ON THE GALACTIC LAW OF ROTATION

L. Mestel

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Summary

A uniformly rotating spherical cloud of nearly uniform density is supposed to contract into a thin disk, with each element of gas conserving its angular momentum. It is shown that if the cloud density is strictly uniform, there exists an axially symmetric disk-like state of centrifugal equilibrium with uniform angular velocity Ω; while if the cloud density decreases slightly (according to a specific law) from centre to surface, disk-like equilibrium is possible with a much more concentrated mass distribution, and with a uniform *rotational* velocity V. The difference between the two cloud density laws is slight, and it is probable that any nearly uniform sphere in uniform rotation can generate (with the same angular momentum constraint) two disks, with rotation laws that approximate respectively to $\Omega = \text{constant}$ and $V = \text{constant}$. The two models are regarded as smoothed-out, zero-order approximations to the barred and the normal spirals: which disk is formed from the collapsing primeval sphere depends on the relative strengths of the perturbations present.

1. *Introduction*.—Recent measurements by radio astronomers of the 21 cm line have greatly extended our knowledge of the rotation of the gas in our Galaxy and in M31 (**1**, **2**). Simultaneously, they have shown that the amount of mass present as atomic hydrogen is only a few per cent of the mass condensed into stars. Unless the amount of molecular hydrogen should turn out to be much greater than the atomic, it is legitimate to ignore the gas as a source of gravitation. Although the magnetic force on the gas may very well be comparable with the self-gravitation of a local condensation such as a spiral arm (**3**), it is almost certainly small compared with the overall gravitational force. Beyond three kiloparsecs from the centre of the Milky Way, the inertia of any radial motion is certainly small compared with the centrifugal force. Equally, the turbulent and thermal motions of the gas and the random motions of the stars make small contributions to momentum balance. We may therefore regard the centrifugal force acting on the gas as a straightforward measure of the radial gravitational field of the stars in or near the galactic plane. Observational support comes from G. and L. Münch (**4**), who report only slight discrepancies between the rotation field of the stars and that of the gas.

A theory of the rotation field of the gas is therefore essentially a theory of the mass distribution of the flat stellar sub-systems. In this paper we are not concerned with constructing a detailed accurate model, such as that of Schmidt for our own Galaxy (**5**). Rather we are interested in seeing whether a simple physical picture of the formation of the flat sub-systems can account for a salient feature of the rotation laws of both galaxies—the wide range over which the rotational *velocity* is roughly uniform. According to Kwee *et al.* (**1**), a value of 200 km/sec

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is correct to within 12 or 13 per cent over the range 1·9–8·2 kpc; between 2·93 and 8·2 kpc, a value of 210 km/sec is correct to about 7 per cent. In fact, the rotational velocity begins to drop off to zero only when the central distance becomes comparable with the thickness of the stellar disk. The suggested revision of the solar distance from Baade's 8·2 kpc to 10 kpc (**6**) does not change this qualitative feature, which appears even more strikingly in M31 (**2**, Fig. 11).

Earlier indications of such a rotation law were in Mayall's measurements (**7**) of the velocities of emission patches in M31. Schwarzschild (**8**) pointed out that they were more consistent with a fairly constant circular velocity—the transition to solid body rotation occurring at less than 1 kpc from the centre—than with the earlier hypothesis of solid body rotation over the bulk of the disk. He also showed that the highly concentrated mass distribution required to yield such a rotation law is consistent with a constant mass to light ratio over the disk—a plausible result, if the disk stars are essentially one stellar population.

More recently, optical measurements of the velocity fields of a number of other disk-like galaxies have been made by the Burbidges and their collaborators (e.g. **9**). A tentative general conclusion from the results to date (**10**) is that the disks fall roughly into two classes; one class with rotation fields similar to those of the Milky Way and M31, and the other with approximate solid body rotation. If, as we shall see, the same picture yields both of these qualitatively distinct models, the question of their mutual stability at once raises itself.

We study the following simple model. A primeval gaseous sphere, of roughly uniform density ρ, and rotating with uniform angular velocity Ω, contracts under its own gravitation, with each element conserving its angular momentum. In spherical polar coordinates (r, θ, ϕ), with origin at the centre of the cloud and axis parallel to the total angular momentum, the ratio of the centrifugal force to the opposing component of gravity is

$$\frac{\Omega^2 r \sin\theta}{Gm(r)\sin\theta/r^2} = \left[\frac{\Omega^2 r^4}{Gm(r)}\right]\frac{1}{r}, \tag{1}$$

where $m(r)$ is the mass within the sphere r, and G the gravitational constant. Since Ωr^2 is constant under spherical contraction with angular momentum conservation, this ratio increases until contraction normal to the axis is halted. However, the component of gravity parallel to the axis is unreduced, and the cloud will flatten.

For the moment, we assume that no sub-condensations are formed at this stage. The cloud as a whole is pictured as collapsing parallel to the rotation axis, with gravitational energy being transformed into macroscopic kinetic energy, to be systematically dissipated in shocks and then radiated away. There is no further supply of kinetic energy (by hypothesis there are as yet no hot stars to stir up the medium), so the flattening must continue until the geometry approximates to that of a thin disk. As the collapse proceeds, the gravitational field perpendicular to the axis changes, and there is simultaneous lateral motion, again with each element conserving its angular momentum, so that centrifugal balance is maintained. We are interested in the possible equilibrium distributions of mass and angular momentum, given the conditions in the primeval sphere, once a highly flattened structure is reached. The gravitational field is then only weakly dependent on the thickness, and the infinitely thin disk is the correct zero-order approximation.

2. *The equilibrium of the isothermal, self-gravitating disk.*—Although the main results of this paper are derived for infinitely thin disks, for completeness we shall discuss pressure equilibrium parallel to the rotation axis, as well as centrifugal equilibrium. The basic approximation assumes the scale of variation perpendicular to the axis large compared with the effective thickness, so that at any point Poisson's equation reduces to

$$\frac{d^2\phi}{dz^2} = -4\pi G\rho, \tag{2}$$

where ϕ is the gravitational potential, ρ the density, and z is distance from the mid-plane, parallel to the axis. We adopt a simple isothermal pressure $p = c^2\rho$; however, the velocity may be regarded as a turbulent rather than a thermal velocity, or as a random stellar velocity. (Extension is in principle straightforward to cases with c dependent on ρ, or with a magnetic pressure dominant.) Thus the equation of hydrostatic support normal to the plane is

$$\frac{1}{\rho}\frac{dp}{dz} = \frac{c^2}{\rho}\frac{d\rho}{dz} = \frac{d\phi}{dz}. \tag{3}$$

Equations (2) and (3) have been solved by Spitzer (**11**) and Ledoux (**12**). If we define

$$\frac{m}{2} = \int_0^z \rho\, dz, \quad \frac{M}{2} = \int_0^\infty \rho\, dz, \tag{4}$$

we find

$$\rho = \bar{\rho}\left[1 - \left(\frac{m}{M}\right)^2\right] = \bar{\rho}\,\mathrm{sech}^2\left(\frac{z}{\bar{z}}\right), \tag{5}$$

and

$$m = M\tanh\left(\frac{z}{\bar{z}}\right), \tag{6}$$

where

$$\bar{\rho} = \frac{\pi G M^2}{2c^2}, \tag{7}$$

and

$$\bar{z} = \frac{M}{2\bar{\rho}} = \frac{c^2}{\pi G M} = \frac{c^2}{2\pi G\bar{\rho}\bar{z}}. \tag{8}$$

The length $\bar{z}$ is the local semi-thickness of a disk with a local density $\bar{\rho}$, independent of z, and with M the local mass per unit area.

The disk approximation is a good one if $\bar{z} \ll R$, the radius of the disk. The order of M is $\mathfrak{M}/\pi R^2$, where $\mathfrak{M}$ is the total mass; hence by (8) we require

$$\frac{G\mathfrak{M}}{R} \gg c^2. \tag{9}$$

If pressure—whether thermal, turbulent or magnetic—in the primeval sphere is not to be too strong for gravity, $G\mathfrak{M}/R$ must certainly exceed c^2; but if the inequality only just holds when centrifugal force becomes important, then equilibrium in the z-direction will be reached after only a slight flattening, and a disk treatment will be invalid. For flattening to be marked, a primeval sphere of given mass must be sufficiently cool; any initial strongly supersonic turbulence must decay; and any magnetic field must either have an energy density much less than the gravitational, or it must be large-scale and directed roughly parallel to the rotation axis, so that there is little magnetic interference with collapse into a disk.

