On the fundamental double four-spiral semigroup

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Abstract

We give a new description of the fundamental double four-spiral semigroup.

The fundamental four-spiral semigroup Sp_4 and the fundamental double four-spiral semigroup DSp_4 were introduced in [1], [3], and [4]. These semigroups are interesting examples of fundamental regular semigroups, and are indispensable building blocks of bisimple, idempotent-generated regular semigroups. Their basic properties are recalled in parts 1 and 2 of this note.

In part 3 we give an alternate construction of DSp_4 in terms of the free semigroup on two generators, as a set of quadruples with a simple, bicyclic-like multiplication. This permits shorter proofs and easier access to the main properties of DSp_4: descriptions of $DSp_4/\mathcal{L}$ and $DSp_4/\mathcal{R}$ (part 4); reduced form of the elements (part 5); and the property of congruences $\mathcal{C} \not\subseteq \mathcal{L}$ that $DSp_4/\mathcal{C}$ is completely simple (part 6).

1. Recall that Sp_4 is the semigroup

$$\begin{aligned} Sp_4 \;\cong\; \langle a,b,c,d;\; & a^2=a, b^2=b, c^2=c, d^2=d, \\ & a=ba, b=ab, b=bc, c=cb, c=dc, d=cd, d=da\rangle \end{aligned}$$

generated by four idempotents a, b, c, d such that $a \,\mathcal{R}\, b \,\mathcal{L}\, c \,\mathcal{R}\, d \leq_L a$. (We denote Green's left preorder $x \in S^1 y$ by $x \leq_L y$). It is shown in [3] that every element of Sp_4 can be written uniquely in reduced form

$$[c](ac)^m[a], \quad [d](bd)^n[b], \quad [c](ac)^m ad(bd)^n[b],$$

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where $m, n \geq 0$ and terms in square brackets may be omitted as long as the remaining product is not empty. Hence Sp_4 has a partition $Sp_4 = A \cup B \cup C \cup D \cup E$, where

$$\begin{aligned} A &= \{ (ac)^m a,\ (bd)^{n+1},\ (ac)^m ad(bd)^n;\ m,n \geq 0 \}, \\ B &= \{ (ac)^{m+1},\ (bd)^n b,\ (ac)^m ad(bd)^n b;\ m,n \geq 0 \}, \\ C &= \{ c(ac)^m,\ d(bd)^n b,\ c(ac)^m ad(bd)^n b;\ m,n \geq 0 \}, \\ D &= \{ d(bd)^n,\ c(ac)^m ad(bd)^n;\ m,n \geq 0 \}, \\ E &= \{ c(ac)^m a;\ m \geq 0 \} \end{aligned}$$

have a number of interesting properties [3].

By Theorem 1 in [2], Sp_4 is a Rees matrix semigroup over the bicyclic semigroup, and can be described up to isomorphism as the set of all quadruples (r, x, y, s) where $r, s \in \{ 0, 1 \}$ and x, y are nonnegative integers, with multiplication
$(r, x, y, s)(t, z, w, u) =$

$$\begin{cases} (r,\ x - y + \max(y, z+1),\ \max(y-1, z) - z + w,\ u) & \text{if } s = 0, t = 1, \\ (r,\ x - y + \max(y, z),\ \max(y, z) - z + w,\ u) & \text{otherwise.} \end{cases}$$

In this form, $a = (0, 0, 0, 0)$, $b = (0, 0, 0, 1)$, $c = (1, 0, 0, 1)$, $d = (1, 0, 1, 0)$, and it is readily verified that

$$\begin{aligned} (ac)^m &= (0, m, 0, 1), & c(ac)^m &= (1, m, 0, 1), \\ (ac)^m a &= (0, m, 0, 0), & c(ac)^m a &= (1, m, 0, 0), \\ (bd)^n &= (0, 0, n, 0), & d(bd)^n &= (1, 0, n+1, 0), \\ (bd)^n b &= (0, 0, n, 1), & d(bd)^n b &= (1, 0, n+1, 1), \end{aligned}$$

$$\begin{aligned} (ac)^m ad(bd)^n &= (0, m+1, n+1, 0), \\ c(ac)^m ad(bd)^n &= (1, m+1, n+1, 0), \\ (ac)^m ad(bd)^n b &= (0, m+1, n+1, 1), \\ c(ac)^m ad(bd)^n b &= (1, m+1, n+1, 1). \end{aligned}$$

