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ON THE GAUSS MAP OF NULL SCROLLS

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Abstract. The purpose of this paper is to characterize a class of non-degenerate ruled surfaces in $\boldsymbol{R}_1^3$, which are said to be null scrolls, satisfying the condition $\Delta\xi = A\xi$, where ξ denote their Gauss maps and $A \in gl(3, \boldsymbol{R})$.

1. Introduction

Let $\boldsymbol{H}^2(-1)$ (resp. $\boldsymbol{S}_1^2(1)$) be the 2-dimensional hyperbolic space of constant curvature -1 (resp. the 2-dimensional de Sitter space of constant curvature 1) in the 3-dimensional Minkowski space $\boldsymbol{R}_1^3$. Let M be a space-like surface (resp. time-like surface) in $\boldsymbol{R}_1^3$ and ξ a unit vector field normal to M. Then, for any point z in M, we regard $\xi(z)$ as a point in $\boldsymbol{H}^2(-1)$ (resp. $\boldsymbol{S}_1^2(1)$) by the parallel translation to the origin in the ambient space $\boldsymbol{R}_1^3$. The map ξ of M into $\boldsymbol{H}^2(-1)$ (resp. $\boldsymbol{S}_1^2(1)$) is called the *Gauss map* of M. In this paper, we give a geometric characterization for a class of non-degenerate ruled surfaces in $\boldsymbol{R}_1^3$ satisfying $\Delta\xi = A\xi (A \in gl(3, \boldsymbol{R}))$.

Let $\boldsymbol{R}^n$ denote the n-dimensional Euclidean space and $\boldsymbol{S}_0^{n-1}(1/r^2)$ the hypersphere of $\boldsymbol{R}^n$ centered at the origin with radius r. In the theory of minimal submanifolds in $\boldsymbol{R}^n$, Takahashi's theorem [11] is one of interesting results. The theorem gives an important relationship between the theory of minimal submanifolds in $\boldsymbol{S}_0^{n-1}(1/r^2)$ $(\subset \boldsymbol{R}^n)$ and that of eigenvalues of the Laplacian. From the viewpoint of this result, Chen [3], [4] generalized the notion of minimal submanifolds in $\boldsymbol{S}_0^{n-1}(1/r^2)$ to that of submanifolds of finite type in $\boldsymbol{R}^n$, and developed the theory of them greatly. Let M be an m-dimensional Riemannian manifold, x an isometric immersion of M into $\boldsymbol{R}^{m+1}$ and Δ the Laplacian of M. Generalizing the notion of minimal submanifolds in $\boldsymbol{S}_0^{n-1}(1/r^2)$ another way,

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Garay [8] also studied hypersurfaces in $\boldsymbol{R}^n$ satisfying the condition $\Delta x = Ax$, where A denotes a constant diagonal $(m+1) \times (m+1)$ matrix.

On the other hand, Chen and Piccinni [2] characterized n-dimensional submanifolds M in $\boldsymbol{R}^m$ satisfying $\Delta G = \lambda G$ $(\lambda \in \boldsymbol{R})$, where $G : M \to G(n, m) \subset \boldsymbol{R}^N$ $(N = {}_mC_n)$ denote the generalized Gauss maps of M. Baikoussis and Blair [1] also characterized surfaces in $\boldsymbol{R}^3$ satisfying $\Delta\xi = A\xi (A \in gl(3, \boldsymbol{R}))$, where ξ denote their Gauss maps.

As a Lorentzian version to [1], in [5] and [6], the first author has considered the Gauss maps ξ of space-likes or time-like surfaces in $\boldsymbol{R}_1^3$ satisfying the following equation

$$\Delta\xi = A\xi, \quad A \in gl(3, \boldsymbol{R}),$$

where $gl(3, \boldsymbol{R})$ denotes the set of all real 3×3-matrices. The first author has proved rigidity theorems only for surfaces of revolution and ruled surfaces along any non-null curve in $\boldsymbol{R}_1^3$.

In this paper let us consider a null curve α with null frame $F = \{X, Y, Z\}$. Then (α, F) is called a *framed null curve with frame F.* A non-degenerate ruled surface M in $\boldsymbol{R}_1^3$ along α parametrized by

$$x(s, t) = \alpha(s) + tY(s)$$

is called a *null scroll.* It is a time-like surface. The purpose of this paper is to give a geometric characterization for null scrolls satisfying $\Delta\xi = A\xi$ in terms of the function k_0 and the third curvature k_3 (See §2).