We now assume that inequality (9) is satisfied so well that the cloud can flatten into a disk which we shall treat as infinitely thin, with an axially symmetric function $M(\varpi')$ representing the mass per unit area at a point distant ϖ' from the centre of rotation. The gravitational potential $\phi(\varpi')$ may then be computed; the equilibrium rotation field is then given by

$$\Omega^2(\varpi') = -\frac{1}{\varpi'}\frac{\partial\phi}{\partial\varpi'}. \tag{10}$$

If M varies with ϖ', so does $\bar{\rho}$, and there is a pressure gradient contribution to (10), but this is smaller than the centrifugal term by the fraction $c^2/\Omega^2\varpi'^2$, and so is negligible in the approximation that ignores the finite thickness of the disk. The most straightforward (though in general not the most convenient) expression for the potential is

$$\phi(\varpi') = 2G\int_0^\pi d\theta \int_0^\infty \frac{M\varpi\, d\varpi}{(\varpi'^2+\varpi^2-2\varpi\varpi'\cos\theta)^{1/2}}. \tag{11}$$

In a disk of finite radius R, the integrand vanishes for $\varpi > R$.

3. *Special cases.*—Consider first the simplest density distribution

$$M = \begin{cases} M_0 = \text{constant} & \varpi \leqslant R_0, \\ 0 & \varpi > R_0. \end{cases} \tag{12}$$

The integral (11) is best dealt with by the transformation

$$\begin{aligned} \varpi\sin\theta &= s\sin\psi, \\ \varpi' - \varpi\cos\theta &= s\cos\psi; \end{aligned} \tag{13}$$

i.e., we use plane polar coordinates centred on the observation point $(\varpi', 0)$. Then

$$\begin{aligned} \phi(\varpi') &= 2GM_0\int_0^\pi d\psi \int_0^{S(R_0)} ds \\ &= 2GM_0\int_0^\pi d\psi\left[\varpi'\cos\psi + (R_0{}^2-\varpi'^2\sin^2\psi)^{1/2}\right] \\ &= 4GM_0R_0\int_0^{\pi/2} d\psi\left[1-\left(\frac{\varpi'}{R_0}\right)^2\sin^2\psi\right]^{1/2}, \end{aligned} \tag{14}$$

and

$$\Omega^2(\varpi') = \frac{4GM_0}{R_0}\int_0^{\pi/2}\frac{\sin^2\psi\, d\psi}{\left[1-\left(\frac{\varpi'}{R_0}\right)^2\sin^2\psi\right]^{1/2}}. \tag{15}$$

For $\omega' \ll R_0$, expansion of the elliptic integral in (15) yields

$$\Omega^2 = \frac{\pi GM_0}{R_0}\left[1+o\left(\frac{\varpi'}{R_0}\right)^2\right] \tag{16}$$

—uniform rotation near the centre. As $\varpi' \to R_0$, ϕ stays finite, but $\partial\phi/\partial\varpi'$ diverges logarithmically: the gravitational pull of an infinitely thin uniform disk is infinite at the edge. At a point ϖ' within the disk, the net attraction is the difference between the inward pull of the matter within ϖ' and the outward pull of the matter beyond ϖ': the two singularities cancel, yielding a finite result.

Any discontinuity in M yields this logarithmic singularity: for Ω to be finite everywhere, M must decrease smoothly to zero at the edge. In particular, the law

$$M = M_0\left[1 - \left(\frac{\varpi'}{R_0}\right)^2\right]^{1/2} \qquad \varpi' \leqslant R_0 \tag{17}$$

is remarkable in yielding a gravitational field that is proportional to ϖ' over the whole of the disk, and so is balanced by a uniform rotation field. This is not easy to show directly from the integral (11). However, the disk (17) is the limit of a uniform spheroid as its minor axis shrinks to zero. A spheroid of density ρ, and of semi-axes R_0, R_0 and $(1-e^2)^{1/2}R_0$, exerts a gravitational force

$$[2\rho R_0(1-e^2)^{1/2}](\pi G/R_0)\{[\sin^{-1}e - e(1-e^2)^{1/2}]/e^3\}\,\varpi'$$

at an equatorial point distant ϖ' from the centre (**13**, p. 64). As $e\to 1$, $\rho\to\infty$, so that $2\rho R_0(1-e^2)^{1/2}\to M_0$, and the angular velocity for equilibrium becomes

$$\Omega_0{}^2 = \frac{\pi^2}{2}\frac{GM_0}{R_0} = \frac{3\pi}{4}\frac{G\mathfrak{M}}{R_0{}^3}, \tag{18}$$

$\mathfrak{M}$ being the mass of the disk (17). A different technique, developed recently by Hunter (**14**), in which Ω and M are expressed as related series of Legendre functions, has (18) as its simplest application.

All the mass distributions that, like (17), are uniform near the centre, have gravitational fields with an important feature in common: in sharp contrast to the field of any *spherically symmetric* mass distribution, the gravitational field at ϖ' *cannot* be computed as if the mass outside ϖ' did not exist. The equipotentials in the region of uniform M must be parallel to the disk: it is only by taking cognizance of the decrease in M as ϖ' approaches R_0 that the equipotentials for the disk (17) can cross the plane, so yielding a ϖ'-component of field*. It is therefore not surprising that R_0 appears in the zero-order term for Ω, so that as $R_0\to\infty$, $\Omega\to 0$.

To elucidate this point, consider the gravitational field at a point ϖ', due to the mass within ϖ' only. At $\varpi'' > \varpi'$, this mass has from (11) a potential ϕ that can be expanded as

$$2G\int_0^{\pi} d\theta \int_0^{\varpi'} \frac{M\varpi\, d\varpi}{\varpi''^2}\left[1 + \sum_{k=1}^{\infty}\left(\frac{\varpi}{\varpi''}\right)^{2k} P_{2k}(\cos\theta)\right], \tag{19}$$

the odd Legendre functions being omitted, as they integrate to zero. The inward gravitational pull at $\varpi'' = \varpi'$ is

$$\left(-\frac{\partial\phi}{\partial\varpi''}\right)_{\varpi''=\varpi'} = G\int_0^{\varpi'}\frac{2\pi M\varpi d\varpi}{\varpi'^2} + \sum_{k=1}^{\infty}\frac{\left[2\pi G(2k+1)\,\alpha_{2k}\left(\int_0^{\varpi'} M\varpi^{2k+1}\,d\varpi\right)\right]}{(\varpi')^{2k+2}}, \tag{20}$$

* I owe this remark to Professor P. A. Sturrock.

where

$$\pi\alpha_{2k} = \int_0^\pi P_{2k}(\cos\theta)\,d\theta = \pi\left[\frac{(2k)!}{2^k(k!)^2}\right]^2 \tag{21}$$

(**15**, p. 1326). The first term in (20) is just the "Keplerian" field—that identical with the field of all the mass within ϖ' concentrated at the centre. The remaining terms form a divergent series; for by Stirling's approximation, $\alpha_{2k} \simeq 1/\pi k$ for large k, and the integral $(2k+1)\int_0^{\varpi'} M\omega^{2k+1}d\omega$ approaches $M(\omega')^{2k+2}$ asymptotically. This is just another example of the logarithmic singularity at a discontinuity in M. However, the mass between ϖ' and the radius R has a potential at a point $\varpi'' < \varpi'$

$$2\pi G\left[\int_{\varpi'}^R M\,d\varpi + \sum_{k=1}^{\infty}\alpha_{2k}\int_{\varpi'}^R M\left(\frac{\varpi''}{\varpi}\right)^{2k} d\varpi\right]; \tag{22}$$

it therefore contributes

$$-2\pi G\sum_{k=1}^{\infty} 2k\,\alpha_{2k}(\varpi')^{2k-1}\int_{\varpi'}^R \frac{M}{\varpi^{2k}}\,d\varpi \tag{23}$$

to the inward field at $\varpi'' = \varpi'$. Again this diverges like $-2GM(\varpi')\log k$, so cancelling the singularity due to the mass within ϖ'. The field at ϖ' due to all the mass is therefore

$$-\frac{\partial\phi}{\partial\varpi'} = G\left[\begin{array}{l}\int_0^{\varpi'} 2\pi M\varpi\,d\varpi/\varpi'^2 \\ +2\pi\sum_{k=1}^{\infty}\alpha_{2k}\left\{\frac{(2k+1)}{(\varpi')^{2k+2}}\int_0^{\varpi'} M\varpi^{2k+1}\,d\varpi - M(\varpi')\right\} \\ +2\pi\sum_{k=1}^{\infty}\alpha_{2k}\left\{M(\varpi') - 2k(\varpi')^{2k-1}\int_{\varpi'}^R \frac{M\,d\varpi}{\varpi^{2k}}\right\}\end{array}\right]. \tag{24}$$

Written this way, the field is the sum of the Keplerian term, and further (finite) contributions from the mass within ϖ', and the mass beyond ϖ'. But although such a break-up of the net field is always possible for any axially-symmetric mass law $M(\varpi')$, it is not in fact useful for the quasi-uniform distributions discussed above. We have seen that these disks have essentially non-Keplerian fields, so that there is nothing to be gained by writing the field as the sum of a Keplerian part, plus the rest. So far from being a perturbation, the series terms in (24) cannot be small compared with the Keplerian term, but must almost cancel it, as is easily verified for the density distributions (12) and (17).

