,. \tag{6.26}$$

This quantity is not even formally dilatation invariant though, as it can be checked studying the behaviour of the volume element $d\Omega(\phi) = \Pi_x d\phi_x$ under dilatations.

We thus see that while the usual canonical quantization as given by (6.26) does not define a formal perturbative expansion which is dilatation invariant, the expression (6.24) has this property. Feynman diagrams are different for the two theories. In particular, the breaking of dilatation invariance is concentrated in the ultra-violet effects, as the structure of the term added to $\mathcal{L}$ tells us.

The presence of the $\ln\phi$ piece has an amusing consequence. Let us first of all regularize the theory by introducing a cut-off $K_M$ *), so $\delta^{(4)}(0) \to K_M^4$. We then look for a constant (co-ordinate independent) minimum $\phi_0$ of $\hat{\mathcal{L}}$ namely of $\lambda/4\ \phi^4 - K_M^4 \ln\phi$. The result is

$$\phi_0^4 = k_M^4 \lambda^{-1} \tag{6.27}$$

We see that the request of formal invariance of the generating functional under dilatation determines a non-zero background value $\phi_0$, as a candidate for a vacuum expectation value for the field. In any case, this discussion shows that perturbing around $\phi = 0$ is not possible in general and some sort of expansion around a background solution is necessary **).

Similar conclusions can be reached when several fields are present. Determining the general expression of the kinetic term for a coupled system of bosons and fermions requires, however, a slightly more complicated discussion which we prefer to leave to a following work. On the other hand, keeping in mind the integration rules for anticommuting fermion variables, it is not hard to be convinced that that the additional factor $\propto \ln\phi$ in $\hat{\mathcal{L}}$ appears with opposite sign for bosons and fermions.

In particular, it can be seen that the invariant volume takes on the form

$$d\Omega_I = d\Omega \exp\left\{ (n_B - \tfrac{1}{2} n_F)\, \delta^{(4)}(0) \int d^4x \ln\phi \right\} \tag{6.30}$$

where $n_B$ is the number of bosonic fields of dimensionality 1 and $n_F$ is the number of fermionic fields of dimensionality 3/2. Recall that a Majorana field counts for four, a Dirac field for eight and auxiliary bosonic fields of dimensionality two do not have any effect.

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*) Working on a lattice, $K_M^{-1} = L$ would correspond to the lattice spacing.

**) A more realistic case is represented by the linear $\sigma$ model[5] in the conformal invariant limit $m \to 0$. One has $\phi \to (\pi_\alpha, \sigma)$ and the above relation becomes $\langle\pi_\alpha\rangle = 0$, $\langle\sigma\rangle = f_\pi \propto k_M/\lambda^{1/4}$ to be compared with the standard one $f_\pi = |m|/\lambda^{1/2}$.

In the case of the elementary Wess-Zumino model[6] with scalar, pseudoscalar and Majorana fields, we have indeed $n_B = 2$ (auxiliary fields do not count for our purpose) and $n_F = 4$ so that the exponent in Eq. (6.30) vanishes. This is a simple example of the "magic" cancellations of divergences dictated by supersymmetry (in the present case the divergences are related to the definition of the vacuum). One of the advantages of supersymmetric theories with respect to the ordinary ones is indeed represented by an improved definition of the volume. We shall come back to this point later.

6.3 Another important case in which one finds deviations from the "naïve" Feynman rules is represented by the theory of pionic systems, based on non-linear realizations of chiral symmetry. For the formal details of the treatment we refer to the classic paper by Gerstein, Jackiw, Lee and Weinberg[7]. Let us just recall that one introduces a pion field $\phi_a$, a = 1, 2, 3, which transforms, in the standard parametrization, as

$$\delta_a^{\pm} \phi_b = \frac{1}{2} \varepsilon_{abc} \phi_c \pm \frac{1}{2} \left\{ \frac{1}{2} \delta_{ab} (1 - \vec{\phi}^2) + \phi_a \phi_b \right\} \tag{6.31}$$

$\delta_a^{\pm}$ refer to the even/odd SU(2) transformations generated by the (commuting) operators $Q_a^{\pm} = 1/2\ (Q_a \pm Q_a^5)$.

In order to build an invariant kinetic term, we first notice that the transformation properties of $\phi_a$ are

$$\delta_a^{\pm} \phi_b = S_{abc}^{\pm} \phi_c \tag{6.32}$$

where

$$S_{abc}^{\pm} = \frac{1}{2} \varepsilon_{abc} \pm \frac{1}{2} (\phi_a \delta_{bc} + \phi_b \delta_{ac} - \phi_c \delta_{ab}) \tag{6.33}$$

As a consequence, one can easily ascertain that

$$\mathcal{T} = \frac{1}{2} k_{ab} \dot{\phi}_a \dot{\phi}_b , \tag{6.34}$$

with

$$k_{ab} = (1 + \vec{\phi}^2)^{-2} \delta_{ab} , \tag{6.35}$$

is the only quadratic form which is invariant under the chiral transformations. We shall therefore choose (6.35) as the appropriate expression for the kinetic term.

In a similar way, we find that the standard Lagrangian $SU(2) \times SU(2)$ invariant is of the form (of course, a well-known result)

$$\mathcal{L} = \frac{1}{2} k_{ab} \partial_\mu \phi_a \partial_\mu \phi_b \,. \tag{6.36}$$

The by now standard steps then lead us to the result

$$\begin{aligned} Z &= \int d\Omega(\phi_a) (\mathrm{Det}\, k_{ab})^{1/2} \exp\left\{-\int d^4x\, \mathcal{L}\right\} \\ &= \int d\hat{\Omega}(\phi_a) \exp\left\{-\int d^4x\, \mathcal{L}\right\} \\ &= \int d\Omega(\phi_a) \exp\left\{-\int d^4x\, \hat{\mathcal{L}}\right\} \end{aligned} \tag{6.37}$$

and

$$\hat{\mathcal{L}} = \mathcal{L} + \delta^{(4)}(0) \ln (1 + \vec{\phi}^2)^3 \tag{6.38}$$

It can be amusing to notice that the invariant volume $d\hat{\Omega}(\phi) = \Pi_{a,x} d\phi_a(x)/(1+\phi^2)^3$ which appears in (6.37) is actually the invariant group measure [see Eq. (4.22)] for the (local) chiral transformations (6.31).

The representation (6.37) defines a quantum field theory invariant under chiral transformations, the "naïve" choice (which ignores $\mathrm{Det}\, K_{ab}$) would not. The result (6.37) is known to be correct. It provides the right perturbative rules and yields a chiral invariant S matrix. However, its standard derivation is rather arduous and the present method has the advantage of reducing the problem of the functional measure to the determination of an invariant kinetic term.

A completely equivalent approach to chiral dynamics is based on the so-called non-linear $\sigma$ model. The $O(4) \sim SU(2) \times SU(2)$ invariance is realized through four scalar fields $\varphi_\alpha(x,t)$ obeying the constraint

$$\frac{1}{2} \Sigma_\alpha \varphi^2_\alpha(x,t) = f^2 \tag{6.39}$$

($f$ is an arbitrary dimensional constant).