THEOREM. *Let M be a null scroll along the framed null curve with proper frame field. Then the Gauss map ξ of M satisfies*

$$\Delta\xi = A\xi, \quad A \in gl(3, \boldsymbol{R})$$

if and only if the mean curvature $H = (k_3/k_0)$ is constant. In this case, A is always equal to a scalar matrix.

A framed null curve (α, F) with the function $k_0 = 1$ and the first curvature $k_1 = 0$ is said to be a *Cartan framed null curve.* Moreover, for a Cartan framed null curve α with Cartan frame $F = \{X, Y, Z\}$ this kind of ruled surface is said to be a *B-scroll* (See Graves [9]).

COROLLARY. *Let M be a B-scroll along the framed null curve (α, F). Then the Gauss map ξ of M satisfies the condition*

$$\Delta\xi = A\xi, \quad A \in gl(3, \boldsymbol{R})$$

if and only if the third curvature k_3 *is constant.*

2. Null scrolls in the Minkowski 3-space

Let us review the terminology and fundamental properties for a null scroll M in $\boldsymbol{R}_1^3$. Here we refer to [7] and [9]. The purpose of this section is to represent the Laplacian Δ on M explicitly in terms of curvatures of the framed null curve, and to calculate the Gaussian curvature K and the mean curvature H of this null scroll.

$\boldsymbol{R}_1^3$ is by definition the 3-dimensional vector space $\boldsymbol{R}^3$ with the inner product of signature (1,2) given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

for any column vectors $x = {}^t(x_1, x_2, x_3)$, $y = {}^t(y_1, y_2, y_3) \in \boldsymbol{R}^3$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of $\boldsymbol{R}_1^3$ given by

$$e_1 = {}^t(1, 0, 0), \quad e_2 = {}^t(0, 1, 0), \quad e_3 = {}^t(0, 0, 1).$$

A basis $F = \{X, Y, Z\}$ of $\boldsymbol{R}_1^3$ is called a (*proper*) *null frame* if it satisfies the following conditions:

$$\langle X, X\rangle = \langle Y, Y\rangle = 0, \quad \langle X, Y\rangle = -1,$$

$$Z = X \times Y = \sum\nolimits_{i=1}^{3} \varepsilon_i \det[X, Y, e_i]e_i,$$

where $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$. Hence we obtain that

$$\langle X, Z\rangle = \langle Y, Z\rangle = 0, \quad \langle Z, Z\rangle = 1.$$

A vector V in $\boldsymbol{R}_1^3$ is said to be *null* if $\langle V, V\rangle = 0$.

Let $\alpha = \alpha(s)$ be a null curve in $\boldsymbol{R}_1^3$, namely, a smooth curve whose tangent vectors $\alpha'(s)$ are null. For a given smooth positive function $k_0 = k_0(s)$ let us put $X = X(s) = k_0^{-1}\alpha'$. Then X is a null vector field along α. Moreover, there exists a null vector field $Y = Y(s)$ along α satisfying $\langle X, Y\rangle = -1$. Here if we put $Z = X \times Y$, then we can obtain a (proper) null frame field $F = \{X, Y, Z\}$ along α. In this case the pair (α, F) is said to be a (*proper*) *framed null curve*. A framed null curve (α, F) satisfies the following, so called the *Frenet equation*:

$$\text{(2.1)} \qquad \begin{cases} X'(s) = k_1(s)X(s) + k_2(s)Z(s), \\ Y'(s) = -k_1(s)Y(s) + k_3(s)Z(s), \\ Z'(s) = k_3(s)X(s) + k_2(s)Y(s), \end{cases}$$

where $k_i = k_i(s)$, $i = 1,2,3$ are smooth functions defined by

$$k_1 = -\langle X', Y\rangle, \quad k_2 = \langle X', Z\rangle, \quad k_3 = \langle Y', Z\rangle.$$

The function k_i is called an *i-th curvature* of the framed curve. It follows from the fundamental theorem of ordinary differential equations that a framed null curve $(\alpha, F) = (\alpha(s), F(s))$ is uniquely determined by the functions k_0 (> 0), k_1, k_2, k_3 and the initial condition.

A framed null curve (α, F) with $k_0 = 1$ and $k_1 = 0$ is called a *Cartan framed null curve* and the frame field F is called a *Cartan frame*.