However, the expansion (24) does suggest that there could be a centrally condensed mass distribution, with the gravitational effect of the latter within ϖ' markedly increased and that of the matter beyond correspondingly decreased, so that the *net* field does approximate to the Keplerian type rather than to the flattened spheroidal type. And in fact, if

$$\begin{aligned} M\varpi &= \gamma = \text{constant} \qquad && \varpi \leqslant R, \\ &= 0 && \varpi > R, \end{aligned} \tag{25}$$

then (24) yields

$$-\frac{\partial \phi}{\partial \varpi'} = \frac{2\pi G\gamma}{\varpi'}\left[1+\sum_{k=1}^{\infty}\alpha_{2k}\left(\frac{\varpi'}{R}\right)^{2k}\right]; \tag{26}$$

near the centre the dominant term is just the Keplerian term (independent of R). Note that even for this density law, there is still a marked difference from a spherically symmetric mass distribution. The field at ϖ' of the matter within ϖ' is *not* Keplerian—in fact the Keplerian term in (20) is swamped by the usual logarithmic singularity. But since there is a compensating singularity in the field of the matter beyond ϖ', we may write the total field, as in (24), as the Keplerian term plus two other finite terms. What is peculiar about the density law (25) is that, near the origin, the last two terms in (24) almost cancel, leaving just the Keplerian term.

If $R \to \infty$, the field is given for all ϖ' by

$$-\frac{\partial \phi}{\partial \varpi'} = 2\pi G M(\varpi') = \frac{2\pi G(M'\varpi')}{\varpi'} = \frac{G\mathfrak{M}(\varpi')}{\varpi'^2}, \tag{27}$$

where $\mathfrak{M}(\varpi')$ is the mass within ϖ'. The centrifugal field maintaining equilibrium is such that

$$V^2(\varpi') = \Omega^2(\varpi')\varpi'^2 = -\varpi'\frac{\partial \phi}{\partial \varpi'} = 2\pi G M'\varpi' = \text{constant} \tag{28}$$

—uniform *rotational* velocity replacing the uniform angular velocity of the flattened uniform spheroid. Because of the greater central condensation, the mass $\mathfrak{M}(\varpi') \propto \varpi'$, as compared with the approximate ϖ'^2 law for the quasi-uniform disk. The singularity in Ω results from the idealization of zero thickness; in a real galaxy the disk approximation breaks down at central distances comparable with the thickness, and the rotational velocity V would necessarily drop steeply to zero at the centre. With R finite, the field departs by about 10 per cent from the simple form (27) near $\varpi'/R \simeq 0{\cdot}6$. As $\varpi' \to R$, the logarithmic singularity re-appears, because of the discontinuity in M at $\varpi' = R$; if the constant γ in (25) is replaced by a function $\gamma(\varpi')$, with $\gamma'(0) = 0$ and $\gamma(R) = 0$, the associated rotation field is everywhere finite (cf. Section 6).

4. *The relation between the quasi-uniform disk, the centrally condensed disk, and the uniformly rotating sphere.*—We have noted that observers report a tentative division of the disk-like galaxies into two classes, one with Ω, and one with V approximately uniform. This is itself a sufficient reason for finding simple mathematical representations of the associated mass distributions. However, it appears that the observations can be given a deeper significance, if one adopts a simple picture of the origin of disk-like galaxies, as outlined in the introduction. Suppose, then, the proto-galactic cloud to be a sphere of uniform density and uniform angular velocity, which contracts as a sphere, with detailed conservation of angular momentum, until centrifugal balance is reached: because of our uniformity assumptions, at all points simultaneously, when the radius of the cloud is R_i, and ρ_i and Ω_i are related (cf. (1)) by

$$\Omega_i^2 = \frac{4\pi}{3}G\rho_i. \tag{29}$$

Subsequent flattening into a disk, without any motion perpendicular to the axis of rotation, leads to the area–density law

$$M = 2\rho_i R_i \left[1 - \left(\frac{\varpi}{R_i} \right)^2 \right]^{1/2}, \tag{30}$$

which by (17) and (18) yields a gravitational field in the plane

$$\frac{\pi^2}{2} \frac{G(2\rho_i R_i)}{R_i} \varpi = (\pi^2 G\rho_i)\varpi \tag{31}$$

—greater than the centrifugal field $\Omega_i^2\varpi$. But by an *homologous* contraction towards the axis, with each element conserving its angular momentum, we have

$$\varpi \to k\varpi, \qquad R_i \to kR_i = R_0,$$

$$\Omega_i \to \frac{\Omega_i}{k^2} = \Omega_0, \qquad 2\rho_i R_i \to \frac{2\rho_i R_i}{k^2} \doteq M_0; \tag{32}$$

equilibrium is reached when

$$\frac{\Omega_i^2}{k^4}(k\varpi) = \frac{\pi^2}{2} \frac{G(2\rho_i R_i)}{k^3 R_i} k\varpi, \tag{33}$$

or when $k = \Omega_i^2/\pi^2 G\rho_i = 4/3\pi$, by (29). Thus the uniformly rotating disk (17) is "derivable" from the primeval sphere with detailed conservation of angular momentum, and

$$R_0 = \frac{4}{3\pi} R_i, \quad \Omega_0 = \frac{9\pi^2}{16} \Omega_i, \quad M_0 = \frac{9\pi^2}{8} \rho_i R_i, \qquad \Omega_0^2 = \frac{\pi^2}{2} \frac{GM_0}{R_0}. \tag{34}$$

The disk radius is about two-fifths of the sphere's radius.

Now suppose this disk is subjected to a violently *non-homologous* transformation

$$\varpi \to \varpi' = \frac{k\omega^2}{R_0} \qquad k = \text{constant}, \tag{35}$$

again subject to the constraint that each element of mass conserves its angular momentum. The rotation field is now

$$\Omega'\varpi' = \frac{\Omega_0 \varpi^2}{\varpi'} = \frac{\Omega_0 R_0}{k} = \left(\frac{\pi^2}{2} \frac{GM_0 R_0}{k^2} \right)^{1/2} = \text{constant}. \tag{36}$$

The new density law is

$$M'\varpi' = M\varpi \frac{d\varpi}{d\varpi'} = \frac{M_0 R_0}{2k} \left[1 - \frac{\varpi'}{kR_0} \right]^{1/2}, \tag{37}$$

or

$$M'\varpi' \simeq \frac{M_0 R_0}{2k} = \text{constant} \tag{38}$$

when $\varpi' \ll kR_0$. Thus, at least for the central regions, we reach density and rotation laws of the forms (25) and (28). For centrifugal equilibrium near the centre, we require by (36), (28) and (38), that

$$\frac{\pi^2}{2}\frac{GM_0R_0}{k^2} = (\Omega'\omega')^2 = 2\pi(GM'\omega') = \frac{2\pi GM_0R_0}{2k}, \tag{39}$$

i.e. $k = \pi/2$. The transformation from the uniformly rotating disk to the centrally condensed disk, with V = constant in its inner regions, is therefore

$$\varpi' = \frac{\pi}{2}\frac{\varpi^2}{R_0}, \qquad R' = \frac{\pi}{2}R_0, \tag{40}$$

with

$$\Omega'\varpi' = (2GM_0R_0)^{1/2}, \qquad M'\varpi' \simeq \frac{M_0R_0}{\pi}. \tag{41}$$

We have then, that this type of disk is also "derivable" approximately, under detailed conservation of angular momentum, from the primeval sphere.