Then

$$\begin{aligned} (r, x, y, s) \in A &\iff r = 0,\ s = 0, \\ (r, x, y, s) \in B &\iff r = 0,\ s = 1, \\ (r, x, y, s) \in C &\iff r = 1,\ s = 1, \\ (r, x, y, s) \in D &\iff r = 1,\ s = 0,\ y > 0, \\ (r, x, y, s) \in E &\iff r = 1,\ s = 0,\ y = 0. \end{aligned}$$

If $\mathcal{C}$ is a proper congruence on Sp_4, then $a\,\mathcal{C}\,ad$, and $Sp_4/\mathcal{C}$ is completely simple. Therefore, when a $\mathcal{D}$-class of a semigroup contains idempotents $e < a$ linked by an E-chain of length 4, then a is contained in a subsemigroup of D which is isomorphic to Sp_4 or to Sp_4^{op} [3].

2. The fundamental double four-spiral semigroup DSp_4 may be defined as the semigroup

$$\begin{aligned} DSp_4 \cong\ & \langle a, b, c, d, e;\ a^2 = a, b^2 = b, c^2 = c, d^2 = d, e^2 = e, \\ & a = ba, b = ab, b = bc, c = cb, c = dc, d = cd, d = de, e = ed, e = ae = ea \rangle \end{aligned}$$

generated by five idempotents a, b, c, d, e such that $a\,\mathcal{R}\,b\,\mathcal{L}\,c\,\mathcal{R}\,d\,\mathcal{L}\,e \leq a$. It is shown in [4] that every element of DSp_4 can be written uniquely in reduced form

$$[c](xc)^m[a], \quad [d](bd)^n[b], \quad [c](xc)^m y(bd)^n[b],$$

where: $m, n \geq 0$; terms in square brackets may be omitted as long as the remaining product is not empty; $(xc)^m$ is short for $xcxc\ldots xc$ where each x stands for either a or e; and y stands for either ad or e. (In [4], x and y are denoted by ∂ and ∂'.) Hence DSp_4 has a partition $DSp_4 = \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D} \cup \overline{E}$, where

$$\begin{aligned}
\overline{A} &= \{\, (xc)^m a,\ (bd)^{n+1},\ (xc)^m y(bd)^n;\ m, n \geq 0 \,\}, \\
\overline{B} &= \{\, (xc)^{m+1},\ (bd)^n b,\ (xc)^m y(bd)^n b;\ m, n \geq 0 \,\}, \\
\overline{C} &= \{\, c(xc)^m,\ d(bd)^n b,\ c(xc)^m y(bd)^n b;\ m, n \geq 0 \,\}, \\
\overline{D} &= \{\, d(bd)^n,\ c(xc)^m y(bd)^n;\ m, n \geq 0 \,\}, \\
\overline{E} &= \{\, c(xc)^m a;\ m \geq 0 \,\}.
\end{aligned}$$

When a $\mathcal{D}$-class of a semigroup contains idempotents $e < a$ linked by an E-chain of length 4, then a and e are contained in a subsemigroup of D which is isomorphic to $DSp_4/\mathcal{C}$ or to $(DSp_4/\mathcal{C})^{\mathrm{op}}$ for some congruence $\mathcal{C} \subseteq \mathcal{L}$ [4].

3. In the above the reduced words $[c](xc)^m[a]$ and $[c](xc)^m y(bd)^n[b]$ are obtained from $[c](ac)^m[a]$ and $[c](ac)^m ad(bd)^n[b]$ by replacing ad or some of the a's by e's. We use sequences of a's and e's as templates to specify which a's remain unchanged and which are replaced by e's.