The potential part is

$$\mathcal{V} = \int d^4x \; \frac{1}{2}(\partial_\mu \varphi_\alpha)^2 \tag{6.40}$$

For the generating functional the expression given by Eq. (3.3) can be used, with the (functional) constraint (6.39). The computation of $\Delta(f)$ is quite easy:

$$F_{\alpha\beta} = \varphi_\alpha \varphi_\beta \; , \qquad \Delta(f) = \Sigma_\alpha \varphi_\alpha^2 \tag{6.41}$$

so $[\Delta(f)]^{1/2}$ is a number that can be dropped from Z. We obtain

$$Z = \int \prod_{\alpha, x} d\varphi_\alpha(x,t)\, \delta\left[\frac{1}{2}\varphi_\alpha^2(x,t) - f^2\right] \exp\{-\mathcal{V}\}. \tag{6.42}$$

It is immediate to verify that (6.42) can be brought to the standard form (6.37). By performing the transformation (f=1)

$$\varphi_a = \frac{2\phi_a}{1+\vec{\phi}^2} \quad a=1,2,3\,; \qquad \varphi_4 = \frac{1-\vec{\phi}^2}{1+\vec{\phi}^2}\,, \tag{6.43}$$

one gets

$$\int d^3\varphi\, d\varphi_4\, \delta(\vec{\varphi}^2 + \varphi_4^2 - 1) F(\vec{\varphi}, \varphi_4) = \int \frac{d^3\phi}{(1+\vec{\phi}^2)^3} F\left[\frac{2\phi_a}{1+\vec{\phi}^2}, \frac{1-\vec{\phi}^2}{1+\vec{\phi}^2}\right] \tag{6.44}$$

which, in the continuum limit, coincides with (6.37).

## 7. - GAUGE THEORIES

One of the testing grounds of our approach is clearly represented by the quantization of Yang-Mills theories[8]; the specific features implied by gauge invariance are well known to require a careful treatment of the problem and lead in particular to the presence in the Feynman functional of a functional determinant, whose origin is quite natural from our point of view and which is already evident in our fundamental formula (5.25).

Thus we shall now treat gauge theories and for simplicity the definite case of an SU(2) gauge group will be considered. It is useful to first recall some notations. We introduce three gauge fields $A_\mu^a$ and the tensors $F_{\mu\nu}{}^a$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon_{abc} A_\mu^b A_\nu^c \tag{7.1}$$

It is customary to define the matrices

$$A_\mu = A_\mu^a \frac{\sigma^a}{2i} \quad , \quad F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2i} \tag{7.2}$$

so that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{7.3}$$

The field transforms according to the law

$$A_\mu \rightarrow A'_\mu = \mathcal{U}^{-1} A_\mu \mathcal{U} + \mathcal{U}^{-1} \partial_\mu \mathcal{U} \tag{7.4}$$

where

$$\mathcal{U} = e^{-i\,\omega_a(x)\sigma_a/2} \equiv e^{\omega(x)} \tag{7.5}$$

is a function of $x_\mu$. For an infinitesimal transformation one has

$$\Delta_\omega A_\mu = \partial_\mu \omega + [A_\mu, \omega] \equiv D_\mu \omega \tag{7.6}$$

or

$$\Delta_\omega A_\mu^a = \partial_\mu \omega^a + \varepsilon_{abc} A_\mu^b \omega^c \equiv D_\mu^{ab} \omega_b \,. \tag{7.6'}$$

(As usual $D_\mu \equiv \partial_\mu + [A_\mu, \,]$ is the covariant derivative.)

As a consequence of these relations, the variation of a functional $F(A)$ under an infinitesimal transformation with gauge function $\omega(x)$ is expressed by

$$\Delta_\omega F(A) = \int d^4x \; \frac{\delta F(A)}{\delta A_\mu^a(x)} \, D_\mu^{ab} \, \omega_b(x) \tag{7.7}$$

whose local version is

$$\frac{\delta}{\delta \omega_a(x)} F(A) = - D_\mu^{ab} \frac{\delta F(A)}{\delta A_\mu^b(x)} \,. \tag{7.7'}$$

The conventional gauge invariant (Euclidean) Lagrangian is, of course,

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a = - \frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu}^2 \,. \tag{7.8}$$

As in the previous sections, we introduce the 5th variable $t$ and consider the fields as functions of the four Euclidean variables $x_\mu$ and of $t$:

$$A_\mu \equiv A_\mu(x,t) \quad , \qquad F_{\mu\nu} \equiv F_{\mu\nu}(x,t) \tag{7.9}$$

The conventional Lagrangian is interpreted as a potential energy term which depends on $A_\nu$ but not on $\dot{A}_\mu$.

In order to get the new Lagrangian, one has to build a gauge invariant kinetic term. We stress that gauge transformations have been defined as usual, Eq. (7.4), and that the gauge matrix $U$ is independent of $t$. The kinetic term is then practically determined by this requirement and by the usual ones we have already discussed (i.e., at most quadratic in the derivatives and Lorentz invariant). We thus write for the kinetic term

$$T = \frac{1}{2}\int d^4x\, \dot{A}^a_\mu \dot{A}^a_\mu = -\int d^4x\, \mathrm{Tr}\, \dot{A}^2_\mu \tag{7.10}$$

Since the gauge transformations are $t$ independent, $\dot{A}^a_\mu$ is a gauge vector ($\dot{A}'_\mu = U^{-1}\dot{A}_\mu U$) and $T$ is gauge invariant. The complete Lagrangian thus has the form*)

$$L = T - \mathcal{V} = \int d^4x \left\{ \frac{1}{2}\dot{A}^a_\mu \dot{A}^a_\mu - \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} \right\} \tag{7.11}$$

A comment is in order about the differences between the present situation and the usual treatment of gauge theories based on the four-dimensional Lagrangian of Eq. (7.8). In the latter case some variables (the gauge degrees of freedom) are absent from $\mathcal{L}$ and further $\mathcal{L}$ is singular because the determinant $\det\{\partial^2\mathcal{L}/\partial\partial_4 A_\mu \partial\partial_4 A_\nu\}$ vanishes, thus preventing us from expressing $\partial A_\mu/\partial x_4$ in terms of the momenta. In the present case some gauge variables will again be absent from $\mathcal{V}$ but their $t$ derivatives appear in $T$: $L$ is not singular

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*) It is interesting to notice that the kinetic term $\propto \dot{A}^2_\mu$ does not preserve scale invariance and so is for the path integral volume. Extra multiplicative factors like $A^2_\mu$, which would give the correct dimensional matching, are forbidden by gauge invariance.

The clash between gauge and conformal invariance can be avoided if non-local terms are used or if other fields are considered (e.g., with a scalar, neutral field $\phi$, the term $\phi^2\dot{A}^2_\mu$ is allowed), in particular if one studies the coupling of Yang-Mills fields to gravitation.

since $\det\{\partial^2\tau/\partial\dot{A}_\mu\partial\dot{A}_\nu\}$ is different from zero. Thus the problem, as we approach it, is simply characterized by a number of ignorable co-ordinates whose momenta are present in the Lagrangian. The situation is thus the same as that discussed in Section 4 with discrete notation, namely a case where an invariance group is present and there are ignorable co-ordinates.

The immediate way of proceeding would be to use directly for the partition function the expression given in Eq. (2.16), namely

$$Z = \int d\Omega(A_\mu^a) \exp\{-\hat{\mathcal{V}}\} \tag{7.12}$$

where, in particular, $\hat{\mathcal{V}} = \mathcal{V}$ since $t_{nm} = \delta_{nm}$.