The transformation (40) is strictly accurate in the limit with $R_0 \to \infty$, $M_0 \to 0$, and M_0R_0 and Ω_0R_0 finite: the infinite disk of density (25) is derivable from the sphere of zero density and angular velocity, infinite radius, mass and angular momentum, but finite rotational velocity at infinity. With R_0 finite, the vanishing of M' at R' avoids the logarithmic singularity in Ω'. However, the transformation (40) applied to the disk (17) yields strict centrifugal equilibrium with the rotation law (41) only near the centre: to achieve equilibrium over the whole of the centrally condensed disk, a modified transformation $\varpi' = (\pi\varpi^2/2R_0)f(\varpi)$ is required, with $f(\varpi)$ varying slowly from unity at the centre. The associated rotation field must therefore depart from the form (41). However, the rotation law V = constant is so simple, that one is at once tempted to look for that area density $M(\varpi)$ that yields a gravitational field strictly $\propto 1/\varpi$ over the whole radius of a *finite* disk; and hence also for the non-uniform primeval sphere from which this new equilibrium state can be derived. No special cosmogonical significance is to be attached to the strictly uniform sphere; in fact one would like to know how variations in the density-rotation fields of the proto-galaxy affect the density-rotation fields of the disks derived as above. But before more progress can be made, we require a more powerful technique for relating the potential and the area-density.

5. *The use of flattened spheroidal shells to describe the gravitational field.*—The straightforward integral (11) for the potential is not very convenient for further study. We now summarise the spheroidal shell technique, employed for describing observed galaxies by Burbidge, Burbidge and Prendergast (**9**), developed for more theoretical studies by Brandt (**16**), and subsequently by Brandt and Belton (**17**), and by the author. The disk is represented not as a set of annuli, but as a superposition of concentric spheroidal shells of varying density; with finite eccentricity, if account is taken of finite thickness, otherwise of zero eccentricity. The simplicity of the method is due to the vanishing of the field interior to a shell, and to its simple form outside a shell (**13**). Thus the mass $\mathfrak{M}(\omega)$ within any radius ϖ is broken up into a part $\mathfrak{M}_1(\varpi)$ due to all the shells that lie completely within ϖ, and the contribution $\mathfrak{M}_2(\varpi)$ of the shells crossing the equator beyond ϖ. Only the mass $\mathfrak{M}_1(\varpi)$ contributes to the field at ϖ. This does not contradict what was said earlier about Keplerian and non-Keplerian disks. The influence of the mass beyond ϖ shows itself, in this representation, in $\mathfrak{M}_1(\varpi)$

being less than $\mathfrak{M}(\varpi)$. Further, the field due to the "gravitating mass" $\mathfrak{M}_1(\varpi)$ is also non-Keplerian—it is in fact *greater* than the field of a mass $\mathfrak{M}_1(\varpi)$ concentrated at the centre (see (46) and (48) below). In the special case with $M\varpi=$ constant, the gravitational field of the mass $\mathfrak{M}_1(\varpi)$ happens to reduce near the centre to the Keplerian field associated with $\mathfrak{M}(\varpi)$—a result that is hidden in the spheroidal representation, but is an immediate deduction from (24).

Consider then a spheroid of semi-major axis a, eccentricity e and density $\rho(a)$. At a point $\varpi > a$ in its equator it has a potential (**13**, p. 176)

$$\phi = \pi G\rho(a)\, a(1-e^2)^{1/2}\left[\frac{2a}{e}\left(1 - \frac{\varpi^2}{2a^2e^2}\right)\sin^{-1}\frac{ae}{\varpi} + \frac{(\varpi^2-a^2e^2)^{1/2}}{e^2}\right]; \tag{42}$$

hence the potential of a spheroidal shell of density $\rho(a)$, defined by similar spheroids a, $a+da$, is

$$\frac{\partial\phi}{\partial a}da = 4\pi G\rho(a)\,(1-e^2)^{1/2}\,\frac{a}{e}\sin^{-1}\frac{ae}{\varpi}\,da, \tag{43}$$

and the (inward) gravitational field at ϖ due to such a shell is

$$-\frac{\partial}{\partial\varpi}\left(\frac{\partial\phi}{\partial a}da\right) = \frac{4\pi G\rho(a)(1-e^2)^{1/2}\,a^2\,da}{\varpi(\varpi^2-a^2e^2)^{1/2}}. \tag{44}$$

Writing now

$$M_0(a) = \lim_{\substack{e\to 1\\ \rho\to\infty}} \left[2\rho(a)\,a\,(1-e^2)^{1/2}\right] \tag{45}$$

—the central mass per unit area of the (infinitely flattened) spheroid from which the shell a has been taken—we have for the gravitational field at ϖ (due just to the shells lying wholly within ϖ),

$$\frac{2\pi G}{\varpi}\int_0^{\varpi}\frac{M_0(a)a\,da}{(\varpi^2-a^2)^{1/2}}. \tag{46}$$

By elementary integration, the mass of a uniform spheroid that lies within the radius $\varpi(<a)$ is

$$\frac{4\pi}{3}\rho(a)\,(1-e^2)^{1/2}\,[a^3-(a^2-\varpi^2)^{3/2}]. \tag{47}$$

The total mass is $4\pi\,\rho(a)\,(1-e^2)^{1/2}\,a^3/3$; the addition of a shell da increases $\mathfrak{M}_1$ by

$$d\mathfrak{M}_1 = 4\pi\,\rho(a)\,(1-e^2)^{1/2}\,a^2\,da = 2\pi\,M_0(a)\,a\,da \tag{48}$$

in the limit (45). The centrifugal field balancing the gravitational field is given from (46) and (48) by

$$\Omega^2(\varpi)\varpi = \frac{V^2}{\varpi} = \frac{G}{\varpi}\int_0^{\varpi}\frac{(d\mathfrak{M}_1/da)\,da}{(\varpi^2-a^2)^{1/2}}, \tag{49}$$

while the contribution of the "external" spheroid shells to the mass within ϖ is

$$\mathfrak{M}_2(\varpi) = \int_{\varpi}^{\infty}\frac{d\mathfrak{M}_1}{da}\left[1-\left\{1-\left(\frac{\varpi}{a}\right)^2\right\}^{1/2}\right]da, \tag{50}$$

by (45), (47) and (48). The integral equation (49) is of Abelian form, and so can be inverted (16) to yield

$$\mathfrak{M}_1(\varpi) = \frac{2}{\pi G}\int_0^{\varpi} \frac{V^2(a)\,a\,da}{(\varpi^2-a^2)^{1/2}}. \tag{51}$$

Thus with $V(a)$ known over the whole disk, $\mathfrak{M}_1(\varpi)$, $\mathfrak{M}_2(\varpi)$, $\mathfrak{M}(\varpi)$ and $M(\varpi)$ can all be found. On the other hand, if we prescribe the total mass function $\mathfrak{M}(\varpi)$ (or equivalently the area density $M(\varpi)$), we can determine its gravitational field, and hence the rotation field. From (50),

$$\frac{1}{\varpi}\frac{d\mathfrak{M}}{d\varpi} = \int_{\varpi}^{\infty} \frac{d\mathfrak{M}_1/da}{a(a^2-\varpi^2)^{1/2}}\,da, \tag{52}$$

which can again be inverted, to yield

$$\frac{1}{\varpi^2}\frac{d\mathfrak{M}_1}{d\varpi} = \frac{2}{\pi}\int_{\varpi}^{\infty} \frac{-d(d\mathfrak{M}/a\,da)/da}{(a^2-\varpi^2)^{1/2}}\,da$$

$$= -\frac{2}{\pi\varpi}\frac{d}{d\varpi}\int_{\varpi}^{\infty} \frac{d\mathfrak{M}/da}{(a^2-\varpi^2)^{1/2}}\,da; \tag{53}$$

the field is then given by (49).

As a check, consider the infinite disk with V^2 constant. From (51) and (50), $\mathfrak{M}_1(\varpi)=2V^2\varpi/G\pi$ and $\mathfrak{M}_2(\varpi)=(2V^2\varpi/G\pi)(\pi/2-1)$; whence $G(\mathfrak{M}_1+\mathfrak{M}_2)/\varpi^2=V^2/\varpi$ —the "Keplerian" relation previously found valid for this rotation field.