Let $F = F^1_{\{a,e\}}$ be the free monoid on $\{\, a, e \,\}$. We write the empty word in F as $\emptyset$ to distinguish it from the number 1. If $X = x_1 x_2 \ldots x_m \in F$ has length $|X| = m \geq 0$, then substituting e's in $[c](ac)^m[a]$ according to X yields

$$X \centerdot [c](ac)^m[a] = [c](x_1c)(x_2c)\ldots(x_mc)[a].$$

If $X \in F$ has length $m+1$, then substituting e's in $[c](ac)^m ad(bd)^n[b]$ according to X yields

$$X \centerdot [c](ac)^m ad(bd)^n[b] = \begin{cases} [c](x_1c)(x_2c)\ldots(x_mc)ad(bd)^n[b] & \text{if } x_{m+1} = a, \\ [c](x_1c)(x_2c)\ldots(x_mc)e(bd)^n[b] & \text{if } x_{m+1} = e. \end{cases}$$

Every reduced word p can then be written uniquely in the form $p = X \centerdot q$, where $X \in F$ has the appropriate length, and q is a reduced word without e's. (If $p = [d](bd)^n[b]$, then $X = \emptyset \in F$ and $q = p$.) Now q is a reduced word for the four-spiral semigroup and can be viewed as a quadruple (r, k, ℓ, s) in which $r, s = 0, 1$ and k, ℓ are nonnegative integers. In all cases k is the number of a's in q which may be replaced by e's; thus $X \centerdot q$ is defined if and only if X has length k. Therefore p is uniquely determined by the quadruple $(r, X; \ell, s)$.

To describe the multiplication on DSp_4 in this form, let X_ℓ denote X with the first ℓ letters removed (added, if $\ell = -1$):

$$X_\ell = \begin{cases} \emptyset & \text{if } \ell \geq m, \\ x_{\ell+1}\ldots x_m & \text{if } 0 \leq \ell < m, \\ aX & \text{if } \ell = -1, \end{cases}$$

where $X = x_1x_2 \ldots x_m \in F$. In each case $|X_\ell| = \max(\ell, m) - \ell$. Inspecting the various products of reduced words $[c](xc)^m[a]$, $[d](bd)^n[b]$, $[c](xc)^m y(bd)^n[b]$ now yields our main result. (In part 5 we give a direct proof, which also establishes that the reduced words are all distinct in DSp_4.)

Main Result. *Up to isomorphism, DSp_4 is the semigroup of all quadruples $(r, X; y, s)$ where $r, s = 0, 1$, y is a nonnegative integer, and $X \in F^1_{\{a,e\}}$, with multiplication $(r, X; y, s)(t, Z; w, u) =$*

$$\begin{cases} (r,\ XZ_{y-1};\ \max(y-1, z) - z + w,\ u) & \text{if } s = 0,\ t = 1, \\ (r,\ XZ_y;\ \max(y, z) - z + w,\ u) & \text{otherwise}, \end{cases}$$

where $z = |Z|$.

4. This main result has a number of easy applications. The projection $\pi : DSp_4 \longrightarrow Sp_4$ may be described by

$$\pi(r, X; y, s) \;=\; (r, |X|, y, s).$$

The partition of DSp_4 into $\overline{A} = \pi^{-1}A$, $\overline{B} = \pi^{-1}B$, etc. is given by:

$$\begin{array}{lcl} (r, X; y, s) \in \overline{A} & \Longleftrightarrow & r = 0,\ s = 0, \\ (r, X; y, s) \in \overline{B} & \Longleftrightarrow & r = 0,\ s = 1, \\ (r, X; y, s) \in \overline{C} & \Longleftrightarrow & r = 1,\ s = 1, \\ (r, X; y, s) \in \overline{D} & \Longleftrightarrow & r = 1,\ s = 0,\ y > 0, \\ (r, X; y, s) \in \overline{E} & \Longleftrightarrow & r = 1,\ s = 0,\ y = 0. \end{array}$$

