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On the gap between the quadratic integer programming problem and its semidefinite relaxation

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Abstract. Consider the semidefinite relaxation (SDR) of the quadratic integer program (QIP): $\gamma := \max\ \{x^T Qx\ :\ x \in \{-1, 1\}^n\} \leq \min\ \{\mathrm{trace}(D)\ :\ D - Q \succeq 0\} =: \bar{\gamma}$ where Q is a given symmetric matrix and D is diagonal. We consider the SDR gap $\bar{\gamma} - \gamma$. We establish the uniqueness of the SDR solution and prove that $\gamma = \bar{\gamma}$ if and only if $\gamma_r := n^{-1} \max\ \{x^T VV^T x : x \in \{-1, 1\}^n\} = 1$ where V is an orthogonal matrix whose columns span the (r–dimensional) null space of $D - Q$ and where D is the unique SDR solution. We also give a test for establishing whether $\gamma = \bar{\gamma}$ that involves 2^{r-1} function evaluations. In the case that $\gamma_r < 1$ we derive an upper bound on γ which is tighter than $\bar{\gamma}$. Thus we show that 'breaching' the SDR gap for the QIP problem is as difficult as the solution of a QIP with the rank of the cost function matrix equal to the dimension of the null space of $D - Q$. This reduced rank QIP problem has been recently shown to be solvable in polynomial time for fixed r.

Key words. Quadratic integer programming – Semidefinite relaxation – Linear matrix inequalities – Zonotopes – Hyperplane arrangements

1. Notation

$\mathcal{R}^n$ denotes the space of real n-dimensional vectors, $\{-1, 1\}^n$ denotes the set of n-dimensional vectors whose entries are either 1 or -1 and $[-1, 1]^n$ denotes the set of n-dimensional vectors whose entries have absolute values less than or equal to 1. $\mathcal{R}^{n\times m}$ denotes the space of $n \times m$ real matrices. For $A \in \mathcal{R}^{n\times m}$, A^T denotes the transpose of A, $\mathrm{trace}(A)$ the sum of the diagonal elements of A and $\mathcal{N}(A)$ the null space of A. If $A = A^T \in \mathcal{R}^{n\times n}$, $\underline{\lambda}(A)$ denotes the smallest eigenvalue of A and we write $A \succeq 0$ if $\underline{\lambda}(A) \geq 0$ and $A \succ 0$ if $\underline{\lambda}(A) > 0$. Corresponding definitions apply to $\bar{\lambda}(A)$, $A \preceq 0$ and $A \prec 0$. The m-dimensional identity matrix is denoted by I_m and the $m \times n$ null matrix is denoted by $0_{m,n}$ (0_m if $m = n$) with the subscripts omitted if they can be inferred from the context. The null set is denoted by ϕ. If $\mathcal{F}$ is a space, $\dim \mathcal{F}$ denotes the dimension of $\mathcal{F}$. $A \in \mathcal{R}^{m\times n}$ is called orthogonal if $A^T A = I_n$. For $A = A^T \in \mathcal{R}^{n\times n}$ the spectral decomposition is the decomposition $A = U\Lambda U^T$ where $U \in \mathcal{R}^{n\times n}$ is orthogonal and $\Lambda \in \mathcal{R}^{n\times n}$ is a diagonal matrix of the eigenvalues of A. A "Schur type argument" refers

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to the fact that for $X = X^T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, with $X_{22} \succ 0$, then $X \succ 0$ if and only if $X_{11} - X_{12}X_{22}^{-1}X_{12}^T \succ 0$.

2. Introduction

In this paper we consider the classical NP–hard unconstrained quadratic integer programming (QIP) problem in $(-1, 1)$ variables

$$\text{(QIP)} \qquad \gamma := \max_{x \in \{-1,1\}^n} x^T Qx \tag{1}$$

for given $Q = Q^T \in \mathcal{R}^{n\times n}$ [10]. The QIP problem has many applications in combinatorial optimization. The QIP can be generalized to the zero–one QIP problem using a simple linear transformation and to problems involving a linear term using a homogenization procedure [20]. Other optimization problems, such as the Maximum–Cut Problem [6], can also be transformed to the QIP problem.

