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ON THE FUNCTIONS OF LITTLEWOOD-PALEY AND MARCINKIEWICZ

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1. Introduction. Let $f(x)$ be a locally integrable function on the real line $\boldsymbol{R}$. The Fourier integral analogue of Marcinkiewicz function [7] is

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 t^{-3} dt\right)^{1/2}$$

where

$$F(x) = \int_0^x f(u)du .$$

We generalize this as follows: for $\alpha > 0$

$$\mu_\alpha(f)(x) = \left\{\int_0^\infty \left|\frac{\alpha}{t}\int_0^\infty \left(1 - \frac{u}{t}\right)^{\alpha-1} (f(x-u) - f(x+u))du\right|^2 \frac{dt}{t}\right\}^{1/2}, \tag{1.1}$$

$\mu_1(f)(x)$ coincides with $\mu(f)(x)$. (1.1) is the one dimensional form of the more general Marcinkiewicz function

$$\mu(f)(x) = \left\{c\int_0^\infty \left|\frac{1}{t}\int_{|u|\leq t} f(x-u)\frac{\Omega(u')}{|u|^{k-1}}du\right|^2 \frac{dt}{t}\right\}^{1/2}$$

where $\Omega(u')/|u|^k$ is the Calderón-Zygmund kernel on k-dimensional space and c is a constant depending on k only, see Stein [8].

On the other hand we have generalized the Littlewood-Paley function as follows

$$g_\beta^*(\phi)(x) = \left\{\frac{1}{\pi}\int_0^\infty y^{2\beta}dy \int_{-\infty}^\infty \frac{|\phi'(t+iy)|^2}{|t-x-iy|^{2\beta}}dt\right\}^{1/2}, \tag{1.2}$$

where $\phi(z) = \phi(x+iy)$ is analytic in the upper half-plane and has boundary value $\phi(x) = \lim_{y\to 0}\phi(x+iy)$. The original Littlewood-Paley function $g^*(\phi)(x)$ in Fourier integral form corresponds to the case $\beta = 1$ in (1.2).

Let $\sigma_\beta(R; x, \beta)$ the R-th (C, β)-mean of Fourier integral of complex valued function $\phi(x)$ and set

$$\tau_\beta(R; x, \phi) = R\frac{d}{dR}\sigma_\beta(R; x, \phi) = \beta\{\sigma_{\beta-1}(R; x, \phi) - \sigma_\beta(R; x, \phi)\} \tag{1.3}$$

and set

$$h_\beta(\phi)(x) = \left(\int_0^\infty \frac{|\tau_\beta(R; x, \phi)|^2}{R} dR\right)^{1/2} . \tag{1.4}$$

Then (1.2) is equivalent to (1.4), that is,

$$Ah_\beta(\phi)(x) \leqq g_\beta^*(\phi)(x) \leqq Bh_\beta(\phi)(x) ,$$

where A and B are constants independent of ϕ and x; see Sunouchi [11]. Here after A and B mean such constants.

Now we consider the functional h_β for imaginary part of ϕ. Let $\bar{\sigma}_\beta(R; x, f)$ the (C, β)-mean of the conjugate Fourier integral of any $f(x)$ and define $\bar{\tau}_\beta(R; x, f)$ and $\bar{h}_\beta(R; x, f)$ analogously to the formula (1.3) and (1.4). We denote by S the Schwartz space on $\boldsymbol{R}$, that is, the space of rapidly decreasing C^∞-functions. Then our main theorem is as follows.

THEOREM 1. *If $\alpha + 1/2 = \beta$ $(\alpha > 0)$, then*

$$A\bar{h}_\beta(f)(x) \leqq \mu_\alpha(f)(x) \leqq B\bar{h}_\beta(f)(x)$$

for any function $f(x) \in S$ and $x \in \boldsymbol{R}$.

One of the inequalities

$$\bar{h}_\beta(f)(x) \leqq A\mu_\alpha(f)(x)$$

is already given by Flett [3] for the functions on the unit circle.

For a variant of this, let $f_\alpha(x)$ be the Riesz potential of $f(x)$, that is, $f_\alpha(x) = \int_{-\infty}^\infty |\xi|^{-\alpha}\hat{f}(\xi)e^{ix\xi}d\xi$ and set

$$D_\alpha(f)(x) = \left(\int_0^\infty \frac{|f_\alpha(x-t) - f_\alpha(x+t)|^2}{t^{1+2\alpha}} dt\right)^{1/2} .$$

THEOREM 2. *If $\alpha + 1/2 = \beta$ $(0 < \alpha < 1)$, then*

$$A\bar{h}_\beta(f)(x) \leqq D_\alpha(f)(x) \leqq B\bar{h}_\beta(f)(x)$$

for $f \in S$ and $x \in \boldsymbol{R}$.

The fact that, for $\alpha + 1/2 > \beta$

$$D_\alpha(f)(x) \leqq B\{\bar{h}_\beta(f)(x) + h_\beta(f)(x)\} = Bh_\beta(\phi)(x) ,$$

is given by Stein [9] for functions of several variables.