6. *Application to the finite disk with constant rotational velocity.*—Consider now the disk of *finite* radius R', maintained in equilibrium by a uniform rotational velocity V. From (51) and (50),

$$\mathfrak{M}_1(\varpi') = \frac{2V^2}{G\pi}\varpi', \tag{54}$$

$$\mathfrak{M}_2(\varpi') = \frac{2V^2}{G\pi}\left\{R' + \left(\frac{\pi}{2}-1\right)\varpi' - \varpi'\sin^{-1}\left(\frac{\varpi'}{R'}\right) - R'\left[1-\left(\frac{\varpi'}{R'}\right)^2\right]^{1/2}\right\}, \tag{55}$$

$$M(\varpi') = \frac{1}{2\pi\varpi'}\frac{d}{d\varpi'}\left(\mathfrak{M}_1+\mathfrak{M}_2\right) = \frac{V^2}{2\pi G\varpi'}\left\{1-\frac{2}{\pi}\sin^{-1}\left(\frac{\varpi'}{R'}\right)\right\}. \tag{56}$$

The prescription of a finite gravitational field at the edge necessarily yields $M(R')=0$; a discontinuity in M would yield a singularity there (cf. Section 3). Beyond R', $\mathfrak{M}_1=\mathfrak{M}(R')=$ constant, and (49) yields

$$\frac{2}{\pi}V^2R'\left[\frac{\sin^{-1}\left(\frac{R'}{\varpi'}\right)}{R'\varpi'}\right] = \frac{G\mathfrak{M}(R')}{\varpi'^2}\left[\left(\frac{\varpi'}{R'}\right)\sin^{-1}\left(\frac{R'}{\varpi'}\right)\right] \tag{57}$$

for the equatorial field outside the disk; as $\varpi'/R'\to\infty$, (57) reduces to the field of a mass $\mathfrak{M}(R')$ at the origin. When $\varpi'/R'\ll 1$, the field is again Keplerian, but departs somewhat from this simple condition for intermediate radii: e.g. at $\varpi'=R'$, the ratio of the Keplerian to the actual gravitational field is, by (57), $2/\pi$.

It is convenient to take this model, with V strictly constant through the disk, as the paradigm of the class with high central condensation, just as the uniformly rotating disk (17) is the obvious paradigm of those with slowly varying density laws. From the point of view of this paper, this turns out to be very reasonable, for we shall now show that it can be derived (with detailed angular momentum conservation) from a uniformly rotating sphere with a spherically symmetric density $\rho(r)$ that is *nearly* (though not strictly) uniform. Thus imagine the disk (56) subjected to a transformation inverse to (35); specifically, let $\varpi' \to \varpi$, where

$$\varpi' = \frac{R'}{R_i^2}\varpi^2, \qquad R' \to R_i. \tag{58}$$

The rotation becomes uniform, of value

$$\Omega = \frac{V\varpi'}{\varpi^2} = \frac{VR'}{R_i^2}, \tag{59}$$

and the area density

$$M(\varpi) = \frac{V^2 R'}{\pi G R_i^2}\left\{1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\varpi}{R_i}\right)^2\right\}. \tag{60}$$

[This disk is *not* in centrifugal equilibrium—the uniform rotation (59) requires a density law (17).] We can now express $M(\varpi')$ in terms of the density $\rho(r)$ in the hypothetical primeval sphere of radius R_i:

$$\frac{M(\varpi)}{2} = \int_{\varpi}^{R_i} \frac{\rho(r) r\, dr}{(r^2 - \varpi^2)^{1/2}}. \tag{61}$$

This can be inverted like (52) to give

$$\begin{aligned}\rho(r) &= \frac{1}{\pi}\int_r^{R_i} \frac{-dM/dx}{(x^2 - r^2)^{1/2}}\, dx \\ &= \frac{4V^2R'}{\pi^3 G R_i^4}\int_r^{R_i} \frac{x\, dx}{(x^2 - r^2)^{1/2}\,[1 - (x/R_i)^4]^{1/2}} \\ &= \frac{2\sqrt{2}V^2R'}{\pi^3 G R_i^3}\int_0^{\pi/2} \frac{d\psi}{[1 - k^2\sin^2\psi]^{1/2}},\end{aligned} \tag{62}$$

where

$$x^2 = R_i^2 - \sin^2\psi\,(R_i^2 - r^2), \tag{63}$$

and

$$k^2 = \tfrac{1}{2}\left[1 - \left(\frac{r}{R_i}\right)^2\right].$$

Thus $\rho'(0) = 0$, and ρ decreases monotonically from the centre. From tables of complete elliptic integrals (**18**) it is found that the integral in (62) decreases from 1·8541 at $r = 0$ to $\pi/2 = 1{\cdot}5708$ at $r = R_i$—a drop of only about one sixth. Thus a uniformly rotating sphere with a slight negative density gradient can yield a disk with uniform rotational velocity. Since $\Omega^2/(4\pi/3)\,G\rho(r)$ increases with r, spherical collapse of such a primeval spherical cloud will be halted first at the edge: the outer regions will begin to flatten while the centre is still contracting quasi-spherically. However, for definiteness we may fix R_i by supposing centrifugal force to balance gravity at the centre—$\Omega^2/(4\pi/3)\; G\rho(0)=1$: whence by (59) and (62)

$$\frac{R'}{R_i}=\frac{8\sqrt{2}\times 1\cdot 854}{3\pi^2}\simeq\frac{2}{3}. \qquad (64)$$

It is of interest to see some consequences of taking this model as a description of our own Galaxy. From (10), we have for the potential at points in the plane, with $\varpi'<R'$,

$$\phi=-V^2\log\left(\frac{\varpi'}{R'}\right)+\text{constant}. \qquad (65)$$

The potential at the edge of the disk is (cf. (42), (45), (48) and (54))

$$G\int_0^{R'}\frac{d\mathfrak{M}_1}{da}\left[\frac{\sin^{-1}\left(\frac{a}{R'}\right)}{a}\right]da=\frac{2V^2}{\pi}\int_0^1\frac{\sin^{-1}u}{u}\,du\simeq 0\cdot 69V^2 \qquad (66)$$

by expansion and term-by-term integration; whence (65) becomes

$$\phi=V^2\left[0\cdot 69-\log\frac{\varpi'}{R'}\right]. \qquad (67)$$

At any point the velocity of escape is $(2\phi)^{1/2}$; hence an observed velocity of escape αV yields an equation for the position ϖ_S' of the Sun*

$$\log\left(\frac{\varpi_S'}{R'}\right)=0\cdot 69-\frac{\alpha^2}{2}. \qquad (68)$$

The marked asymmetry in the distribution of high velocity stars near the Sun begins at about 63 km/sec relative to the Sun: stars moving with this relative velocity in the direction of the orbital motion of the Sun have just about enough energy to escape (**19**). If we adopt from observation pairs of values for V and ϖ_S', we may estimate ϖ_S'/R', and hence find R', and also (from (54)) the mass $\mathfrak{M}_g$ in the disk.

TABLE I

V km/sec	ϖ_S' kpc	α	ϖ_S'/R'	R' kpc	$\mathfrak{M}_g$ $10^{11}\odot$
200	8·2	4/3	0·82	10	0·6
240	10	5/4	0·91	11	0·95

Both estimates put the Sun near the galactic "edge", especially the revised figures of 240 km/sec and 10 kpc (**6**). It would be of interest to know how sensitive these values are to conditions in the primeval sphere. For example, if the disk were formed from a sphere with a density varying markedly from centre to radius, then the gravitational field would depart from V^2/ϖ' well within the radius of the disk. The consequent increase in the mass and radius of the disk could yield smaller values for ϖ_S'/R'.

* This application was suggested by Professor M. Schwarzschild.

7. *Disks related by detailed equality of angular momentum.*—The difference between the density law (62) and uniform density is small, and certainly not cosmogonically significant. However, there is some imprecision about the model of Section 6, in that the primeval sphere is not in centrifugal balance under uniform rotation. Equally, the disk (60)—derived from (56) by the non-homologous expansion (58)—is not in equilibrium under the uniform rotation which this expansion yields (cf. (59)). Even though the constraint of detailed angular momentum conservation cannot be expected to hold strictly (cf. Section 9), it is of interest to compare two disks representative of the two types, which are related by detailed angular momentum equality. The obvious cases to consider are either the quasi-uniform disk related to the model (56), or the centrally condensed disk related to the uniformly rotating model (17). As this latter model is derivable from the uniformly rotating, uniformly dense sphere, this is the case considered.