Up to isomorphism, $\overline{A}$ consists of all pairs (X, y), with multiplication $(X, y)(Z, w) = (XZ_y;\ \max(y, z) - z + w)$. Then

$$(XZ_{y-1};\ \max(y-1, z) - z + w) \;=\; (X, y)(a, 0)(Z, w)$$

and the main result describes DSp_4 as a 2×2 Rees matrix semigroup over $\overline{A}$, with sandwich matrix $\begin{pmatrix} (\emptyset, 0) & (a, 0) \\ (\emptyset, 0) & (\emptyset, 0) \end{pmatrix}$, equivalently, $\begin{pmatrix} a & aca \\ a & a \end{pmatrix}$.

It is readily verified that, if $r = 1$, $s = 0$, then (r, X, y, s) is idempotent if and only if $y = |X| + 1$; otherwise (r, X, y, s) is idempotent if and only if $y = |X|$.

Green's relations on DSp_4 may be described by:

$$\begin{array}{lcl} (r, X, y, s) \leq_R (t, Y, z, u) & \Longleftrightarrow & r = t \text{ and } X \leq_R Y \text{ in } F, \\ (r, X, y, s) \leq_L (t, Y, z, u) & \Longleftrightarrow & s = u \text{ and } y \geq z. \end{array}$$

Thus $\mathcal{L}$ contains the congruence induced by $\pi : DSp_4 \longrightarrow Sp_4$, and the partially ordered set $\Lambda = DSp_4/\mathcal{L}$ is isomorphic to $Sp_4/\mathcal{L}$ and consists of two unrelated ω-chains. For $DSp_4/\mathcal{R}$ we note that $X \leq_R Y$ in F if and only if X is a prefix of Y. Thus $R_\emptyset > R_a, R_e$; $R_a > R_{aa}, R_{ae}$; $R_e > R_{ea}, R_{ee}$, etc.; and $F/\mathcal{R}$ is a complete (upside down) binary tree in which every element covers two elements and (except for $R_\emptyset$) is covered by one element. Thus $I = DSp_4/\mathcal{R}$ consists of two unrelated complete binary trees. Our main result describes DSp_4 as $I \times \Lambda$ with a suitable multiplication.

5. We now give a direct proof of the main result. This proof also establishes that the reduced words $[c](xc)^m[a]$, $[d](bd)^n[b]$, $[c](xc)^m y(bd)^n[b]$ are all distinct in DSp_4.

First we verify that the quadruples in the statement constitute a semigroup D. We use the mapping $\pi : D \longrightarrow Sp_4$, $\pi(r, X; y, s) = (r, |X|, y, s)$, which is a homomorphism since $|XZ_y| = |X| - y + \max(y, |Z|)$ and $|XZ_{y-1}| = |X| - y + 1 + \max(y - 1, |Z|) = |X| - y + \max(y, |Z| + 1))$.

Let $(r, A; b, s)$, $(t, C; d, u)$, $(v, E; f, w) \in D$ and

$$(r, A; b, s)(t, C; d, u) = (r, G; h, u), \qquad (r, G; h, u)(v, E; f, w) = (r, K; l, w),$$
$$(t, C; d, u)(v, E; f, w) = (t, I; j, w), \qquad (r, A; b, s)(t, I; j; w) = (r, M; n, w)$$

We want to show that $(r, K; l, w) = (r, M; n, w)$. Since Sp_4 is a semigroup, we have $l = n$, and need only show that $K = M$.

There are four cases to consider. If $(s, t), (u, v) \neq (0, 1)$, then $G = AC_b$, $h = \max(b, c) - c + d$, where $c = |C|$, $K = GE_h$, $I = CE_d$, and $M = AI_b$. Thus $K = AC_bE_h$ and $M = A(CE_d)_b$. If C has length $c \geq b$, then $h = d$, $(CE_d)_b = C_bE_d = C_bE_h$, and $K = M$. If C has length $c < b$, then $h = b - c + d$, $(CE_d)_b = (E_d)_{b-c} = E_{d+b-c} = E_h$, and $K = AE_h = M$.