Corresponding to $h_\beta(f)(x)$, we consider

$$\delta_\alpha(f)(x) = \left(\int_0^\infty t \left|\frac{d}{dt} M_\alpha(t; x, f)\right| dt\right)^{1/2}$$

where

$$M_\alpha(t; x, f) = \frac{\alpha}{t}\int_{|u| \leqq t} \left(1 - \frac{|u|}{t}\right)^{\alpha-1} f(x-u)du .$$

THEOREM 3. *If* $\alpha - (1/2) = \beta$ $(\alpha > 0)$, *then*

$$Ah_\beta(f)(x) \leqq \delta_\alpha(f)(x) \leqq Bh_\beta(f)(x)$$

for $f \in S$ *and* $x \in \boldsymbol{R}$.

In the last section, an analogous relation to the Littlewood-Paley function $g(f)(x)$, is also established. In particular the relationship between

$$\delta_0(f)(x) = \left(\int_0^\infty t\left|\frac{d}{dt}\{f(x-t)+f(x+t)\}\right|^2 dt\right)^{1/2}$$

and $g(f)(x)$ is clarified. This question is proposed as problem 6(a) of Stein-Wainger [10, p. 1289] for several variables case.

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2. Notations. We suppose throughout this paper $f(x)$ belongs to the class S. We write for a fixed x_0,

$$\phi(t) = \phi(t; x_0, f) = f(x_0 - t) + f(x_0 + t)$$

and

$$\psi(t) = \psi(t; x_0, f) = f(x_0 - t) - f(x_0 + t)\ .$$

For $\alpha > 0$ and $t \geqq 0$, set

$$\phi_\alpha(t) = \phi_\alpha(t; x_0, f) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1}\phi(u)du\ , \tag{2.1}$$

and

$$\psi_\alpha(t) = \psi_\alpha(t; x_0, f) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1}\psi(u)du\ . \tag{2.2}$$

The generalized Marcinkiewicz function and a variant are written as

$$\mu_\alpha(f)(x_0) = \left(\int_0^\infty \left|\psi_\alpha(t; x_0, f)\right|^2 \frac{dt}{t}\right)^{1/2}\ , \tag{2.3}$$

$$\delta_\alpha(f)(x_0) = \left(\int_0^\infty t\left|\frac{d}{dt}\phi_\alpha(t; x_0, f)\right|^2 dt\right)^{1/2}\ . \tag{2.4}$$

For the Cesàro-Riesz mean of $f(x)$, we introduce the well-known Young function. Let

$$\gamma_\alpha(x) + i\bar{\gamma}_\alpha(x) = \int_0^1 (1-t)^{\alpha-1}e^{ixt}dt$$

where $\alpha > 0$, $x \geqq 0$, then it is known [1],

$$\gamma_\alpha(x) \sim x^{-p} \quad \text{as} \quad x \to \infty\,, \qquad \text{where} \quad p = \text{Min}\,(2, \alpha)\,.$$

Then the R-th Cesàro-Riesz mean of of the β-th order for Fourier integral of $f(x)$ is

$$\sigma_\beta(R) = \sigma_\beta(R; x_0, f) = c\int_0^\infty \phi(u) R\gamma_{\beta+1}(Ru)du \tag{2.5}$$

and for the conjugate Fourier integral of $f(x)$ is

$$\bar{\sigma}_\beta(R) = \bar{\sigma}_\beta(R; x_0, f) = c'\int_0^\infty \psi(u) R\bar{\gamma}_{\beta+1}(Ru)du \tag{2.6}$$

where c and c' are constants. Then we have

$$h_\beta(f)(x_0) = \left(\int_0^\infty |\sigma_{\beta-1}(R) - \sigma_\beta(R)|^2 \frac{1}{R} dR\right)^{1/2} \tag{2.7}$$

and

$$\bar{h}_\beta(f)(x_0) = \left(\int_0^\infty |\bar{\sigma}_{\beta-1}(R) - \bar{\sigma}_\beta(R)|^2 \frac{1}{R} dR\right)^{1/2}. \tag{2.8}$$

3. Proof of Theorem 1. Let $f \in S$ and fix a point x_0 in $\boldsymbol{R}$. By the change of the variables $u = e^{-y}$ and $t = e^{-x}$ (2.3) becomes

$$\mu_\alpha(f)(x_0) = \left[\int_{-\infty}^\infty \left|\int_x^\infty \alpha e^{(x-y)}(1 - e^{(x-y)})^{\alpha-1} \psi(e^{-y}) dy\right|^2 dx\right]^{1/2}. \tag{3.1}$$

If we rewrite

$$K_\alpha(x) = \begin{cases} \alpha e^x (1 - e^x)^{\alpha-1}\,, & x \leqq 0 \\ 0 \,, & x > 0 \end{cases} \tag{3.2}$$

and $\Psi(x) = \psi(e^{-x})$, then (3.1) becomes

$$\mu_\alpha(f)(x_0) = \left(\int_{-\infty}^\infty |(\Psi * K_\alpha)(x)|^2 dx\right)^{1/2}. \tag{3.3}$$