We therefore require the form of the function $f(\varpi) \equiv g(\varpi')$ in the modification to the transformation (40)—

$$\varpi' = \frac{\pi}{2}\frac{\varpi^2}{R_0} f(\varpi) = \frac{\pi}{2}\frac{\varpi^2}{R_0} g(\varpi'),$$
$$f(0) = g(0) = 1. \tag{69}$$

The new rotation law, replacing (41), is

$$\Omega' \varpi' = \frac{\Omega_0 \varpi^2}{\varpi'} = \frac{(2GM_0R_0)^{1/2}}{g(\varpi')}, \tag{70}$$

and the new density law

$$M'\varpi' = \frac{M_0R_0}{\pi}\left\{\left(1 - \frac{2\varpi'}{\pi R_0 g(\varpi')}\right)^{1/2}\left(1 - \frac{\varpi'}{g}\frac{dg}{d\varpi'}\right)\Big/ g(\varpi')\right\}. \tag{71}$$

The mass of the original disk within ϖ is, by (45) and (47),

$$\frac{2\pi}{3} M_0 R_0^2 \left[1 - \left\{1 - \left(\frac{\varpi}{R_0}\right)^2\right\}^{3/2}\right], \tag{72}$$

so that the new disk has a mass function

$$\mathfrak{M}(\varpi') = \frac{2\pi}{3} M_0 R_0^2 \left[1 - \left\{1 - \frac{2\varpi'}{\pi R_0 g(\varpi')}\right\}^{3/2}\right]. \tag{73}$$

This mass can be split up by (51) and (50) into

$$\mathfrak{M}_1(\varpi') = \frac{2}{\pi G}\int_0^{\varpi'} \frac{2GM_0R_0}{g^2(a)}\frac{a\,da}{(\varpi'^2 - a^2)^{1/2}}$$
$$= \frac{4}{\pi} M_0 R_0 \int_0^{\varpi'} \frac{a\,da}{g^2(a)\,(\varpi'^2 - a^2)^{1/2}} \tag{74}$$

and

$$\mathfrak{M}_2(\varpi') = \int_{\varpi'}^{R'} \frac{d\mathfrak{M}_1}{da}\left[1 - \left\{1 - \left(\frac{\varpi'}{a}\right)^2\right\}^{1/2}\right] da, \tag{75}$$

with

$$R' = \frac{\pi}{2} R_0 g(R'). \tag{76}$$

Writing

$$\frac{\varpi}{R_0} = u, \qquad \frac{\varpi'}{R_0} = u'$$

$$H(u') = \int_0^{u'} \frac{x\,dx}{g^2(x)\,(u'^2 - x^2)^{1/2}}, \tag{77}$$

we have

$$\mathfrak{M}_1(\varpi') = \frac{4}{\pi} M_0 R_0^2 H(u'), \tag{78}$$

and

$$\mathfrak{M}_2(\varpi') = \frac{4}{\pi} M_0 R_0^2 \int_{u'}^{R'/R_0} \frac{dH}{dx} \left[1 - \left\{ 1 - \left(\frac{u'}{x} \right)^2 \right\}^{1/2} \right] dx. \tag{79}$$

When $\varpi' = R'$, $\mathfrak{M}_1(R') = \mathfrak{M}(R')$, so that by (72) and (78),

$$H\left(\frac{R'}{R_0}\right) = \frac{\pi^2}{6}. \tag{80}$$

Hence the integral equation for $g(u')$ becomes

$$\int_{u'}^{r'} \frac{dH}{dy} \left[1 - \left(\frac{u'}{y} \right)^2 \right]^{1/2} dy = \frac{\pi^2}{6} \left[1 - \frac{2u'}{\pi g(u')} \right]^{3/2}, \tag{81}$$

with

$$r' = \frac{R'}{R_0} = \frac{\pi}{2} g(r'), \tag{82}$$

and H given by (77).

Unfortunately, considerable difficulty is met in trying to solve (81). It can readily be shown that g cannot be analytic at the origin. Even with the help of an electronic computer, no convergent iterative procedure for g has as yet been found. If, as a first guess, we put $H(u') = u'$—corresponding to $g = 1$ in (77)—then (81) yields a new function $g(u')$ that increases from unity at $u' = 0$ to $2r'/\pi = \pi/3 \simeq 1{\cdot}0472$ at $u' = r' = \pi^2/6$ (cf. (80)). This discrepancy of less than 5 per cent encourages one to believe that this zero-order approximation is reasonably good; however, successive iterations yield a series of functions g that depart more and more from unity. Other trial functions yield even smaller discrepancies between the two g functions. For example, if $H(x)$ is taken to be $x + \alpha x^2$, and α chosen as 0·0904, then g from (77) decreases monotonically from unity at the centre to 0·9256 at $r' = 1{\cdot}454$; while g from (81) has the same terminal values, but in between is *less*, though by not more than about 2 per cent. (The analytical difficulties show themselves at once; with $H = x + \alpha x^2$, (77) yields $g = [1 + 4\alpha u'/\pi]^{1/2}$, whereas (82) has $(g - 1)$ behaving like $-u' \log u'$ near the origin.) The fact that the two curves for g have now crossed over does suggest that the correct function g is close to but less than unity, and that the non-dimensional radius r' lies between the first estimate $\pi^2/6$, and the second, 1·454; but as numerical convergence has not yet been demonstrated, the conclusions are still tentative.

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It is of interest to compare the density laws

$$M_1(\varpi') = \frac{V^2}{2\pi G}\frac{\left\{1-\dfrac{\varpi'}{R'}\right\}^{1/2}}{\varpi'}, \tag{83}$$

and

$$M_2(\varpi') = \frac{V^2}{2\pi G}\frac{\left\{1-\dfrac{2}{\pi}\sin^{-1}\left(\dfrac{\varpi'}{R'}\right)\right\}}{\varpi'}: \tag{56}$$

the first resulting from applying the transformation (40) to the uniformly rotating disk (17), and the other being the disk that is in equilibrium under uniform rotational velocity V. M_1 and M_2 agree at the centre and at the edge, diverge in between; the ratio $(M_1-M_2)/M_2$ increases monotonically to about 0·11 at R'. This very moderate discrepancy is a further indication that the equilibrium model (56), with V constant, is a reasonably good approximation to the disk (83), which is in strict equilibrium only at the centre.

8. *Stability*.—The search for the model of Section 7 is not made merely for the sake of completeness. It is true that the model of Section 6, with V constant, is a satisfactory paradigm, and it is hoped to discuss its stability in a subsequent paper. However, there is also the question of the stability of the quasi-uniform disks; in particular, the model (17) that rotates as a rigid body. The time-dependent virial theorem (**20**) for rotating disks is

$$\tfrac{1}{2}\ddot{I} = 2T + v, \tag{84}$$

where for the present problem

$$I = 2\pi\int_0^{R_0} M\varpi^3\,d\varpi \tag{85}$$

—the "moment of inertia about the origin"; T is the kinetic energy, and v the gravitational potential energy. In equilibrium, the whole of T is the rotational kinetic energy:

$$T = \pi\int_0^{R_0} M\Omega^2\varpi^3\,d\varpi, \tag{86}$$

and $2T+v=0$. Further, under a first-order perturbation from an equilibrium state, $\delta(T+v)$ is of the second order. Thus if an element of mass moves from its initial position ϖ to ϖ', where

$$\varpi' = \varpi\,[1-\zeta(\varpi)], \tag{87}$$

we have

$$\tfrac{1}{2}\delta\ddot{I} = \delta T + o\,(\zeta^2). \tag{88}$$

When applied to the uniformly rotating disk, this yields

$$\frac{d^2}{dt^2}\left\{\int_0^{R_0} M\varpi^3\zeta\,d\varpi\right\} = -\Omega_0^{\,2}\int_0^{R_0} M\varpi^3\zeta\,d\varpi. \tag{89}$$

Use has been made here of the conservation by each element of its angular momentum. Thus we have that any normal mode $\zeta \propto e^{i\sigma t}$ is either stable, with $\sigma = \pm \Omega_0$, or unstable, but with

$$\int_0^{R_0} M\varpi^3 \zeta \, d\varpi = 0 : \tag{90}$$

i.e., for a mode to be unstable, not only must $\delta(T+v)$ vanish to the first order—a necessary condition for the zero-order state to be one of equilibrium—but so must δT and therefore also δv individually.

Now consider the relation between the uniformly rotating disk and the centrally condensed disk (related by detailed equality of angular momentum). If the transformation (69) were such that each element *moves in*—i.e. if $g(\varpi')$ decreases sufficiently fast with ϖ'—then we could be sure that whatever instabilities exist for uniformly rotating disks, no small perturbation could transform it into the centrally condensed model: for no perturbation, with ζ positive for all ϖ, can make the integral (90) vanish. Thus the variation of the function g from unity is important: if g were unity, then by (69), elements nearer the centre than $2R_0/\pi$ would move in, those beyond would move out, and no conclusion could be drawn from the above argument as to the *dynamical* (as opposed to the *kinematical*) derivability of the centrally condensed disk from the uniformly rotating disk. And since numerical estimates do suggest that $g(\varpi')$ does not in fact drop much below unity—probably by less than 10 per cent—the virial theorem argument does no more than show that an unstable mode must satisfy the integral relation (90).