The other cases are similar. If $(s, t) = (0, 1)$ and $(u, v) \neq (0, 1)$, then $G = AC_{b-1}$, $h = \max(b - 1, c) - c + d$, $K = GE_h$, $I = CE_d$, and $M = AI_{b-1}$. If $b > 0$, then replacing b by $b - 1$ in the above yields $K = M$. If $b = 0$, then $h = d$, $G = AaC$, and $K = AaCE_h = AaI = M$.

If $(s, t) \neq (0, 1)$ and $(u, v) = (0, 1)$, then $G = AC_b$, $h = \max(b, c) - c + d$, $K = GE_{h-1}$, $I = CE_{d-1}$, and $M = AI_b$. If C has length $c \geq b$, then $h = d$; if $d > 0$, then

$$I_b = (CE_{d-1})_b = C_bE_{d-1} = C_bE_{h-1}$$

and $K = M$; if $d = 0$, then

$$I_b = (CE_{d-1})_b = (CaE)_b = C_baE = C_bE_{h-1}$$

and $K = M$. If C has length $c < b$, then $h = b - c + d > 0$; if $d > 0$, then

$$I_b = (CE_{d-1})_b = (E_{d-1})_{b-c} = E_{d-1+b-c} = E_{h-1}$$

and $K = AE_{h-1} = M$; if $d = 0$, then

$$I_b = (CE_{d-1})_b = (CaE)_b = E_{b-c-1} = E_{d-1+b-c} = E_{h-1}$$

and $K = AE_{h-1} = M$.

If $(s, t) = (u, v) = (0, 1)$, then $G = AC_{b-1}$, $h = \max(b-1, c)-c+d$, $K = GE_{h-1}$, $I = CE_{d-1}$, and $M = AI_{b-1}$. If $b > 0$, then replacing b by $b-1$ in the previous case yields $K = M$. If $b = 0$, then $h = d$; if $d > 0$, then $K = AaCE_{h-1} = AaI = M$; if $d = 0$, then $K = GaE = AaCaE = AaI = M$. Thus $K = M$ in all cases and D is a semigroup.

Let

$$\alpha = (0, \emptyset; 0, 0), \qquad \beta = (0, \emptyset; 0, 1),$$
$$\gamma = (1, \emptyset; 0, 1), \qquad \delta = (1, \emptyset; 1, 0), \qquad \epsilon = (0, e; 1, 0).$$

It is immediate that α, β, γ, δ, and ϵ satisfy all the defining relations of DSp_4 (α, β, γ, δ, and ϵ are idempotent, and $\alpha \mathcal{R} \beta \mathcal{L} \gamma \mathcal{R} \delta \mathcal{L} \epsilon \leq \alpha$): their images a, b, c, d, and $ad \in Sp_4$ under $\pi : D \longrightarrow Sp_4$ have these properties, and the second components (most of which are empty words) cooperate. Hence there is a homomorphism $\varphi : DSp_4 \longrightarrow D$ such that $\varphi(a) = \alpha$, $\varphi(b) = \beta$, $\varphi(c) = \gamma$, $\varphi(d) = \delta$, and $\varphi(e) = \epsilon$. We show that φ is an isomorphism.