In (2.6) and (2.8), we set $u = e^{-y}$ and $R = e^x$, then (2.8) is

$$\begin{aligned} \bar{h}_\beta(f)(x_0) &= \left[c'\int_0^\infty \left|\int_0^\infty \psi(u) R\{\bar{\gamma}_\beta(Ru) - \bar{\gamma}_{\beta+1}(Ru)\} du\right|^2 \frac{dR}{R}\right]^{1/2} \\ &= \left[c'\int_{-\infty}^\infty \left|\int_{-\infty}^\infty \psi(e^{-y}) e^{x-y}\{\bar{\gamma}_\beta(e^{x-y}) - \bar{\gamma}_{\beta+1}(e^{x-y})\} dy\right|^2 dx\right]^{1/2}. \end{aligned}$$

If we rewrite

$$\bar{K}_\beta^*(x) = e^x(\bar{\gamma}_\beta(e^x) - \bar{\gamma}_{\beta+1}(e^x)) = e^x \gamma'_\beta(e^x) \tag{3.4}$$

and $\Psi(x) = \psi(e^{-x})$, then

$$\bar{h}_\beta(f)(x_0) = \left(\int_{-\infty}^\infty |(\Psi * \bar{K}_\beta^*)(x)|^2 dx\right)^{1/2}. \tag{3.5}$$

To compare (3.3) with (3.5), we apply Fourier transform method.

Since $\psi(u) = f(x_0 - u) - f(x_0 + u) \in S$, $\Psi(x) = \psi(e^{-x}) = 0(e^{-x})$ as $x \to \infty$, and $0(e^{-|x|})$ as $x \to -\infty$. Since $K_\alpha(x)$ is integrable on $(-\infty, \infty)$, $(\Psi * K_\alpha)(x)$ is an ordinary convolution. However, since

$$\bar{\gamma}_\beta(e^x) \sim e^{-\beta x} \quad \text{as} \quad x \to \infty \qquad (0 < \beta \leqq 1)$$

we have

$$\bar{K}^*_\beta(x) \sim e^x \cdot e^{-\beta x} = e^{(1-\beta)x} \quad \text{as} \quad x \to \infty .$$

But

$$\bar{K}^*_\beta(x) = 0(e^x) \quad \text{as} \quad x \to -\infty \qquad (0 < \beta \leqq 1) .$$

If $1/2 < \beta \leqq 1$, then $1 - \beta \geqq 0$ and $\bar{K}^*_\beta(x)$ is locally integrable, but not integrable on $(-\infty, \infty)$. In fact this is the most interesting case. Hence we have to consider a distributional Fourier transform. As we shall show at (3.11), the Fourier transform $\hat{\bar{K}}^*_\beta(\xi)$ belongs to the class L^∞ and evidently $\hat{\Psi}(\xi) \in L \cap L^\infty$. Accordingly we can apply convolution rule to $(\Psi * \bar{K}^*_\beta)(x)$, see Katznelson [5, p. 151, Lemma].

Now we take the complex Fourier transform of kernels (3.2) and (3.4). Let $s = \zeta - i\xi$, where ζ is a complex number. Then

$$\text{(3.6)} \qquad \int_{-\infty}^{\infty} K_\alpha(x) e^{sx} dx = \alpha \int_{-\infty}^{0} e^{(s+1)x}(1 - e^x)^{\alpha-1} dx = \alpha \int_0^1 (1 - t)^{\alpha-1} t^s dt$$
$$= \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \qquad (\alpha > 0,\ \mathrm{Re}\, s > -1) .$$

Let $\theta(x) \in S$, and consider Parseval's formula:

$$\langle \bar{K}^*_\beta(x) e^{\zeta x}, \theta(x) \rangle = 2\pi \left\langle \int_{-\infty}^{\infty} \bar{K}^*_\beta(x) e^{\zeta x} \cdot e^{-i\xi x} dx, \hat{\theta}(\xi) \right\rangle .$$

Then both sides are analytic functions of a complex variable ζ in an appropriate domain. Therefore we can calculate the distributional Fourier transform of $\bar{K}^*_\beta(x)$ by analytic continuation method, see [4, p. 171]. We have

$$\text{(3.7)} \qquad \int_{-\infty}^{\infty} \bar{K}^*_\beta(x) e^{sx} dx = \int_{-\infty}^{\infty} e^x \gamma'_\beta(e^x) e^{sx} dx$$
$$= c' \frac{\Gamma(\beta) s}{\Gamma(\beta - s + 1) \sin \pi s/2}$$
$$(\beta > 0,\ \beta + 1 > \mathrm{Re}\, s,\ 2 > \mathrm{Re}\, s > -2) .$$

Accordingly we have, from (3.6) and (3.7),

$$\text{(3.8)} \qquad \hat{K}_\alpha(\xi) = \frac{\Gamma(\alpha + 1)\Gamma(1 - i\xi)}{\Gamma(\alpha + 1 - i\xi)} \quad (\alpha > 0)$$

and

$$\hat{\bar{K}}_\beta^*(\xi) = \frac{c'\Gamma(\beta)(i\xi)}{\Gamma(\beta + i\xi + 1)\sin \pi i\xi/2} \quad (\beta > 0)\,; \tag{3.9}$$

