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Igor V. Evstigneev, Klaus Schürger, Michael Taksar

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## On the fundamental theorem of asset pricing: random constraints and bang-bang no-arbitrage criteria

by

**Igor V. Evstigneev, Klaus Schürger, Michael I. Taksar**

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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# ON THE FUNDAMENTAL THEOREM OF ASSET PRICING: RANDOM CONSTRAINTS AND BANG-BANG NO ARBITRAGE CRITERIA

Igor V. Evstigneev [a], Klaus Schürger [b] and Michael I. Taksar [c]

**Abstract**

The paper generalizes and refines the Fundamental Theorem of Asset Pricing of Dalang, Morton and Willinger in the following two respects: (a) the result is extended to a model with general portfolio constraints; (b) versions of the no arbitrage criterion based on the bang-bang principle in control theory are developed.

*Key Words:* no arbitrage criteria, portfolio constraints, supermartingale measures, bang-bang control

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[a] School of Economic Studies, University of Manchester, Oxford Road, Manchester, M13 9PL, UK. E-mail: igor.evstigneev@man.ac.uk. (Corresponding author.)

[b] Department of Statistics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. E-mail: schuerg@finasto.uni-bonn.de.

[c] Mathematics Department, University of Missouri, Columbia, MO 65211, USA. E-mail: taksar@math.missouri.edu.

# 1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T$ a family of $\sigma$-subalgebras of the $\sigma$-algebra $\mathcal{F}$. We denote by $L^0(\mathcal{F}_t, R^d) = L^0(\Omega, \mathcal{F}_t, \mathbb{P}, R^d)$ the linear space (of equivalence classes) of $\mathcal{F}_t$-measurable $d$-dimensional random vectors endowed with the topology of convergence in measure. For $x = (x^1, ..., x^d) \in L^0(\mathcal{F}_t, R^d)$, we write $x \in L^p(\mathcal{F}_t, R^d)$ $(p \in [1, \infty))$ or $x \in L^\infty(\mathcal{F}_t, R^d)$ if the random variable $|x|^p = (|x^1| + ... + |x^d|)^p$ has finite expectation $\mathbb{E}|x|^p$ or is essentially bounded, respectively. If $d = 1$, we omit " $R^d$ " in the notation and write $L^p(\mathcal{F}_t)= L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $p$.

Let $\mathcal{C}_t \subseteq L^0(\mathcal{F}_t, R^d)$ $(t = 0, ..., T-1)$ be non-empty sets and $x_t \in L^0(\mathcal{F}_t, R^d)$ $(t = 0, 1, ..., T)$ random vectors. For $t = 1, 2, ..., T$, define

$$\mathcal{R}_t = \{\sum_{m=1}^{t} h_{m-1} x_m : \ h_m \in \mathcal{C}_m, \ m = 0, 1, ..., t-1\}, \tag{1.1}$$

where $h_{m-1}x_m = \sum_{i=1}^d h_{m-1}^i x_m^i$ for $h_{m-1} = (h_{m-1}^1, ..., h_{m-1}^d)$ and $x_m = (x_m^1, ..., x_m^d)$. Let $L_+^0 = L_+^0(\mathcal{F})$ denote the cone of non-negative elements in $L^0 = L^0(\mathcal{F})$. Consider the condition:

(NA) $\mathcal{R}_T \cap L_+^0 = \{0\}$.

This work is aimed at the development and refinement of the following result of Dalang, Morton and Willinger (1990), playing an important role in models of securities markets.

**Theorem 1.1.** *Let $\mathcal{C}_t = L^0(\mathcal{F}_t, R^d)$ $(t = 0, ..., T-1)$. Then condition (NA) holds if and only if there exists a strictly positive random variable $\lambda \in L^\infty(\Omega, \mathcal{F})$ such that $\mathbb{E}\lambda = 1$, $\mathbb{E}\lambda|x_t| < \infty$ $(t = 0, ..., T)$ and*

$$\mathbb{E}(\lambda x_t | \mathcal{F}_{t-1}) = 0 \tag{1.2}$$

*almost surely for all $t = 1, ..., T$.*

In models of securities markets, vectors $h_t = (h_t^1, ..., h_t^d) \in L^0(\mathcal{F}_t, R^d)$ represent *portfolios* of $d$ *assets* at time $t = 0, 1, ..., T-1$. The number $h_t^i$

indicates the amount of asset $i$ in the portfolio $h_t$. Sequences $(h_0, ..., h_{T-1})$, $h_t \in L^0(\mathcal{F}_t, R^d)$, are interpreted as *investment strategies*. Those strategies which satisfy the constraints $h_t \in \mathcal{C}_t$ $(t = 0, ..., T-1)$ are *admissible*. The random vectors $x_t$ describe the increments

$$x_t = s_t - s_{t-1} \; (t \geq 1), \; x_0 = s_0, \tag{1.3}$$

of the *price vectors* $s_t \in L^0_+(\mathcal{F}_t, R^d)$, $t = 0, 1, ..., T$, that are supposed to be given in the model. The $i$th coordinate of $s_t = (s_t^1, ..., s_t^d)$ specifies the price of one unit of asset $i$ at time $t$. The amount $\sum_{m=1}^{t} h_{m-1} x_m$ $(t = 1, ..., T)$ is the *net gain* from the strategy $(h_0, ..., h_{T-1})$ over the time interval $0, ..., t$. If the investor's wealth at time 0 is $w_0$ $[\in L^0(\mathcal{F}_0)]$, then the investor's *wealth at time* $t$ can be expressed as

$$w_t = w_0 + \sum_{m=1}^{t} h_{m-1} x_m. \tag{1.4}$$

This formula presumes that there are no external sources of funding (the assumption of *self-financing*) and no consumption, so that the increment $w_t - w_{t-1}$ of wealth in each time period between $t-1$ and $t$ depends only on the price change $x_t = s_t - s_{t-1}$ and the portfolio $h_{t-1}$ held during this period. Condition (NA) is interpreted as the *absence of arbitrage* over the time horizon $0, ..., T$: there is no investment strategy allowing to gain a non-negative amount almost surely and a strictly positive amount with positive probability. If $\lambda > 0$, $\mathbb{E}\lambda = 1$ and $\mathbb{E}\lambda|x_t| < \infty$, then, as is easily seen, property (1.2) holds if and only if the price process $s_t$ is a martingale with respect to the filtration $\mathcal{F}_0 \subseteq ... \subseteq \mathcal{F}_T$ and the probability $\mathbb{P}^\lambda(d\omega) := \lambda(\omega)\mathbb{P}(d\omega)$, i.e., $\mathbb{P}^\lambda$ is an *equivalent martingale measure*. Equivalent martingale measures play a key role in the design of pricing rules for derivative assets. This has led to the term "Fundamental Theorem of Asset Pricing" (FTAP), that is often associated with Theorem 1.1 and its variants. Such results go back to the seminal work of Harrison and Kreps (1979), Harrison and Pliska (1981) and Kreps (1981); for introductory expositions see Pliska (1997) and Björk (1998) (discrete- and continuous-time models, respectively). An account of current research in the field is given in the survey by Kabanov (2001).

We develop the above theorem in the following two directions. (a) We consider proper subsets $\mathcal{C}_t$ in $L^0(\mathcal{F}_t, R^d)$, i.e., we deal with *portfolio constraints*. (b) We show that the function $\lambda$, appearing in (1.2), can be selected from some special functional classes that are much narrower than the totality of all strictly positive elements in $L^\infty(\Omega, \mathcal{F})$. These classes are described in terms of conditionally finite-valued random variables (see below).

Models with portfolio constraints, in discrete and continuous time, have been considered by many authors – see, in particular, Cvitanić and Karatzas (1993), Karatzas and Kou (1996), Jouini and Kallal (1995), Schürger (1996), Föllmer and Kramkov (1997), Brannath (1997), Pham and Touzi (1999), Pham (2000), Carassus, Pham and Touzi (2001) and references therein. In the previous studies aimed at generalizations of FTAP, the main focus has been on constraints of the form $h_t \in G$ almost surely (a.s.) or

$$(s_t^1 h_t^1, ..., s_t^d h_t^d) \in G \text{ (a.s.)}, \tag{1.5}$$

where $G$ is a non-random set in $R^d$ and $(s_t^1, ..., s_t^d)$ is the vector of prices at time $t$. In this paper, we analyze systematically restrictions of a more general type, defined in terms of fairly general random sets $G_t(\omega)$ adapted to the given filtration $(\mathcal{F}_t)$.[1] Under such restrictions, the set of admissible portfolios might depend on random factors in a way more complex than (1.5) (for example, short sales of an asset might be allowed or not depending on whether the price of the asset decreases or grows). The results we obtain appear to be final in the framework under consideration.

The second of our themes, (b), is entirely new in the present context.

[1] Versions of FTAP involving similar constraints have been considered in the unpublished work of Brannath (1997). However, the approach and the structure of the results in that work are substantially different from those in the present paper.

Our main result along this line shows that the function $\lambda$ involved in (1.2) can be selected from a class of functions of the form $\lambda = \lambda_0 ... \lambda_T$, where the conditional distribution of $\lambda_t$ given $\mathcal{F}_{t-1}$ is concentrated on a finite set. The cardinality of the set can be restricted: it is sufficient to consider distributions concentrated on not more than $d+1$ points, where $d$ is the number of assets in the market. There is a parallelism between this refinement of FTAP and a number of known results in control theory and statistics that demonstrate the possibility of achieving the objectives of control or optimization by using not all admissible strategies but only those belonging to some finite set. The minimum necessary number of elements in this set can usually be estimated based on the dimensionality of the problem. The related theory and techniques are usually referred to as *bang-bang control* (see Sonnenborn and Van Vleck 1965, Hermes and LaSalle 1969, and Artstein 1980). Because of the similarity of the results and the underlying methodology (centering around Lyapounov's and Carathéodory's theorems) we associate the term "bang-bang no arbitrage criteria" with those refinements of the conventional no arbitrage criteria we consider in this work.

The Dalang–Morton–Wilinger (1990) theorem has attracted attention of many researchers. During the last decade, several different methods for proving the theorem have been proposed – see Schachermayer (1992), Kabanov and Kramkov (1994), Rogers (1994), Jacod and Shiryaev (1998), and Kabanov and Stricker (2001). Our approach to the subject is close to the original one, as suggested by Dalang, Morton and Wilinger (1990). We reduce the problem under study to the analysis of "conditional" versions of property (NA), that are formulated in terms of conditional distributions given the $\sigma$-algebras $\mathcal{F}_t$. The technical tools we employ are measurable selection theorems and convex analysis in spaces $L^0$ with measures depending on parameters.

The paper is organized as follows. In Section 2, we state and discuss the main results. Sections 3 – 5 focus on various aspects of the model at hand, aiming basically (but not only) at the preparation for the proof of the main theorem (Theorem 2.1). This proof is given in Section 6. Section 7 provides equivalent formulations of the main hypotheses. Two appendices, I and II, assemble several general facts of measure theory and functional analysis exploited in this work.

## 2 The main results

Suppose that, for each $t = 0, 1, ..., T-1$ and $\omega \in \Omega$, we are given a closed cone[2] $H_t(\omega) \subseteq R^d$ and a set $M_t(\omega) \subseteq R^d$ $(\omega \in \Omega)$ satisfying the following condition:

(M) For each $a \in H_t(\omega)$, there exists a number $r > 0$ such that $ra \in M_t(\omega)$.
According to this condition, the set

$$G_t(\omega) := H_t(\omega) \cap M_t(\omega) \tag{2.1}$$

generates the cone $H_t(\omega)$, and $0 \in G_t(\omega)$. Clearly (M) is fulfilled, in particular, if 0 belongs to the interior of $M_t(\omega)$.

We will assume that the graphs $\{(\omega, a) : a \in M_t(\omega)\}$ and $\{(\omega, a) : a \in H_t(\omega)\}$ of the multivalued mappings $\omega \mapsto M_t(\omega)$ and $\omega \mapsto H_t(\omega)$ are $\mathcal{F}_t \times \mathcal{B}(R^d)$-measurable. Here and in what follows, $\mathcal{B}(\cdot)$ stands for the Borel $\sigma$-algebra in a topological space. The assumption imposed means that $M_t(\omega)$ and $H_t(\omega)$ are *$\mathcal{F}_t$-measurable random sets.*

We will examine the model described in the previous section in terms of the random vectors $x_t \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, R^d)$, $t = 0, 1, 2, ..., T$, and the constraint sets $\mathcal{C}_t$, $t = 0, 1, ..., T-1$, assuming that $\mathcal{C}_t$ are defined by

$$\mathcal{C}_t = \{h(\cdot) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, R^d) : h(\omega) \in G_t(\omega) \text{ (a.s.)}\}. \tag{2.2}$$

[2]By a cone we mean a set containing with each vector $a$ the vector $ra$ where $r$ is any non-negative number (convexity is not included in this definition).

The sets $G_t(\omega) = H_t(\omega) \cap M_t(\omega)$ include constraints of two types. The cones $H_t(\omega)$ can be defined, in particular, in terms of linear inequalities of the form $\alpha_t^{ij} h_t^i + \beta_t^{ij} h_t^j \geq 0$, where $\alpha_t^{ij}, \beta_t^{ij}$ are some given $\mathcal{F}_t$-measurable random variables and $h_t^i, h_t^j$ $(i, j = 1, 2, ..., d)$ are positions of the portfolio $h_t = (h_t^1, ..., h_t^d)$. By an appropriate choice of $\alpha_t^{ij}, \beta_t^{ij}$, one can establish bounds on the proportions between coordinates $h_t^i$ and $h_t^j$ of the vector $h_t = (h_t^1, ..., h_t^d)$. If $\beta_t^{ij} = 0$ and $\alpha_t^{ij} = \tau_t^i$, where $\tau_t^i \in \{-1, 0, +1\}$, the above inequalities specify conditions $\tau_t^i h_t^i \geq 0$ on the signs of $h_t^i$ $(i = 1, 2, ..., d)$, that may incorporate, for example, short selling restrictions depending on the random situation $\omega$. The sets $M_t(\omega)$ allow to impose upper bounds $h_t^i \leq \beta_t^i$ for long positions of the portfolio and lower bounds $\alpha_t^i \leq h_t^i$ for its short positions $[\alpha_t^i, \beta_t^i \in L^0(\mathcal{F}_t)$, $\alpha_t^i < 0 < \beta_t^i]$. One can consider analogous constraints defined in terms of the values $s_t^i h_t^i$ of assets (expressed in terms of the current prices $s_t^i$), rather than their physical units $h_t^i$. Further examples of $M_t(\omega)$ include constraints of the form $\sum_{i \in J} s_t^i h_t^i \leq \beta_{Jt}$ or $\alpha_{Jt} \leq \sum_{i \in J} s_t^i h_t^i$, where $J$ is a subset of $\{1, 2, ..., d\}$ and $\alpha_{Jt} < 0 < \beta_{Jt}$ are $\mathcal{F}_t$-measurable random variables.