For any symmetric $Q, D \in \mathcal{R}^{n\times n}$ and $x \in \mathcal{R}^n$ the following identity

$$x^T Qx = -\left(\text{trace}(D) - x^T Dx\right) - x^T(D - Q)x + \text{trace}(D) \tag{2}$$

can be easily verified. Then for all $x \in \{-1, 1\}^n$ and diagonal D such that $D - Q \succeq 0$ we have $x^T Qx \le \text{trace}(D)$ and so $\gamma = \max_{x \in \{-1,1\}^n} x^T Qx \le \bar{\gamma}$ where

$$\text{(SDR)} \qquad \bar{\gamma} := \min_{\substack{D \text{ is diagonal} \\ D - Q \succeq 0}} \text{trace}(D) \tag{3}$$

so that $\bar{\gamma}$ is an upper bound on γ. The semidefinite relaxation problem in (3) is a semidefinite minimization which is a class of convex optimization problems. It can be solved efficiently using interior point algorithms [17]. The SDR of the QIP problem is well known in the literature; see [1, 4, 14, 20, 24], and [19] and the references therein. Here our contribution is an investigation of the optimality properties of the SDR with a view to reducing the relaxation gap $\bar{\gamma} - \gamma$. Other approaches that address this problem can be found in [11, 16, 26].

In Section 3 we establish the uniqueness of the solution of the SDR problem. Section 4 gives necessary and sufficient conditions for the absence of the relaxation gap. These, together with a recent result on reduced rank QIP problems [2], are used in Section 5 to derive an algorithm for reducing the relaxation gap. Section 6 outlines a polynomial–time algorithm for solving a reduced rank QIP problem; this involves the equivalent combinatorial problem of identifying the extreme points of a zonotope. Finally, Section 7 presents a numerical example which illustrates our results and outlines the main conclusions of the work.

3. Optimality properties of the SDR problem

The next result gives necessary conditions for the optimality of the SDR problem. They are needed to establish uniqueness and to investigate the relaxation gap.

Lemma 1. *Let D be a minimizer for the SDR problem so that D is diagonal,* $\operatorname{trace}(D) = \bar{\gamma}$ *and $D - Q \succeq 0$. Then*

1. $\dim \mathcal{N}(D - Q) \geq 1$, *(equivalently, $\underline{\lambda}(D - Q) = 0$) so that*

$$D - Q = [\, V \quad V_+ \,] \begin{bmatrix} 0_r & 0 \\ 0 & \Lambda_+ \end{bmatrix} [\, V \quad V_+ \,]^T \tag{4}$$

is a spectral decomposition for $D - Q$ for some orthogonal $[\, V \quad V_+ \,] \in \mathcal{R}^{n \times n}$ where $V \in \mathcal{R}^{n \times r}$, $r = \dim \mathcal{N}(D - Q) \geq 1$ and $\Lambda_+ \succ 0$.
2. *There does not exist diagonal $Z \in \mathcal{R}^{n \times n}$ such that* $\operatorname{trace}(Z) = 0$ *and $V^T Z V \prec 0$.*
3. *Every row of V has (Euclidean) norm at least $1/\sqrt{n}$. In particular, none of the rows of V is zero.*

Proof. 1. A proof can be found in Lemma 2.1 in [20]. Here we give an alternative algebraic proof. Suppose on the contrary that $D - Q \succ 0$ and let $\lambda_n := \underline{\lambda}(D - Q) > 0$. Pick ϵ such that $0 < \epsilon < \lambda_n$. Then $\underline{\lambda}(D - \epsilon I - Q) = \lambda_n - \epsilon > 0$ so that $D - \epsilon I - Q \succ 0$, $D - \epsilon I$ is diagonal and $\operatorname{trace}(D - \epsilon I) = \bar{\gamma} - n\epsilon$ contradicting the optimality of D.