Both $\hat{K}_\alpha(\xi)$ and $\hat{\bar{K}}_\beta^*(\xi)$ have no zero on $\xi \in (-\infty, \infty)$ and finite. By the asymptotic formula of the Gamma function

$$|\Gamma(a + i\xi)| \sim (2\pi)^{-1/2} e^{-\pi|\xi|/2} |\xi|^{a-(1/2)}\,, \qquad a \in (-\infty, \infty)$$

as $|\xi| \to \infty$, we have as $|\xi| \to \infty$,

$$|\hat{K}_\alpha(\xi)| \sim \frac{c\,|\xi|^{1-(1/2)}}{|\xi|^{\alpha+1-(1/2)}} \sim c\,|\xi|^{-\alpha} \tag{3.10}$$

and as $|\xi| \to \infty$,

$$|\hat{\bar{K}}_\beta^*(\xi)| \sim c' \frac{|\xi|}{|\xi|^{1+\beta-(1/2)} e^{-\pi|\xi|/2} e^{\pi|\xi|/2}} \sim c'\,|\xi|^{(1/2)-\beta}\,. \tag{3.11}$$

Hence $|\hat{K}_\alpha(\xi)/\hat{\bar{K}}_\beta^*(\xi)|$ is bounded if $\alpha + 1/2 = \beta$ $(\alpha > 0)$. Thus

$$\begin{aligned}
0 &\leqq \int_{-\infty}^{\infty} |(\Psi * K_\alpha)(x)|^2 dx = (2\pi)\int_{-\infty}^{\infty} |\hat{\Psi}(\xi)\hat{K}_\alpha(\xi)|^2 d\xi \\
&= (2\pi)\int_{-\infty}^{\infty} \left|\hat{\Psi}(\xi)\cdot \hat{\bar{K}}_\beta^*(\xi)\cdot \frac{\hat{K}_\alpha(\xi)}{\hat{\bar{K}}_\beta^*(\xi)}\right|^2 d\xi \\
&\leqq c\int_{-\infty}^{\infty} |\hat{\Psi}(\xi)\hat{\bar{K}}_\beta^*(\xi)|^2 d\xi \\
&= c'\int_{-\infty}^{\infty} |(\Psi * \bar{K}_\beta^*)(x)|^2 dx\,,
\end{aligned}$$

provided that the last term is finite. Since the proof of converse part is done similarly, Theorem is proved completely.

4. Proof of Theorem 3.

From (2.1),

$$\phi_\alpha(t) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} \phi(u)du = \alpha\int_0^1 (1 - v)^{\alpha-1}\phi(tv)dv\,,$$

and since $f(x_0 - u) + f(x_0 + u) \in S$,

$$\begin{aligned}
\frac{d}{dt}\phi_\alpha(t) &= \alpha\int_0^1 (1 - v)^{\alpha-1}\cdot v\phi'(tv)dv \\
&= \alpha\cdot\frac{1}{t^2}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} u\phi'(u)du
\end{aligned} \tag{4.1}$$

where $\phi'(u) = -\{f'(x_0 - u) - f'(x_0 + u)\}$.

We set as in §3 $u = e^{-y}$, $t = e^{-x}$, then, by (2.4), we have

$$\{\delta_\alpha(f)(x_0)\}^2 = \int_0^\infty \frac{1}{t}\left|t\frac{d}{dt}\phi_\alpha(t)\right|^2 dt \tag{4.2}$$
$$= \int_{-\infty}^\infty \left|\alpha \int_x^\infty \phi'(e^{-y}) \cdot e^{-y} \cdot e^{(x-y)}(1 - e^{x-y})^{\alpha-1} dy\right|^2 dx .$$

We set $\chi(x) = e^{-x}\phi'(e^{-x})$ and

$$K_\alpha(x) = \begin{cases} \alpha e^x(1 - e^x)^{\alpha-1} , & x \leqq 0 \\ 0 , & x > 0 . \end{cases}$$

Then

$$\{\delta_\alpha(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K_\alpha)(x)|^2 dx .$$

On the other hand, by (2.5)

$$\sigma_\beta(R) = c\int_0^\infty \phi(u)R\gamma_{\beta+1}(Ru)du$$

and

$$\sigma'_\beta(R) = c\int_0^\infty \phi'(u)u\gamma_{\beta+1}(Ru)du . \tag{4.3}$$

By the definition (2.7)

$$\{h_\beta(f)(x_0)\}^2 = c'\int_0^\infty \frac{1}{R}|R\sigma'_\beta(R)|^2 dR$$
$$= c'\int_0^\infty \frac{1}{R}\left|R\int_0^\infty \phi'(u)\cdot u\gamma_{\beta+1}(Ru)du\right|^2 dR .$$

We set $u = e^{-y}$, $R = e^x$, then

$$\{h_\beta(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K^*_\beta)(x)|^2 dx \tag{4.4}$$

where $\chi(x) = e^{-x}\phi'(e^{-x})$ and $K^*_\beta(x) = e^x\gamma_{\beta+1}(e^x)$.