Since this work was done, a full treatment of the instabilities of the uniformly rotating disk has been given by Hunter (**14**). He confirms the above conclusions for unstable, axially symmetric modes: the only stable mode is the homologous contraction or expansion, which is clearly an oscillation with the frequency $\sigma = \Omega_0$. The axially symmetric unstable mode with the slowest growth rate is found to have a ζ with just one node; it may be presumed that if this mode were the only one present, the disk would spontaneously transform into the centrally condensed disk. It may readily be verified that if $g(\varpi')$ is less than unity, the energy released by the contracting inner parts exceeds that absorbed by the expanding outer parts, as must be the case if the condensed disk is to be dynamically accessible from the quasi-uniform disk. Further, if $g(\varpi')$ is close to unity, the net energy released is small compared with the rotational or gravitational energy; thus if this energy were not dissipated—e.g. if it stayed as random stellar energy—the resulting pressure would not contribute much to momentum balance, (as is indeed assumed in all the above treatment). Any frictional transport of angular momentum can only assist this instability, as the expanding outer parts gain angular momentum at the expense of the more rapidly rotating inner parts.

However, Hunter also shows that the modes of smaller scale—both axially and non-axially symmetric—have systematically faster growth-rates. In a disk of zero thickness there is no lower limit to the length-scale of unstable perturbations, for the stabilizing effect of the rotation field decreases with decreasing scale. But in a real galaxy, the same pressure effects—whether gaseous or stellar—that halt contraction parallel to the axis will also stabilize perturbations of scale comparable with the thickness (**21**). One may anticipate that once a uniformly rotating disk-like galaxy has been able to form, the fastest-growing perturbations will be those that form rings and spirals, rather than the particular one that yields the centrally condensed disk with a monotonically decreasing density.

9. *The collapse of the primeval sphere.*—We now return to the primeval quasi-uniform sphere, and suppose that centrifugal force has halted the contraction normal to the rotation axis, so that further collapse is initially one-dimensional. We assume the collapse is nearly isothermal: for example, if the gas is nearly pure atomic hydrogen, collisional ionization and radiative recombination prevent the temperature from either rising much above or falling much below about $1\cdot5 \times 10^4$ °K. As the sphere flattens, the gravitational field gradually changes its properties from those pertaining to a sphere to those pertaining to a disk, and we may expect to appear, one by one, the analogues of some of the instabilities found by Hunter (14) for the infinitely thin, uniformly rotating disk. The modes of largest scale should appear first, as it is for them that the disk approximation first becomes adequate; but as the flattening increases, smaller modes appear, with growth-times that are systematically shorter as pressure becomes progressively ignorable. By analogy with Hunter's first axially-symmetric mode, the first instability that turns up can concentrate mass towards the axis, yielding the analogue of our centrally condensed disk, with a violently non-uniform rotation law. (There is also the possibility—apparently not so far examined in the literature—of the same instability but with opposite sign, yielding ultimately a model with most of the mass concentrated into a self-gravitating torus, rotating about an axis perpendicular to the plane through the axis of the torus.)

We have seen that if instabilities did not occur until the sphere has flattened into the uniformly rotating disk, then it is the smaller-scale modes with their faster growth rates that would dominate, and presumably transform the axially symmetric disk into a barred spiral. (Once such a non-axially symmetric equilibrium state has been constructed, its stability would have to be established against perturbations tending to concentrate the central mass.) The question then arises: what decides whether the collapsing primeval sphere becomes respectively a quasi-uniform disk or a centrally-condensed disk? In our picture, what is it that sometimes allows the first instability that appears during the collapse enough time to grow, before the analogues of Hunter's other instabilities can manifest themselves and alter sufficiently the zero-order state? Clearly, this problem can only be formulated at this stage—its solution must await treatment of the instabilities of the collapsing sphere. Perhaps spheres with greater initial central condensation may in fact end up as centrally condensed disks, and those more or less uniform, as quasi-uniform disks. There seems no reason at this stage to invoke radically different pictures for the formation of the two types.

At this point we digress to discuss the hypothesis of detailed angular momentum conservation—implicit also in Hunter's treatment of axially-symmetric modes. As long as the collapsing sphere is rotating nearly uniformly, there is no reason to question this constraint; but as soon as violent shears develop, one may ask whether any physical process is able at any stage of the collapse to redistribute angular momentum to any significant extent. Since Ω increases towards the centre, any such transfer of angular momentum will be outwards, so that relaxation of the constraint will only increase the central condensation, and thereby sharpen the divergence between the two disks. However, in fact the conservation hypothesis turns out to be a reasonable first approximation, at least after a fair degree of flattening has occurred.

Suppose—as is very plausible—the gas in the primeval sphere is turbulent, with a mean random velocity v_t and a mean eddy size λ, so that the kinematic eddy viscosity is $\simeq \lambda v_t$. The time for turbulent friction to destroy a gradient of angular velocity is roughly

$$\frac{1}{v_t\lambda}\left|\frac{\Omega}{\nabla^2\Omega}\right| \simeq \frac{\varpi^2}{v_t\lambda}. \tag{91}$$

This must be compared with the time of collapse of the sphere. If the turbulence is weak, gravitation dominates, and the time of collapse from a density ρ is $\simeq (G\rho)^{-1/2}$. The frictional time-scale (91) is longer than this by a factor

$$\left(\frac{\varpi}{\bar{z}}\right)\left(\frac{\varpi}{\lambda}\right)\left[G\rho^2\bar{z}^2/\rho v_t^2\right]^{1/2} = \left(\frac{\varpi}{\bar{z}}\right)\left(\frac{\varpi}{\lambda}\right), \tag{92}$$

where by (8) $\bar{z}$ is approximately the thickness of a disk supported normal to its plane by the weak turbulent pressure ρv_t^2. Since at any epoch λ is at the most of the order of the instantaneous z-dimension, the ratio (92) is always greater than unity, and increases steadily during the collapse.

On the other hand, suppose that the turbulence is always strong enough to be in approximate instantaneous balance with gravity, so that the actual thickness is $\simeq \bar{z} = v_t(G\rho)^{-1/2}$. This will be the case if the kinetic energy of collapse continuously feeds the turbulence, ultimately to be dissipated in small eddies. The collapse time is then not shorter than the decay-time of the turbulence; this can hardly exceed the corresponding time for ordinary incompressible turbulence (**22**), which is of order $\lambda/v_t = (\lambda/\bar{z})(G\rho)^{-1/2}$—certainly no longer than the free-fall time. We conclude that (92) is always a fair estimate of the ratio of the frictional time-scale to the collapse time: in the initial stages, the axial condensation may be assisted by some outward turbulent transfer of angular momentum, but in the later stages the turbulent friction is negligible. (In fact, the dissipation of kinetic energy in this highly supersonic flow can occur through shock formation rather than via a hierarchy of eddies; if the shocks are roughly axially symmetric, the dissipation of energy required for disk formation will not be associated with even a moderate transfer of angular momentum by eddy viscosity.)

Now suppose the primeval sphere has a large-scale magnetic field $\mathbf{H}$. If its direction is roughly parallel to the rotation axis, there is no interaction with the rotation field; but if perpendicular, Alfvén waves travelling for the time $(G\rho)^{-1/2}$ would tend to iron out non-uniformities of rotation over a distance

$$\frac{H}{(4\pi\rho)^{1/2}}(G\rho)^{-1/2} = \left(\frac{H^2}{4\pi G\rho^2}\right)^{1/2}. \tag{93}$$

By (8), this is at most of order $\bar{z}$—less, if the magnetic pressure is not the dominant force opposing the z-component of gravity. Thus if the field is able to interfere seriously with conservation of angular momentum—i.e. if the distance (93) is of the order of the radius of the cloud—then the field must be strong enough to prevent collapse into a disk, so that the present theory is in any case not relevant (cf. Section 10).

10. *The formation of stellar sub-systems.*—We now focus attention on the case for which axial condensation does occur along with the flattening of the sphere, so that ultimately a disk is formed with a roughly constant rotational velocity over its

bulk, as in the Milky Way or M31. As the sphere flattens, the degree of axial condensation steadily increases, transforming more gravitational energy into kinetic energy of ϖ-motion. For flattening and axial condensation to be permanent, the z-energy and the associated ϖ-energy must be dissipated and radiated away. Dissipation—e.g. through shock formation—will be efficient, as long as sub-condensations have not formed; but once proto-star clusters have separated out, they will tend to preserve the kinetic energy of the gas from which they are formed. In general terms, one may describe the process of star formation as the means by which a self-gravitating gas is able to cut off the continuous release and rapid dissipation of gravitational energy: as soon as blobs with small collisional cross-section are formed, dissipation of energy by direct collision becomes negligible, and only elastic scattering occurs between any two blobs.