2. Suppose on the contrary that such a Z exists and choose $\alpha > 0$ (sufficiently small) such that $\Lambda_+ - \alpha {V_+}^T Z V_+ \succ 0$ and

$$-V^T Z V - \alpha V^T Z V_+ (\Lambda_+ - \alpha V_+^T Z V_+)^{-1} V_+^T Z V \succ 0.$$

This is possible since $\Lambda_+ \succ 0$ and $V^T Z V \prec 0$. A Schur type argument gives

$$\begin{bmatrix} V^T \\ V_+^T \end{bmatrix} (D - \alpha Z - Q) \begin{bmatrix} V & V_+ \end{bmatrix} = \begin{bmatrix} -\alpha V^T Z V & -\alpha V^T Z V_+ \\ -\alpha V_+^T Z V & \Lambda_+ - \alpha V_+^T Z V_+ \end{bmatrix} \succ 0.$$

Thus $D - \alpha Z - Q \succ 0$. Moreover, $\operatorname{trace}(D - \alpha Z) = \operatorname{trace}(D)$ since $\operatorname{trace}(Z) = 0$. Thus $D - \alpha Z$ is optimal and this contradicts Part 1 and proves the result.

3. Suppose on the contrary that $V = [\, V_{11}^T \quad V_{21}^T \,]^T$ where $V_{11} \in \mathcal{R}^{1 \times r}$ and $\|V_{11}\|^2 < \frac{1}{n}$ (if necessary, rearrange the rows of V). Then

$$V_{11}^T V_{11} - \frac{1}{n} I_r \prec 0. \tag{5}$$

Define

$$Z = \begin{bmatrix} n - 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}.$$

Then Z is diagonal and $\operatorname{trace}(Z) = 0$. However, $V^T Z V = -I_r + n V_{11}^T V_{11} \prec 0$ from (5). This contradicts Part 2. □

The next result establishes the uniqueness of the SDR optimal solution.

Theorem 1. *The minimizer for the SDR problem is unique.*

Proof. Let D be a minimizer for the SDR problem and let $D - Q$ have a spectral decomposition (4). Suppose that Z is diagonal, $\mathrm{trace}(Z) = 0$ and

$$D - Z - Q \succeq 0 \tag{6}$$

so that $D - Z$ is another minimizer. We prove the theorem by establishing that $Z = 0$. Now, (6) and (4) imply that

$$\begin{bmatrix} V^T \\ V_+^T \end{bmatrix} (D - Z - Q) \begin{bmatrix} V & V_+ \end{bmatrix} = \begin{bmatrix} -V^T ZV & -V^T ZV_+ \\ -V_+^T ZV & \Lambda_+ - V_+^T ZV_+ \end{bmatrix} \succeq 0, \tag{7}$$

so that $V^T ZV \preceq 0$. It follows from Part 2 of Lemma 1 that $\bar{\lambda}(V^T ZV) = 0$. Let $1 \leq \dim \mathcal{N}(V^T ZV) = r_1 \leq r$. If $r_1 = r$ (i.e. $V^T ZV = 0$) we are done since in this case $ZV = 0$ from (7) and, since Z is diagonal and none of the rows of V is zero from Part 3 of Lemma 1, $Z = 0$. If $1 \leq r_1 < r$ we proceed as follows. Introduce an orthogonal transformation $U \in \mathcal{R}^{r\times r}$ if necessary on V so that $V := VU = [\,V_1 \;\; V_2\,]$ with $V_1 \in \mathcal{R}^{n\times r_1}$ and such that

$$V^T ZV = [\,V_1 \quad V_2\,]^T Z\, [\,V_1 \quad V_2\,] = \begin{bmatrix} 0_{r_1} & 0 \\ 0 & V_2^T ZV_2 \end{bmatrix} \tag{8}$$

with $V_2^T ZV_2 \prec 0$ (e.g. $U^T V^T ZVU$ is in spectral form). Then $ZV_1 = 0$ from (7) and (8). Next, we prove that $Z = 0$ by showing that none of the rows of V_1 is zero. Choose $\alpha > 0$, sufficiently small, such that $\Lambda_+ - \alpha V_+^T ZV_+ \succ 0$ and

$$-V_2^T ZV_2 - \alpha V_2^T ZV_+ (\Lambda_+ - \alpha V_+^T ZV_+)^{-1} V_+^T ZV_2 \succ 0.$$