Since $\phi'(e^{-x}) = f'(x_0 - e^{-x}) - f'(x_0 + e^{-x})$, $\chi(x)$ behaves better than $\Psi(x)$ in §3. However since

$$K^*_\beta(x) \sim e^x \cdot e^{-(\beta+1)x} = e^{-\beta x} \quad \text{as} \quad x \to \infty ,$$

$K^*_\beta(x)$ behaves for $0 \geqq \beta > -1/2$, as if $\bar{K}^*_\beta(x)$ in (3.4). Hence all things go analogously as in §3.

The complex Fourier transform of $K_\alpha(x)$ is

$$\alpha\int_{-\infty}^0 e^{sx}e^x(1 - e^x)^{\alpha-1}dx = \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \quad (\alpha > 0, \operatorname{Re} s > -1)$$

and that of $K_\beta^*(x)$ is

$$\int_{-\infty}^{\infty} e^{sx} e^x \gamma_{\beta+1}(e^x)dx = \int_0^\infty t^s \gamma_{\beta+1}(t)dt$$
$$= \frac{\pi}{2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-s)\cos \pi s/2}$$
$$(\beta+1>0,\ \beta+1>\operatorname{Re} s,\ -1<\operatorname{Re} s<1)\ .$$

Since $\hat{K}_\alpha(\xi) = \Gamma(\alpha+1)\Gamma(1-i\xi)/\Gamma(\alpha+1-i\xi)$, $|\hat{K}_\alpha(\xi)| \sim c|\xi|^{-\alpha}$ and since $\hat{K}_\beta^*(\xi) = \Gamma(\beta+1)/(\Gamma(\beta+1-i\xi)\cos \pi i\xi/2)$, $|\hat{K}_\beta^*(\xi)| \sim c'|\xi|^{-\beta-(1/2)}$.

If $\beta > -1/2$, then $\hat{K}_\beta^*(\xi)$ is bounded, and the theorem is proved for $\alpha = \beta + (1/2)$.

REMARK. If $\alpha > 1$, we may eliminate differentiability of $f(x)$ in (4.1) by a partial integration. However $0 < \alpha \leqq 1$ case, we define $\phi'_\alpha(t)$ by (4.1) and $\sigma'_\beta(R)$ by (4.3) respectively assuming differentiability of $f(x)$.

5. Proof of Theorem 2. For a proof of Theorem, we need two lemmas.

LEMMA 1. *For f in S we have*

$$f_\alpha(x-t) - f_\alpha(x+t) = c\int_0^\infty \xi^{-\alpha} \sin \xi t d\xi \int_0^\infty \psi(u; x, f) \sin u\xi du\ .$$

PROOF. By definition of Riesz potential, we have

$$f_\alpha(x-t) - f_\alpha(x+t) = c\int_{-\infty}^{\infty} |\xi|^{-\alpha} \hat{f}(\xi) e^{ix\xi}(e^{-it\xi} - e^{it\xi})d\xi$$
$$= -2ic\int_{-\infty}^{\infty} |\xi|^{-\alpha} \sin t\xi \cdot \hat{f}(\xi)e^{ix\xi}d\xi\ .$$

Since $|\xi|^{-\alpha}\sin t\xi$ is odd, we may take the odd part of

$$\hat{f}(\xi)e^{ix\xi} = \int_{-\infty}^{\infty} f(x+u)e^{iu\xi}du\ ,$$

which implies the lemma.

LEMMA 2. *If $\alpha + 1/2 = \beta$ $(0 < \alpha < 1)$, then for $f \in S$ we have*

$$\left(\int_0^\infty \frac{|\psi(u)|^2}{u^{2\alpha}} \frac{du}{u}\right)^{1/2} \sim \left(\int_0^\infty R\left|\frac{d}{dR}\bar{\sigma}_{\beta,\alpha}(R)\right|^2 dR\right)^{1/2}$$

where

$$\bar{\sigma}_{\beta,\alpha}(R) = \bar{\sigma}_{\beta,\alpha}(R; x_0, f)$$

is the (C, β)-mean of the Fourier integral

$$\int_0^\infty \xi^\alpha \sin \xi t d\xi \int_0^\infty \psi(u; x, f) \sin u\xi du\ .$$