This equation is, in a certain sense, physically acceptable even if one has to perform a summation on equivalent configurations related to gauge equivalent choices of the variables. For instance, in versions of gauge theories like QCD on a lattice, some field theoretical quantities of interest are computed just using the discrete analogue of (7.12).

However, in the continuum limit, the presence of a local group of gauge transformations leads to an infinite number of equivalent configurations. These infinite factors are quite troublesome and do not allow the simple introduction of a consistent perturbation expansion. One thus has to resort to the formalism developed in Section 4 which allows us to obtain a final result where those redundant infinite sums are fully limited. As we shall see, this procedure will give rise to the celebrated Faddeev-Popov determinant factors, valid for any choice of gauge.

In order to reach such a form for the generating functional, Eq. (4.25) has to be adapted to the present case. The main task is therefore to establish the analogues of the relations which have been used in that part.

We first perform the change of co-ordinates which separates the dynamical variables in a set of gauge independent fields $B_\mu$ (the analogues of the Q's), constrained by some gauge fixing conditions, and the remaining, ignorable gauge variables (the $\phi$'s). This is achieved by suitably generalizing the relation (7.4). We write

$$A_\mu = M^{-1} B_\mu M + M^{-1} \partial_\mu M \tag{7.13}$$

The matrix $M$ will contain the gauge variables and as such is $x$ and $t$ dependent.

We stress that Eq. (7.12) must not be confused with a gauge transformation but defines a new, inequivalent set of fields*). The new dynamical variables are thus $B_\mu$ and $M$ and the redundancy of co-ordinates is eliminated by imposing suitable constraint conditions over $B_\mu$. The constraint is, in general, given by a (functional) relation of the form

$$F\{B_\mu^a(x)\} = 0 \tag{7.14}$$

as, for instance, in the customary Lorentz or axial gauge $\partial^\mu B_\mu^a = 0$ or $B_3^a = 0$. On the other hand, the matrix $M$ contains the Lagrange co-ordinates related to the local gauge group. They depend on a continuous index $y_\mu$, which specifies the point where the transformation is applied, and on a discrete one $\alpha$. They depend explicitly on $t$ since they will be considered as (ignorable) dynamical variables. We shall dictate these quantities by $\phi_\alpha(y,t)$.

On the other hand, if we want to preserve in the continuum limit the distinctive rôle of the parameter $t$, it is meaningful to keep the dependence on $x_\mu$ and on $t$ well separated and think of the matrix field $M$ as a functional of the $\phi_\alpha$'s. In particular, in correspondence with Eq. (7.5), we can write

$$M(x,t) = \int d^4y\, \delta^{(4)}(x-y)\, e^{-i\phi_a(y,t)\sigma_a/2}$$

In view of this fact, it will be convenient in the following to express the indices $(y,\alpha)$ by means of a single "discrete" index "$i$". So we can simply write

$$M(x,t) \equiv M[x, \phi_i(t)] \; . \tag{7.15}$$

We now wish to express our Lagrangian in terms of the new variables. For the potential term we simply have

$$\mathcal{V}(A) = \mathcal{V}(B) = \frac{1}{4g^2}\int d^4x\, F_{\mu\nu}^a(B) F_{\mu\nu}^a(B) \, .$$

For the computation of the kinetic term, we evaluate the "time" derivative of $A_\mu$. A simple calculation produces the following result

$$\dot{A}_\mu = M^{-1}\Pi_\mu M \tag{7.16}$$

where $\Pi_\mu$ is defined by the relation

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*) Under a gauge transformation parametrized by a matrix $U(x)$ one has

$$B_\mu \to B'_\mu = B_\mu \; , \qquad M \to M' = M\mathcal{U} \, .$$

$$\Pi_\mu = \dot{B}_\mu - D^B_\mu \left( M \frac{\partial M^{-1}}{\partial t} \right) \tag{7.17}$$

and $D^B_\mu$ is the covariant derivative in the $B_\mu$ basis, $D^B_\mu \equiv \partial_\mu + [B_\mu, ]$. This enables us to write

$$T = \frac{1}{2} \int d^4x\, \Pi^a_\mu \Pi^a_\mu = - \int d^4x\, \mathrm{Tr}\, \Pi^2_\mu \tag{7.18}$$

By further inserting the representation (7.14) into Eq. (7.17) one obtains

$$\Pi_\mu = \dot{B}_\mu - \Sigma_i D^B_\mu \left( M \frac{\partial M^{-1}}{\partial \phi_i} \right) \dot{\phi}_i \equiv \dot{B}_\mu - E^i_\mu \dot{\phi}_i \tag{7.19}$$

where the "sum" over $i$ corresponds to an ingegral on $y$ and to a summation on $\alpha$.

The final step is to compute $\partial B_\mu / \partial \phi_i$ at fixed $A_\mu$ [compare with (4.7)]:

$$\left.\frac{\partial B_\mu}{\partial \phi_i}\right)_A = \frac{\partial}{\partial \phi_i} \left( M A_\mu M^{-1} + M \partial_\mu M^{-1} \right) \Big|_A =$$

$$= \partial_\mu \left( M \frac{\partial M^{-1}}{\partial \phi_i} \right) + \left[ B_\mu, M \frac{\partial M^{-1}}{\partial \phi_i} \right] = D_\mu \left( M \frac{\partial M^{-1}}{\partial \phi_i} \right) \equiv E^i_\mu \tag{7.20}$$

Thus we can write

$$\Pi_\mu = \dot{B}_\mu - \Sigma_i \left.\frac{\partial B_\mu}{\partial \phi_i}\right)_A \dot{\phi}_i \tag{7.21}$$

(with the above understanding on the meaning of $\Sigma_i$).

This completes the check that the case of gauge theories can be formulated in a way which is completely analogous to the discrete treatment. One can therefore exploit the final form for $Z$ which, for gauge fields, reads

$$Z = \int d\Omega [B^a_\mu(x)] \exp\left\{ - \int d^4y\, \mathcal{L} \right\} . \tag{7.22}$$

$$\times \prod_x \delta\left[F(B_\mu^a(x))\right] \mathrm{Det}\left[\frac{\delta}{\delta\omega(x)} \Delta_\omega F(B)\right] \tag{7.22 cont.}$$

given the particularly simple form of the kinetic term matrix, i.e., $t_{ij} \to \delta_{\mu\nu}\delta_{ab}$ $[\delta/\delta_\omega \Delta_\omega F$ can be computed using Eq. (7.7')]. This result is completely equivalent to the Faddeev-Popov prescription for quantizing gauge fields[9].

We remind the reader that in the FP approach one works with the gauge-dependent field $A_\mu(x)$ and that the determinant of interest involves the infinitesimal gauge transform of the constraint $F(A)$, i.e., Det $(\delta/\delta\omega\ \Delta F)$. This is identical in form to the determinant in our formula for $Z$ once $A \to B$, which confirms the equivalence of the two methods.

## 8. - THEORY OF GRAVITATION

Quantum gravity bears a close resemblance to non-Abelian gauge theories and its functional quantization will proceed in this case in the same manner as for Yang-Mills fields. The relevant expressions will be somewhat more cumbersome due to the nature of the invariance properties of the theory[10].

We shall formulate the theory in terms of the "vierbein" fields $V_\mu^a(x)$ related to the metric tensor by

$$g_{\mu\nu}(x) = \Sigma_a V_\mu^a(x) V_\nu^a(x) \tag{8.1}$$

It is well known that in this approach to gravitation there are two invariance principles to be considered, namely under arbitrary changes of co-ordinates, $x^\mu \to x'^\mu(x)$, and under local Lorentz transformations (actually four-dimensional rotations in Euclidean space).