This is possible since $\Lambda_+ \succ 0$ and $V_2^T ZV_2 \prec 0$. A Schur type argument gives

$$\Lambda_{1,+} := \begin{bmatrix} V_2^T \\ V_+^T \end{bmatrix} (D - \alpha Z - Q) \begin{bmatrix} V_2 & V_+ \end{bmatrix} = \begin{bmatrix} -\alpha V_2^T ZV_2 & -\alpha V_2^T ZV_+ \\ -\alpha V_+^T ZV & \Lambda_+ - \alpha V_+^T ZV_+ \end{bmatrix} \succ 0.$$

Define $V_{1,+} = [\,V_2 \;\; V_+\,]$. Then $(D - \alpha Z)$ is diagonal, $\mathrm{trace}(D - \alpha Z) = \bar{\gamma}$ and

$$(D - \alpha Z) - Q = [\,V_1 \quad V_{1,+}\,] \begin{bmatrix} 0_{r_1} & 0 \\ 0 & \Lambda_{1,+} \end{bmatrix} [\,V_1 \quad V_{1,+}\,]^T,$$

so that $D - \alpha Z$ is optimal and V_1 spans $\mathcal{N}(D - \alpha Z - Q)$. Hence none of the rows of V_1 is zero from Lemma 1 which, together with $ZV_1 = 0$ proves $Z = 0$. □

Remark 1. The SDR problem in (3) is related to the minimum trace factor analysis problem, for which the existence and uniqueness of the optimal solution was proved in [21–23] utilizing Lagrange multipliers. Here, our proof is algebraic and is given in a form suitable for analyzing the relaxation gap considered next. □

4. The gap between the QIP and SDR problems

In this section we derive necessary and sufficient conditions for the absence of a relaxation gap. Since the optimal solution D of the SDR problem is unique, $\mathcal{N}(D-Q)$ and $r = \dim \mathcal{N}(D-Q)$ are well defined and depend only on Q. We start with the following general necessary and sufficient conditions for $\gamma = \bar{\gamma}$.

Lemma 2. *Let D be the (unique) minimizer for the SDR problem and let $D-Q$ have a spectral decomposition (4). Then the following statements are equivalent:*

1. $\gamma = \bar{\gamma}$.
2. $\mathcal{N}(D-Q) \cap \{-1, 1\}^n \neq \phi$.
3. $Vy \in \{-1, 1\}^n$ *for some* $y \in \mathcal{R}^r$.

Proof. $(2 \rightarrow 1)$: Suppose that $x \in \mathcal{N}(D-Q) \cap \{-1, 1\}^n$. Then $\operatorname{trace}(D) - x^T Dx = 0$ and $x^T(D-Q)x = 0$. Substituting the optimal D in the global identity (2) gives $x^T Qx = \operatorname{trace}(D) = \bar{\gamma}$ and hence $\gamma = \bar{\gamma}$ since $\bar{\gamma}$ is an upper bound on γ.

$(1 \rightarrow 2)$: Suppose that $\gamma = \bar{\gamma}$. Consider again (2) when $x \in \{-1, 1\}^n$. Then the first term in the RHS of (2) is zero, while the second term $x^T(D-Q)x \geq 0$ in view of the condition $D - Q \succeq 0$. Thus $\gamma = \bar{\gamma}$ implies that $x^T(D-Q)x = 0$ and hence $(D-Q)x = 0$. Thus $x \in \mathcal{N}(D-Q)$.

$(2 \leftrightarrow 3)$: This follows from the fact that the columns of V span $\mathcal{N}(D-Q)$. □

Lemma 2 suggests a simple test for the absence of the relaxation gap.

