

ON THE FUNDAMENTAL LEMMA OF NEYMAN AND PEARSON¹

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1. Summary and introduction. The following lemma proved by Neyman and Pearson [1] is basic in the theory of testing statistical hypotheses:

LEMMA. Let $f_1(x), \dots, f_{m+1}(x)$ be $m + 1$ Borel measurable functions defined over a finite dimensional Euclidean space R such that $\int_R |f_i(x)| dx < \infty$ ($i = 1, \dots, m + 1$). Let, furthermore, c_1, \dots, c_m be m given constants and \mathcal{S} the class of all Borel measurable subsets S of R for which

$$(1.1) \quad \int_S f_i(x) dx = c_i \quad (i = 1, \dots, m).$$

Let, finally, \mathcal{S}_0 be the subclass of \mathcal{S} consisting of all members S_0 of \mathcal{S} for which

$$(1.2) \quad \int_{S_0} f_{m+1}(x) dx \geq \int_S f_{m+1}(x) dx \quad \text{for all } S \text{ in } \mathcal{S}.$$

If S is a member of \mathcal{S} and if there exist m constants k_1, \dots, k_m such that

$$(1.3) \quad f_{m+1}(x) \geq k_1 f_1(x) + \dots + k_m f_m(x) \quad \text{when } x \in S,$$

$$(1.4) \quad f_{m+1}(x) \leq k_1 f_1(x) + \dots + k_m f_m(x) \quad \text{when } x \notin S,$$

then S is a member of \mathcal{S}_0 .

The above lemma gives merely a sufficient condition for a member S of \mathcal{S} to be also a member of \mathcal{S}_0 . Two important questions were left open by Neyman and Pearson: (1) the question of existence, that is, the question whether \mathcal{S}_0 is non-empty whenever \mathcal{S} is non-empty; (2) the question of necessity of their sufficient condition (apart from the obvious weakening that (1.3) and (1.4) may be violated on a set of measure zero).

The purpose of the present note is to answer the above two questions. It will be shown in Section 2 that \mathcal{S}_0 is not empty whenever \mathcal{S} is not empty. In Section 3, a necessary and sufficient condition is given for a member of \mathcal{S} to be also a member of \mathcal{S}_0 . This necessary and sufficient condition coincides with the Neyman-Pearson sufficient condition under a mild restriction.

2. Proof that \mathcal{S}_0 is not empty whenever \mathcal{S} is not empty. Each function $f_i(x)$ determines a finite measure μ_i given by the equation

$$(2.1) \quad \mu_i(S) = \int_S f_i(x) dx \quad (i = 1, 2, \dots, m + 1).$$

¹ The main results of this paper were obtained by the authors independently of each other using entirely different methods.

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Let μ be the vector measure with the components μ_1, \dots, μ_{m+1} ; i.e., for any measurable set S the value of $\mu(S)$ is the vector $(\mu_1(S), \dots, \mu_{m+1}(S))$. Thus, for each S the value of $\mu(S)$ can be represented by a point in the $m + 1$ -dimensional Euclidean space E . A point $g = (g_1, \dots, g_{m+1})$ of E is said to belong to the range of the vector measure μ if and only if there exists a measurable subset S of R such that $\mu(S) = g$.

It was proved by Lyapunov [2] (see also [4]) that the range M of μ is a bounded, closed and convex subset of E . Let L be the line in E which is parallel to the $(m + 1)$ -th axis and goes through the point $(c_1, c_2, \dots, c_m, 0)$. Suppose that S is not empty. Then the intersection M^* of L with M is not empty. Because of Lyapunov's theorem, M^* is a finite closed interval (which may reduce to a single point). There exists a subset S of R such that $\mu(S)$ is equal to the upper end point of M^* . Clearly, S is a member of S_0 .

3. Necessary and sufficient condition that a member of S be also a member of S_0 . Let $\nu(S)$ be the vector measure with the components $\nu_1(S), \dots, \nu_m(S)$. According to the aforementioned theorem of Lyapunov, the range N of ν is a bounded, closed and convex subset of the m -dimensional Euclidean space.

By the dimension of a convex subset Q of a finite dimensional Euclidean space we shall mean the dimension of the smallest dimensional hyperplane that contains Q . A point q of a convex set Q is said to be an interior point of Q if there exists a sphere V with center at q and positive radius such that $V \cap \Pi \subset Q$, where Π is the smallest dimensional hyperplane containing Q . Any point q that is not an interior point of Q will be called a boundary point. We shall now prove the following theorem.