The combined action of these transformations gives, for the fundamental fields,

$$V_\mu^a(x) \longrightarrow V_\mu^{a'}(x) = L_{ab}[x'(x)]\ V_\nu^b[x'(x)] \frac{\partial x'^\nu}{\partial x^\mu} \tag{8.2}$$

($L_{ab}$ is the orthogonal matrix defining the original Lorentz transformation). It is particularly useful to define the behaviour of the system under the infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \tag{8.3}$$

$$L_{ab} \simeq \delta_{ab} + \omega_{ab}(x)$$

where $\varepsilon^{\mu}(x)$, $\omega_{ab}(x)$ are arbitrary, infinitesimal functions.

The corresponding variation of the vierbein fields is determined by the ten functions $\varepsilon^{\mu}(x)$, $\omega_{ab}(x)$, namely

$$\Delta V_{\mu}^{a}(x) \equiv V_{\mu}^{a\prime}(x) - V_{\mu}^{a}(x) = \varepsilon^{\lambda}\partial_{\lambda}V_{\mu}^{a}(x) + \tag{8.4}$$

$$+ V_{\lambda}^{a}(x)\partial_{\mu}\varepsilon^{\lambda} + \omega_{ab}(x)V_{\mu}^{b}(x)$$

and for an arbitrary functional $F(V_{\mu}^{a})$ we shall find

$$\Delta F(V_{\mu}^{a}) = \int d^4x \, \frac{\delta F}{\delta V_{\mu}^{a}} \, \Delta V_{\mu}^{a}(x) \tag{8.5}$$

According to our general procedure, we now introduce the "time" variable $t$ and consider the fields as functions of the four Euclidean variables $x_{\mu}$ and of $t$:

$$V_{\mu}^{a} \equiv V_{\mu}^{a}(x,t) \,, \qquad g_{\mu\nu} \equiv g_{\mu\nu}(x,t) \tag{8.6}$$

For the conventional Lagrangian, to be interpreted as potential energy, we shall take the familiar form*), allowing the possibility of a cosmological term

$$\mathcal{V} = \int d^4x \, \mathcal{L} = -\frac{1}{4}\int d^4x \sqrt{g}\,(R + 2\Lambda) \tag{8.7}$$

$\mathcal{V}$ depends on $V_{\mu}^{a}$, $\partial_{\nu}V_{\mu}^{a}$ but not on $\dot{V}_{\mu}^{a}$. Such a dependence is, of course, introduced by the kinetic term. Since only $t$ independent transformations are considered, $\dot{V}_{\mu}^{a}$ will transform in the same way as $V_{\mu}^{a}$. For the kinetic part one can thus write the expression

$$T = \frac{1}{2}\int d^4x \sqrt{g}\, T_{ab}^{\mu\nu}\, \dot{V}_{\mu}^{a}\, \dot{V}_{\nu}^{b} \tag{8.8}$$

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*) $\mathcal{V}$ can be explicitly written in terms of the vierbein fields. By a partial integration one finds

$$\mathcal{V} = \int d^4x \left\{ \begin{pmatrix} \mu_1 & \nu_1 & a_1 \\ \mu_2 & \nu_2 & a_2 \end{pmatrix} \partial_{\mu_1}V_{\nu_1}^{a_1}\,\partial_{\mu_2}V_{\nu_2}^{a_2} + \frac{1}{2}\Lambda\sqrt{g} \right\}$$

where the dimensionless operator $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ is given for instance in Ref. 11).

where $T^{ab}_{\mu\nu}$ is a local function of $V^\mu$, which transforms as a second order controvariant tensor, namely

$$T^{\mu\nu\,\prime}_{ab}(x') = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\rho}\, T^{\lambda\rho}_{cd}(x)\, L_{ac}(x)\, L_{bd}(x) . \qquad (8.9)$$

According to our general rules of the game, the tensor $T^{\mu\nu}_{ab}$ will enter into the final result only through its determinant. It is easy to show that the form of such a determinant depends only on the transformation properties of $T^{\mu\nu}_{ab}$ and not on its detailed form*).

One thus obtains

$$\det T^{\mu\nu}_{ab} = \text{const.}\,(g)^{-4} = \text{const}\,(V)^{-8}$$

where

$$V = \det V_\mu{}^a = \sqrt{g} \qquad (8.10)$$

and the Lorentz index "a" has been taken into account in computing the power of the determinant. Thus

$$\det\left(\sqrt{g}\; T^{\mu\nu}_{ab}\right) = \text{const.}\; g^4 = \text{const.}\; V^8 \qquad (8.11)$$

As in the previous case of QCD we start by accepting the redundancy implied by the summation on gauge equivalent configurations. This is acceptable when the number of equivalent configurations is finite, as in the case of a lattice formulation of quantum gravity. Applying the standard procedure to the complete Lagrangian

$$L = T - \mathcal{V}$$

we obtain

$$\mathcal{Z} = \int d\hat{\Omega}(V_\mu{}^a)\exp\{-\mathcal{V}\} = \int d\Omega\,(V_\mu{}^a)\exp\left\{-\int d^4x\;\hat{\mathcal{L}}\right\} \qquad (8.12)$$

with

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*) The most general form of $T^{ab}_{\mu\nu}$ is easily seen to be

$$T^{\mu\nu}_{ab} = c_1\delta_{ab}\, g^{\mu\nu} + c_2 V^\mu_a V^\nu_b + c_3 V^\nu_a V^\mu_b$$

$$d\hat{\Omega}(V_\mu^a) = \prod_{x,a} dV_\mu^a(x)\ V^4(x) \tag{8.13}$$

or

$$\hat{\mathcal{L}} = \mathcal{L} - 4\delta^{(4)}(0)\ \ln V \tag{8.14}$$

One first of all notices that in order to achieve an invariant volume $d\hat{\Omega}(V)$, the multiplicative factors $\det V_\mu^a(x)$ are required. A similar phenomenon has appeared in the discussion of scale invariant theories where, for a spinless field, $d\hat{\Omega}(\phi) = \pi_x \phi(x) d\phi(x)$.

Clearly, the analogy is not accidental since dilatations

$$V_\mu^a(x) \longrightarrow V_\mu^{a\prime}(x) = c V_\mu^a(cx) \tag{8.15}$$

are a particular case of co-ordinate reparametrizations and one may indeed be easily convinced that the volume $d\hat{\Omega}(V)$ is invariant under (8.15).

As in the corresponding case of a scalar field, the effect of the extra factor $[V(x)]^4$ is to produce a vanishing contribution to the configuration $V = 0$. We are thus forced to base a perturbative treatment of the theory on the expansion around a non-vanishing classical background, i.e.,

$$V_\mu^a(x) = [V_\mu^a(x)]_0 + V_\mu^{\prime a}(x) \tag{8.16}$$

In particular, if one considers the effective Lagrangian $\hat{\mathcal{L}}$ in Eq. (8.14), it is interesting to seek a constant solution which minimizes the "potential" $\Lambda V(x) - 8\delta^{(4)}(0)\ \ln V(x)$. Since in this case

$$[V_\mu^a(x)]_0 = c\delta_{\mu a} \tag{8.17}$$

the following result is easily obtained

$$c^4 = 8\ k^4{}_M / \Lambda \tag{8.18}$$

namely a relation between the lattice spacing $k_M^{-1}$, the cosmological constant and the vacuum expectation value $C$ of the vierbein field (to be identified with $C = G_N^{-1/2}$, $G_N$ being the Newton constant).