As in [3] it follows from the defining relations that every element of DSp_4 can be written in reduced form: $X \centerdot [c](ac)^m[a] = [c](xc)^m[a]$, $[d](bd)^n[b]$, or $X \centerdot [c](ac)^m ad(bd)^n[b] = [c](xc)^m y(bd)^n[b]$. To prove that φ is an isomorphism (and that the reduced words are all distinct in DSp_4) we evaluate φ at all reduced words. First,

$$\begin{aligned}\varphi(ac) &= (0,\emptyset;0,0)(1,\emptyset;0,1) = (0,a;0,1),\\ \varphi(ec) &= (0,e;1,0)(1,\emptyset;0,1) = (0,e;0,1),\end{aligned}$$

and $(0,X;0,1)(0,Y;0,1) = (0,XY;0,1)$. By induction, $\varphi(X \centerdot (ac)^m) = (0,X;0,1)$. Also $\varphi(bd) = (0,\emptyset;0,1)(1,\emptyset;1,0) = (0,\emptyset;1,0)$ and, by induction, $\varphi(bd)^n = (0,\emptyset;n,0)$. Hence, for all $m,n > 0$:

$$\begin{aligned}\varphi(X \centerdot (ac)^m) &= (0,X;0,1),\\ \varphi(X \centerdot c(ac)^m) &= (1,\emptyset;0,1)(0,X;0,1) = (1,X;0,1),\\ \varphi(X \centerdot (ac)^m a) &= (0,X;0,1)(0,\emptyset;0,0) = (0,X;0,0),\\ \varphi(X \centerdot c(ac)^m a) &= (1,X;0,1)(0,\emptyset;0,0) = (1,X;0,0),\\ \varphi((bd)^n) &= (0,\emptyset;n,0),\\ \varphi(d(bd)^n) &= (1,\emptyset;1,0)(0,1;n,0) = (1,\emptyset;n+1,0),\\ \varphi((bd)^n b) &= (0,\emptyset;n,0)(0,\emptyset;0,1) = (0,\emptyset;n,1),\\ \varphi(d(bd)^n b) &= (1,\emptyset;n+1,0)(0,\emptyset;0,1) = (1,\emptyset;n+1,1).\end{aligned}$$

These equalities actually hold for all $m,n \geq 0$ as long as φ is not applied to empty products. If now $X = Ya$ has length $m+1$ and ends with a, then $X \centerdot [c](ac)^m ad(bd)^n[b] = (Y \centerdot [c](ac)^m a)(d(bd)^n[b])$ and

$$\begin{aligned}\varphi(X \centerdot (ac)^m ad(bd)^n) &= (0,Y;0,0)(1,\emptyset;n+1,0) = (0,X;n+1,0),\\ \varphi(X \centerdot c(ac)^m ad(bd)^n) &= (1,Y;0,0)(1,\emptyset;n+1,0) = (1,X;n+1,0),\\ \varphi(X \centerdot (ac)^m ad(bd)^n b) &= (0,Y;0,0)(1,\emptyset;n+1,1) = (0,X;n+1,1),\\ \varphi(X \centerdot c(ac)^m ad(bd)^n b) &= (1,Y;0,0)(1,\emptyset;n+1,1) = (1,X;n+1,1),\end{aligned}$$

for all $m,n \geq 0$. If on the other hand $X = Ye$ has length $m+1$ and ends with e, then $X \centerdot [c](ac)^m ad(bd)^n[b] = (Y \centerdot [c](ac)^m)e(bd)^n[b]$. If $m > 0$:

$$\begin{aligned}\varphi((Y \centerdot (ac)^m)e) &= (0,Y;0,1)(0,e;1,0) = (0,X;1,0),\\ \varphi((Y \centerdot c(ac)^m)e) &= (1,Y;0,1)(0,e;1,0) = (1,X;1,0),\end{aligned}$$

These equalities also hold if $m = 0$. Hence we obtain, for all $m \geq 0$, $n > 0$:

$$\begin{aligned}\varphi(X \centerdot (ac)^m ad(bd)^n) &= (0,X;1,0)(0,\emptyset;n,0) = (0,X;n+1,0),\\ \varphi(X \centerdot (ac)^m ad(bd)^n b) &= (0,X;1,0)(0,\emptyset;n,1) = (0,X;n+1,1),\\ \varphi(X \centerdot c(ac)^m ad(bd)^n) &= (1,X;1,0)(0,\emptyset;n,0) = (1,X;n+1,0),\\ \varphi(X \centerdot c(ac)^m ad(bd)^n b) &= (1,X;1,0)(0,\emptyset;n,1) = (1,X;n+1,1).\end{aligned}$$

These equalities also hold if $n = 0$.