Corollary 1. *Let all variables be as in Lemma 2. By rearranging the rows of V if necessary, let $V = [\, V_{11}^T \;\; V_{21}^T \,]^T$ with $V_{11} \in \mathcal{R}^{r \times r}$ nonsingular. Then $\gamma = \bar{\gamma}$ if and only if $V_{21}V_{11}^{-1}z \in \{-1, 1\}^{n-r}$ for some $z \in \{-1, 1\}^r$.*

Proof. Note that the required rearrangement of the rows of V is possible since V is orthogonal and hence has rank r. From Lemma 2, $\gamma = \bar{\gamma}$ if and only if there exists $y \in \mathcal{R}^r$ such that $[\, V_{11}^T \;\; V_{21}^T \,]^T y = [\, I_r \;\; (V_{21}V_{11}^{-1})^T \,]^T V_{11}y \in \{-1, 1\}^n$. That is, $\gamma = \bar{\gamma}$ if and only if there exists $y \in \mathcal{R}^r$ such that $V_{11}y \in \{-1, 1\}^r$ and $V_{21}V_{11}^{-1}V_{11}y \in \{-1, 1\}^{n-r}$ and the result follows by setting $z = V_{11}y$. □

Remark 2. The test involves enumerating all $z \in \{-1, 1\}^r$ and evaluating $V_{21}V_{11}^{-1}z$ for each. There are 2^r such z (actually, 2^{r-1} since only one of z and $-z$ need be tested). This is useful when r is sufficiently small for this calculation to be feasible. Our experience indicates that r is generally much smaller than n. □

The next result gives another necessary and sufficient condition for the absence of the relaxation gap. It will prove useful for reducing this gap.

Corollary 2. *Let all variables be as in Lemma 2. Then $\gamma = \bar{\gamma}$ if and only if $\max_{x \in \{-1, 1\}^n} x^T VV^T x = n$.*

Proof. Note that for any $x \in \{-1, 1\}^n$, $x^T x = n$ and $x^T V V^T x \leq x^T x = n$ since V is orthogonal. Thus $\max_{x \in \{-1,1\}^n} x^T V V^T x \leq n$. Suppose there exists $x \in \{-1, 1\}^n$ such that $x^T V V^T x = n$. Then, $\|[\, V \quad V_+ \,]^T x\| = \|x\| = \sqrt{n} = \|V^T x\|$. Hence $\|V_+^T x\| = 0$ so that $V_+^T x = 0$ and so $x \in \mathcal{N}(D - Q)$. Thus $\gamma = \bar{\gamma}$ from Lemma 2. Conversely suppose that $\gamma = \bar{\gamma}$. Then Lemma 2 implies there exists $y \in \mathcal{R}^r$ such that $Vy \in \{-1, 1\}^n$. Since $y^T V^T V y = n$ and $V^T V = I_r$, then $y^T y = n$. Setting $x = Vy$, we have $x^T V V^T x = y^T y = n$, as required. □

A simple sufficient condition for $\gamma = \bar{\gamma}$ is that $\dim \mathcal{N}(D - Q) = 1$.

Corollary 3. *Let all variables be as defined in Lemma 2. Then $\gamma = \bar{\gamma}$ if $r = 1$.*

Proof. Let $x = \sqrt{n} V \in \mathcal{R}^n$ so that $x \in \mathcal{N}(D - Q)$. We prove the lemma by showing that $x \in \{-1, 1\}^n$. Since $V^T V = 1$ and every row of V has norm at least $1/\sqrt{n}$ from Part 3 of Lemma 1, it follows that every row of V has norm equal to $1/\sqrt{n}$ and so $x \in \{-1, 1\}^n$ and the corollary follows from Lemma 2. □

Remark 3. An alternative proof of this result was given in [20] using properties of the trust region subproblem. □

5. Reducing the relaxation gap

In this section we consider the problem of reducing the relaxation gap. That is, we seek an upper bound on γ that is tighter than $\bar{\gamma}$.