We plan to present a detailed physical discussion of this result in a separate work. Let us close this discussion by noticing that, as in the case of the Wess-Zumino supersymmetry, $N = 1$ supergravity[12)] leads to a complete cancellation between boson (graviton) and fermion (gravitino) effects so that in Eq. (8.14) $\hat{\mathcal{L}}$ coincides with $\mathcal{L}$ .

Let us now concentrate on the treatment of quantum gravity in the full continuum limit. As for Yang-Mills theories, this requires taking care of the integration over gauge equivalent configurations. In order to dispose of this difficulty, one has to exploit the approach described in Section 4 and already used for gauge fields.

The first step is to use for the fields $V_\mu^a(x)$ a representation which displays the dependence on the gauge parameters, to be then considered as dynamical variables. This can be done bearing in mind the transformation law, Eq. (8.2): we put

$$V_\mu^a(x,t) = M_{ab}(x,t)\, \omega_\nu^b[f,t]\, \frac{\partial f^\nu}{\partial x^\mu} \tag{8.19}$$

where $M_{ab}$ is a real orthogonal matrix and $f^\mu$ an arbitrary controvariant vector function.

Furthermore, the correct generalization of the procedure developed in the discrete case suggests that we should consider, as we already did for Yang-Mills theories, $f^\mu$ and $M_{ab}$ as functionals of the gauge variables characterizing changes of co-ordinates and local Lorentz transformations. Thus one may write

$$f^\mu(x,t) = \int d^4y\, \delta^{(4)}(x-y)\, f^\mu(y,t)$$

and express this in the form

$$f^\mu(x,t) = f^\mu[x, \phi_i(t)] \tag{8.20}$$

where the index "i" includes both the discrete $(\mu)$ and the continuous $(y)$ ones. Similarly, we shall use

$$M_{ab}(x,t) = M_{ab}[x, \theta_j(t)] \tag{8.21}$$

and again "j" runs continuous (y) and discrete (ab) indices*).

*) Several, more explicit representations for $M$ can easily be offered according to the possible parametrizations for the Lorentz generators. For instance,

$$M(x,t) = \int d^4y\, \delta^{(4)}(x-y)\, e^{i\sigma_{ab}\omega_{ab}(y,t)}$$

The quantities $w_\nu{}^b[f,t]$, $\phi_i(t)$, $\theta_j(t)$ will now play the rôle of Lagrangian co-ordinates. There will be an unavoidable redundancy of which we dispose by an appropriate set of constraint conditions (ten actually)

$$F\,[\,w_\mu{}^a(x)\,] = 0 \tag{8.22}$$

The tensor $g_{\mu\nu}(x,t)$ will be written in terms of the new variables as

$$\begin{aligned} g_{\mu\nu}(x,t) &= \Sigma_a\, V_\mu^a(x,t)\, V_\nu^a(x,t) = \\ &= \frac{\partial f^\lambda}{\partial x^\mu}\, C_{\lambda\rho}\, \frac{\partial f^\rho}{\partial x^\nu} \end{aligned} \tag{8.23}$$

with

$$C_{\lambda\rho} = \Sigma_b\, w_\lambda^b\,[\,f,t)\,]\, w_\rho^b\,[\,f,t\,]\,. \tag{8.24}$$

We can now proceed along standard lines and by some slightly cumbersome calculations we obtain (as usual!)

$$\dot{V}_\mu^a(x,t) = \eta_{ab}\,\Pi_\nu{}^b[\,f,t\,]\,\frac{\partial f^\nu}{\partial x^\mu} \tag{8.25}$$

where

$$\Pi_\mu^a = \frac{\partial w_\mu^a}{\partial t} - \Sigma_j \left.\frac{\partial w_\mu^a}{\partial \theta_j}\right)_{\text{fixed}\,\nu} \dot{\theta}_j - \Sigma_j \left.\frac{\partial w_\mu^a}{\partial \phi_j}\right)_{\text{fixed}\,\nu} \dot{\phi}_j \tag{8.26}$$

These relations are the exact analogues of (7.14) and (7.21) for gauge theories.

We finally substitute $\dot{V}_\mu^a$ as given by (8.25) in the starting expression (8.8) of the kinetic term. Taking into account the transformation properties of the tensor $T^{\mu\nu}_{ab}$ and the invariance of the volume element $\sqrt{g}\, d^4x$, this gives

$$T = \frac{1}{2}\int d^4f\,\sqrt{C(f)}\;T^{\lambda\rho}_{ab}(f,t)\,\Pi^a_\lambda(f,t)\,\Pi^b_\rho(f,t) \tag{8.27}$$

where

$$C(f) = \det C_{\mu\nu}(f) \tag{8.28}$$

Furthermore

$$\mathcal{V}(v_\mu^a) = \mathcal{V}(w_\mu^a) = -\frac{1}{4}\int d^4 \xi \sqrt{c(\xi)}\ [R(c) + 2\Lambda] \tag{8.29}$$

The Einstein theory of gravitation is thus shown to have been cast in the standard form (4.11). One can follow the general procedure which leads to the final form for $Z$

$$Z = \int d\hat{\Omega}(w_\mu^a)\ \exp\{-\mathcal{V}(w)\} \cdot \\ \cdot \prod_{x,a} \delta[F(w_\mu^a(x))]\ \mathrm{Det}\left[\frac{\delta}{\delta\eta}\Delta_\eta F(w)\right] \tag{8.30}$$

$\Delta_\eta F(w)$ being the variation of the constraint function $F(w)$ consequent to the transformation of the field $w_\mu^a$ determined by the ten functions $\eta_i(x)$. This formula represents the prescription to quantize gravitation à la FP.

## 9. - NEW CANONICAL FORMALISM

### 9.1 Introductory Remarks

This chapter is devoted to a reflection about the similarities between Quantum Field Theory and Classical Statistical Mechanics that could lead to possible useful developments. To start with, let us remind the reader that in the whole course of our treatment, the introduction of the "time" $t$ has been interpreted as a technical step in order to define the form of the measure in our functional integral. As we have seen in the course of the entire paper, this procedure has allowed us, on the one hand, to determine the invariance properties of the measure with respect to the different groups of the various theories and, at the same time, it has allowed us to control the transformation properties for general transformations of Lagrangian variables.

A more radical point of view is also possible, namely to assume as fundamental and exploit completely the equivalence of Quantum Field Theory in four-dimensional Euclidean space-time and Classical Statistical Mechanics where the fields depend on the four-dimensional Euclidean $x_\mu$ and a fifth parameter $t$. This equivalence has been used in the so-called stochastic approach to Quantum Field Theory, a new and original point of view.

Most of this chapter is dedicated to a detailed display of the classical canonical formalism (in the fifth parameter $t$) underlying the classical statistical mechanics treatment. At the end we shall rapidly hint at possible new ways suggested by these developments.