Inspection shows that every element of D is the image under φ of a reduced word, and that distinct reduced words in DSp_4 have distinct images under φ. This implies that the reduced words are all distinct in DSp_4, and that φ is an isomorphism, which completes the proof.

6. Finally we use our main result to prove the following congruence property: that if $\mathcal{C}$ is a congruence on DSp_4 and $\mathcal{C} \nsubseteq \mathcal{L}$ then $DSp_4/\mathcal{C}$ is completely simple. This property implies that a $\mathcal{D}$-class of a semigroup which contains idempotents $e < a$ linked by an E-chain of length 4 must also contain a subsemigroup which is isomorphic to $DSp_4/\mathcal{C}$ or to $(DSp_4/\mathcal{C})^{\mathrm{op}}$ for some congruence $\mathcal{C} \subseteq \mathcal{L}$ [4].

For the proof we identify our two descriptions of DSp_4, so that $a = (0, \emptyset; 0, 0)$, $b = (0, \emptyset; 0, 1)$, $c = (1, \emptyset; 0, 1)$, $d = (1, \emptyset; 1, 0)$, $e = (0, e; 1, 0)$, and, from the previous proof, $X \centerdot (ac)^m = (0, X; 0, 1)$, etc.

Lemma 1. *Let $\mathcal{C}$ be a congruence on DSp_4. If $(bd)^k \,\mathcal{C}\, (bd)^m b$, then $a \,\mathcal{C}\, (bd)^m b$. If $a \,\mathcal{C}\, (bd)^m b$, then $a \,\mathcal{C}\, ad \,\mathcal{C}\, e$.*

Proof. If $(0, \emptyset; k, 0) = (bd)^k \;\mathcal{C}\; (bd)^m b = (0, \emptyset; m, 1)$, then $k > 0$,

$$\begin{aligned} (0, \emptyset; k, 0)(0, \emptyset; 0, 0) &= (0, \emptyset; k, 0), \\ (0, \emptyset; m, 1)(0, \emptyset; 0, 0) &= (0, \emptyset; m, 0), \\ (0, \emptyset; k, 0)(1, \emptyset; 0, 0) &= (0, \emptyset; k-1, 0), \\ (0, \emptyset; m, 1)(1, \emptyset; 0, 0) &= (0, \emptyset; m, 0), \end{aligned}$$

so that $(0, \emptyset; k, 0) \,\mathcal{C}\, (0, \emptyset; m, 0) \,\mathcal{C}\, (0, \emptyset; k-1, 0)$ and, by induction, $a = (0, \emptyset; 0, 0) \,\mathcal{C}\, (0, \emptyset; k, 0) = (bd)^k \,\mathcal{C}\, (bd)^m b$.

In turn, $a \,\mathcal{C}\, (bd)^m b$ implies $b = ab \;\mathcal{C}\; (bd)^m b \;\mathcal{C}\; a$, $c = dac \;\mathcal{C}\; dbc = db \;\mathcal{C}\; da = d$, $ad \;\mathcal{C}\; bd \;\mathcal{C}\; bc = b \;\mathcal{C}\; a$, and $a \;\mathcal{C}\; ad = ade = ace \;\mathcal{C}\; bce = e$. ∎

Lemma 2. *Let $\mathcal{C}$ be a congruence on DSp_4. If $\mathcal{C} \nsubseteq \mathcal{L}$, then $a \,\mathcal{C}\, ad \,\mathcal{C}\, e$.*

Proof. Assume $\mathcal{C} \nsubseteq \mathcal{L}$, so that there exists $p = (r, K; l, s)$ and $q = (t, M; n, u) \in DSp_4$ such that $p \,\mathcal{C}\, q$ but not $p \,\mathcal{L}\, q$. Then $(l, s) \neq (n, u)$.