The fundamental assumptions under which our results are obtained are concerned with the random cones $H_t(\omega)$ and the vectors $x_t$. We shall not need any conditions on the sets $M_t(\omega)$ except for those introduced above. To formulate the assumptions, denote by $P_\omega^t(\Gamma)$ $(\omega \in \Omega,\ \Gamma \in \mathcal{B}(R^d))$ the conditional distribution of the random vector $x_{t+1}(\omega)$ given the $\sigma$-algebra $\mathcal{F}_t$, $t = 0, 1, ..., T-1$ (see Appendix I). Put $B := R^d$, $\mathcal{B} := \mathcal{B}(R^d)$ and, for each $\omega \in \Omega$, define

(2.3)

$$X_t(\omega) := \{v(\cdot) \in L^0(B, \mathcal{B}, P_\omega^t) : v(b) = ab\ P_\omega^t\text{-a.e. for some } a \in H_t(\omega)\}.$$

The cone $X_t(\omega)$ is the image of the cone $H_t(\omega)$ under the linear mapping of $R^d$ into $L^0(B, \mathcal{B}, P_\omega^t)$ that transforms a vector $a \in R^d$ into the element $v_a(\cdot)$ of $L^0(B, \mathcal{B}, P_\omega^t)$ for which $v_a(b) = ab$ $P_\omega^t$-almost everywhere ($P_\omega^t$-a.e.) on $B$.

If $X$ and $Y$ are sets in a linear space, we write $X \pm Y := \{x \pm y :$

$x \in X, y \in Y\}$. The main assumptions are as follows.

(X.1) The set $X_t(\omega)$ is closed in $L^0(B, \mathcal{B}, P^t_\omega)$ with respect to convergence in measure.

(X.2) The set $X_t(\omega) - L^0_+(B, \mathcal{B}, P^t_\omega)$ is convex.

Conditions (X.1) and (X.2) are supposed to hold for each $t = 0, 1, ..., T-1$ and for $\omega \in \Omega_t$, where $\Omega_t$ is an $\mathcal{F}_t$-measurable set with $\mathbb{P}(\Omega_t) = 1$. Clearly (X.2) holds if the cone $H_t(\omega)$, and hence the cone $X_t(\omega)$, are convex. We will present equivalent versions of assumptions (X.1) and (X.2), as well as conditions sufficient for their validity, after the formulation of the main result, Theorem 2.1 below.

Let us introduce the classes of random variables that are involved in our refinement of Theorem 1.1. For each $t = 1, 2, ..., T$ and $k = 1, 2, ...$, let $\Lambda_t(k)$ denote the set of random variables $\lambda \in L^0(\mathcal{F}_t)$ representable in the form

$$\lambda(\omega) = f(\omega, x_t(\omega)), \tag{2.4}$$

where $f(\omega, b)$ is a real-valued function of $\omega \in \Omega$ and $b \in R^d$ satisfying the following conditions:

(f.1) the function $f(\omega, b)$ is $\mathcal{F}_{t-1} \times \mathcal{B}(R^d)$-measurable;

(f.2) there exist strictly positive $\mathcal{F}_{t-1}$-measurable random variables $c_1(\omega)$, ..., $c_k(\omega)$ such that

$$f(\omega, b) \in \{c_1(\omega)\} \cup ... \cup \{c_k(\omega)\} \text{ for each } b \in B \text{ and } \omega \in \Omega. \tag{2.5}$$

It can be shown (see Proposition 7.4) that a random variable $\lambda$ of the form (2.4) coincides a.s. with a random variable $\lambda' \in \Lambda_t(k)$ if and only if the conditional distribution of $\lambda$ given $\mathcal{F}_{t-1}$ is concentrated a.s. on a finite set in $(0, \infty)$ containing not more than $k$ elements.

For a random variable $\lambda \in \Lambda_t(k)$, we write $\lambda \in \Lambda^\infty_t(k)$ $(t = 1, 2, ..., T)$ if $\lambda$ is bounded. For $t = 0$, we define $\Lambda^\infty_0(k)$ as the class consisting of strictly positive constants. We denote by $\Lambda^\infty(k)$ the class of random variables $\lambda(\omega)$

with $\mathbb{E}\lambda = 1$ that can be represented as $\lambda = \lambda_0\lambda_1...\lambda_T$, where $\lambda_t \in \Lambda_t^\infty(k)$ for each $t = 0, 1, ..., T$.

The main results are contained in the following theorem.

**Theorem 2.1.** *Let condition (NA) hold. Then there is a strictly positive random variable* $\lambda \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\mathbb{E}\lambda = 1$,

$$\mathbb{E}\lambda|x_t| < \infty,\ t = 0, ..., T, \tag{2.6}$$

*and*

$$h\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1}) \leq 0 \text{ (a.s.)},\ h \in \mathcal{C}_{t-1},\ t = 1, 2, ..., T. \tag{2.7}$$

*If, additionally,*

$$\mathbb{E}(|x_t||\mathcal{F}_{t-1}) < \infty \text{ (a.s.)},\ t = 1, 2, ..., T, \tag{2.8}$$

*then there exists* $\lambda \in \Lambda^\infty(d+1)$ *satisfying (2.6) and (2.7). If the conditional distribution* $P_\omega^t(\cdot)$ *is atomless for each* $t = 0, 1, ..., T-1$ *and almost all* $\omega \in \Omega$, *then one can replace in the foregoing assertion* $d+1$ *by* $2$.

*Conversely, if there is a random variable* $\lambda > 0$ *with properties (2.7) and*

$$\mathbb{E}(\lambda|x_t||\mathcal{F}_{t-1}) < \infty \text{ (a.s.)}, t = 1, ..., T, \tag{2.9}$$

*then condition (NA) holds.*

**Remark 2.1.** Put

$$\mathcal{H}_t := \{h(\cdot) \in L^0(\mathcal{F}_t, R^d) : h(\omega) \in H_t(\omega) \text{ (a.s.)}\} \tag{2.10}$$

and observe that condition (2.7) is equivalent to the following one

$$h\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1}) \leq 0 \text{ (a.s.)},\ h \in \mathcal{H}_{t-1},\ t = 1, 2, ..., T. \tag{2.11}$$

Indeed, (2.11) implies (2.7) since the sets $\mathcal{H}_t$ are larger than the original constraint sets $\mathcal{C}_t$ (functions $h$ in $\mathcal{C}_t$ satisfy the additional restriction $h(\omega) \in M_t(\omega)$ a.s.). The converse implication holds because, for any $h \in$

$\mathcal{H}_t$, there exists a real-valued $\mathcal{F}_t$-measurable function $\rho(\omega) > 0$ such that $\rho(\omega)h(\omega) \in M_t(\omega)$ (a.s.), and, consequently, $\rho h \in \mathcal{C}_t$. This assertion follows from condition (M); the proof can easily be conducted by using a measurable selection argument, see Theorem AI.2 in Appendix I. Thus we can replace (2.7) by (2.11) in the formulation of Theorem 2.1. This observation means that the main role in the characterization of the no arbitrage property is played by the constraints specified by the sets $H_t(\omega)$ (rather than $M_t(\omega)$), as long as the sets $M_t(\omega)$ satisfy condition (M). An intuition for this fact is as follows: under assumption (M), property (NA) depends only on the structure of admissible portfolios $h_t$ "in a neighborhood of zero", which is determined by $H_t(\omega)$.

**Remark 2.2.** Let $\lambda > 0$ be a random variable with $\mathbb{E}\lambda = 1$ satisfying (2.6). Fix a version of the conditional expectation $\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1})$. Then inequalities (2.7), or equivalent inequalities (2.11), hold if and only if

$$\max_{a \in H_{t-1}(\omega)} a\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1}) = 0 \text{ (a.s.).} \tag{2.12}$$

The "if" assertion is straightforward; "only if" obtains by using measurable selection (see Appendix I, Theorem AI.2). Property (2.12) means that the random vector $\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1})$ belongs almost surely to the polar of the cone $H_{t-1}(\omega)$. If $H_{t-1}(\omega) = R^d$ for all $\omega$, then (2.12) reduces to (1.2). If $H_{t-1}(\omega) = R^d_+$ for all $\omega$, then (2.12) is equivalent to $\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1}) \le 0$ (a.s.). Thus, if $x_t$ is defined by (1.3), and (2.6) holds, the last inequality says that the process $s_t, t = 0, ..., T$, is a supermartingale with respect to the measure $\mathbb{P}^\lambda(d\omega) = \lambda(\omega)\mathbb{P}(d\omega)$ (cf. Jouini and Kallal 1995 and Schürger 1996). Note that $\mathbb{E}\lambda|s_t| < \infty$ by virtue of (2.6) and (1.3).

**Remark 2.3.** Under the assumptions (1.3) and (2.6), property (2.7) can be interpreted as follows. If we change the original measure $\mathbb{P}$ by $\mathbb{P}^\lambda(d\omega) := \lambda(\omega)\mathbb{P}(d\omega)$, we obtain that, *under the equivalent measure* $\mathbb{P}^\lambda$, *the wealth process* $w_t$ (see (1.4)) *is a generalized supermartingale for any admissible trading strategy* $(h_0, ..., h_{T-1})$. The term "generalized" points to the fact that the random variables $w_t$ are not necessarily integrable with respect to $\mathbb{P}^\lambda$, although the conditional expectations of $w_t$ given $\mathcal{F}_{t-1}$ are well-defined and finite (which follows from (2.6)). To guarantee the integrability of the random variables $w_t$ defined by (1.4) it is sufficient to assume that the vectors $h_t$ are bounded and $\mathbb{E}\lambda|w_0| < \infty$. For extensions of the above supermartingale property to models of a more general type (involving transaction costs) see Kabanov and Stricker (2001a), Evstigneev and Taksar (2000), and Schachermayer (2001).

**Remark 2.4.** Define

$$\mathcal{X}_t = \{w(\cdot) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}) : w(\omega) = h(\omega)x_t(\omega) \text{ (a.s.)}, h(\cdot) \in \mathcal{H}_{t-1}\} \tag{2.13}$$

$(t = 1, 2, ..., T)$ and consider the following condition:

(NA$_t$) $\mathcal{X}_t \cap L^0_+ = \{0\}$.

This condition may be regarded as a local (at time $t$) version of the no arbitrage hypothesis (NA). Note, however, that $\mathcal{X}_t$ is defined in terms of the cones $\mathcal{H}_{t-1}$, rather than the constraint sets $\mathcal{C}_{t-1}$ involved in (NA). But, if we replace $\mathcal{H}_{t-1}$ by $\mathcal{C}_{t-1}$ in (2.13), this will lead to a condition equivalent to (NA$_t$), because, for any $h \in \mathcal{H}_{t-1}$, there is $\rho > 0$, $\rho \in L^0(\mathcal{F}_{t-1})$ satisfying $\rho h \in \mathcal{C}_{t-1}$ (see Remark 2.1). Further, observe that (NA) implies (NA$_t$) for each $t = 1, 2, ..., T$. To show this for some given $t = t_0$ it is sufficient to consider strategies $(h_t)$ for which $h_t = 0$, $t \neq t_0 - 1$. On the other hand, in the course of the proof of Theorem 2.1 (see Section 6), we will show that (NA) implies the existence of a random variable $\lambda$ with properties (2.6) and (2.7) by using not the hypothesis (NA) itself, but only its consequence – condition (NA$_t$), $t = 1, ..., T$. According to the last assertion of Theorem 2.1, the existence of such a random variable is sufficient for (NA). Consequently, *the validity of (*NA*) is equivalent to the validity of (*NA$_t$*) for all* $t = 1, ..., T$.

**Remark 2.5.** Fix some $t = 0, ..., T-1$. We will prove in Section 7 that condition (X.1) (resp. (X.2)) holds for all $\omega \in \Omega$, except for an $\mathcal{F}_t$-measurable set of measure zero, if and only if condition ($\mathcal{X}$.1) (resp. ($\mathcal{X}$.2)) below is satisfied.

($\mathcal{X}$.1) The cone $\mathcal{X}_{t+1}$ is closed in $L^0(\Omega, \mathcal{F}_{t+1}, \mathbb{P})$ under convergence in measure (or, equivalently, under convergence almost surely).

($\mathcal{X}$.2) The set $\mathcal{X}_{t+1} - L^0_+(\Omega, \mathcal{F}_{t+1}, \mathbb{P})$ is convex.

Although properties ($\mathcal{X}$.1) and ($\mathcal{X}$.2) do not use in their formulations conditional distributions, it is generally more convenient to deal with the original versions of the assumptions – (X.1) and (X.2) – rather than with their "unconditional" versions ($\mathcal{X}$.1) and ($\mathcal{X}$.2). The reason for this is the fact that the set $X_t(\omega)$ involved in the former two conditions is finite-dimensional: it is contained in a finite-dimensional subspace $V_t(\omega)$ of $L^0(B, \mathcal{B}, P^t_\omega)$ – the image of $R^d$ under the linear mapping transforming a vector $a \in R^d$ into the function $v_a(b) = ab$ ($P^t_\omega$-a.e.). By the definition of $X_t(\omega)$, the mapping $a \mapsto v_a(\cdot)$ transforms $H_t(\omega)$ into $X_t(\omega)$, and, since $H_t(\omega)$ is closed for each $\omega$, we immediately obtain two important cases where we can guarantee the closedness of $X_t(\omega)$ and hence the validity of (X.1):

(P) The cone $H_t(\omega)$ is polyhedral, i.e., it is a conic convex hull of a finite set of vectors in $R^d$.

(V) The dimension $\dim V_t(\omega)$ of the linear space $V_t(\omega) \subseteq L^0(B, \mathcal{B}, P^t_\omega)$ is equal to $d$.