## 9.2 Scalar Field

Let us start from a simple theory with a single scalar field. The new Lagrangian is

$$L = \int d^4x \, \frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \mathcal{V} \tag{9.1}$$

where $\mathcal{V}$ is the customary Lagrangian:

$$\mathcal{V} = \int d^4x \, \mathcal{L} = \int d^4x \left\{ \frac{1}{2} (\partial_\mu\varphi)^2 - \mathcal{U} \right\}. \tag{9.2}$$

The dependence of $\varphi$ on $t$ is determined by the classical equation of motion ensuing from the Lagrangian (9.1):

$$\ddot{\varphi} - \Box\varphi - \frac{\partial \mathcal{U}}{\partial\varphi} = 0 \tag{9.3}$$

The construction of the constants of the evolution in $t$ due to Poincaré invariance of the theory is a simple exercise. Indeed, let us define

$$P_\mu = \int d^4x \, \frac{\partial\varphi}{\partial t} \frac{\partial\varphi}{\partial x_\mu} \tag{9.4}$$

$$L_{\mu\nu} = \int d^4x \, \frac{\partial\varphi}{\partial t} (x_\mu\partial_\nu - x_\nu\partial_\mu)\varphi \tag{9.5}$$

We have of course

$$\frac{dP_\mu}{dt} = 0 \, , \qquad \frac{d\dot{L}_{\mu\nu}}{dt} = 0 \tag{9.6}$$

The Poisson brackets algebra of these constants of motion is indeed the Poincaré one. To see this, we must first introduce the canonical picture:

$$p(x,t) = \frac{\delta L}{\delta\dot{\varphi}} = \frac{\partial\varphi}{\partial t} \tag{9.7}$$

$$H = \int d^4x \left\{ \frac{1}{2} p^2 + \mathcal{V} \right\} = \int d^4x \left\{ \frac{1}{2} p^2 + \frac{1}{2} (\partial_\mu\varphi)^2 - \mathcal{U} \right\}$$

We now introduce the classical fundamental (equal $t$) Poisson brackets*)

*) $$[A,B]_{PB} = \int d^4x \left( \frac{\delta A}{\delta\varphi(x)} \frac{\delta B}{\delta p(x)} - \frac{\delta A}{\delta p(x)} \frac{\delta B}{\delta\varphi(x)} \right)$$

$$[\varphi(x,t),\ p(y,t)]_{PB} = \delta^{(4)}(x-y)$$

$$[\varphi(x,t),\ \varphi(y,t)]_{PB} = 0 \qquad (9.8)$$

$$[p(x,t),\ p(y,t)]_{PB} = 0$$

We then see easily that

$$[P_\mu,\ \varphi(x,t)]_{PB} = -\partial_\mu \varphi(x,t) \qquad (9.9)$$

Namely, $P_\mu$ generates translations and

$$[L_{\mu\nu},\ \varphi(x,t)]_{PB} = -(x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi(x,t) \qquad (9.10)$$

namely, $L_{\mu\nu}$ generates rotations of the scalar field. It is also a routine exercise to check that the Poisson brackets of the ten quantities $P_\mu$ and $L_{\mu\nu}$ realize the Poincaré algebra.

## 9.3 Gauge Field Theory

Let us now pass to the more interesting case of gauge field theories. We rewrite the main formulae for the case of $SU(2)$ that, as usual, we take as a model. The new Lagrangian is

$$L = \int d^4x\ \frac{1}{2}\left(\frac{\partial A^a_\mu}{\partial t}\right)^2 - \mathcal{V} \qquad (9.11)$$

where $\mathcal{V}$ is the customary (Euclidean) Lagrangian

$$\mathcal{V} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (9.12)$$

From (9.11) we have the evolution equation in $t$

$$\frac{\partial^2 A^a_\mu}{\partial t^2} - D^{ab}_\lambda F^b_{\lambda\mu} = 0 \qquad (9.13)$$

The conjugate momenta are given by

$$p^a_\mu = \frac{\partial A^a_\mu}{\partial t} \qquad (9.14)$$

It is worthwhile to observe that in the five-dimensional $(x_\mu, t)$ formalism all the conjugate momenta are perfectly defined, namely the "kinetic term" is non-singular. We point out once more that in the present frame the effect of the gauge symmetry is that some dynamical variables become ignorable ones. In the simple example of electrodynamics we have

$$L = \frac{1}{2}\int d^4x \left(\frac{\partial A_\mu}{\partial t}\right)^2 - \frac{1}{4}\int d^4x\, F_{\mu\nu}F_{\mu\nu}$$

We see that all the Lagrangian co-ordinates admit a well-defined conjugate momentum. The Lagrangian co-ordinate $\partial_\mu A_\mu$ is absent from the "potential" $1/4\, F_{\mu\nu}F_{\mu\nu}$ as a consequence of gauge invariance; however, its conjugate momentum can be defined. Also the conjugate momentum to $A_4$ exists, it is simply $\dot{A}_4$.

We now go back to SU(2) and write down the constants of motion due to Poincaré invariance:

$$P_\mu = \int d^4x\, \frac{\partial A^a_\lambda}{\partial t}\frac{\partial A^a_\lambda}{\partial x_\mu} \tag{9.15}$$

$$J_{\mu\nu} = \int d^4x\, \frac{\partial A^a_\lambda}{\partial t}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)A^a_\lambda + \int d^4x\, \frac{\partial A^a_\lambda}{\partial t}\Sigma^{\lambda\rho}_{\mu\nu}A^a_\rho \tag{9.16}$$

where

$$\Sigma^{\lambda\rho}_{\mu\nu} = \frac{1}{2}\left(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}\right).$$

As we are accustomed to from the scalar case, introducing now the Poisson fundamental brackets

$$[A^a_\mu(x,t), p^b_\nu(y,t)]_{PB} = \delta_{ab}\delta_{\mu\nu}\delta^{(4)}(x-y).$$

the quantities (9.15) and (9.16) form the Poincaré algebra.

This is nothing new with respect to the scalar case. Now, however, we want to introduce the generators of the local SU(2) invariance transformations. Let us have an infinitesimal transformation given by the SU(2) matrix

$$\mathcal{U} \simeq 1 - i\, \frac{\sigma_a}{2} \omega_a(x) \tag{9.18}$$

The variations of $A_\mu^a$ and $p_\mu^a$ are given by

$$\begin{aligned} \delta A_\mu^a &= \frac{1}{g} \partial_\mu \omega_a - \varepsilon_{abc}\, \omega_b A_\mu^c\,, \\ \delta p_\mu^a &= -\, \varepsilon_{abc}\, \omega_b\, p_\mu^c\,. \end{aligned} \tag{9.19}$$

The generator of this transformation is now given by

$$G_\omega = \int d^4x \;\; G_a(x)\, \omega_a(x) \tag{9.20}$$

where

$$G_a(x) = \frac{1}{g}\, D_\mu^{ab}\, p_\mu^b \tag{9.21}$$

It is indeed easy to check that the Poisson brackets $[G_\omega, A_\mu^a]$ and $[G_\omega\, p_\mu^a]$ reproduce the variations (9.19). Clearly the expression (9.21) represents the generator for an infinitesimal gauge transformation located at $x_\mu$. Thus, general SU(2) gauge transformations are just canonical transformations. The algebra of the generators is

$$[G_f, G_g]_{PB} = G_h \tag{9.22}$$

where

$$\vec{h} = \vec{f} \times \vec{g} \tag{9.23}$$

In particular, for the local generators we have

$$[G_a(x), G_b(y)]_{PB} = \varepsilon_{abc}\, G_c(x)\, \delta^{(4)}(x-y)\,. \tag{9.24}$$