It follows from Corollary 2 that $\gamma = \bar{\gamma}$ if and only if the maximum of the reduced rank QIP (RRQIP)

$$\text{(RRQIP)} \qquad \gamma_r := \frac{1}{n} \max_{x \in \{-1,1\}^n} x^T V V^T x \tag{9}$$

is equal to 1. Suppose that $\gamma_r < 1$. Then $\gamma < \bar{\gamma}$ and the question arises as to whether γ_r can induce an upper bound on γ that is tighter than $\bar{\gamma}$. The following result derives such a bound.

Lemma 3. *Let all variables be as in Lemma 2 and suppose that $\gamma_r < 1$. Then*

$$\gamma \leq \bar{\gamma} - n(1 - \gamma_r)\underline{\lambda}(\Lambda_+) < \bar{\gamma}. \tag{10}$$

Proof. Let $x \in \{-1, 1\}^n$. Then $n = x^T x = x^T V V^T x + x^T V_+ V_+^T x$, since $V V^T + V_+ V_+^T = I_n$ so that

$$x \in \{-1, 1\}^n \Rightarrow x^T V_+ V_+^T x = n - x^T V V^T x. \tag{11}$$

Now,

$$\gamma := \max_{x \in \{-1,1\}^n} x^T Q x = \max_{x \in \{-1,1\}^n} -\Big(\mathrm{trace}(D) - x^T D x\Big) - x^T (D - Q) x + \bar{\gamma} \tag{12}$$

$$= \max_{x \in \{-1,1\}^n} \bar{\gamma} - x^T (D - Q) x \tag{13}$$

$$= \max_{x \in \{-1,1\}^n} x^T \left(\frac{\bar{\gamma}}{n} I - (D - Q) \right) x \tag{14}$$

$$= \max_{x \in \{-1,1\}^n} x^T \, [\, V \quad V_+ \,] \begin{bmatrix} \frac{\bar{\gamma}}{n} I_r & 0 \\ 0 & \frac{\bar{\gamma}}{n} I - \Lambda_+ \end{bmatrix} [\, V \quad V_+ \,]^T x \tag{15}$$

$$\leq \max_{x \in \{-1,1\}^n} \frac{\bar{\gamma}}{n} x^T V V^T x + \left(\frac{\bar{\gamma}}{n} - \underline{\lambda}(\Lambda_+) \right) x^T V_+ V_+^T x \tag{16}$$

$$= n \left(\frac{\bar{\gamma}}{n} - \underline{\lambda}(\Lambda_+) \right) + \underline{\lambda}(\Lambda_+) \max_{x \in \{-1,1\}^n} x^T V V^T x = \bar{\gamma} - n \underline{\lambda}(\Lambda_+)(1 - \gamma_r) \tag{17}$$

where we have used

- the identity (2) in (12),
- the fact that $\bar{\gamma} = \text{trace}(D) = x^T D x$ for all $x \in \{-1, 1\}^n$ in (13),
- the fact that $x^T x = n$ for all $x \in \{-1, 1\}^n$ in (14),
- the spectral decomposition (4) in (15),
- the fact that $z^T Z z \leq \bar{\lambda}(Z) z^T z$ for all $z \in \mathcal{R}^n$ and $Z \in \mathcal{R}^{n \times n}$ in (16),
- (11) and (9) in (17).

This proves the first inequality in (10). The second inequality follows from the fact that $\underline{\lambda}(\Lambda_+) > 0$ and the assumption that $\gamma_r < 1$. □

It follows that the relaxation upper bound $\bar{\gamma}$ can be 'breached' provided that there exists a simple solution to the RRQIP in (9) (in fact, an upper bound on γ_r is sufficient provided it is less than 1). Although this is similar to the original QIP, the difference is that the matrix $V V^T$ in the cost function in (9) has a potentially low rank. The question then arises as to whether low rank quadratic integer programming problems are any easier to solve than full rank problems.