THEOREM 3.1. *If (c_1, \dots, c_m) is an interior point of N , then a necessary and sufficient condition for a member S of S to be a member of S_0 is that there exist m constants k_1, \dots, k_m such that (1.3) and (1.4) hold for all x except perhaps on a set of measure zero.*

PROOF. The Neyman-Pearson lemma cited in Section 1 states that our condition is sufficient. Thus, we merely have to prove the necessity of our condition. Assume that (c_1, \dots, c_m) is an interior point of N . Let c^* be the largest value for which $(c_1, \dots, c_m, c^*) \in M$, and c^{**} the smallest value for which

$$(c_1, \dots, c_m, c^{**}) \in M.$$

We shall first consider the case when $c^* = c^{**}$. Let $(\bar{c}_1, \dots, \bar{c}_m)$ be any other interior point of N . We shall show that there exists exactly one real value \bar{c} such that $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}) \in M$. For suppose that there are two different values \bar{c}^* and \bar{c}^{**} such that both $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}^*)$ and $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}^{**})$ are in M . Since (c_1, \dots, c_m) and $(\bar{c}_1, \dots, \bar{c}_m)$ are interior points of N , there exists a point (c'_1, \dots, c'_m) in N such that (c_1, \dots, c_m) lies in the interior of the segment determined by (c'_1, \dots, c'_m) and $(\bar{c}_1, \dots, \bar{c}_m)$. There exists a real value c' such that $(c'_1, \dots, c'_m, c') \in M$. Consider the convex set T determined by the 3 points: $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}^*)$, $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}^{**})$ and (c'_1, \dots, c'_m, c') . Obviously, $T \subset M$. But T contains points (c_1, \dots, c_m, h) and (c_1, \dots, c_m, h') with

$h \neq h'$, contrary to our assumption that $c^* = c^{**}$. Thus, for any interior point $(\bar{c}_1, \dots, \bar{c}_m)$ of N there exists exactly one real value \bar{c} such that $(\bar{c}_1, \dots, \bar{c}_m, \bar{c}) \in M$. Since M is closed and convex, this remains true also when $(\bar{c}_1, \dots, \bar{c}_m)$ is a boundary point of N . Thus, there exists a single valued function $\varphi(g_1, \dots, g_m)$ such that $g_{m+1} = \varphi(g_1, \dots, g_m)$ holds for all points $g = (g_1, \dots, g_m, g_{m+1})$ in M . Since M is convex, φ must be linear; i.e., $\varphi(g_1, \dots, g_m) = \sum_{i=1}^m k_i g_i + k_0$.

Since the origin is obviously contained in M , we have $k_0 = 0$. Thus, we have $g_{m+1} = \sum_{i=1}^m k_i g_i$ for all points g in M . But then $f_{m+1}(x) = \sum_{i=1}^m k_i f_i(x)$ must hold for all x , except perhaps on a set of measure zero. Thus, for any subset S of R , the inequalities (1.3) and (1.4) are fulfilled for all x , except perhaps on a set of measure zero. This completes the proof of our theorem in the case when $c^* = c^{**}$.

We shall now consider the case when $c^{**} < c^*$. Let c be any value between c^{**} and c^* ; i.e., $c^{**} < c < c^*$. We shall show that (c_1, \dots, c_m, c) is an interior point of M . For this purpose, consider a finite set of points $c^i = (c_1^i, \dots, c_m^i)$ in N ($i = 1, \dots, n$) such that c^1, \dots, c^n are linearly independent, the simplex determined by c^1, \dots, c^n has the same dimension as N and contains the point (c_1, \dots, c_m) in its interior. Such points c^i in N obviously exist. There exist real values h_i ($i = 1, \dots, n$) such that $(c_1^i, \dots, c_m^i, h_i) \in M$ ($i = 1, \dots, n$). Let T be the smallest convex set containing the points $(c_1^i, \dots, c_m^i, h_i)$ ($i = 1, \dots, n$), (c_1, \dots, c_m, c^*) and $(c_1, \dots, c_m, c^{**})$. Clearly, the dimension of T is the same as that of M and (c_1, \dots, c_m, c) is an interior point of T . Thus, (c_1, \dots, c_m, c) is an interior point of M . The point (c_1, \dots, c_m, c^*) is obviously a boundary point of M . Let $g = (g_1, \dots, g_{m+1})$ be the generic designation of a point in the $m + 1$ -dimensional Euclidean space E . Since (c_1, \dots, c_m, c^*) is a boundary point of M , there exists an m -dimensional hyperplane Π through (c_1, \dots, c_m, c^*) such that Π contains only boundary points of M and M lies entirely on one side of Π .³ Let the equation of Π be given by

$$(3.1) \quad k_{m+1} g_{m+1} - \sum_{i=1}^m k_i g_i = k_{m+1} c^* - \sum_{i=1}^m k_i c_i.$$