Assume $s \neq u$, say, $s = 0$ and $u = 1$. If $y \geq |K|, |M|$, then

$$\begin{aligned} (0, \emptyset; y, 1)(r, K; l, s) &= (0, \emptyset; k, s), \\ (0, \emptyset; y, 1)(t, M; n, u) &= (0, \emptyset; m, u), \end{aligned}$$

and $(0, \emptyset; k, 0) \,\mathcal{C}\, (0, \emptyset; m, 1)$ for some $k, m \geq 0$. Thus $(bd)^k \,\mathcal{C}\, (bd)^m b$ if $k > 0$, $a \,\mathcal{C}\, (bd)^m b$ if $k = 0$. By Lemma 1, $a \,\mathcal{C}\, ad \,\mathcal{C}\, e$.

Now assume $s = u$, so that, say, $l < n$. Then

$$\begin{aligned} (r, K; l, s)(s, a^l; 0, 0) &= (r, K; 0, 0), \\ (t, M; n, u)(s, a^l; 0, 0) &= (t, M; n-l, 0), \end{aligned}$$

and $(r, K; 0, 0) \,\mathcal{C}\, (t, M; m, 0)$ where $m = n - l > 0$. For any $y \leq |K|, |M|$,

$$\begin{aligned} (0, \emptyset; y, 0)(r, K; 0, 0) &= (0, K_y; 0, 0), \\ (0, \emptyset; y, 0)(t, M; m, 0) &= (0, M_y; m, 0), \end{aligned}$$

and $(0, K_y; 0, 0) \,\mathcal{C}\, (0, M_y; m, 0)$.

If $|K| \leq |M|$, then $y = |K|$ yields $(0, \emptyset; 0, 0) \,\mathcal{C}\, (0, M_y; m, 0)$. Then $a \;\mathcal{C}\; M_y \centerdot (ac)^k ad(bd)^{m-1}$ for some $k \geq 0$. Since $de = d$ it follows that $a \;\mathcal{C}\; ae = e$, and $ad \;\mathcal{C}\; ed = e$.

If $|K| > |M|$, then $y = |M|$ yields $(0, K_y; 0, 0) \,\mathcal{C}\, (0, \emptyset; m, 0)$. Then $K_y \centerdot (ac)^k \,\mathcal{C}\, (bd)^m$ for some $k = |K| - |M| > 0$. Since $cb = c$ it follows that $(bd)^m \;\mathcal{C}\; (bd)^m b$. Again $a \,\mathcal{C}\, ad \,\mathcal{C}\, e$ by Lemma 1. ∎

We can now show that $DSp_4/\mathcal{C}$ is completely simple when $\mathcal{C} \not\subseteq \mathcal{L}$ is a congruence on DSp_4. We see on the reduced forms of the elements that the subsemigroup $T = \langle a, b, c, d \rangle$ of DSp_4 is isomorphic to Sp_4. Since $a\,\mathcal{R}\,b\,\mathcal{L}\,c\,\mathcal{R}\,d\,\mathcal{L}\,ad \leq a$ holds in T, there is a homomorphism $\tau : DSp_4 \longrightarrow T$ such that $\tau x = x$ for $x = a, b, c, d$ and $\tau e = ad$. By Lemma 2, $a\,\mathcal{C}\,ad\,\mathcal{C}\,e$. Therefore $\tau p\,\mathcal{C}\,p$ for all $p \in DSp_4$. Hence every $\mathcal{C}$-class intersects T, and $DSp_4/\mathcal{C}$ is a homomorphic image of $T/\mathcal{C}$. But $T/\mathcal{C}$ is completely simple, since $T \cong Sp_4$ and $a\,\mathcal{C}\,ad$ shows that $\mathcal{C}$ is not the equality on T. Therefore $DSp_4/\mathcal{C}$ is completely simple.

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