Remark 4. As pointed out to us by an anonymous reviewer, the main results of the paper can also be obtained by considering the dual of the SDR problem (3), given as:

$$\text{(DSDR)} \qquad \max_{\substack{X_{ii} = 1 \text{ for all } i \\ X \succeq 0}} \text{trace}(QX).$$

For example, from SDP duality we know that the SDR and DSDR problems have the same optimal value ($\bar{\gamma}$ in our notation) and that $\text{trace}((D - Q)X) = 0$ for the optimal D and X. Hence, if V is an orthogonal matrix spanning the null-space of $D - Q$, then X must be of the form $X = V \tilde{X} V^T$. Now, the SDR is exact (zero duality gap $\bar{\gamma} - \gamma$) if there exists $x \in \{-1, 1\}^n$ such that $X = x x^T$ is optimal for the DSDR problem. From the optimality conditions, this means that $x = V y$ and hence the SDR is exact if there exists $x = V y \in \{-1, 1\}^n$, proving the equivalence of (1) and (3) in Lemma 2. To prove Corollary 3 in the dual setting, suppose that x is a general vector in $\{-1, 1\}^n$. Decompose x along $\mathcal{R}(V)$ and $\mathcal{R}(V)^\perp$ (where V is an orthogonal matrix whose columns span $\mathcal{N}(D - Q)$ for D optimal), i.e. write $x = V y + w$ where $V^T w = 0$. We then have:

$$x^T V V^T x = \|y\|^2 = \|V y\|^2 = \|x\|^2 - \|w\|^2 = n - \|w\|^2 \leq n$$

with equality only if $w = 0$, i.e. $x = Vy$, which means that the SDR problem is exact. Thus the SDR is exact if and only if

$$\max_{x \in \{-1, 1\}^n} x^T V V^T x = n.$$

The bound given in Lemma 3 also follows using similar arguments. □

6. A polynomial time solution to the RRQIP problem

The RRQIP problem in (9) has been considered in a modified form (V replaced by V^T and $x \in \{0, 1\}^n$) in [2]. In our notation, [2] argue that the solution of the problem reduces to the enumeration of the extreme points of the zonotope

$$\mathcal{Z} = \{V^T x : x \in [-1, 1]^n\} \tag{18}$$

since

$$n\gamma_r = \max_{x \in \{-1, 1\}^n} x^T V V^T x = \max_{x \in [-1, 1]^n} x^T V V^T x = \max_{z \in \mathcal{Z}} z^T z$$

and the last maximization is achieved at an extreme point of $\mathcal{Z}$ since $\mathcal{Z}$ is convex. Different versions of the RRQIP problem have also been considered in [12, 13, 15] in connection with real and complex structured singular value problems in robust control applications. The problem of enumerating the extreme points of $\mathcal{Z}$ is well known, see [5, 7, 8, 18, 25], although the treatment is given in the dual setting of finding arrangements of hyperplanes. It is shown in [2] that this mapping gives the number of vertices of $\mathcal{Z}$ as $n_{\mathcal{Z}} \leq 2\sum_{i=0}^{r-1}\binom{n-1}{i}$ and allows their enumeration in an $O(n^{r-1})$ algorithm for $r \geq 3$ and $O(n^r)$ algorithm for $r \leq 2$.

Remark 5. The success of our method depends on two factors. The first involves the dimension r of $\mathcal{N}(D - Q)$, which defines the complexity of the RRQIP. Even if r is small (so that the RRQIP problem is tractable), the reduction in the upper bound $n(1 - \gamma_r)\underline{\lambda}(\Lambda_+)$ may be small if either $\underline{\lambda}(\Lambda_+)$ is small or γ_r is close to one (of course, if $\gamma_r = 1$, then $\gamma = \bar{\gamma}$). Moreover, the problem of determining the numerical nullity of $D - Q$ for large scale problems may be ill-posed. □