Since Π contains only boundary points of M , and since (c_1, \dots, c_m, c) is not a boundary point when $c^{**} < c < c^*$, the hyperplane Π cannot be parallel to the $(m + 1)$ -th coordinate axis; i.e., $k_{m+1} \neq 0$. We can assume without loss of generality that $k_{m+1} = 1$. Since M lies entirely on one side of Π , and since for $(g_1, \dots, g_m, g_{m+1}) = (c_1, \dots, c_m, c^{**})$ the left hand member of (3.1) is smaller than the right hand member, we must have

$$(3.2) \quad g_{m+1} - \sum_{i=1}^m k_i g_i \leq c^* - \sum_{i=1}^m k_i c_i$$

for all $g \in M$. Let S be a subset of R such that

³ This follows from well known results on convex bodies. See, for example, [3], p. 6.

$$(3.3) \quad (\mu_1(S), \dots, \mu_m(S), \mu_{m+1}(S)) = (c_1, \dots, c_m, c^*).$$

It can easily be seen that (3.2) and (3.3) can be fulfilled simultaneously only if S satisfies the conditions (1.3) and (1.4) for all x , except perhaps on a set of measure zero. This completes the proof of our theorem.

It remains to investigate the case when (c_1, \dots, c_m) is a boundary point of N . For this purpose, we shall introduce some definitions and prove some lemmas.

Let $\xi = (\xi_1, \dots, \xi_m)$ be an m -dimensional vector with real valued components at least one of which is not zero. We shall say that ξ is maximal relative to the point $c = (c_1, \dots, c_m)$ if

$$(3.4) \quad \sum_{i=1}^m \xi_i g_i \leq \sum_{i=1}^m \xi_i c_i$$

for all points (g_1, \dots, g_m) in N .

We shall say that a set $\{\xi^i\} (i = 1, 2, \dots, r; r > 1)$ of vectors is maximal relative to the point $c = (c_1, \dots, c_m)$ if the set $\{\xi^i\} (i = 1, \dots, r - 1)$ is maximal relative to c , not all components of ξ^r are zero and

$$(3.5) \quad \sum_{j=1}^m \xi_j^r g_j \leq \sum_{j=1}^m \xi_j^r c_j$$

holds for all points (g_1, \dots, g_m) of N for which

$$(3.6) \quad \sum_{j=1}^m \xi_j^i g_j = \sum_{j=1}^m \xi_j^i c_j \quad (i = 1, \dots, r - 1).$$

A set of vectors $\{\xi^i\} (i = 1, \dots, r)$ is said to be a complete maximal set relative to $c = (c_1, \dots, c_m)$ if $\{\xi^i\} (i = 1, 2, \dots, r)$ is maximal relative to c and no vector ξ^{r+1} exists such that ξ^{r+1} is linearly independent of the sequence (ξ^1, \dots, ξ^r) and $(\xi^1, \dots, \xi^r, \xi^{r+1})$ is maximal relative to c .

LEMMA 3.1. *If $c = (c_1, \dots, c_m)$ is a boundary point of N , then there exists a positive integer r and a set $\{\xi^1, \dots, \xi^r\}$ of vectors that is a complete maximal set relative to c .*

PROOF. Since c is a boundary point of N , there exists an $(m - 1)$ -dimensional hyperplane Π through c such that N lies entirely on one side of Π .³ Let the equation of Π be given by

$$\sum_{i=1}^m \xi_i g_i = \sum_{i=1}^m \xi_i c_i.$$

Since N lies entirely on one side of Π , either $\sum_{i=1}^m \xi_i g_i \geq \sum_{i=1}^m \xi_i c_i$ for all points (g_1, \dots, g_m) in N , or $\sum_{i=1}^m \xi_i g_i \leq \sum_{i=1}^m \xi_i c_i$ for all (g_1, \dots, g_m) in N . We put $\xi^1 = -\xi$ if $\sum \xi_i g_i \geq \sum \xi_i c_i$ for all points (g_1, \dots, g_m) in N . Otherwise, we put $\xi^1 = \xi$. Clearly, ξ^1 is maximal relative to c . If ξ^1 is not a complete maximal set relative to c , there exists a vector ξ^2 such that ξ^2 is linearly independent of

ξ^1 and (ξ^1, ξ^2) is maximal relative to c . If (ξ^1, ξ^2) is not a complete maximal set, we can find a vector ξ^3 such that ξ^3 is linearly independent of (ξ^1, ξ^2) and (ξ^1, ξ^2, ξ^3) is a maximal set relative to c , and so on. Continuing this procedure, we shall arrive at a set (ξ^1, \dots, ξ^r) ($r \leq m$) that is a complete maximal set relative to c . This completes the proof of Lemma 3.1.