## 9.4 Gravity

Finally, we report on the case of gravity. The Lagrangian is

$$L = \frac{1}{2}\int d^4x \sqrt{g}\; T^{\mu\nu}_{ab}\, \dot{V}^a_\mu\, \dot{V}^b_\nu - \mathcal{V}$$

$$\mathcal{V} = -\frac{1}{4}\int d^4x\, \sqrt{g}\, (R + 2\Lambda) \tag{9.25}$$

and the matrix $T^{\mu\nu}_{ab}$ is given by

$$T^{\mu\nu}_{ab} = c_1 \delta_{ab}\, g^{\mu\nu} + c_2 V^\mu_a V^\nu_b + c_3 V^\mu_b V^\nu_a \tag{9.26}$$

The evolution equation in $t$ is

$$\frac{\delta T}{\delta V^a_\mu} - \frac{\partial}{\partial t}\frac{\delta T}{\delta \dot{V}^a_\mu} - E_\mu = 0 \tag{9.27}$$

$$\left( T = \frac{1}{2}\int d^4x \sqrt{g}\; T^{\mu\nu}_{ab}\, \dot{V}^a_\mu\, \dot{V}^b_\nu \right)$$

where $E_\mu$ is the usual Einstein term of the equation of motion

$$E_\mu = \frac{\delta \mathcal{V}}{\delta V^a_\mu} \tag{9.28}$$

We now introduce the momenta*)

$$p^\mu_a = \frac{\delta T}{\delta \dot{V}^a_\mu} = \sqrt{g}\; T^{\mu\nu}_{ab}\, \dot{V}^b_\nu \,. \tag{9.29}$$

We see again that all the momenta are perfectly defined. Thus reparametrization invariance also leads to ignorable co-ordinates.

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*) Note that $p^\mu$ is not a vector, but rather a density. The quantity $p^\mu/\sqrt{g}$ is, of course, a controvariant vector.

The fundamental Poisson brackets are

$$[V_\mu^a(x,t),\ p^{\nu b}(y,t)]_{PB} = \delta_{ab}\,\delta_{\mu\nu}\,\delta^{(4)}(x-y) \tag{9.30}$$

We may now give an expression for the generator of an infinitesimal reparametrization defined by $x^\mu \to x^\mu + f^\mu(x)$

$$L_f = \int d^4x\ p^{\mu a}\left(f^\lambda \partial_\lambda V_\mu^a + V_\lambda^a \partial_\mu f^\lambda\right) \tag{9.31}$$

It is easy to prove that the action of $L_f$ on $V_\mu^a$ and $p^{\mu a}$ reproduces their infinitesimal gauge transformations

$$\delta V_\mu^a = -\left(f^\lambda \partial_\lambda V_\mu^a + V_\lambda^a \partial_\mu f^\lambda\right)$$

$$\delta\left(\frac{p^{\mu a}}{\sqrt{g}}\right) = -\,f^\lambda \partial_\lambda\left(\frac{p^{\mu a}}{\sqrt{g}}\right) + \frac{p^{\lambda a}}{\sqrt{g}}\,\partial_\lambda f^\mu \tag{9.32}$$

The algebra of the generators (9.31) is

$$[L_f, L_g]_{PB} = L_h \tag{9.33}$$

where

$$h^\mu = f^\lambda \partial_\lambda g^\mu - g^\lambda \partial_\lambda f^\mu\ . \tag{9.34}$$

## 9.5 A Possible Scenario (New Quantization)

The analogy of Quantum Field Theory with classical statistical theory in 4 + 1 dimensions suggests a new point of view whose implications we have not yet worked out in detail. We shall illustrate these thoughts on the simple case of a scalar field theory with the Hamiltonian [see Eq. (9.7)]

$$H = \int d^4x \left\{\frac{1}{2}p^2 + \frac{1}{2}(\partial_\mu\varphi)^2 - \mathcal{U}(\varphi)\right\}\ . \tag{9.7}$$

The idea is to substitute for the <u>classical</u> statistical theory in 4 + 1 dimensions the corresponding <u>quantum</u> theory. This leads to the following well-known steps. p and $\varphi$ become operators whose commutator is given by the corresponding principle:

$$[\varphi(x,t),\ p(y,t)] = i b\, \delta^{(4)}(x-y)\,. \tag{9.35}$$

Here b is a new constant with dimensions of a length*). The new generating functional is formally defined by the trace

$$Z = \mathrm{Tr}\ e^{-bH}\,. \tag{9.36}$$

The old results of customary Quantum Field Theory are recovered by letting

$$b \to 0$$

Of course, at this point the problem of the invariance of the volume is solved since there is no volume left! Symmetries are implemented by unitary transformations that leave the form (9.36) invariant.

The problem of whether this point of view is useful is still under investigation. We only hint here at the fact that this treatment leads to a theory that has some similarities with the Kaluza-Klein approach, since the theory contains an infinite number of scalar particles with masses $M_n = 2\pi/b\ n$ $(n=0,1,\ldots)$ [13]. The interesting property can be seen by applying standard techniques of finite temperature Quantum Field Theory[14]. Indeed, the function Z given by Eq. (9.36) can in turn be represented by a functional integral:

$$Z = \int_{\text{periodic}} d\Omega(\varphi)\ \exp\left\{-\int_0^b L\, dt\right\} \tag{9.37}$$

where

$$L = \int d^4x \left\{\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\partial_\mu\varphi)^2 + \mathcal{U}(\varphi)\right\}$$

The functional integration is performed on all functions of five variables that are periodic in t between 0 and b. Thus t has the circle topology.

In this case the variable t takes on a dynamical rôle. Let us expand $\varphi(x_\mu,t)$ as

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*) The variable t is now considered to have the dimension of a length so that $p(x,t) = \dot{\varphi}$ has dimensionality -2.

$$\varphi(x,t) = \sum_m \phi_m(x)\, e^{\frac{2\pi i m}{b}t} \tag{9.38}$$

and the new action

$$A = \int dt\, L$$

can be represented as

$$A = \int d^4x\, \sum_m \left\{ \left(\frac{2\pi}{b}\right)^2 m^2 \phi_m^2(x) + \frac{1}{2}\left(\partial_\mu \phi_m\right)^2 \right\} - \int d^4x\, \mathcal{U}(\varphi)\,. \tag{9.39}$$

Thus the theory represents a tower of particles with growing mass. The limit $b \to 0$ sends away all the particles except the fundamental one.

We see that starting from a completely different point of view we have entered a popular game - the study of a theory with higher dimensionality (in our case $4 + 1$) in which one of them has been compactified. Work is in progress to investigate this alternative approach to compactification.

## 10. - CONCLUDING REMARKS

In this paper we have developed a simple determination of the path integral expression of the physical amplitude without appealing to the usual canonical formalism. Our treatment will exhibit at all stages four-dimensional rotational invariance in Euclidean space time.

It is well known that in many cases a simple Lagrangian formulation of the action integral leads to incorrect results which are embodied in the so-called "naîve Feynman rules". The main result of this paper is that, in general, the naîve rules are incorrect just because the elementary integration volume does not satisfy the appropriate invariance requirements of the physical problems. This is particularly evident in the case of a chiral invariant theory as shown in Section 6.

Our procedure to build a volume element with the required invariance behaviour consists of a representation of the functional integral à la Gibbs, in analogy with Statistical Mechanics, which is based on the canonical formalism in a new time variable. The key point is the selection of an invariant kinetic term (in the new time): this is indeed shown to lead to an integration volume in the field variables which has the required invariance properties.