Note that the finite set involved in (P) might depend on $\omega$, and the number of elements in it might be different for different $\omega$.

**Remark 2.6.** Clearly, condition (V) can be restated as follows: If $a$ is a non-zero element of $R^d$, then $v_a(\cdot)$ is a non-zero element of $L^0(B, \mathcal{B}, P^t_\omega)$. Under (V), the mapping $a \mapsto v_a(\cdot)$ is a linear homeomorphism of $R^d$ onto $V_t(\omega)$, and so the closedness of the set $H_t(\omega)$ implies the closedness of its image, $X_t(\omega)$. One can formulate an equivalent version of (V) that does not involve conditional distributions. It can be shown (see Proposition 7.3) that (V) holds for all $\omega$, except for an $\mathcal{F}_t$-measurable set of measure zero, if and only if the following requirement is fulfilled:

($\mathcal{V}$) If $g \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ and $\mathbb{P}\{g x_{t+1} = 0\} = 1$, then $\mathbb{P}\{g = 0\} = 1$.

It can easily be proved that $(\mathcal{V})$ is satisfied, in particular, if $\mathbb{E}|x_{t+1}|^2 < \infty$ and the conditional covariance matrix

$$\mathbb{E}\{[x^i_{t+1} - \mathbb{E}(x^i_{t+1}|\mathcal{F}_t)][x^j_{t+1} - \mathbb{E}(x^j_{t+1}|\mathcal{F}_t)]|\mathcal{F}_t\},\ i, j = 1, ..., d, \tag{2.14}$$

of the vector $x_{t+1} = (x^1_{t+1}, ..., x^d_{t+1})$ $(t = 0, ..., T-1)$ is non-degenerate with probability one. The same assertion is true, if, instead of (2.14), we consider the conditional covariance matrix of $\gamma_{t+1}x_{t+1}$, where $\gamma_{t+1} > 0$ is a random variable in $L^0(\Omega, \mathcal{F}_{t+1}, \mathbb{P})$ such that $\mathbb{E}|\gamma_{t+1}x_{t+1}|^2 < \infty$. Further, suppose the random vectors $x_t$, $t = 0, ..., T$, are defined through $s_t$ by (1.3), and $\mathbb{E}|s_t|^2 < \infty$ (or, equivalently, $\mathbb{E}|x_t|^2 < \infty$) for all $t$. Consider the conditional covariance matrix $(\mu^{ij}_t)$ of the vector $s_{t+1} = (s^1_{t+1}, ..., s^d_{t+1})$ given $\mathcal{F}_t$. Clearly the matrix $(\mu^{ij}_t)$ coincides with (2.14) with probability one. Thus, if the determinant $\det(\mu^{ij}_t)$ of the matrix $(\mu^{ij}_t)$ is non-zero almost surely, then conditions $(\mathcal{V})$, (V), and, consequently, (X.1) hold. If $\det(\mu^{ij}_t) \neq 0$ (a.s.), $M_t(\omega) = R^d$ for all $\omega$, and $H_t(\omega)$ is of the form (1.5), where $G$ is a closed convex cone in $R^d$, assertion (i) of Theorem 2.1 follows from a result of Pham and Touzi (1999), Theorem 4.2. A version of this result is obtained by Carassus, Pham and Touzi (2001), Theorem 3.2. It should be emphasized that the requirement of non-degeneracy of $(\mu^{ij}_t)$, as well as requirements (V) and $(\mathcal{V})$, are not needed if the cone $H_t(\omega)$ is polyhedral (see condition (P) above).

**Remark 2.7.** One can consider an extension of the model at hand in which the portfolio constraints at time $t$ depend on the sign of the wealth process $w_t$. More precisely, suppose there are two constraint sets $\mathcal{C}^{(j)}_t$, $j = 1, 2$, defined in terms of $M^{(j)}_t(\omega)$ and $H^{(j)}_t(\omega)$, $j = 1, 2$, exactly as the sets $\mathcal{C}_t$. Assume that the constraint $h_t \in \mathcal{C}_t$ is replaced by the following ones: $h_t \in \mathcal{C}^{(1)}_t$ if $w_t > 0$; $h_t \in \mathcal{C}^{(2)}_t$ if $w_t < 0$; $h_t = 0$ if $w_t = 0$ (cf. Karatzas and Kou 1996). It can be shown that such a model can be reduced to the one studied in this work with $M_t(\omega) = M^{(1)}_t(\omega) \cup M^{(2)}_t(\omega)$ and $H_t(\omega) = H^{(1)}_t(\omega) \cup H^{(2)}_t(\omega)$. This reduction has been carried out – in a somewhat different setting – by Carassus, Pham and Touzi (2001). An analogous reduction can be performed by using the same considerations in the framework adopted in this paper. Our results imply the versions of FTAP obtained in Carassus, Pham and Touzi (2001) and permit to replace the counterpart of condition ($\mathcal{V}$) imposed in the paper cited by weaker assumptions similar to (X.1) or ($\mathcal{X}$.1).

# 3 Local and conditional no arbitrage

Fix some $t = 1, 2, ..., T$. Propositions 3.1 and 3.2 below are auxiliary results that will be applied to the sets $\mathcal{X}_t$ and $X_{t-1}(\omega)$ defined in terms of the cone $H_{t-1}(\omega)$ (see (2.13) and (2.3)). For the proofs of these propositions, however, we do not need the assumption that $H_{t-1}(\omega)$ is a cone. It is sufficient to assume only that $H_{t-1}(\omega)$ *is a non-empty* $\mathcal{F}_{t-1}$*-measurable random set.* We write $B := R^d$, $\mathcal{B} = \mathcal{B}(R^d)$, $P_\omega = P_\omega^{t-1}$, $E_\omega = E_\omega^{t-1}$ ($E_\omega$ is the integral with respect to $P_\omega$) and denote by $\mathcal{F}_{t-1}^{\mathbb{P}}$ the completion of $\mathcal{F}_{t-1}$ with respect to $\mathbb{P}$.

**Proposition 3.1.** *Let* $f(\omega, b)$, $\omega \in \Omega$, $b \in B$, *be an* $\mathcal{F}_{t-1} \times \mathcal{B}$*-measurable real-valued function. Then the set*

$$\Omega_f := \{\omega \in \Omega : \ f(\omega, \cdot) \in X_{t-1}(\omega)\}$$

*is measurable with respect to* $\mathcal{F}_{t-1}^{\mathbb{P}}$*. The following two conditions are equivalent:*

*(F.1) The random variable* $f(\omega, x_t(\omega))$ *belongs to the class* $\mathcal{X}_t$*;*

*(F.2)* $\mathbb{P}(\Omega_f) = 1$.

*Proof.* Consider the set $\Delta_f$ of $(\omega, a) \in \Omega \times R^d$ for which $a \in H_{t-1}(\omega)$ and $E_\omega|f(\omega, b) - ab| = 0$. We have $\Delta_f \in \mathcal{F}_{t-1} \times \mathcal{B}(R^d)$ and $\Omega_f = \mathrm{pr}_\Omega \Delta_f$. Consequently (see Theorem AI.2), $\Omega_f \in \mathcal{F}_{t-1}^{\mathbb{P}}$.

*(F.1)⇒(F.2).* According to (F.1) and (2.13), there exists an $\mathcal{F}_{t-1}$-measurable function $h(\omega)$ such that $h(\omega) \in H_{t-1}(\omega)$ (a.s.) and $f(\omega, x_t(\omega)) = h(\omega)x_t(\omega)$ (a.s.). The last equality implies $0 = \mathbb{E}|f(\omega, x_t(\omega)) - h(\omega)x_t(\omega)| = \mathbb{E}E_\omega|f(\omega, b) - h(\omega)b|$. Consequently, $\mathbb{P}(\Omega') = 1$, where $\Omega' := \{\omega : E_\omega|f(\omega, b) - h(\omega)b| =$

$0\} \in \mathcal{F}_{t-1}$. Further, the set $\Omega'' := \{\omega : h(\omega) \in H_{t-1}(\omega)\}$ belongs to $\mathcal{F}_{t-1}$ and has full measure. Thus $\mathbb{P}(\Omega_f) = 1$ because $\Omega' \cap \Omega'' \subseteq \Omega_f$.

*(F.2)⇒(F.1).* By applying Theorem AI.2 to the set $\Delta_f \in \mathcal{F}_{t-1} \times \mathcal{B}(R^d)$, we construct a measurable mapping $h : (\Omega, \mathcal{F}_{t-1}) \to (R^d, \mathcal{B}(R^d))$ such that $h(\omega) \in H_{t-1}(\omega)$ and $E_\omega|f(\omega, b) - h(\omega)b| = 0$ for all $\omega$ in a set $\tilde{\Omega} \in \mathcal{F}_{t-1}$ having the same measure $\mathbb{P}$ as $\Omega_f = \text{pr}_\Omega \Delta_f$. Since $\mathbb{P}(\Omega_f) = 1$, we have $\mathbb{P}(\tilde{\Omega}) = 1$. Then $h(\omega) \in H_{t-1}(\omega)$ (a.s.) and $E_\omega|f(\omega, b) - h(\omega)b| = 0$ (a.s.). Therefore $\mathbb{E}|f(\omega, x_t(\omega)) - h(\omega)x_t(\omega)| = \mathbb{E}E_\omega|f(\omega, b) - h(\omega)b| = 0$, and so $f(\omega, x_t(\omega)) = h(\omega)x_t(\omega)$ (a.s.), which shows that the random variable $f(\omega, x_t(\omega))$ belongs to the class $\mathcal{X}_t$. □

**Proposition 3.2.** *The set $\Omega^*$ of those $\omega \in \Omega$ for which*

$$X_{t-1}(\omega) \cap L^0_+(B, \mathcal{B}, P_\omega) = \{0\} \tag{3.1}$$

*is measurable with respect to the completion $\mathcal{F}^{\mathbb{P}}_{t-1}$ of the $\sigma$-algebra $\mathcal{F}_{t-1}$.*

*Let $0 \in H_{t-1}(\omega)$ for all $\omega$. Then condition ($\text{NA}_t$) holds if and only if $\mathbb{P}(\Omega^*) = 1$.*

Property (3.1) is a "conditional" version of ($\text{NA}_t$): it is stated in terms of the conditional probabilities $P_\omega = P^{t-1}_\omega$.

*Proof of Proposition 3.2.* The complement $\Omega_* := \Omega \backslash \Omega^*$ of $\Omega^*$ can be represented as the projection on $\Omega$ of the set $\Delta_*$ of those $(\omega, a) \in \Omega \times R^d$ for which $a \in H_{t-1}(\omega)$, $P_\omega\{b : ab \geq 0\} = 1$ and $P_\omega\{b : ab > 0\} > 0$. We have $\Delta_* \in \mathcal{F}_{t-1} \times \mathcal{B}(R^d)$, and so, by virtue of Theorem AI.2, $\Omega_*$ and $\Omega^*$ belong to $\mathcal{F}^{\mathbb{P}}_{t-1}$.

*"Only if".* Let ($\text{NA}_t$) hold. Suppose $\mathbb{P}(\Omega^*) < 1$ and hence $\mathbb{P}(\Omega_*) > 0$. By using Theorem AI.2, we construct a measurable mapping $h : (\Omega, \mathcal{F}_{t-1}) \to (B, \mathcal{B})$ such that $(\omega, h(\omega)) \in \Delta_*$ for all $\omega \in \Omega'$, where $\Omega' \in \mathcal{F}_{t-1}$, $\Omega' \subseteq \Omega_*$ and $\mathbb{P}(\Omega') = \mathbb{P}(\Omega_*)$. By redefining $h(\omega)$ as 0 outside $\Omega'$, we get $h(\omega) \in H_{t-1}(\omega)$ for all $\omega$, and $P_\omega\{b : h(\omega)b \geq 0\} = 1$ and $P_\omega\{b : h(\omega)b > 0\} > 0$ for $\omega \in \Omega'$. Put $y(\omega) = h(\omega)x_t(\omega)$. Then $y \in \mathcal{X}_t$, and $y(\omega) = 0$ for $\omega \in \Omega \backslash \Omega'$. Further,

$$\mathbb{P}\{\omega \in \Omega' : \ h(\omega)x_t(\omega) \geq 0\} = \mathbb{E}\chi_{\Omega'} P_\omega\{b : h(\omega)b \geq 0\} = \mathbb{P}(\Omega')$$

and so $y(\omega) = h(\omega)x_t(\omega) \geq 0$ (a.s.). Finally, since $\mathbb{P}(\Omega') = \mathbb{P}(\Omega_*) > 0$, we obtain

$$\mathbb{P}\{\omega \in \Omega' : \ y(\omega) > 0\} = \mathbb{E}\chi_{\Omega'} P_\omega\{b : h(\omega)b > 0\} > 0,$$

which contradicts ($\mathrm{NA}_t$).

*"If"*. Suppose ($\mathrm{NA}_t$) does not hold, i.e., there exists a function $h(\cdot)$ in $\mathcal{H}_{t-1}$ such that $h(\omega)x_t(\omega) \geq 0$ (a.s.) and $\mathbb{P}\{\omega \in \Omega : \ h(\omega)x_t(\omega) > 0\} > 0$. Then $\mathbb{E}P_\omega\{b : h(\omega)b \geq 0\} = \mathbb{P}\{\omega : h(\omega)x_t(\omega) \geq 0\} = 1$, and so $P_\omega\{b : h(\omega)b \geq 0\} = 1$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Analogously, $\mathbb{E}P_\omega\{b : h(\omega)b > 0\} = \mathbb{P}\{\omega : h(\omega)x_t(\omega) > 0\} > 0$, which implies that, with positive probability $\mathbb{P}$, we have $P_\omega\{b : h(\omega)b > 0\} > 0$. Consequently, $X_{t-1}(\omega) \cap L^0_+(B, \mathcal{B}, P_\omega) \neq \{0\}$ with positive probability, and so $\mathbb{P}(\Omega^*) < 1$. □

# 4 A separation theorem in $L^1$.

In this section, $(B, \mathcal{B}, P)$ is any probability space.