LEMMA 3.2. *If (ξ^1, \dots, ξ^r) is a maximal set of vectors relative to $c = (c_1, \dots, c_m)$ and if $\nu(S) = c$, then the following two conditions are fulfilled for all x (except perhaps on a set of measure zero):*

a) *If x is a point in R for which $\sum_{j=1}^m \xi_j^i f_j(x) = 0$ for $i = 1, 2, \dots, u - 1$ and $\sum_{j=1}^m \xi_j^u f_j(x) > 0$ ($u = 1, 2, \dots, r$), then $x \in S$.*

b) *If x is a point of R for which $\sum_{j=1}^m \xi_j^i f_j(x) = 0$ for $i = 1, 2, \dots, u - 1$ and $\sum_{j=1}^m \xi_j^u f_j(x) < 0$, then $x \notin S$.*

PROOF. Assume that (ξ^1, \dots, ξ^r) is maximal relative to c . Then, ξ^1 is maximal relative to c . This implies that for all x (except perhaps on a set of measure zero) the following condition holds: $x \in S$ when $\sum_{j=1}^m \xi_j^1 f_j(x) > 0$ and $x \notin S$ when $\sum_{j=1}^m \xi_j^1 f_j(x) < 0$. Thus, conditions (a) and (b) of our lemma must be fulfilled for $u = 1$. We shall now show that if (a) and (b) hold for $u = 1, \dots, v$ then (a) and (b) must hold also for $u = v + 1$. For this purpose, consider the set R' of all points x for which $\sum_{j=1}^m \xi_j^i f_j(x) = 0$ for $i = 1, \dots, v$. If R is replaced by R' , then ξ^{v+1} is maximal relative to $c' = (c'_1, \dots, c'_m)$ where $c'_i = \int_{R'} f_i(x) dx$ and $S' = S \cap R'$. Hence, for any x in R' (except perhaps on a set of measure zero) the following condition holds: $x \in S$ when $\sum_{j=1}^m \xi_j^{v+1} f_j(x) > 0$ and $x \notin S$ when $\sum_{j=1}^m \xi_j^{v+1} f_j(x) < 0$. But this implies that (a) and (b) hold for $u = v + 1$. This completes the proof of our lemma.

LEMMA 3.3. *Let (ξ^1, \dots, ξ^r) be a complete maximal set of vectors relative to $c = (c_1, \dots, c_m)$, and let T be the set of all points $g = (g_1, \dots, g_m)$ of N for which $\sum_{j=1}^m \xi_j^i g_j = \sum_{j=1}^m \xi_j^i c_j$ for $i = 1, 2, \dots, r$. Then T is a bounded, closed and convex set and c is an interior point of T .*

PROOF. Clearly, T is a bounded, closed and convex set. Suppose that c is a boundary point of T . Then there exists a hyperplane Π of dimension $m - 1$ such that Π goes through c , Π contains only boundary points of T and T lies entirely on one side of Π^3 . Let the equation of Π be given by

$$\sum_{j=1}^m \xi_j g_j = \sum_{j=1}^m \xi_j c_j,$$

where ξ is independent of ξ^1, \dots, ξ^r . Since T lies on one side of Π , we have either $\sum_{j=1}^m \xi_j g_j \geq \sum_{j=1}^m \xi_j c_j$ for all $g = (g_1, \dots, g_m)$ in T , or $\sum_{j=1}^m \xi_j g_j \leq \sum_{j=1}^m \xi_j c_j$ for all g in T . Let $\xi_j^{r+1} = \xi_j (j = 1, \dots, m)$ in the latter case, and $\xi_j^{r+1} = -\xi_j$ in the former case. Then $\sum_{j=1}^m \xi_j^{r+1} g_j \leq \sum_{j=1}^m \xi_j^{r+1} c_j$ for all g in T . But then $(\xi^1, \dots, \xi^r, \xi^{r+1})$ is a maximal set relative to c , contrary to our assumption that (ξ^1, \dots, ξ^r) is a complete maximal set. Thus, c must be an interior point of T and our lemma is proved.