Our treatment of theories with a local invariance group like Yang-Mills gauge theory and Einstein gravitation follows again from group theoretical considerations. The existence of an invariance group leads to the presence of ignorable co-ordinates, the dependence on which is trivial. The integration over those quantities can be performed a priori and the ensuing formula turns out to be equivalent to the Faddeev-Popov recipe.

The results obtained by means of our procedure do coincide in general with the canonical ones, showing a beautiful consistency of the over-all quantum field theoretical picture*).

There is, however, a discrepancy between the two approaches in the cases where the invariance group of the theory contains transformations involving explicitly the time variable. This is the case of conformal invariance and is also the case of the invariance group of the Einstein theory of gravitation. In our opinion, this is a hint at the fact that the problem of quantization in both cases is still unsettled. In this context it is amusing to notice that for both supersymmetric conformal invariant and Einstein invariant (supergravity) theories the canonical (in the usual sense) and the invariant volume do indeed coincide. This suggests that supersymmetry could be a fundamental ingredient in the formulation of future field theoretical models.

Let us conclude by recalling that there are two kinds of features of the functional approach which may lead to deviations from the naïve rules, those which are already manifest when the functional integration is still represented by a discrete sum and others which arise when the continuum limit is approached, as required in the treatment of Quantum Field Theory.

In this work our attention has been concentrated on the class of problems just described but we are well aware of the crucial rôle implied by the transition to the limit of the continuum, which we have tacitly assumed to be straightforward. Actually, it can be shown that phenomena like the chiral and conformal

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*) This is particularly important because, as emphasized by E.S. Fradkin, the canonical formulation ensures the over-all unitarity of the perturbation expansion.

anomalies for the divergence of the axial vector current and for the trace of energy-momentum tensor, respectively, require a careful treatment of the functional integral based on appropriate regularization and limiting procedures. We hope to be able to tackle these problems in a future work and we believe that our approach may represent a useful tool for their study.

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APPENDIX A

We want to supply a quick proof of the result (4.22) for the invariant group measure. Let us indicate by $\phi_1,\dots,\phi_R$ the $R$ parameters which characterize the group transformations and by $\delta_\alpha\phi_\beta$ the "$\alpha$" infinitesimal variation.

The volume element

$$d\Omega(\phi) = \prod_{1}^{R}{}_{\beta}\, d\phi_\beta \tag{A.1}$$

is not invariant and its variation is easily computed. With

$$\phi' = \phi + \delta_\alpha\phi$$

one has

$$d\Omega(\phi') = \prod d\phi' = \prod d\phi \cdot \det\left(\frac{\partial\phi'}{\partial\phi}\right) \simeq$$
$$\simeq d\Omega \det\left(1+\frac{\partial}{\partial\phi}\delta_\alpha\phi\right) \simeq d\Omega\left(1+\mathrm{tr}\,\frac{\partial}{\partial\phi}\delta_\alpha\phi\right)$$

namely

$$\delta_\alpha d\Omega(\phi) = d\Omega(\phi)\sum_\beta \frac{\partial}{\partial\phi_\beta}\delta_\alpha\phi_\beta \, . \tag{A.2}$$

In order to compensate for this non-vanishing variation, one has to introduce $\det(\delta_\alpha\phi_\beta)$ for which it is easily found that

$$\det(\delta\phi) \xrightarrow[\alpha]{} \det(\delta_\beta\phi'_\gamma) = \det(\delta_\beta\phi_\gamma + \delta_\beta\delta_\alpha\phi_\gamma)$$

$$\simeq \det\left\{\delta_\beta\phi_\gamma + \frac{\partial}{\partial\phi_\delta}(\delta_\alpha\phi_\gamma)\delta_\beta\phi_\delta\right\} \simeq$$

$$\simeq \det(\delta_\beta\phi_\gamma)\left(1+\mathrm{tr}\,\frac{\partial}{\partial\phi}\delta_\alpha\phi\right)$$

i.e.,

$$\delta_\alpha \det(\delta\phi) = \det(\delta\phi)\sum_\beta\frac{\partial}{\partial\phi_\beta}\delta_\alpha\phi_\beta \tag{A.3}$$

Comparing (A.2) and (A.3) confirms the invariance of

$$d\Omega_I = d\Omega(\phi)/\det(\delta\phi) \, .$$

APPENDIX B

In the text the behaviour of the integration measure under dilatations has often been mentioned and we consider it useful to offer an intuitive argument about this point.

Consider first the case of a scalar field of dimension +1. It is convenient to introduce the operator $\chi$ with dimension d

$$\chi = \phi^d \quad , \quad \delta\chi = \varepsilon\,(x\cdot\partial + d)\chi \tag{B.1}$$

and to study the behaviour of the volume element $d\Omega(x) = \pi_x d\chi(x,t)$. For its variation we shall exploit the continuum version of Eq. (A.2) where the trace operation is now defined as

$$\mathrm{Tr}\,\frac{\partial}{\partial\chi}\,\delta\chi \equiv \int d^4x\, d^4y\, \delta^{(4)}(x-y)\,\frac{\partial}{\partial\chi(x)}\,\delta\chi(y) \tag{B.2}$$

(the $\partial/\partial\chi$ derivative is actually understood in a functional sense).

Thus

$$\begin{aligned}\delta\, d\Omega(\chi) &= d\Omega(\chi)\,\varepsilon\int d^4x\, d^4y\, \delta^{(4)}(x-y)\,\frac{\delta}{\delta\chi(y)}\,(x\cdot\partial + d)\chi(x) \\ &= \varepsilon\, d\Omega(\chi)\,(d-2)\,\delta^{(4)}(0)\int d^4x\,.\end{aligned} \tag{B.3}$$

which shows (if one tolerates the presence of the ill-defined quantities on the right-hand side) that $d\Omega(\chi)$ is not scale invariant unless $d = 2$. This corresponds to

$$d\Omega(\chi) = d\Omega(\phi^2) = \prod_x \phi(x,t)\, d\phi(x,t)\,. \tag{B.4}$$

confirming the general result of Section 6.

Analogous considerations can be developed for other fields with the important remark that, in the case of fermions, the rules for the integration of anticommuting variables require that the Jacobian determinant be written in the denominator instead of the numerator. This leads to an opposite sign in the formula (B.3) for the scale variation of the volume and hints at the possibility of a cancellation (i.e., of an over-all invariance of the volume) when several fields are present.

In particular, ascribing canonical dimensionality $d_B = 1$, $d_F = 3/2$ (units of mass) to the field operators, one finds

$$\delta\, d\Omega = \varepsilon\, d\Omega \left( \tfrac{1}{2} n_F - n_B \right) \delta^{(4)}(0) \int d^4x \tag{B.5}$$

where $n_B$, $n_F$ is the number of bosonic and fermionic integration variables, respectively. It is clear that bosonic auxiliary fields of dimensionality +2 do not contribute to Eq. (B.5).

It is instructive, for the sake of completeness, to consider the case of a Weyl transformation

$$\delta \chi = (\varepsilon\, d) \chi \tag{B.6}$$

for which the size of the field $\chi$ is rescaled but the position $x$ is left unchanged. In this case it is easy to see that the invariant volume is

$$d\Omega(\chi) = \prod_x \phi^{-1}(x\,t)\, d\phi(x\,t) \tag{B.7}$$

This Weyl invariant volume is the one considered by some authors both for the scalar theory and for gravitation[16).

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