**Theorem 4.1.** *Let $W$ be a closed cone contained in a $d$-dimensional linear subspace $L$ of $L^1 = L^1(B, \mathcal{B}, P)$. Let the set $W - L^1_+$ be convex. Then the following conditions are equivalent.*

*(W.1) $W \cap L^1_+ = \{0\}$.*

*(W.2) There exist numbers $0 < c_1 \leq ... \leq c_{d+1} \leq 1$ and a measurable function $\mu(b)$ taking values in the set $\{c_1, ..., c_{d+1}\}$ such that $E\mu w \leq 0$ for all $w \in W$.*

*If the measure $P$ is atomless, we can replace $d+1$ by 2 in (W.2).*

The validity of assertion (W.2) means that we can separate $W$ and $L^1_+$ by a linear functional $l(x)$ of $x \in L^1$ such that $l(x) = E\mu x$, where the function $\mu$ takes on not more than $d+1$ values $c_i \in (0, 1]$.

*Proof of Theorem 4.1.* The implication (W.2)⇒(W.1) is straightforward. Let us prove that (W.1) implies (W.2). Define $Z := W - L^1_+$. Clearly $Z \cap L^1_+ = \{0\}$. Let us show that $Z$ is $L^1$-closed. Suppose $z_k = w_k - u_k \to z$

in $L^1$, where $w_k \in W$ and $u_k \in L^1_+$. Then the sequence $w_k$ is bounded in the norm $||\cdot||$ of the space $L^1$. Indeed, if this is not so, then, by passing to a subsequence, we obtain $\gamma_k := ||w_k|| \to \infty$. This yields $w'_k - u'_k \to 0$, where $w'_k := w_k\gamma_k^{-1}$ and $u'_k := u_k\gamma_k^{-1}$. Since $W$ is a cone, we have $w'_k \in W$ and since $W$ is contained in a finite-dimensional space, we can select from the bounded sequence $w'_k$ a subsequence $w'_{k_i}$ converging to some $w \in L^1$ in $||\cdot||$. Then $w \in W$ because $W$ is closed and $||w|| = 1$ because $||w'_{k_i}|| = 1$. As $w'_{k_i} - u'_{k_i} \to 0$, we find that $u'_{k_i} \to u \in L^1_+$ and $u = w$. Thus $0 \neq w \in W \cap L^1_+$, which is a contradiction. Consequently, the sequence $w_k$ is bounded. By passing to subsequences, we obtain $w_k \to w \in W$ and $u_k \to u \in L^1_+$ (again, we use here the fact that $W$ is a closed set contained in a finite-dimensional subspace of $L^1$). This proves that $z = w - u \in W - L^1_+$, which establishes the closedness of $Z$.

Denote by $V$ the $L^1$-closure of the convex hull $coW$ of $W$. The set $V$ is contained in the same finite-dimensional space $L$ as $W$. Observe that $coW \subseteq Z$ since $Z$ is convex, and so $V \subseteq Z$ because $Z$ is closed. Consequently, $V \cap L^1_+ = \{0\}$.

Define $\Sigma := \{y \in L^1 : y \geq 0,\ Ey = 1\}$, where $Ey = \int y(b)P(db)$. Let us show that the $L^1$-distance $\rho(V, \Sigma)$ between $V$ and $\Sigma$ is strictly positive. First observe that $\rho(V, \Sigma) \leq \rho(0, 1)$, where 0 and 1 are constant functions regarded as elements $L^1$, and so $\rho(V, \Sigma) \leq 1$. Further, note that if $E|w| > 2$, then $\rho(w, \Sigma) \geq 1$. Indeed, if $y \in \Sigma$, then $\rho(w, y) = E|w - y| \geq E|w| - E|y| > 2 - 1 = 1$. Consequently, $\rho(V, \Sigma) = \rho(V(2), \Sigma)$, where $V(2)$ is the intersection of $V$ with the ball $\mathbb{B}(2) := \{y \in L^1 : E|y| \leq 2\}$ of radius 2. The set $V(2)$ is compact because it is a closed subset of $L^1$ contained in a finite-dimensional subspace. Since $V \cap L^1_+ = \{0\}$, the intersection of $V(2)$ and the closed set $\Sigma$ is empty. Consequently, $\rho(V(2), \Sigma) > 0$, and so $\rho(V, \Sigma) > 0$.

Define $\kappa = \rho(V, \Sigma)/2$ and put $V_\kappa := V + \mathbb{B}(\kappa)$ of $V$. We have $V_\kappa \cap \Sigma = \emptyset$. Since the interior of $V_\kappa$ is non-empty, we can separate the convex sets $V_\kappa$ and $\Sigma$ by a non-zero continuous linear functional $l(y)$ on $L^1$: $l\,(y) \leq l\,(y')$, $y \in V_\kappa$,

$y' \in \Sigma$. Since $l \neq 0$ and $\mathbb{B}(\kappa) \subseteq V_\kappa$, we have $\delta := \sup\{l(y) : y \in V_\kappa\} > 0$. Thus

$$l(y) \leq \delta \leq l(y'), \; y \in V_\kappa, \; y' \in \Sigma. \tag{4.1}$$

Every continuous linear functional $l$ on $L^1$ can be represented in the form $l(y) = E\alpha_0 y$, where $\alpha_0(b)$ is a bounded measurable function. The second inequality in (4.1) yields $\alpha_0(b) \geq \delta \; (> 0)$ $P$-almost everywhere (a.e.), while the first implies $\alpha_0(b) \leq \delta/\kappa$ (a.e.), because $\mathbb{B}(\kappa) \subseteq V_\kappa$. By setting $\alpha(b) = \kappa\delta^{-1}\alpha_0(b)$, we get $\kappa \leq \alpha(b) \leq 1$ (a.e.). From (4.1), we can see that $E\alpha y \leq \kappa$ for each $y \in W \subseteq V_\kappa$, which implies $E\alpha y \leq 0$, $y \in W$, because $W$ is a cone. We may assume that the inequalities $\kappa \leq \alpha(b) \leq 1$ hold for all $b$ (this can be obtained by modifying the function $\alpha$ on a set of measure zero).

Consider a basis $x_1, ..., x_d$ in the $d$-dimensional linear space $L$ containing $W$. Denote by $x$ the vector function $x = (x_1, ..., x_d)$ and by $G$ the set of those vectors $g = (g_1, ..., g_d) \in R^d$ for which $gx := g_1x_1 + ... + g_dx_d \in W$. Then $W = \{gx : g \in G\}$. We have constructed a bounded function $\alpha$ such that $\alpha(b) \geq \kappa > 0$ and $gE\alpha x = E\alpha gx \leq 0$ for all $g \in G$. Since $x_i \in L^1$, $i = 1, 2, ..., d$, the vector function $x = (x_1, ..., x_d)$ belongs to $L^1(B, \mathcal{B}, P, R^d)$. Consequently, we can apply Proposition AII.1 (see Appendix II), according to which there exists a function $\beta(b)$ taking not more than $d + 1$ different values $\kappa \leq r_1 \leq ... \leq r_{d+1}$ such that $E\beta x = E\alpha x$. Define $\mu(b) = \beta(b)/r_{d+1}$. The function $\mu(b)$ takes values in the set $\{c_1, ..., c_{d+1}\}$, where $c_i = r_i/r_{d+1}$, and we have $0 < c_1 \leq ... \leq c_{d+1} = 1$. Finally,

$$E\mu gx = gE\mu x = gE\beta x/r_{d+1} = gE\alpha x/r_{d+1} \leq 0, \; g \in G,$$

which implies $E\mu w \leq 0$, $w \in W$, because $W = \{gx : g \in G\}$. According to Proposition AII.1, if $P$ is atomless, we can replace $d + 1$ by 2 in the last argument. □

# 5 Two-stage model

In this section, we fix $t \in \{1, 2, ..., T\}$ and consider only two moments of time: $t-1$ and $t$ (two-stage model). In accordance with hypotheses (X.1) and (X.2), we assume that $X_{t-1}(\omega)$ is closed under convergence $P_\omega$-a.e. and $X_{t-1}(\omega) - L^0_+(B, \mathcal{B}, P_\omega)$ is convex for all $\omega \in \Omega_{t-1}$, where $\Omega_{t-1} \in \mathcal{F}_{t-1}$ and $\mathbb{P}(\Omega_{t-1}) = 1$. Additionally, we postulate that $\mathbb{E}(|x_t||\mathcal{F}_{t-1}) < \infty$ (a.s.). This implies $E^{t-1}_\omega|b| < \infty$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore we may suppose that $E^{t-1}_\omega|b| < \infty$ for all $\omega \in \Omega_{t-1}$ (the set $\Omega_{t-1}$ can be replaced by a smaller $\mathcal{F}_{t-1}$-measurable set of full measure). As before, we write $P_\omega := P^{t-1}_\omega$ and $E_\omega := E^{t-1}_\omega$.

**Theorem 5.1.** *Let condition ($NA_t$) hold. Then for each strictly positive bounded random variable $\gamma$ there exists a random variable $\lambda \in \Lambda^\infty_t(d+1)$ such that $\mathbb{E}\lambda[|x_t| + |x_{t-1}|] < \infty$ and*

$$h\mathbb{E}(\lambda\gamma x_t|\mathcal{F}_{t-1}) \leq 0 \text{ (a.s.)},\ h \in \mathcal{H}_{t-1}.$$

*If $P_\omega$ is atomless for almost all $\omega$, then $\lambda$ can be selected in the class $\Lambda^\infty_t(2)$.*

*Proof.* Let $\mathcal{F}_{t-1} \vee \sigma(x_t)$ denote the $\sigma$-algebra generated by $\mathcal{F}_{t-1}$ and $x_t$. Consider the random variable $\hat{\gamma} = \mathbb{E}[\gamma|\mathcal{F}_{t-1} \vee \sigma(x_t)] > 0$. We can represent it as $\hat{\gamma}(\omega) = g(\omega, x_t(\omega))$ (a.s.), where $g(\omega, b)$ ($b \in B = R^d$) is a bounded strictly positive $\mathcal{F}_{t-1} \times \mathcal{B}$-measurable function [$\mathcal{B} = \mathcal{B}(R^d)$]. For each $\omega \in \Omega$, denote by $W_{t-1}(\omega)$ the set consisting of functions $w(b)$ on $B$ of the form $w(b) = g(\omega, b)v(b)$, where $v(\cdot) \in X_{t-1}(\omega)$. Since $g(\omega, b)$ is bounded and any function $v(\cdot)$ in $X_{t-1}(\omega)$ is of the form $ab$ ($a \in H_{t-1}(\omega)$), condition $E_\omega|b| < \infty$, $\omega \in \Omega_{t-1}$, implies $W_{t-1}(\omega) \subseteq L^1(B, \mathcal{B}, P_\omega)$, $\omega \in \Omega_{t-1}$.

By virtue of Proposition 3.2, it follows from (NA$_t$) that the intersection of $X_{t-1}(\omega)$ and $L^0_+(B, \mathcal{B}, P_\omega)$ is $\{0\}$ for all $\omega \in \Omega'$, where $\Omega' \in \mathcal{F}_{t-1}$ and $\mathbb{P}(\Omega') = 1$. Since $g(\omega, b) > 0$, we have $W_{t-1}(\omega) \cap L^0_+(B, \mathcal{B}, P_\omega) = \{0\}$ for all $\omega \in \Omega'$. We may assume without loss of generality that $\Omega_{t-1} \subseteq \Omega'$ (we can always replace $\Omega_{t-1}$ by $\Omega_{t-1} \cap \Omega'$). For $\omega \in \Omega_{t-1}$, the cone $X_{t-1}(\omega)$ is closed under convergence $P_\omega$-a.e., and the cone $X_{t-1}(\omega) - L^0_+(B, \mathcal{B}, P_\omega)$ is convex. From this, and since $g(\omega, b) > 0$, we obtain that the set $W_{t-1}(\omega)$ is closed under convergence $P_\omega$-a.e. and the set $W_{t-1}(\omega) - L^1_+(B, \mathcal{B}, P_\omega)$ – the intersection of $L^1(B, \mathcal{B}, P_\omega)$ and $W_{t-1}(\omega) - L^0_+(B, \mathcal{B}, P_\omega)$ – is convex for each $\omega \in \Omega_{t-1}$. Furthermore, $W_{t-1}(\omega)$ is contained in the $d$-dimensional linear subspace of $L^1(B, \mathcal{B}, P_\omega)$ spanned on the functions $g(\omega, b)b^j$, $j = 1, 2, ..., d$, where $b = (b^1, ..., b^d)$. Thus, we can apply Theorem 4.1, from which it follows that, for each $\omega \in \Omega_{t-1}$, there exists a Borel function $\mu(b)$, $b \in B$, with at most $k = d + 1$ values $0 < c_1 \leq ... \leq c_k \leq 1$ ($k = 2$ when $P_\omega$ is atomless) satisfying

$$E_\omega \mu(b) g(\omega, b) ab \leq 0, \; a \in H_{t-1}(\omega). \tag{5.1}$$

Note that if $P_\omega$ is atomless for almost all $\omega$, it can be assumed that this is so for all $\omega \in \Omega_{t-1}$.