THEOREM 3.2. *If $c = (c_1, \dots, c_m)$ is a boundary point of N and if (ξ^1, \dots, ξ^r) is a complete maximal set of vectors relative to c , then a necessary and sufficient condition for a member S of \mathcal{S} to be a member of \mathcal{S}_0 is that there exist m constants k_1, \dots, k_m such that for all x in R' (except perhaps on a set of measure zero) the inequalities (1.3) and (1.4) hold, where R' is the set of all points x for which*

$$\sum_{j=1}^m \xi_j^i f_j(x) = 0 \quad \text{for } i = 1, 2, \dots, r.$$

PROOF. Suppose that $c = (c_1, \dots, c_m)$ is a boundary point of N and that (ξ^1, \dots, ξ^r) is a complete maximal set of vectors relative to c . Let R^* be the set of all points x for which the following two conditions hold: (1) $\sum_{j=1}^m \xi_j^i f_j(x) \neq 0$ for at least one value i ; (2) $\sum_{j=1}^m \xi_j^i f_j(x) > 0$ where i is the smallest integer for which $\sum_{j=1}^m \xi_j^i f_j(x) \neq 0$. For any member S of \mathcal{S} let S^* denote the intersection of S with $R - R'$. It follows from Lemma 3.2 that $R^* - R^* \cap S^*$ and $S^* - R^* \cap S^*$ are sets of measure zero. Thus

$$(3.7) \quad \int_{S^*} f_i(x) dx = \int_{R^*} f_i(x) dx \quad (i = 1, \dots, m + 1)$$

for all $S \in \mathcal{S}$. Let

$$(3.8) \quad f_i^*(x) = f_i(x) \quad \text{for } x \in R' \quad (i = 1, \dots, m + 1)$$

and

$$(3.9) \quad f_i^*(x) = 0 \quad \text{for } x \in R - R' \quad (i = 1, 2, \dots, m + 1).$$

Let, furthermore,

$$(3.10) \quad c_i^* = c_i - \int_{R^*} f_i(x) dx \quad (i = 1, \dots, m)$$

Let $\mu^*, \nu^*, M^*, N^*, \mathcal{S}^*$ and \mathcal{S}_0^* have the same meaning with reference to the functions $f_1^*(x), \dots, f_{m+1}^*(x)$ and the point $c^* = (c_1^*, \dots, c_m^*)$ as $\mu, \nu, M, N, \mathcal{S}$ and \mathcal{S}_0 have with reference to the functions $f_1(x), \dots, f_{m+1}(x)$ and the point $c = (c_1, \dots, c_m)$.

It follows from Lemma 3.2 that for any subset S of R for which $\nu(S)$ is a point of the set T defined in Lemma 3.3 we have

$$\int_S f_i(x) dx = \int_S f_i^*(x) dx + \int_{R^*} f_i(x) dx \quad (i = 1, \dots, m + 1).$$

Since the range of $\nu^*(S)$ is equal to N^* even when S is restricted to subsets S for which $\nu(S) \in T$, the set N^* is obtained from the set T by a translation. The same translation brings the point $c = (c_1, \dots, c_m)$ into $c^* = (c_1^*, \dots, c_m^*)$. It then follows from Lemma 3.3 that c^* is an interior point of N^* . Application of Theorem 3.1 gives the following necessary and sufficient condition for a member S of S^* to be a member of S_0^* : There exist m constants k_1, \dots, k_m such that for all x (except perhaps on a set of measure zero)

$$(3.11) \quad f_{m+1}^*(x) \geq k_1 f_1^*(x) + \dots + k_m f_m^*(x) \quad \text{when } x \in S$$

and

$$(3.12) \quad f_{m+1}^*(x) \leq k_1 f_1^*(x) + \dots + k_m f_m^*(x) \quad \text{when } x \notin S.$$

It follows from (3.8) and (3.9) that (3.11) and (3.12) are equivalent to

$$(3.13) \quad f_{m+1}(x) \geq k_1 f_1(x) + \dots + k_m f_m(x) \quad \text{when } x \in S \cap R'$$

and

$$(3.14) \quad f_{m+1}(x) \leq k_1 f_1(x) + \dots + k_m f_m(x) \quad \text{when } x \in (R - S) \cap R'.$$

Theorem 3.2 follows from this and the fact that every member S of S is a member of S^* and that a member S of S is a member of S_0^* if and only if S is a member of S_0 .

It may be of interest to note that if the set R' is of measure zero, the members of S can differ from each other only by sets of measure zero; i.e., S consists essentially of one element. This is an immediate consequence of Lemma 3.2.

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