7. Numerical example and Conclusions

In this section the main results of the paper are illustrated by means of a simple numerical example. All results were obtained with a numerical accuracy of 10^{-12} (although for simplicity they are reported here truncated to four decimal places). Consider the symmetric positive-definite matrix:

$$Q = \begin{bmatrix} 3.2368 & 1.8290 & -0.8646 & -0.4653 & 1.8528 \\ 1.8290 & 5.2224 & -0.6692 & -1.4772 & -0.3376 \\ -0.8646 & -0.6692 & 4.7965 & -1.8219 & -1.6190 \\ -0.4653 & -1.4772 & -1.8219 & 1.4809 & 0.2846 \\ 1.8528 & -0.3376 & -1.6190 & 0.2846 & 4.6822 \end{bmatrix}$$

generated randomly using Matlab. Direct enumeration over $\{-1, 1\}^5$ gives $\gamma = 32.7409$. The semidefinite relaxation bound obtained using the Linear Matrix Inequality (LMI) toolbox was found as $\overline{\gamma} = 34.5436$ and the corresponding matrix D has diagonal elements $(7.1588,\ 7.6476,\ 8.3973,\ 3.4801,\ 7.8598)$. The eigenvalues of $D - Q$ were $\lambda(D - Q) = \{0.0000,\ 0.0000,\ 3.2898,\ 5.4270,\ 6.4081\}$ giving $r = 2$. The spectral decomposition of $D - Q$ shows that the rows of

$$V^T = \begin{bmatrix} 0.5287 & 0.3780 & -0.5131 & 0.1424 & 0.5423 \\ -0.2183 & -0.5561 & -0.2809 & 0.7378 & 0.1410 \end{bmatrix}$$

form an orthonormal basis of $\mathcal{N}(D - Q)$. The norms of the rows of V were: $\{0.5720,\ 0.6724,\ 0.5849,\ 0.7514,\ 0.5603\}$ which are all greater than $1/\sqrt{5}$, illustrating Part (3) of Lemma 1. The reduced rank integer quadratic optimisation problem $\max_{x \in \{-1,1\}^5} x^T V V^T x$ was next solved by direct enumeration. The maximum cost was $4.5773 < n = 5$ which indicates the existence of a non-zero duality gap $\bar{\gamma} - \gamma$. The improved bound on γ evaluated via (10) was obtained as 33.1528, narrowing the gap to approximately 23% of its original range.

To test the efficiency of polynomial-time algorithms in solving reduced-rank integer quadratic optimisation problems the "reverse-enumeration" algorithm [3, 9] was programmed and tested in Matlab. The algorithm identifies the extreme points of zonotope $\mathcal{Z}$ defined in (18); see [3, 9] for details. Our programme was extensively tested for small and medium–size problems and found to perform well. In this example the 10 extreme points of $\mathcal{Z}$ were identified correctly and are shown in Figure 1 along with the remaining 22 interior points. Thus the improved upper bound can be obtained using only five extreme–point evaluations.

There are some issues related to this work which we intend to pursue in the future. These include: (a) Investigation of numerical issues associated with the technique for

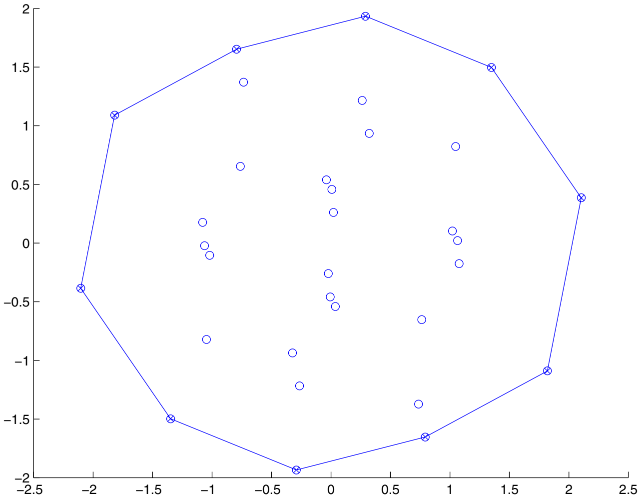

Fig. 1. Zonotope and its extreme points

large-scale problems; (b) Investigating potential implications of the reduced–rank integer quadratic theory in graph theory, especially in connection to the “max–cut” problem; (c) Detailed investigation of the reverse–enumeration algorithm and similar techniques for the efficient solution of the reduced–rank integer–quadratic problem; and, (d) Extension of our results for the efficient solution of the structured–singular value problem arising in robust control.

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