Let $\{h^{(m)}_{t-1}(\cdot)\}_{m=1}^\infty$ be a sequence of $\mathcal{F}_{t-1}$-measurable vector functions and $\Omega'' \in \mathcal{F}_{t-1}$ a set such that $\mathbb{P}(\Omega'') = 1$ and, for all $\omega \in \Omega''$, the sequence of points $\{h^{(m)}_{t-1}(\omega)\}$ is dense $H_{t-1}(\omega)$ (see Theorem AI.2). Again, without loss of generality, we may assume that $\Omega_{t-1} \subseteq \Omega''$. Consider the Borel function $\psi(r, b)$ of $r \in I := [0, 1]$ and $b \in B$ described in Theorem AI.3 and define the set $\Delta$ consisting of $(\omega, r, c_1, ..., c_k)$ such that $\omega \in \Omega$, $r \in I$, $0 < c_1 \leq ... \leq c_k \leq 1$,

$$E_\omega \psi(r, b) g(\omega, b) h^{(m)}_{t-1}(\omega) b \leq 0, \; m = 1, 2, ..., \tag{5.2}$$

$$P_\omega\{b : \min_{i=1,...,k} |\psi(r, b) - c_i| = 0\} = 1. \tag{5.3}$$

The last condition means that $\psi(r, \cdot)$ coincides $P_\omega$-a.e. with a function taking its values in $\{c_1, ..., c_k\}$. We can see from (5.2) and (5.3) that $\Delta \in \mathcal{F}_{t-1} \times \mathcal{B}(R^{k+1})$. We claim that $\Omega_{t-1}$ is contained in the projection of $\Delta$ on $\Omega$. Indeed, if $\omega \in \Omega_{t-1}$, then, as we have shown above, there exists a Borel function $\mu(\cdot) = \mu^\omega(\cdot)$ on $B$ taking on $k$ values $0 < c_1 \leq ... \leq c_k \leq 1$ ($c_i = c_i^\omega$) and satisfying (5.1). By using the property of the function $\psi$ described in Theorem AI.3, we conclude that there exists $r = r^\omega \in I$ such that $\psi(r, b) = \mu(b)$ for $P_\omega$-almost all $b$. For $r = r^\omega$, condition (5.2) follows from (5.1) and requirement (5.3) from the fact that the values of $\mu(b)$ are in $\{c_1, ..., c_k\}$. We now can apply the measurable selection theorem (see Theorem AI.2), and construct an $\mathcal{F}_{t-1}$-measurable mapping $\omega \mapsto \xi(\omega) = (r(\omega), c_1(\omega), ..., c_k(\omega))$ such that $(\omega, \xi(\omega)) \in \Delta$ for all $\omega$ in an $\mathcal{F}_{t-1}$-measurable set $\hat{\Omega} \subseteq \Omega_{t-1}$ with $\mathbb{P}(\hat{\Omega}) = 1$.

For $\omega \in \hat{\Omega}$, define $\phi(\omega, b) = \psi(r(\omega), b)$ if $\psi(r(\omega), b) \in \{c_1(\omega), ..., c_k(\omega)\}$ and put $\phi(\omega, b) = c_k(\omega)$ otherwise. Set $\phi(\omega, b) = 1$ for all $b \in B, \omega \in \Omega\backslash\hat{\Omega}$, and redefine $c_i(\omega)$ by setting $c_i(\omega) = 1$ for all $i = 1, ..., k$ and $\omega \in \Omega\backslash\hat{\Omega}$. Then $\phi(\omega, b) \in \{c_1(\omega), ..., c_k(\omega)\}$ for all $\omega$ and $b$. For $\omega \in \hat{\Omega}$, we have $\phi(\omega, b) = \psi(r(\omega), b)$ $P_\omega$-a.e. (see (5.3)) and

$$E_\omega \phi(\omega, b) g(\omega, b) h_{t-1}^{(m)}(\omega) b \leq 0, \; m = 1, 2, ..., \tag{5.4}$$

(see (5.2)), which yields

$$E_\omega \phi(\omega, b) g(\omega, b) ab \leq 0, \; a \in H_{t-1}(\omega), \; \omega \in \hat{\Omega}. \tag{5.5}$$

Define

$$\theta(\omega) = 1 + E_\omega\{\phi(\omega, b)[|b| + |x_{t-1}(\omega)|]\}, \; f(\omega, b) = \phi(\omega, b)/\theta(\omega) \tag{5.6}$$

and $\lambda(\omega) = f(\omega, x_t(\omega))$. Then $f(\omega, b) \in \{c_1(\omega)/\theta(\omega), ..., c_k(\omega)/\theta(\omega)\}$ for each $\omega$ and $b$. Consequently, $f$ meets requirements (f.1) and (f.2) in Section 2, and so $\lambda \in \Lambda_t(k)$. Further, we have $\lambda \leq 1$ and

$$\mathbb{E}\lambda(|x_t| + |x_{t-1}|) = \mathbb{E}E_\omega\{\frac{\phi(\omega, b)}{\theta(\omega)}[|b| + |x_{t-1}(\omega)|]\} \leq 1.$$

Finally, we can replace $\phi$ by $f$ in (5.5), which yields

$$h\mathbb{E}(\lambda\gamma x_t|\mathcal{F}_{t-1}) = h\mathbb{E}(\lambda\hat{\gamma} x_t|\mathcal{F}_{t-1}) = E_\omega f(\omega, b) g(\omega, b) h(\omega) b \leq 0 \text{ (a.s.)}$$

for any $h \in \mathcal{H}_{t-1}$. □

# 6 Proof of the main theorem

*Proof of Theorem 2.1.* Let us start with proving the second assertion of the theorem. As we have shown in Remark 2.4, (NA) implies ($\mathrm{NA}_t$) for each $t = 1, 2, ..., T$. Put $k = d + 1$ or $k = 2$, if $P_\omega^t$ is atomless for all $t$ and almost all $\omega$. By virtue of ($\mathrm{NA}_t$) and Theorem 5.1, the following assertion is valid:

(*) For each $t = 1, ..., T$ and each strictly positive bounded random variable $\gamma$, there exists a random variable $\lambda \in \Lambda_t^\infty(k)$ such that $\mathbb{E}\lambda[|x_t|+|x_{t-1}|] < \infty$ and $h\mathbb{E}(\lambda\gamma x_t|\mathcal{F}_{t-1}) \le 0$ (a.s.), $h \in \mathcal{H}_{t-1}$.

We will prove by way of induction with respect to $m$ (from $m = T$ to $m = 1$) the following assertion:

($\mathcal{A}_m$) There exists a function $\lambda$ of the form $\lambda = \lambda_m\lambda_{m+1}...\lambda_T$, where $\lambda_t \in \Lambda_t^\infty(k)$, such that $\mathbb{E}\lambda|x_t| < \infty$ $(t = m-1, ..., T)$ and

$$h\mathbb{E}(\lambda x_t|\mathcal{F}_{t-1}) \le 0, \; h \in \mathcal{H}_{t-1}, \; t = m, ..., T. \tag{6.1}$$

For $m = T$, this assertion follows from (*) with $\gamma = 1$. For $m = 1$, this assertion yields the desired result (by choosing a constant $\lambda_0 > 0$, we can obtain $\mathbb{E}\lambda = 1$). Suppose ($\mathcal{A}_m$) holds for some $1 < m \le T$. Let us prove ($\mathcal{A}_{m-1}$).

Consider a function $\lambda$ with properties described in ($\mathcal{A}_m$) and put $\gamma = \mathbb{E}(\lambda|\mathcal{F}_{m-1})$. By virtue of (*), there exists a random variable $\lambda_{m-1} \in \Lambda_{m-1}^\infty(k)$ such that $\mathbb{E}\lambda_{m-1}[|x_{m-1}| + |x_{m-2}|] < \infty$ and $h\mathbb{E}[\lambda_{m-1}\gamma x_{m-1}|\mathcal{F}_{m-2}] \le 0$, $h \in \mathcal{H}_{m-2}$. From the first of these two inequalities, we obtain

$$\mathbb{E}\lambda_{m-1}\lambda[|x_{m-1}| + |x_{m-2}|] = \mathbb{E}\lambda_{m-1}\mathbb{E}(\lambda|\mathcal{F}_{m-1})[|x_{m-1}| + |x_{m-2}|] =$$

$$\mathbb{E}\lambda_{m-1}\gamma[|x_{m-1}| + |x_{m-2}|] \le ||\gamma||_\infty \, \mathbb{E}\lambda_{m-1}[|x_{m-1}| + |x_{m-2}|] < \infty, \tag{6.2}$$

and from the second we get

$$h\mathbb{E}[\lambda_{m-1}\lambda x_{m-1}|\mathcal{F}_{m-2}] = h\mathbb{E}[\lambda_{m-1}\mathbb{E}(\lambda|\mathcal{F}_{m-1})x_{m-1}|\mathcal{F}_{m-2}] =$$

(6.3) $$h\mathbb{E}[\lambda_{m-1}\gamma x_{m-1}|\mathcal{F}_{m-2}] \le 0,\ h \in \mathcal{H}_{m-2}.$$

By virtue of (6.1), we have

$$h\mathbb{E}[\lambda_{m-1}\lambda x_t|\mathcal{F}_{t-1}] = \lambda_{m-1}h\mathbb{E}[\lambda x_t|\mathcal{F}_{t-1}] \le 0,\ h \in \mathcal{H}_{t-1},$$

for each $t \ge m$. It remains to observe that the function $\lambda_{m-1}\lambda = \lambda_{m-1}\lambda_m...\lambda_T$ satisfies $\mathbb{E}\lambda_{m-1}\lambda|x_t| < \infty$ for all $t = m-2, ..., T$ in view of (6.2) and by virtue of the induction hypothesis and the boundedness of $\lambda_{m-1}$. The function $\lambda_{m-1}\lambda$ possesses all the properties needed for $(\mathcal{A}_{m-1})$.

Thus we have established the second assertion of the theorem; let us deduce from it the first one. Let $\pi > 0$ be a bounded random variable such that $\mathbb{E}\pi = 1$ and $\mathbb{E}\pi|x_t| < \infty$ for all $t$. Consider the analogous model in which the probability $\mathbb{P}$ is replaced by the equivalent probability $\widetilde{\mathbb{P}}(d\omega) = \pi(\omega)\mathbb{P}(d\omega)$, but the other data ($x_t$, $M_t$ and $H_t$) are the same. In this model, condition (NA) holds since $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ have the same sets of measure zero, and, additionally, (2.8) is fulfilled. Furthermore, the modified model satisfies requirements (X.1) and (X.2) because the conditional distributions $\widetilde{P}^t_\omega(db)$ and $P^t_\omega(db)$ for the measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent for almost all $\omega$. Therefore, by virtue of the part of the proof we have just completed, there exists a bounded strictly positive random variable $\tilde{\lambda}$ such that $\widetilde{\mathbb{E}}\tilde{\lambda}|x_t| < \infty$ for all $t = 0, ..., T$ and $h\widetilde{\mathbb{E}}(\tilde{\lambda}x_t|\mathcal{F}_{t-1}) \le 0$ (a.s.) for all $h \in \mathcal{C}_{t-1}$ and $t = 1, 2, ..., T$. We have $\widetilde{\mathbb{E}}(\tilde{\lambda}x_t|\mathcal{F}_{t-1}) = \mathbb{E}(\pi\tilde{\lambda}x_t|\mathcal{F}_{t-1})/\mathbb{E}(\pi|\mathcal{F}_{t-1})$ and $\widetilde{\mathbb{E}}\tilde{\lambda}|x_t| = \mathbb{E}\pi\tilde{\lambda}|x_t|$, from which it follows that the bounded strictly positive random variable $\lambda := \pi\tilde{\lambda}/\mathbb{E}\pi\tilde{\lambda}$ possesses all the properties listed in the first assertion of Theorem 2.1.

To prove the last assertion of the theorem let us show by induction that $\mathcal{R}_m \cap L^0_+ = \{0\}$ for $m = 0, 1, ..., T$, where $\mathcal{R}_0 = \{0\}$ and $\mathcal{R}_m$ ($m \ge 1$) is defined by (1.1). For $m = 0$, the statement holds trivially. Assuming that it holds for some $0 \le m < T$, let us prove it for $m + 1$. Suppose the contrary: there exists $\xi \in \mathcal{R}_{m+1} \cap L^0_+$ with $\mathbb{E}\xi > 0$. Then $\lambda \sum_{t=1}^m h_{t-1}x_t + h_m\lambda x_{m+1} = \lambda\xi$ for some $h_0 \in \mathcal{C}_0, ..., h_m \in \mathcal{C}_m$. Let us show that the random variable $\zeta := \sum_{t=1}^{m} h_{t-1}x_t$ is nonnegative (a.s.) and $\mathbb{E}\zeta > 0$. Suppose that either $\mathbb{P}\{\zeta < 0\} > 0$ or $\zeta = 0$ (a.s.). Define $\Gamma = \{\omega : \zeta(\omega) < 0\}$ $(\in \mathcal{F}_m)$ in the former case and $\Gamma = \Omega$ in the latter. Then the random variable $\theta := \chi_\Gamma h_m \lambda x_{m+1}$ satisfies $\theta = \chi_\Gamma \lambda \xi - \chi_\Gamma \lambda \zeta$, from which it follows that $\theta \geq 0$ (a.s.) and $\mathbb{E}\theta > 0$. On the other hand, $\mathbb{E}(\theta|\mathcal{F}_m) = h_m \chi_\Gamma \mathbb{E}(\lambda x_{m+1}|\mathcal{F}_m) \leq 0$ by virtue of (2.9) and (2.7). This is a contradiction. Thus $\zeta \geq 0$ (a.s.) and $\mathbb{E}\zeta > 0$, which, in turn, contradicts the induction hypothesis. □

**Remark 6.1.** We note that, in general, condition (2.8) involved in Theorem 2.1 cannot be dropped. Suppose $H_t = M_t = R^d$. For the expression $h\mathbb{E}(\lambda x_T|\mathcal{F}_{T-1}) \leq 0$ in (2.7) to be well-defined for all $h$, we must have $\mathbb{E}(\lambda|x_T||\mathcal{F}_{T-1}) < \infty$ (a.s.). If $\lambda = \lambda_0 ... \lambda_T$, where $\lambda_t \in \Lambda_t(k)$, we obtain $\mathbb{E}(\lambda_T|x_T||\mathcal{F}_{T-1}) < \infty$ (a.s.). Consequently,

$$f_*(\omega)\mathbb{E}(|x_T||\mathcal{F}_{T-1}) \leq \mathbb{E}(f(\omega, x_T)|x_T||\mathcal{F}_{T-1}) = \mathbb{E}(\lambda_T|x_T||\mathcal{F}_{T-1}) < \infty \text{ (a.s.)},$$

where $f_*(\omega) = \min_i(c_i(\omega)) > 0$ (see (2.5)). From this we conclude that $\mathbb{E}(|x_T||\mathcal{F}_{T-1}) < \infty$.

# 7 Equivalent versions of the main assumptions

The purpose of this section is to establish the equivalence of "conditional" and "unconditional" forms of the main assumptions introduced in Section 2. We fix some $t = 1, 2, ..., T$ and examine the sets $\mathcal{X}_t$ and $X_{t-1}(\omega)$ defined in terms of $H_{t-1}(\omega)$ by (2.13) and (2.3). Throughout the section, we assume that $H_{t-1}(\omega)$ *is an* $\mathcal{F}_{t-1}$*-measurable random set closed and non-empty for each* $\omega \in \Omega$. When needed, it is additionally assumed that $H_{t-1}(\omega)$ is a cone. As before, we use the notation $B := R^d$, $\mathcal{B} = \mathcal{B}(R^d)$, $P_\omega = P_\omega^{t-1}$ and $E_\omega = E_\omega^{t-1}$. We define $\rho(\omega, y, y') = E_\omega\{|y - y'|(1 + |y - y'|)^{-1}\}$, where $y, y' \in L^0(B, \mathcal{B}, P_\omega)$. The functional $\rho(\omega, y, y')$ is a metric in $L^0(B, \mathcal{B}, P_\omega)$

inducing the topology of convergence in measure.