PROOF. We set $u = e^{-y}$. Then

$$\int_0^\infty \frac{|\psi(u)|^2}{u^{2\alpha}} \frac{du}{u} = \int_{-\infty}^\infty |\Theta(y)|^2 dy \ , \tag{5.1}$$

where

$$\Theta(y) = \psi(e^{-y})e^{\alpha y} \ .$$

On the other side

$$\bar{\sigma}_{\beta,\alpha}(R) = c\int_0^\infty \psi(u)\left\{R\int_0^1 (1-z)^\beta (Rz)^\alpha \sin(Rzu)dz\right\}du \ ,$$

where

$$\int_0^1 (1-z)^\beta z^\alpha \sin(zu)dz \qquad (\beta > -1, \alpha > -1)$$

is Kummer's confluent hypergeometric function. Set $u = e^{-y}$, $R = e^x$, then

$$\int_0^\infty R\left|\frac{d}{dR}\bar{\sigma}_{\beta,\alpha}(R)\right|^2 dR = \int_0^\infty \frac{1}{R}|\bar{\sigma}_{\beta-1,\alpha}(R) - \bar{\sigma}_{\beta,\alpha}(R)|^2 dR \tag{5.2}$$
$$= c\int_{-\infty}^\infty |(\Theta * \bar{K}^*_{\beta,\alpha})(x)|^2 dx$$

where

$$\bar{K}^*_{\beta,\alpha}(x) = e^{(\alpha+1)x}\int_0^1 (1-z)^{\beta-1}z^{\alpha+1}\sin(e^x z)dz$$

The complex Fourier tranform of $\bar{K}^*_{\beta,\alpha}(x)$ is

$$\int_{-\infty}^\infty \bar{K}^*_{\beta,\alpha}(x)e^{sx}dx = \frac{\Gamma(\beta)\Gamma(1-s)\Gamma(1+\alpha+s)\cos\{(\alpha+s)\pi/2\}}{\Gamma(\beta+1-s)}$$
$$(0 < \alpha < 1, 1/2 < \beta < 3/2, 1-\alpha > \operatorname{Re} s > -(1+\alpha))$$

which is analytic in the strip near the line $\operatorname{Re} s = 0$ and has no zero on $\operatorname{Re} s = 0$. Furthermore we have

$$|\hat{\bar{K}}^*_{\beta,\alpha}(\xi)| \sim c\frac{e^{-(\pi|\xi|/2)}|\xi|^{1-(1/2)}e^{-(\pi|\xi|/2)}|\xi|^{\alpha+1-(1/2)}e^{\pi|\xi|/2}}{e^{-(\pi|\xi|/2)}|\xi|^{\beta+1-(1/2)}}$$
$$= c|\xi|^{-\beta+\alpha+(1/2)} \quad \text{as} \quad |\xi| \to \infty \ .$$

Furthermore $\hat{\bar{K}}^*_{\beta,\alpha}(\xi)$ is bounded on $\xi \in (-\infty, \infty)$. Thus any necessary condition analogous to §3 are satisfied. Comparing (5.1) with (5.2) we get the lemma.

Theorem 2 is obvious from Lemmas 1 and 2.

6. We consider here the Abel summability analogue of the preceeding sections. Let $f(x) \in S$ and the Poisson and conjugate Poisson integral of $f(x)$ be

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(u)}{(x-u)^2 + y^2} du = \frac{1}{\pi}\int_{0}^{\infty} \frac{y\phi(u; x, f)}{u^2 + y^2} du ,$$

$$\bar{u}(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-u)f(u)}{(x-u)^2 + y^2} du = \frac{1}{\pi}\int_{0}^{\infty} \frac{u\psi(u; x, f)}{u^2 + y^2} du .$$

The Littlewood-Paley function $g(f)(x)$ is defined by

$$g(f)(x) = \left\{\int_0^{\infty} y\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right)dy\right\}^{1/2} .$$

But since

$$\left|\frac{\partial u(x, y)}{\partial x}\right|^2 = \left|\frac{\partial \bar{u}(x, y)}{\partial y}\right|^2 ,$$

$$g(f)(x) = \left\{\int_0^{\infty} y\left(\left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial \bar{u}}{\partial y}\right|^2\right)dy\right\}^{1/2} .$$

We separate the real and imaginary part and set

$$h(f)(x_0) = \left(\int_0^{\infty} y \left|\frac{\partial u(x_0, y)}{\partial y}\right|^2 dy\right)^{1/2} , \tag{6.1}$$

$$\bar{h}(f)(x_0) = \left(\int_0^{\infty} y \left|\frac{\partial \bar{u}(x_0, y)}{\partial y}\right|^2 dy\right)^{1/2} . \tag{6.2}$$

We use notations $\phi(u) = \phi(u; x_0, f)$ and $\psi(u) = \psi(u; x_0, f)$. We set $R = y^{-1}$, and write

$$a(R) = a(R; x_0, f) = \frac{1}{\pi}\int_0^{\infty} \frac{R\phi(u)}{1 + (uR)^2} du \tag{6.3}$$

and

$$\bar{a}(R) = \bar{a}(R; x_0, f) = \frac{1}{\pi}\int_0^{\infty} \frac{R^2 \cdot u\psi(u)}{1 + (uR)^2} du . \tag{6.4}$$

Then

$$h(f)(x_0) = \left(c\int_0^{\infty} R\left|\frac{da(R)}{dR}\right|^2 dR\right)^{1/2}$$

and

$$\bar{h}(f)(x_0) = \left(c'\int_0^{\infty} R\left|\frac{d\bar{a}(R)}{dR}\right|^2 dR\right)^{1/2} .$$