Now suppose that as the sphere flattens and condenses axially, at each stage there occurs some proto-star formation. Condensations formed before centrifugal force becomes important will have ϖ- and z-motions large compared with their rotatory motion. Later condensations will have a mean rotatory motion associated with the axially condensed gas from which they form, and associated ϖ- and z-motions. The bulk of the gas breaks up into stars (the disk population (23)) only after it has become highly flattened, with the associated strong axial condensation—i.e. as described by the zero-order theory of this paper. After this, the remaining gas moves essentially under the gravitational field of the stars already formed—its mass is too small for its integrated self-gravitation to be important. Further, the gas now has a supply of energy in the ultra-violet light from the hot stars (24), to offset further dissipation: the situation is now as described in "contemporary" cosmical gas dynamics.

What we have outlined is essentially a dynamical picture of the formation of stellar sub-systems. The earliest condensations, with high z-energy, are identified with the old Type II population—e.g. the globular clusters; those with systematically less z-energy and z-dispersion are the intermediate sub-systems and the disk population. The point to be emphasized is that the increase of mean rotational speed with degree of flattening arises through the axial condensation of the flattened gas masses, with each element conserving its angular momentum about the galactic axis. Lindblad's explanation (25) of the differing mean rotations of the different sub-systems is quasi-static. If a sub-system with a high velocity dispersion did have the same mean rotation as the disk population, it would contain some stars able to escape from the galaxy; the consequent removal of angular momentum would cut down the mean rotational speed of the sub-system. The argument of the present Section is complementary: all the gas is gravitationally bound, and there is no reason why any one part should acquire an excess of kinetic energy at the expense of the rest. High rotational energies arise from the release of gravitational energy through axial condensation, and this occurs simultaneously with the flattening.

For a blob to be able to separate out from a gaseous background, it must not have too much spin about its centre of mass. Consider a spherical blob, of mass greater than the limit (set by the virial theorem) for gravitational binding at the assumed constant temperature, and with not too great internal turbulence. The centrifugal force due to its *orbital* motion is sufficient to prevent it from falling in indefinitely towards the centre of mass of the cloud—the axial condensation following on the flattening of the cloud fixes the limit of the permanent inward motion of any element. But if such a sphere of radius r separates out, the ratio of its *rotatory* centrifugal force to its own gravitational force is

$$\Omega^2 r/[G(4\pi/3)\rho r^3/r^2] = \Omega^2/(4\pi/3)G\rho.$$

At the instant when a uniform sphere just reaches centrifugal balance, this ratio is unity: thus the same condition that halts isotropic collapse of the hypothetical uniform primeval sphere, also prevents spherical sub-condensation. Under the subsequent flattening, however, this ratio becomes systematically smaller: flattening of the proto-galactic sphere by e.g. a factor 10 will allow a sub-sphere to form and *contract isotropically* by the same factor 10, before centrifugal force of spin becomes comparable with the self-gravitation of the sub-sphere.

The subsequent evolution of such a rapidly spinning sub-condensation could begin by repeating on a smaller scale the early evolution of the proto-galactic sphere—flattening, further sub-sub-condensation etc. There still remains the problem of how *stars*—quasi-spherical bodies, with masses of solar order—can reach main sequence densities in spite of their initially high spin. But for the present paper, this problem can be ignored; the important point is that an increase in ρ through motion parallel to the rotation axis does allow the formation of proto-star-clusters. What is not obvious is how this occurs before *all* the gas has formed a disk—i.e. why the spheroidal sub-systems do separate out. Perhaps Jeans-type pressure waves parallel to the axis cause local density increases by gravitational instability, followed by more or less isotropic collapse. It would be necessary to show that such local sub-condensations can separate out, in the time available, against the collapsing background (cf. Hunter's treatment of the analogous problem in spherical geometry (**26**)). It may be that only blobs with abnormally low vorticity can in fact separate out, retaining the kinetic energy associated with this time for formation. A prediction of the fraction of gas condensing at each stage would depend on hypotheses about the turbulence in the cloud, and is a formidable problem.

The picture of the formation of successive sub-systems during the collapse of a rotating primeval sphere implies that even the least flattened sub-systems—e.g. the globular clusters—should have a net orbital angular momentum. According to Hogg (**25**), Mayall estimates a difference of 80 or 100 km/sec between the motion of the Sun relative to the galactic centre and its motion relative to the globular clusters. No doubt the revised values (**6**) for the solar rotation and the solar distance will affect this estimate; however, as an example, we accept 100 km/sec, so that the system of globular clusters has an orbital angular velocity of

$$\simeq (10^7 \text{ cm/sec})/10 \text{ kpc} \simeq 3{\cdot}3 \times 10^{-16}/\text{sec}.$$

This should be compared with the angular velocity of the primeval sphere at the radius R_i at which isotropic contraction is halted. We adopt the model of Section 6, with V taken as 240 km/sec, $R' = 11$ kpc and $\mathfrak{M} = 9{\cdot}5 \times 10^{10}\odot$. By (64) $R_i \simeq (3/2)R' = 16{\cdot}5$ kpc, whence $\Omega_i \simeq (G\mathfrak{M}/R_i^3)^{1/2} \simeq 3{\cdot}2 \times 10^{-16}$. The closeness of the agreement is fortuitous; in fact, since the globular clusters form not a spherical but an oblate system, with an axial ratio of about 2 : 1 (**6**; **23**, p. 419), one would expect them to rotate more rapidly than the primeval sphere.

Support for the picture of a rapid collapse of the proto-galaxy comes from the work of Eggen, Lynden-Bell and Sandage (**27**), who have found correlations between the ultraviolet excesses of dwarf stars, and the eccentricities of their orbits and their z-motions. They estimate the scale of the collapse—measured from the formation of the oldest stars—as at least 10 in the radial direction and 25 in the z-direction. These figures are larger than the estimate of Section 6, but the ratio (64) relates the radius of the ultimate disk to that of the primeval sphere at the time of approximate centrifugal balance: the conclusion to be drawn is that these very old stars were formed before centrifugal force became important. The correlation found between large ultraviolet excess and small angular momentum suggests that these stars formed from matter nearer the centre of the proto-galaxy—where perhaps densities were somewhat higher than the mean—whereas the bulk of the gas, with higher angular momentum, had to flatten and condense axially before it could break up into stars.

An adequate theory of galaxy formation must also account for the ellipticals. Unless we ascribe to these a much lower angular momentum than to the disk-like galaxies, so that there is no obvious asymmetry leading to flattening, there must be some distinguishing physical feature which facilitates early star formation in ellipticals. Whereas in a disk-like galaxy most of the gravitational energy released during the collapse is dissipated, so that most of the stars formed have small z-motions, in ellipticals much more of the gravitational energy is preserved as kinetic energy of z-motion. One possibility is a strong magnetic field directed perpendicular to the rotation axis. A moderate degree of flattening will increase the magnetic pressure sufficiently to balance the z-component of gravity, while again centrifugal force prevents collapse in the two other directions (cf. Section 9). Such an oblate body could be in equilibrium even if at zero "temperature"—the thermal and turbulent energies are just small perturbations. Twisting of the magnetic field by non-uniform rotation would generate magnetic stresses, which would transport angular momentum, and so cause motion along the field lines (**28**). Once sufficiently dense blobs had been formed, they would tend to break off from the magnetic field and fall under the overall gravitational field. The essential difference from the earlier discussion is that the process takes place in its own time-scale—there is no demand that the sub-condensations form within the free-fall time of the proto-galaxy, since by hypothesis the magnetic field will not allow collapse of the quasi-uniform initial gas. It is at least possible that by the time the dense proto-star-clusters have formed as above and have begun to fall towards the galactic equator, their mutual collision cross-sections may be small enough for subsequent dissipation of energy to be negligible, so that they retain a high z-dispersion. As more and more of the gas condensed into stars, the gravitational force density on the remaining gas would decline, and the compressed magnetic field lines would leak out of the galaxy. Further discussion is beyond the scope of this paper.

Finally, we should note that there are problems underlying all of the discussion—the origin of the proto-galactic cloud, its angular momentum, and its magnetic field—problems that may very well be only partly separable from the cosmological problem.

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Dept. of Applied Mathematics and Theoretical Physics,
Free School Lane,
Cambridge:

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