We begin with two auxiliary results, Lemmas 7.1 and 7.2 below, that will be used in this section. Fix some $\mathcal{F}_{t-1}$-measurable vector function $h_*(\omega)$ satisfying $h_*(\omega) \in H_{t-1}(\omega)$ for almost all $\omega$ (such a function exists by virtue of Theorem AI.2). For each $N = 1, 2, ...$, define $H_{t-1}^N(\omega) = \{a \in H_{t-1}(\omega) : |a - h_*(\omega)| \leq N\}$. Consider the class $\mathcal{H}_{t-1}^N$ of $\mathcal{F}_{t-1}$-measurable functions $h(\cdot)$ for which $h(\omega) \in H_{t-1}^N(\omega)$ (a.s.). Put $\mathcal{X}_t^N = \{h(\omega)x_t(\omega) : \ h(\cdot) \in \mathcal{H}_{t-1}^N\}$. Let $\{h_k^N(\omega)\}_{k=1}^\infty$ be a sequence of functions $h_k^N \in \mathcal{H}_{t-1}^N$ and $\Omega_H$ an $\mathcal{F}_{t-1}$-measurable subset of $\Omega$ such that $\mathbb{P}(\Omega_H) = 1$, $h_*(\omega) \in H_{t-1}(\omega)$ for $\omega \in \Omega_H$, and, for each $N = 1, 2, ...$ and each $\omega \in \Omega_H$, the points $h_1^N(\omega), h_2^N(\omega), ...$ form a dense subset of $H_{t-1}^N(\omega)$ (see Theorem AI.2). Consider the continuous mapping $a \mapsto v_a(\cdot)$ of $R^d$ into $L^0(B, \mathcal{B}, P_\omega)$ defined by $v_a(b) = ab$ ($P_\omega$-a.e.). The mapping $a \mapsto v_a(\cdot)$ transforms the compact set $H_{t-1}^N(\omega)$ into a compact set in $L^0(B, \mathcal{B}, P_\omega)$, that we will denote by $X_{t-1}^N(\omega)$, and the sequence $\{h_k^N(\omega)\}$ dense in $H_{t-1}^N(\omega)$ into the sequence $v_k^N(\omega, b) := h_k^N(\omega)b$ dense in $X_{t-1}^N(\omega)$. Put $\rho_k^N(\omega, y(\cdot)) = \rho(\omega, y(\cdot), v_k^N(\omega, \cdot))$.

**Lemma 7.1.** *Let $\omega \in \Omega_H$. Then a function $y(\cdot) \in L^0(B, \mathcal{B}, P_\omega)$ does not belong to the set $X_{t-1}(\omega)$ if and only if we have $\inf_k \rho_k^N(\omega, y(\cdot)) > 0$ for each $N \in \{1, 2, ...\}$.*

*Proof.* The assertion of the lemma follows from the following facts: (a) $X_{t-1}(\omega) = \cup_N X_{t-1}^N(\omega)$; (b) $X_{t-1}^N(\omega)$ is compact; (c) for $\omega \in \Omega_H$, the sequence $v_k^N(\omega, \cdot)$, $k = 1, 2, ...$, is dense in $X_{t-1}^N(\omega)$, and so $\rho(\omega, y(\cdot), X_{t-1}^N(\omega)) = \inf_k \rho_k^N(\omega, y(\cdot))$. □

Let $\psi_j(b)$ ($j = 1, 2, ...$) be a sequence of bounded Borel functions on $B$ satisfying the following condition:

($\psi$) For each $\omega$, the sequence $\{\psi_j(\cdot)\}$ is dense in $L^0(B, \mathcal{B}, P_\omega)$.

(To construct $\{\psi_j(\cdot)\}$, consider a countable dense subset in the nonnegative cone in space of continuous functions on $[0, 1]$ and a Borelian isomorphism between $B$ and $[0,1]$.) Define $Z_{t-1}(\omega) := X_{t-1}(\omega) - L^0_+(B,\mathcal{B},P_\omega)$ and

$$\rho^N_{k,j}(\omega, y(\cdot)) = \rho(\omega, y(\cdot), v^N_k(\omega,\cdot) - \psi_j(\cdot)).$$

**Lemma 7.2.** *Let $\omega \in \Omega_H$. Then a function $y(\cdot) \in L^0(B,\mathcal{B},P_\omega)$ does not belong to $Z_{t-1}(\omega)$ if and only if $\inf_{k,j} \rho^N_{k,j}(\omega, y(\cdot)) > 0$ for each $N \in \{1,2,...\}$.*

*Proof.* We have $Z_{t-1}(\omega) = \cup_N Z^N_{t-1}(\omega)$, where $Z^N_{t-1}(\omega) := X^N_{t-1}(\omega) - L^0_+(B,\mathcal{B},P_\omega)$. Since $X^N_{t-1}(\omega)$ is compact, $Z^N_{t-1}(\omega)$ is closed. Thus $y(\cdot) \notin Z_{t-1}(\omega)$ if and only if $\rho(\omega, y(\cdot), Z^N_{t-1}(\omega)) > 0$ for each $N \in \{1,2,...\}$. On the other hand, $\rho(\omega, y(\cdot), Z^N_{t-1}(\omega)) = \inf_{k,j} \rho^N_{k,j}(\omega, y(\cdot))$ because $\{v^N_k(\omega,\cdot) - \psi_j(\cdot)\}$ is dense in $Z^N_{t-1}(\omega)$. □

The proposition below establishes the equivalence of assumptions (X.1) and ($\mathcal{X}$.1).

**Proposition 7.1.** *The set*

$$\overline{\Omega} := \{\omega \in \Omega : X_{t-1}(\omega) \text{ is closed in } L^0(B,\mathcal{B},P_\omega)\}$$

*is measurable with respect to $\mathcal{F}^{\mathbb{P}}_{t-1}$. The following conditions are equivalent:*

*(i) $\mathcal{X}_t$ is closed in $L^0(\Omega,\mathcal{F}_t,\mathbb{P})$;*

*(ii)* $\mathbb{P}(\overline{\Omega}) = 1$.

*Proof.* For each sequence $\mathbf{a} = (a_1, a_2, ...)$, $a_j \in R^d$, and each $b \in B = R^d$, define $Y_*(\mathbf{a}, b) := \liminf\{a_j b\}$ and $Y^*(\mathbf{a}, b) := \limsup\{a_j b\}$. Put $Y(\mathbf{a}, b) = Y^*(\mathbf{a}, b)$ if $-\infty < Y_*(\mathbf{a}, b) = Y^*(\mathbf{a}, b) < +\infty$ and $Y(\mathbf{a}, b) = 0$ otherwise. Consider the set $\mathbf{H}(\omega)$ of those sequences $\mathbf{a} = (a_1, a_2, ...)$ for which $a_j \in H_{t-1}(\omega)$, $j \in \{1, 2, ..., \}$. A function $y(\cdot) \in L^0(B,\mathcal{B},P_\omega)$ belongs to the closure of the set $X_{t-1}(\omega)$ with respect to convergence in measure $P_\omega$ (or, equivalently, with respect to convergence $P_\omega$-a.e.) if and only if there is a sequence $\mathbf{a} = (a_1, a_2, ...) \in \mathbf{H}(\omega)$ such that $Y(\mathbf{a}, b) = y(b)$ $P_\omega$-a.e. and $P_\omega\{b : -\infty < Y_*(\mathbf{a}, b) = Y^*(\mathbf{a}, b) < +\infty\} = 1$.

Denote by $\underline{\Omega}$ the complement of $\overline{\Omega}$, i.e., the set of those $\omega \in \Omega$ for which $X_{t-1}(\omega)$ is not closed in $L^0(B,\mathcal{B},P_\omega)$ with respect to $P_\omega$-a.e. convergence. Define $\mathbf{R} := R^d \times R^d \times ...$ and $\Omega^0 = \underline{\Omega} \cap \Omega_H$. By virtue of Lemma 7.1, $\Omega^0$ is the projection on $\Omega$ of the set $\Delta^0$ in $\Omega \times \mathbf{R}$ consisting of pairs $(\omega, \mathbf{a})$ which satisfy

$$(7.1) \quad \omega \in \Omega_H,\ \mathbf{a} \in \mathbf{H}(\omega),\ P_\omega\{b : -\infty < Y_*(\mathbf{a}, b) = Y^*(\mathbf{a}, b) < +\infty\} = 1,$$

$$(7.2) \qquad \inf_k \rho_k^N(\omega, Y(\mathbf{a}, \cdot)) > 0 \text{ for each } N \in \{1, 2, ...\}.$$

We can see that $\Delta^0 \in \mathcal{F}_{t-1} \times \mathcal{B}(\mathbf{R})$, and so, by virtue of Theorem AI.2, the projection $\Omega^0 = \mathrm{pr}_\Omega \Delta^0$ is $\mathcal{F}_{t-1}^{\mathbb{P}}$-measurable. Consequently, $\underline{\Omega} \in \mathcal{F}_{t-1}^{\mathbb{P}}$ because $\Omega^0 = \underline{\Omega} \cap \Omega_H$, $\Omega_H \in \mathcal{F}_{t-1}$ and $\mathbb{P}(\Omega_H) = 1$.

*(i)⇒(ii).* Suppose $\mathbb{P}(\overline{\Omega}) < 1$. Then $\mathbb{P}(\underline{\Omega}) > 0$ and $\mathbb{P}(\Omega^0) > 0$ because $\Omega^0 = \underline{\Omega} \cap \Omega_H$ and $\mathbb{P}(\Omega_H) = 1$. By virtue of Theorem AI.2, there exists a set $\hat{\Omega} \in \mathcal{F}_{t-1}$, $\hat{\Omega} \subseteq \Omega^0$, and a measurable mapping $\mathbf{h}(\omega) = (h_1(\omega), h_2(\omega), ...)$ of $(\Omega, \mathcal{F}_{t-1})$ into $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that $\mathbb{P}(\hat{\Omega}) = \mathbb{P}(\Omega^0)$ $(> 0)$ and $(\omega, \mathbf{h}(\omega)) \in \Delta^0$ for each $\omega \in \hat{\Omega}$. Redefine $\mathbf{h}(\omega)$ as $(h_1^1(\omega), h_1^1(\omega), ...)$ outside $\hat{\Omega}$. Then, for all $\omega \in \Omega$, the sequence of functions $y_j(\omega, b) := h_j(\omega)b$ converges to $y(\omega, b) := Y(\mathbf{h}(\omega), b)$ $P_\omega$-a.e. (see (7.1)). Put $w_j(\omega) = h_j(\omega)x_t(\omega)$ and $w(\omega) = Y(\mathbf{h}(\omega), x_t(\omega))$. We have

$$\mathbb{E}\frac{|w_j(\omega) - w(\omega)|}{1 + |w_j(\omega) - w(\omega)|} = \mathbb{E}E_\omega \frac{|h_j(\omega)b - Y(\mathbf{h}(\omega), b)|}{1 + |h_j(\omega)b - Y(\mathbf{h}(\omega), b)|} \to 0,$$

and so $w_j(\omega) \to w(\omega)$ in measure. For each $j$, we have $w_j(\cdot) \in \mathcal{X}_t$ because $h_j(\omega) \in H_{t-1}(\omega)$ (a.s.). On the other hand, the function $w(\omega) = Y(\mathbf{h}(\omega), x_t(\omega))$ does not belong to $\mathcal{X}_t$. Indeed, the function $f(\omega, b) := Y(\mathbf{h}(\omega), b)$ is $\mathcal{F}_{t-1} \times \mathcal{B}$-measurable and $f(\omega, \cdot) \notin X_{t-1}(\omega)$ for all $\omega$ in the set $\Omega^0 \in \mathcal{F}_{t-1}$ having positive measure $\mathbb{P}$ (see (7.2)). According to Proposition 3.1, this means that $w(\cdot) \notin \mathcal{X}_t$. Thus $\mathcal{X}_t$ is not closed with respect to convergence in measure.

*(ii)⇒(i).* Suppose $P(\overline{\Omega}) = 1$. Consider a sequence $w_j(\omega)$ of functions in $\mathcal{X}_t$ converging to a function $w(\cdot) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ for $\mathbb{P}$-almost all $\omega$. We wish to show that $w \in \mathcal{X}_t$. Since $w_j(\cdot) \in \mathcal{X}_t$, we have $w_j(\omega) = h_j(\omega)x_t(\omega)$

(a.s.), where $h_j(\omega)$, $j = 1, 2, ...,$ are $\mathcal{F}_{t-1}$-measurable functions satisfying $h_j(\omega) \in H_{t-1}(\omega)$ for all $j$ and all $\omega$ in a set $\Omega' \in \mathcal{F}_{t-1}$ with $\mathbb{P}(\Omega') = 1$. Put $f_j(\omega, b) = h_j(\omega)b$ and denote by $f(\omega, b)$ the function which is equal to $\lim f_j(\omega, b)$ when the sequence $\{f_j(\omega, b)\}$ converges and which is equal to zero otherwise. The function $f(\omega, b)$ is $\mathcal{F}_{t-1} \times \mathcal{B}$-measurable, and $w(\omega) = f(\omega, x_t(\omega))$ (a.s.). Further, we have

$$\mathbb{E}\rho(\omega, f_j(\omega, \cdot), f(\omega, \cdot)) = \mathbb{E}E_\omega \frac{|f_j(\omega, b) - f(\omega, b)|}{1 + |f_j(\omega, b) - f(\omega, b)|} = \mathbb{E}\frac{|w_j - w|}{1 + |w_j - w|} \to 0.$$

By passing to a subsequence, we obtain that, for all $\omega$ in a set $\tilde{\Omega} \in \mathcal{F}_{t-1}$ with $\mathbb{P}(\tilde{\Omega}) = 1$, $\rho(\omega, f_j(\omega, \cdot), f(\omega, \cdot)) \to 0$, and so $f_j(\omega, \cdot) \to f(\omega, \cdot)$ in measure $P_\omega$ for all $\omega \in \tilde{\Omega}$. We may assume without loss of generality that $\tilde{\Omega}$ is contained in $\Omega'$ and $\overline{\Omega}$. Now, if $\omega \in \tilde{\Omega}$, then $X_{t-1}(\omega)$ is closed, $f_j(\omega, \cdot) \in X_{t-1}(\omega)$ and $f_j(\omega, \cdot) \to f(\omega, \cdot)$. Consequently, $f(\omega, \cdot) \in X_{t-1}(\omega)$ for all $\omega$ in an $\mathcal{F}_{t-1}$-measurable set of full measure. By virtue of Proposition 3.1, the function $w(\omega)$, coinciding with $f(\omega, x_t(\omega))$ (a.s.), belongs to the class $\mathcal{X}_t$. □

In the next proposition, *we assume that $H_{t-1}(\omega)$ is a cone.* The result below implies that assumptions (X.2) and ($\mathcal{X}$.2) are equivalent.