Now we consider $\bar{h}(f)(x_0)$. By definition

$$\{\bar{h}(f)(x_0)\}^2 = c\int_0^\infty \frac{1}{R}\left|\int_0^\infty \psi(u)\cdot\frac{2R^2u}{(1+R^2u^2)^2}du\right|^2 dR .$$

We set $u = e^{-y}$, $R = e^x$ and $\Psi(y) = \psi(e^{-y})$, $\bar{K}^*(x) = (e^{2x}/(1+e^{2x})^2)$. Then

$$\{\bar{h}(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\Psi * \bar{K}^*)(x)|^2 dx . \tag{6.5}$$

The convolution is obviously well-defined. The complex Fourier transform of $\bar{K}^*(x)$ is

$$\begin{aligned}\int_{-\infty}^\infty e^{sx}\bar{K}^*(x)dx &= \int_{-\infty}^\infty \frac{e^{(s+2)x}}{(1+e^{2x})^2}dx \\ &= \int_0^\infty \frac{t^{s+1}}{(1+t^2)^2}dt \\ &= \frac{s}{2}\cdot\frac{\pi}{\sin \pi s/2} .\end{aligned}$$

For an Abel analogue of Marcinkiewicz function, we set

$$\psi_a(t) = \frac{1}{t}\int_0^\infty \left(\frac{u}{t}\right)^{1/2} e^{-u/t}\psi(u)du , \tag{6.6}$$

(see Levinson [6]) and

$$\mu_a(f)(x_0) = \left(\int_0^\infty |\psi_a(t)|^2\frac{dt}{t}\right)^{1/2} . \tag{6.7}$$

Set $t = e^{-x}$, $u = e^{-y}$, then

$$\{\mu_a(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\Psi * K)(x)|^2 dx \tag{6.8}$$

where

$$K(x) = e^{(1+1/2)x}\exp(-e^x)$$

The complex Fourier transform of $K(x)$ is

$$\begin{aligned}\int_{-\infty}^\infty e^{sx}K(x)dx &= \int_{-\infty}^\infty e^{(s+3/2)x}\exp(-e^x)dx \\ &= \int_0^\infty t^{(s+1/2)}e^{-t}dt = \Gamma\left(\frac{3}{2}+s\right) .\end{aligned}$$

Therefore

$$\hat{\bar{K}}^*(\xi) = \frac{-i\xi}{2}\cdot\frac{\pi}{\sin\pi(-i\xi)/2} ; \qquad \hat{\bar{K}}^*(\xi) \sim \frac{c|\xi|}{e^{\pi|\xi|/2}} \quad \text{as} \quad |\xi| \to \infty$$

and

$$\hat{K}(\xi) = \Gamma(3/2 - i\xi) ; \qquad \hat{K}(\xi) \sim c' \frac{|\xi|}{e^{\pi|\xi|/2}} \quad \text{as} \quad |\xi| \to \infty .$$

Thus we get the following theorem.

THEOREM 4. *For $f(x) \in S$, and $x_0 \in \boldsymbol{R}$,*

$$A\bar{h}(f)(x_0) \leqq \mu_a(f)(x_0) \leqq B\bar{h}(f)(x_0) .$$

If we set

$$\psi_a^*(t) = t \int_0^\infty \left(\frac{t}{u}\right)^{1/2} e^{-t/u} \psi(u) \frac{du}{u^2}$$

and change $t = e^{-x}$, $u = e^{-y}$, then the corresponding kernel is

$$K_a^*(x) = e^{-(1+1/2)x} \exp(-e^{-x}) .$$

Since

$$\int_{-\infty}^\infty e^{sx} K_a^*(x) = \Gamma\left(\frac{3}{2} - s\right), \qquad \hat{K}_a^*(\xi) = \Gamma\left(\frac{3}{2} + i\xi\right)$$

equals asymptotically to that of $\psi_a(t)$ as $|\xi| \to \infty$. Therefore we have

THEOREM 4′. *For $f(x) \in S$ and $x_0 \in \boldsymbol{R}$,*

$$A\bar{h}(f)(x_0) \leqq \mu_a^*(f)(x_0) \leqq B\bar{h}(f)(x_0) ,$$

where

$$\mu_a^*(f)(x_0) = \left\{\int_0^\infty \frac{1}{t} \left| t \int_0^\infty \left(\frac{t}{u}\right)^{1/2} e^{-t/u} \psi(u) \frac{du}{u^2} \right|^2 dt \right\}^{1/2} .$$

For the real part function $h(f)(x_0)$, we consider the following function. Let

$$\phi_a(t) = \frac{1}{t} \int_0^\infty \left(\frac{u}{t}\right)^{-1/2} e^{-u/t} \phi(u) du \tag{6.9}$$

and

$$\delta_a(f)(x_0) = \left(\int_0^\infty t \left| \frac{d}{dt} \phi_a(t) \right|^2 dt \right)^{1/2} . \tag{6.10}$$

Moreover put

$$\phi_a^*(t) = t \int_0^\infty \left(\frac{t}{u}\right)^{-1/2} e^{-t/u} \phi(u) \frac{du}{u^2}$$

and

$$\delta_a^*(f)(x_0) = \left(\int_0^\infty t\left|\frac{d}{dt}\phi_a^*(t)\right|^2 dt\right)^{1/2} .$$

Then we have

THEOREM 5. *For $f(x) \in S$ and $x_0 \in \boldsymbol{R}$,*

$$Ah(f)(x_0) \leqq \delta_a(f)(x_0) \leqq Bh(f)(x_0)$$

and

$$Ah(f)(x_0) \leqq \delta_a^*(f)(x_0) \leqq Bh(f)(x_0)$$

PROOF. By definition

$$\{h(f)(x_0)\}^2 = c\int_0^\infty \frac{1}{R}\left|R\frac{d}{dR}a(R)\right|^2 dR .$$

By (6.3) we have $(d/dR)a(R) = c\int_0^\infty \{u\phi'(u)/(1+(Ru)^2)\}du$. Now set $u = e^{-y}$, $R = e^x$, $\chi(x) = e^{-x}\phi'(e^{-x})$ and $K^*(x) = e^x/(1+e^{2x})$, then

$$\{h(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K^*)(x)|^2 dx . \tag{6.11}$$

On the other hand, since

$$\phi_a(t) = \frac{1}{t}\int_0^\infty \phi(u)\left(\frac{u}{t}\right)^{-1/2} e^{-u/t}du = \int_0^\infty \phi(tv)v^{-1/2}e^{-v}dv ,$$

$$\phi_a'(t) = \int_0^\infty \phi'(tv)v^{1-(1/2)}e^{-v}dv = \int_0^\infty \phi'(u)\left(\frac{u}{t}\right)^{1/2} e^{-u/t}du .$$

Set $t = e^{-x}$, $u = e^{-y}$, $\chi(x) = e^{-x}\phi'(e^{-x})$ and

$$K(x) = e^{x/2}\exp(-e^x)$$

then

$$\{\delta_a(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K)(x)|^2 dx . \tag{6.12}$$

The Fourier transforms of the kernels (6.11) and (6.12) are

$$\hat{K}(\xi) = c\Gamma\left(-i\xi + \frac{1}{2}\right) \quad \text{and} \quad \hat{K}^*(\xi) = \frac{\pi}{2\cos\pi(-i\xi)/2} ,$$

respectively. Hence we get the first part of Theorem. By the same method we can prove the another part.

7. Here we give some corollaries of the above theorems. Fefferman [2] proves that $\bar{h}_\beta(f)(x)$ and $h_\beta(f)(x)$ is of weak type (p, p) for $1 < p < 2$ and $\beta = (1/p)$. We assume this results in the sequel. In fact he proved the theorem in several variables form.

COROLLARY 1. *For* $\alpha = (1/p) - (1/2)$ *and* $1 < p < 2$, *the operator* $\mu_\alpha(f)(x)$ *is of weak type* (p, p).

This is given from Theorem 1. $\alpha = (1/p) - (1/2)$, so if $\alpha = (1/2)$ then $\mu_{1/2}(f)(x)$ is of strong type (p, p) for any p $(1<p<2)$. Zygmund [12] proved that $\mu(f)(x) = \mu_1(f)(x)$ is of strong type (p, p) for any $p > 1$.

COROLLARY 2. *For* $\alpha = (1/p) - (1/2)$ and $1 < p < 2$, *the operator* $D_\alpha(f)(x)$ *has weak type* (p, p).

This comes from Theorem 3. Fefferman [2] remarks that this corollary is established by the same method to proof of $g_\beta^*(f)(x)$.

COROLLARY 3. *For* $\alpha = (1/p) + (1/2)$ and $1 < p < 2$, *the operator* $\delta_\alpha(f)(x)$ *has weak type* (p, p).

Since, $h_\beta(f)(x)$ is of weak type (p, p) for $1 < p < 2$, the corollary comes from Theorem 2.

COROLLARY 4. *Let*

$$\delta_0(f)(x) = \left(\int_0^\infty t\left|\frac{d}{dt}\{f(x-t) + f(x+t)\}\right|^2 dt\right)^{1/2} .$$

Then, for $\alpha_1 > \alpha_2 > 0$

$$h(f)(x) \sim \delta_\alpha(f)(x) \prec \delta_{\alpha_1}(f)(x) \prec \delta_{\alpha_2}(f)(x) \prec \delta_0(f)(x)$$

and for $\beta_1 > \beta_2 > -1/2$

$$h(f)(x) \prec h_{\beta_1}(f)(x) \prec h_{\beta_2}(f)(x) \sim \delta_{\beta_2+1/2}(f)(x) \prec \delta_0(f)(x) ,$$

where $\prec$ *means that if the right side is finite then the left side is finite.*

A comparison each other of Fourier transform of corresponding kernels and Theorems 3 and 5 yield the corollory.

This is an answer of Problem 6 (a) of Stein-Wainger [11, p. 1289] in one dimensional form.

REMARK. Several variables analogues in spherical sense of the above theorems will appear in the forthcoming paper.

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