**Proposition 7.2.** *The set*

$$\Omega_c := \{\omega \in \Omega : X_{t-1}(\omega) - L^0_+(B, \mathcal{B}, P_\omega) \text{ is a convex subset of } L^0(B, \mathcal{B}, P_\omega)\}$$

*is measurable with respect to $\mathcal{F}^{\mathbb{P}}_{t-1}$. The following conditions are equivalent:*

*(c.1) $\mathcal{X}_t - L^0_+(\Omega, \mathcal{F}_t, \mathbb{P})$ is a convex subset of $L^0(\Omega, \mathcal{F}_t, \mathbb{P})$;*

*(c.2) $\mathbb{P}(\Omega_c) = 1$.*

*Proof.* Since $H_{t-1}(\omega)$ is a cone, the set $Z_{t-1}(\omega)$ is not convex if and only if there exist $a$ and $a'$ such that

$$a, a' \in H_{t-1}(\omega),\ v_a(\cdot) + v_{a'}(\cdot) \notin Z_{t-1}(\omega). \tag{7.3}$$

Consider the set $\Delta$ consisting of those $(\omega, a, a')$ for which $\omega \in \Omega_H$ and relations (7.3) hold. Then $\text{pr}_\Omega \Delta$ coincides with the intersection of $\Omega_H$ and

$\Omega \backslash \Omega_c$. We have $\Omega_H \in \mathcal{F}_{t-1}$ and $\mathbb{P}(\Omega_H) = 1$; consequently, to prove the $\mathcal{F}^{\mathbb{P}}_{t-1}$-measurability of $\Omega_c$ it is sufficient to verify that $\mathrm{pr}_\Omega \Delta \in \mathcal{F}^{\mathbb{P}}_{t-1}$. By virtue of Lemma 7.2, we have $v_a(\cdot) + v_{a'}(\cdot) \notin Z_{t-1}(\omega)$ if and only if $\inf_{k,j} \rho^N_{k,j}(\omega, v_a(\cdot) + v_{a'}(\cdot)) > 0$ for each $N \in \{1, 2, ...\}$. By using this and the definition of $\Delta$, we conclude that $\Delta \in \mathcal{F}_{t-1} \times \mathcal{B} \times \mathcal{B}$, and so $\mathrm{pr}_\Omega \Delta \in \mathcal{F}^{\mathbb{P}}_{t-1}$ by virtue of Theorem AI.2.

*(c.1)⇒(c.2).* Suppose $\mathbb{P}(\Omega_c) < 1$. Then $\mathbb{P}(\mathrm{pr}_\Omega \Delta) > 0$. By applying Theorem AI.2 to the set $\Delta$ defined above, we construct a set $\Omega_\Delta \in \mathcal{F}_{t-1}$ and an $\mathcal{F}_{t-1}$-measurable mapping $\omega \mapsto (h(\omega), h'(\omega))$ such that $\Omega_\Delta \subseteq \mathrm{pr}_\Omega \Delta$, $\mathbb{P}(\Omega_\Delta) = \mathbb{P}(\mathrm{pr}_\Omega \Delta) > 0$, and, for each $\omega \in \Omega_\Delta$ we have $h(\omega), h'(\omega) \in H_{t-1}(\omega)$ and $v(\omega, \cdot) + v'(\omega, \cdot) \notin Z_{t-1}(\omega)$, where $v(\omega, b) = h(\omega)b$ and $v'(\omega, b) = h'(\omega)b$. Let us redefine $h, h'$ as 0 outside $\Omega_\Delta$. Then the random variables $w(\omega) := v(\omega, x_t(\omega))$ and $w'(\omega) := v'(\omega, x_t(\omega))$ belong to the cone $\mathcal{X}_t$, however, the sum $w + w'$ does not belong to $\mathcal{X}_t - L^0_+(\Omega, \mathcal{F}_t, \mathbb{P})$. Indeed, suppose the contrary: $h(\omega)x_t(\omega) + h'(\omega)x_t(\omega) \le h''(\omega)x_t(\omega)$ (a.s.), where $h'' \in \mathcal{H}_{t-1}$. The last inequality implies $P_\omega\{b : h(\omega)b + h'(\omega)b \le h''(\omega)b\} = 1$ for $\mathbb{P}$-almost all $\omega$. Consequently, $v(\omega, \cdot) + v'(\omega, \cdot) \in Z_{t-1}(\omega)$ for $\mathbb{P}$-almost all $\omega$, which is a contradiction.

*(c.2)⇒(c.1).* It suffices to show that, for any $h, h' \in \mathcal{H}_{t-1}$, there is $h'' \in \mathcal{H}_{t-1}$ satisfying $hx_t + h'x_t \le h''x_t$ (a.s.). Let $\check{\Omega} \in \mathcal{F}_{t-1}$ be a set such that $\mathbb{P}(\check{\Omega}) = 1$ and $h(\omega), h'(\omega) \in H_{t-1}(\omega)$ for $\omega \in \check{\Omega}$. Since $\mathbb{P}(\Omega_c) = 1$, we may assume without loss of generality that $\check{\Omega} \subseteq \Omega_c$. By the definition of $\Omega_c$, for each $\omega \in \Omega_c$, there exists a vector $a \in H_{t-1}(\omega)$ possessing the following property:

$$P_\omega\{b : h(\omega)b + h'(\omega)b \le ab\} = 1. \tag{7.4}$$

By applying Theorem AI.2 to the set $\check{\Delta}$ of $(\omega, a)$ satisfying $\omega \in \check{\Omega}$, $a \in H_{t-1}(\omega)$ and (7.4), we construct a function $h''(\cdot) \in \mathcal{H}_{t-1}$ for which $P_\omega\{b : h(\omega)b + h'(\omega)b \le h''(\omega)b\} = 1$ (a.s.). This yields $hx_t + h'x_t \le h''x_t$ (a.s.). □

**Proposition 7.3.** *Condition* (V) *holds for all* $\omega$*, except for an* $\mathcal{F}_t$-

*measurable set of measure zero, if and only if requirement ($\mathcal{V}$) is fulfilled.*

*Proof.* Denote by $e_i$ the vector in $R^d$ whose coordinates are equal to zero, except for the $i$th coordinate which is equal to one. The mapping $a \mapsto v_a(b)$ transforms the basis $\{e_1, ..., e_d\}$ of $R^d$ into the set $\{v^i(b) = b^i,\ i = 1, 2, ..., d\}$ of elements of $L^0(B, \mathcal{B}, P_\omega)$ that is a basis of $V_t(\omega)$. Consequently, the set $\Omega^V$ of those $\omega \in \Omega$ for which $\dim V_t(\omega) < d$ can be represented as $\mathrm{pr}_\Omega \Delta^V$, where

$$(7.5) \qquad \Delta^V := \{(\omega, r^1, ..., r^d) \in \Omega \times (R^d \backslash \{0\}) : E_\omega |r^1 b^1 + ... + r^d b^d| = 0\}.$$

By virtue of Theorem AI.2, $\Omega^V = \mathrm{pr}_\Omega \Delta^V \in \mathcal{F}^{\mathbb{P}}_{t-1}$. Therefore condition (V) holds for all $\omega$, except for an $\mathcal{F}_t$-measurable set of measure zero, if and only if $\mathbb{P}(\Omega^V) = 0$.

Assume $\mathbb{P}(\Omega^V) > 0$. Then, by virtue of Theorem AI.2, there is an $\mathcal{F}_{t-1}$-measurable mapping $g(\omega) = (g^1(\omega), ..., g^d(\omega))$ and a set $\Omega^g \in \mathcal{F}_{t-1}$ such that $\mathbb{P}(\Omega^g) > 0$ and $(\omega, g(\omega)) \in \Delta^V$ for $\omega \in \Omega^g$. By redefining $g(\omega)$ as 0 outside $\Omega^g$, we obtain $\mathbb{P}\{g \neq 0\} > 0$ and

$$(7.6) \qquad \mathbb{E}|g x_{t+1}| = \mathbb{E} E_\omega |g^1(\omega) b^1 + ... + g^d(\omega) b^d| = 0,$$

which contradicts ($\mathcal{V}$).

Suppose ($\mathcal{V}$) does not hold: there exists $g \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $g x_{t+1} = 0$ (a.s.) and $\mathbb{P}\{g \neq 0\} > 0$. Then, by virtue of (7.5) and (7.6), $\{\omega : g(\omega) \neq 0\} \subseteq \mathrm{pr}_\Omega \Delta^V = \Omega^V$, and so $\mathbb{P}(\Omega^V) > 0$. □

We now provide equivalent definitions of the class $\Lambda_t(k)$. The proposition below gives necessary and sufficient conditions for a class of equivalent random variables measurable with respect to the $\sigma$-algebra $\mathcal{F}_{t-1} \vee \sigma(x_t)$ to contain a representative belonging to $\Lambda_t(k)$.

**Proposition 7.4.** *Let $\lambda$ be a random variable measurable with respect to $\mathcal{F}_{t-1} \vee \sigma(x_t)$. Then the following assertions are equivalent:*

*($\lambda$.1) There is a random variable $\lambda' \in \Lambda_t(k)$ such that $\lambda = \lambda'$ (a.s.).*

*($\lambda$.2) The conditional distribution of $\lambda(\omega)$ given $\mathcal{F}_{t-1}$ is concentrated (for $\mathbb{P}$-almost all $\omega$) on some finite set in $(0, +\infty)$ containing not more than $k$ elements.*

*($\lambda$.3) There exist strictly positive $\mathcal{F}_{t-1}$-measurable random variables* $c_1(\omega), ...,$ $c_k(\omega)$ such that

$$\mathbb{P}\{\omega : \lambda(\omega) \in \{c_1(\omega)\} \cup ... \cup \{c_k(\omega)\}\} = 1. \tag{7.7}$$

*Proof.* Let $\Pi_\omega(dr)$ denote the conditional distribution of $\lambda(\omega)$ given $\mathcal{F}_{t-1}$.

*($\lambda$.1)⇒($\lambda$.2).* The conditional distributions of $\lambda$ and $\lambda'$ coincide a.s., and so we may suppose without loss of generality that $\lambda(\omega) = \lambda'(\omega)$ for all $\omega$. Assume that requirements (2.4), (f.1) and (f.2) are satisfied and denote $C(\omega) := \{c_1(\omega)\} \cup ... \cup \{c_k(\omega)\}$. Then we have

$$\mathbb{E}\Pi_\omega(C(\omega)) = \mathbb{P}\{\lambda(\omega) \in C(\omega)\} = \mathbb{P}\{f(\omega, x_t(\omega)) \in C(\omega)\} =$$

$$\mathbb{E}\Pi_\omega\{f(\omega, b) \in C(\omega)\} = 1. \tag{7.8}$$

Consequently, $\Pi_\omega(C(\omega)) = 1$ (a.s.).

*($\lambda$.2)⇒($\lambda$.3).* Consider the set $\Delta^\lambda$ of $(\omega, r_1, ..., r_k)$ such that $\omega \in \Omega$, $r_1, ..., r_k \in (0, +\infty)$, and $\Pi_\omega(\{r_1\} \cup ... \cup \{r_k\}) = 1$. We have $\Delta^\lambda \in \mathcal{F}_{t-1} \times \mathcal{B}(R^k)$, and so, by virtue of Theorem AI.2, $\text{pr}_\Omega \Delta^\lambda \in \mathcal{F}^{\mathbb{P}}_{t-1}$, $\mathbb{P}(\text{pr}_\Omega \Delta^\lambda) = 1$, and there exist $\mathcal{F}_{t-1}$-measurable $c_1(\omega), ..., c_k(\omega) > 0$ for which $\Pi_\omega(C(\omega)) = 1$ (a.s.), where $C(\omega) = \{c_1(\omega)\} \cup ... \cup \{c_k(\omega)\}$. Consequently, $\mathbb{P}\{\lambda(\omega) \in C(\omega)\} = \mathbb{E}\Pi_\omega(C(\omega)) = 1$, which proves ($\lambda$.3).

*($\lambda$.3)⇒($\lambda$.1).* Since $\lambda$ is $\mathcal{F}_{t-1} \vee \sigma(x_t)$-measurable, we have $\lambda(\omega) = f(\omega, x_t(\omega))$ for some $\mathcal{F}_{t-1} \times \mathcal{B}$-measurable $f(\omega, b)$. Then the third and the second equalities in (7.8) hold, from which we obtain, in view of (7.7), that $\Pi_\omega\{f(\omega, b) \in C(\omega)\} = 1$ (a.s.). Define $f'(\omega, b) = f(\omega, b)$ if $f(\omega, b) \in C(\omega)$ and $f'(\omega, b) = c_1(\omega)$ otherwise. Put $\lambda'(\omega) = f'(\omega, x_t(\omega))$. Then $\lambda' = \lambda$ (a.s.) and $\lambda' \in \Lambda_t(k)$. □

## Appendix I. Standard spaces, conditional distributions and measurable choice

A measurable space $(B, \mathcal{B})$ is called *standard* if it is isomorphic to a Borel subset of a complete separable metric space with Borel measurable structure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(B, \mathcal{B})$ a standard measurable space.

**Theorem AI.1.** *For each $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$ and each measurable mapping $x : (\Omega, \mathcal{F}) \to (B, \mathcal{B})$, there exists a function $P_\omega(A)$ of $\omega \in \Omega$ and $A \in \mathcal{B}$ satisfying the following conditions:*

*(c.1) for each $\omega \in \Omega$, $P_\omega(A)$ is a probability measure on $\mathcal{B}$;*

*(c.2) for each $A \in \mathcal{B}$, $P_\omega(A)$ is a $\mathcal{G}$-measurable function of $\omega$;*

*(c.3) for each non-negative $\mathcal{G} \times \mathcal{B}$-measurable function $f(\omega, b)$, we have*

$$\text{(AI.1)} \qquad \mathbb{E}[f(\omega, x(\omega))|\mathcal{G}] = E_\omega f(\omega, \cdot) \ [:= \int P_\omega(db) f(\omega, b) \ ] \ \text{(a.s.)}.$$

Note that if formula (AI.1) is valid for all nonnegative $\mathcal{G} \times \mathcal{B}$-measurable functions $f(\omega, b)$, it extends to all $\mathcal{G} \times \mathcal{B}$-measurable functions $f(\omega, b)$ for which, with probability 1, at least one of the random variables $\mathbb{E}\{\max[0, f(\omega, x(\omega))]|\mathcal{G}\}$ and $\mathbb{E}\{\min[0, f(\omega, x(\omega))]|\mathcal{G}\}$ is finite. If conditions (c.1)–(c.3) hold, then the probability measure $P_\omega(A)$ depending on $\omega$ is called the *conditional distribution of the random element $x(\omega)$ given the $\sigma$-algebra $\mathcal{G}$*. The conditional distribution is defined uniquely up to $\mathbb{P}$-equivalence. For a proof of Theorem AI.1 see, e.g., Arkin and Evstigneev (1987), Appendix II.

Let $(\Omega, \mathcal{F})$ be a measurable space. For each probability measure $\mathbb{P}$ on $\mathcal{F}$, we denote by $\mathcal{F}^{\mathbb{P}}$ the *completion* of $\mathcal{F}$ with respect to $\mathbb{P}$, i.e., the $\sigma$-algebra of those sets $\Gamma \subseteq \Omega$ for which there exist $\Gamma_1, \Gamma_2 \in \mathcal{F}$ such that $\Gamma_1 \subseteq \Gamma \subseteq \Gamma_2$ and $\mathbb{P}(\Gamma_1) = \mathbb{P}(\Gamma_2)$. The measure $\mathbb{P}$ extends uniquely from $\mathcal{F}$ to $\mathcal{F}^{\mathbb{P}}$.

In Theorem AI.2 below, $(B, \mathcal{B})$ is a standard space, $(\Omega, \mathcal{F})$ an arbitrary measurable space and $\mathbb{P}$ a probability on $\mathcal{F}$. For a set $\Delta \subseteq \Omega \times B$, we write $\Delta(\omega) = \{b \in B : (\omega, b) \in \Delta\}$.

**Theorem AI.2.** *For each set* $\Delta \in \mathcal{F} \times \mathcal{B}$*, the projection* $\mathrm{pr}_\Omega \Delta$ *is* $\mathcal{F}^{\mathbb{P}}$*-measurable. There exists a set* $\Omega' \in \mathcal{F}$ *and a measurable mapping* $\xi : (\Omega, \mathcal{F}) \to (B, \mathcal{B})$ *such that* $\Omega' \subseteq \mathrm{pr}_\Omega \Delta$*,* $\mathbb{P}(\Omega') = \mathbb{P}(\mathrm{pr}_\Omega \Delta)$ *and* $(\omega, \xi(\omega)) \in \Delta$ *for all* $\omega \in \Omega'$*. Moreover, there exists a countable family of measurable mappings* $\xi_i : (\Omega, \mathcal{F}) \to (B, \mathcal{B})$ $(i = 1, 2, ...)$ *such that, for all* $\omega \in \Omega'$*, the sequence* $\{\xi_i(\omega)\}_{i=1}^{\infty}$ *is dense in* $\Delta(\omega)$*.*

If $\Omega'$ is a subset of $\Omega$ and $\xi : \Omega \to B$ is a mapping satisfying $\xi(\omega) \in \Delta(\omega)$ for $\omega \in \Omega'$, then we say that $\xi$ is a *selector of the multivalued mapping* $\omega \mapsto \Delta(\omega)$ *on the set* $\Omega'$. By virtue of Theorem AI.2, for each $\Delta \in \mathcal{F} \times \mathcal{B}$, there exists an $\mathcal{F}$-measurable mapping $\xi(\omega)$ that is a selector of $\omega \mapsto \Delta(\omega)$ on some $\mathcal{F}$-measurable subset $\Omega'$ of $\mathrm{pr}_\Omega \Delta$ having the same measure as $\mathrm{pr}_\Omega \Delta$. A proof of Theorem AI.2 is given, e.g., in Arkin and Evstigneev (1987), Appendix I.

**Theorem AI.3.** *For any standard space* $(B, \mathcal{B})$*, there exists a real-valued function* $\psi(r, b)$ *of* $r \in [0, 1]$ *and* $b \in B$ *measurable with respect to* $\mathcal{B}([0, 1]) \times \mathcal{B}$ *and possessing the following property. For each finite measure* $P$ *on* $\mathcal{B}$ *and for each* $\mathcal{B}$*-measurable real-valued function* $f(b)$*, there exists* $r \in [0, 1]$ *such that* $\psi(r, b) = f(b)$ *for* $P$*-almost all* $b \in B$*.*

This result establishes the existence of a "universal" jointly measurable function $\psi(r, b)$ on $[0, 1] \times B$ parametrizing all equivalence classes of measurable functions on $B$ with respect to all finite measures: any such class contains a representative of the form $\psi(r, \cdot)$, where $r$ is some number in $[0, 1]$ (compare with Natanson 1961, Chapter 15, Section 3, Theorem 4).

*Proof of Theorem AI.3.* Any standard space is isomorphic either to a finite or countable set with the $\sigma$-algebra of all its subsets, or to the segment $I := [0, 1]$ equipped with the Borel measurable structure (see, e.g., Dynkin and Yushkevich 1979, Appendix I). If $B = \{b_j\}_{i \in J}$ is finite or countable, we can define $\psi(r, b_j) = \gamma_j(r)$, where $\gamma(r) = (\gamma_j(r))_{j \in J}$ is a mapping of $I$ into $\Pi_{j \in J} R^1$ defining a Borelian isomorphism of the two spaces. Suppose $B$ is uncountable. Then we may assume without loss of generality that

$(B, \mathcal{B}) = (I, \mathcal{B}(I))$. Let $(C, \mathcal{C})$ be the space of continuous functions on $I$ and $\mathcal{C}$ the Borel $\sigma$-algebra on $C$ generated by the uniform metric. Consider the standard space $\mathbf{C} := C \times C \times ...$ endowed with the product measurable structure and, for each $\mathbf{c} = (c_1, c_2, ...) \in \mathbf{C}$, define $\phi_*(\mathbf{c}, b) = \liminf c_i(b)$, $\phi^*(\mathbf{c}, b) = \limsup c_i(b)$ $(b \in B = I)$ and

$$\phi(\mathbf{c}, b) = \begin{cases} \phi_*(\mathbf{c}, b), & \text{if } \phi_*(\mathbf{c}, b) = \phi^*(\mathbf{c}, b) \in (-\infty, +\infty), \\ 0, & \text{otherwise.} \end{cases}$$

For each $i$, the function $\phi_i(\mathbf{c}, b) := c_i(b)$ is jointly measurable in $\mathbf{c} = (c_1, c_2, ...)$ and $b$ (since $c_i(b)$ is continuous in $c_i \in C$ and continuous in $b \in B$), and so $\phi(\mathbf{c}, b)$ is jointly measurable. Let $P$ be a finite measure on $\mathcal{B} = \mathcal{B}(I)$ and $f(b)$ a function measurable with respect to $\mathcal{B}$. Consider a sequence of continuous functions $\mathbf{c} = (c_i)$ and a Borel set $B_0$ with $P(B_0) = 0$ such that $c_i(b) \to f(b)$ for all $b \in B_1 := B \setminus B_0$. Then $f(b) = \phi(\mathbf{c}, b)$ for $b \in B_1$. To complete the proof it remains to define $\psi(r, b)$ as $\phi(\delta(r), b)$, where $\delta : r \to \mathbf{C}$ is a Borelian isomorphism. $\square$

## Appendix II. A corollary to Carathéodory's and Lyapounov's theorems

This appendix contains a result, Proposition AII.1 below, that is obtained by combining the Carathéodory theorem in convex analysis (see, e.g., Rockafellar 1970, Chapter IV) and the Lyapounov theorem on the convexity of the range of an atomless vector measure (e.g., Ioffe and Tihomirov 1979, Chapter 8). For similar results see Artstein (1980) and references therein.

**Proposition AII.1.** *Let $(B, \mathcal{B}, P)$ be a probability space and let $x(b)$ be a vector function in $L^1(B, \mathcal{B}, P, R^d)$ $(d \geq 1)$. Then, for each bounded measurable function $\alpha : B \to [r_*, \infty)$ $(r_* \in R^1)$, there exists a measurable function $\beta : B \to [r_*, \infty)$, taking on not more than $d+1$ values, such that*

$$E\alpha x = E\beta x.$$

*If the measure $P$ is atomless, then one can select $\beta$ so that, for each $b$, $\beta(b) \in \{r_*, r^*\}$ where $r^* :=$ess sup $\alpha$ (hence $\beta$ takes on at most two values).*

The proof of the above proposition relies upon the following lemma.

**Lemma AII.1.** *Let $J$ be a finite or countable set, $w_j, j \in J$, a family of vectors in $R^d$, and $\gamma_j, j \in J$, a family of real numbers satisfying $\gamma_j \geq 0$, $\sum \gamma_j < \infty$, and $\sum \gamma_j |w_j| < \infty$. Then there exist numbers $\delta_j \geq 0, j \in J$, such that not more than $d$ of them are not equal to zero and*

$$\sum_{j \in J} \delta_j w_j = \sum_{j \in J} \gamma_j w_j.$$

*Proof.* We may assume without loss of generality that $\gamma_j > 0$ for all $j$ and $\sum \gamma_j = 1$. Observe that the vector $w := \sum \gamma_j w_j$ is a convex combination of a finite number of $w_j$. This is clear if $d = 1$. Suppose this assertion is established for $d-1$ and let us prove it for $d$. Assume the contrary: $w \notin W$, where $W$ is the convex hull of $w_j$, $j \in J$. By the separation theorem for convex sets in $R^d$, there exist a linear functional $l$ such that $l(w) \geq l(w_j)$ for all $j$. We have

$$0 = l(w - \sum \gamma_j w_j) = \sum \gamma_j l(w - w_j),\ l(w - w_j) \geq 0,$$

which yields $l(w - w_j) = 0$ for all $j$. Consequently, all the points $w_j$ are contained in the hyperplane $\{v : l(v) = l(w)\}$ of dimension $d-1$, and so $w \in W$ by virtue of the induction hypothesis. This contradicts the assumption. It now remains to refer to the Carathéodory theorem (see, e.g., Rockafellar 1970, Chapter IV, Corollary 17.1.2), from which it follows that if $w$ is a non-negative linear combination of a finite number of vectors $w_j \in R^d$, then $w$ is a non-negative linear combination of at most $d$ of the vectors $w_j$. $\square$

*Proof of Proposition AII.1.* The assertion concerning the atomless case is a direct consequence (and in fact an equivalent form) of the Lyapounov theorem; it is proved, e.g., in Ioffe and Tihomirov (1979), Section 8.2, Theorems 1 and 2.

When dealing with the general case, we will assume $r_* = 0$: this does not lead to a loss in generality. Consider disjoint measurable sets $B_0 \subseteq B$ and $B_i \subseteq B$, $i \in I \subseteq \{1, 2, ...\}$, such that $B_i, i \in I$, are atoms of the measure $P$, $B_0$ does not contain atoms of $P$, and $B = \cup_{j \in J} B_j$, where $J = I \cup \{0\}$. We will assume that at least one atom exists ($I \neq \emptyset$), while the set $B_0$ might be empty. By using the result concerning the atomless case, we obtain $E\alpha x \chi_{B_0} = E r \chi_\Gamma x$ for some measurable set $\Gamma \subseteq B_0$ and some number $r \geq 0$. Let $r_i \geq 0$, $i \in I$, be numbers and $x_i \in R^d$, $i \in I$, vectors such that $\alpha(b) = r_i$ and $x(b) = x_i$ for almost all $b \in B_i$, $i \in I$. Put $r_0 = r/P(B_0)$, and $x_0 = E\chi_\Gamma x$ (if $P(B_0) = 0$, we define $r_0$ as 0). Then

$$E\alpha x = E r \chi_\Gamma x + \sum_{i \in I} E\alpha x \chi_{B_i} = r_0 x_0 P(B_0) + \sum_{i \in I} r_i x_i P(B_i) = \sum_{i \in J} r_j x_j P(B_j).$$

Since $E\alpha|x| < \infty$, we have $\sum_{i \in J} r_j |x_j| P(B_j) < \infty$, and since $\alpha$ is bounded, we have $\sum_{j \in J} r_j P(B_j) < \infty$. By applying Lemma AII.1 with $\gamma_j = r_j P(B_j)$ and $w_j = x_j$, we find numbers $\delta_j \geq 0, j \in J$, such that not more than $d$ of $\delta_j$

are not equal to zero and

$$E\alpha x = \sum_{i\in J} r_j x_j P(B_j) = \sum_{j\in J} \gamma_j w_j = \sum_{j\in J} \delta_j w_j.$$

We define

$$\beta(b) = \begin{cases} \delta_i/P(B_i), & \text{if } b \in B_i,\ i \in I, \\ \delta_0 \chi_\Gamma(b), & \text{if } b \in B_0. \end{cases}$$

The function $\beta(b)$ takes on not more than $d+1$ values (either 0, or $\delta_0$, or $\delta_i/P(B_i),\ i \in I$), and we have

$$E\alpha x = \sum_{j\in J} \delta_j w_j = \delta_0 x_0 + \sum_{i\in I} \delta_i x_i =$$

$$E\delta_0 \chi_\Gamma x + \sum_{i\in I} \frac{\delta_i}{P(B_i)} x_i P(B_i) = E\beta x.$$

□

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