Let $(\alpha, F) = (\alpha(s), F(s))$ be a null curve with frame $F = \{X, Y, Z\}$. A ruled surface M along α parametrized by

$$x(s,t) = \alpha(s) + tY(s), \qquad s \in I,\ t \in J$$

is called a *null scroll*. It is a time-like surface. Furthermore, for a Cartan framed null curve α with Cartan frame $F = \{X, Y, Z\}$ the ruled surfaces is called a *B-scroll*.

From the Frenet equation (2.1), the natural frame $\{x_s, x_t\}$ on the null scroll M is obtained by

$$x_s = k_0 X - k_1 t Y + k_3 t Z, \quad x_t = Y,$$

and the first fundamental form g on M is given by

$$g = g_{11}(ds)^2 + 2g_{12}\, ds \cdot dt + g_{22}(dt)^2,$$

$$g_{11} = 2k_0 k_1 t + k_3^2 t^2, \quad g_{12} = -k_0, \quad g_{22} = 0.$$

Hence the null scroll M is a time like surface, namely, $\det g < 0$ everywhere on M. Let g^{ij} $(i,j = 1,2)$ denote the components of the inverse matrix g^{-1}:

$$g^{-1} = -\frac{1}{k_0^2}\begin{pmatrix} 0 & k_0 \\ k_0 & (k_3 t)^2 + 2k_0 k_1 t \end{pmatrix}. \tag{2.2}$$

One can show that the Laplacian Δ of M is expressed as

$$\begin{aligned} \Delta &= -\frac{1}{\sqrt{|\mathfrak{G}|}} \sum\nolimits_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{|\mathfrak{G}|}\, g^{ij} \frac{\partial}{\partial x_j}\right) \\ &= -\frac{1}{k_0}\left[\frac{\partial}{\partial s}\left(-\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial t}\left\{\left(-\frac{\partial}{\partial s}\right) - \left(\frac{k_3^2 t^2 + 2k_0 k_1 t}{k_0}\right)\frac{\partial}{\partial t}\right\}\right] \\ &= \frac{2}{k_0}\frac{\partial^2}{\partial s \partial t} + \frac{2}{k_0^2}(k_3^2 t + k_0 k_1)\frac{\partial}{\partial t} + \frac{1}{k_0^2}(k_3^2 t^2 + 2k_0 k_1 t)\frac{\partial^2}{\partial t^2}, \end{aligned} \tag{2.3}$$

where $\mathfrak{G}$ denotes the determinant of (g_{ij}).

Let ξ be the unit normal vector field on the null scroll M in $\boldsymbol{R}_1^3$ defined by

$$\xi = -\frac{k_3}{k_0} tY - Z. \tag{2.4}$$

Then, it is a space-like normal vector field to M. Thus, for any point x in M, we can regard $\xi(s)$ as a point in $\boldsymbol{S}_1^2(1)$ by the parallel translation to the origin in the ambient space $\boldsymbol{R}_1^3$. The map ξ of M into $\boldsymbol{S}_1^2(1)$ is called the *Gauss map* of M in $\boldsymbol{R}_1^3$. So, the components h_{ij}, $i, j = 1, 2$, of the second fundamental form of M in $\boldsymbol{R}_1^3$ are given by

$$h_{12} = g(x_{st}, \xi) = -k_3, \quad h_{22} = g(x_{tt}, \xi) = 0$$

since $x_{st} = x_{ts} = Y' = -k_1 Y + k_3 Z$, $x_{tt} = 0$. Accordingly, the Gaussian curvature K and the mean curvature H of the null scroll M is given by respectively

$$H = \frac{1}{2}\Sigma_{ij} g^{ij} h_{ij} = g^{12} h_{12} = \frac{k_3}{k_0},$$

and

$$K = \frac{-h_{11}h_{22} + h_{12}^2}{g_{11}g_{22} - g_{12}^2} = -\left(\frac{k_3}{k_0}\right)^2.$$

From the last formula we can assert

PROPOSITION. *A null scroll M along the framed null curve α in $\boldsymbol{R}_1^3$ is flat if and only if the third curvature k_3 of α vanishes identically.*

3. Proof of Theorem

In this section, let us prove the Theorem in the introduction.

Since $\xi = -(k_3/k_0)tY - Z$, by applying the Frenet equation (2.1), the Laplacian of ξ is calculated as follows:

$$\begin{aligned} \Delta\xi &= \frac{2}{k_0}\frac{\partial^2 \xi}{\partial s \partial t} + \frac{2}{k_0^2}(k_3^2 t + k_0 k_1)\frac{\partial \xi}{\partial t} + \frac{1}{k_0^2}(k_3^2 t^2 + 2k_0 k_1 t)\frac{\partial^2 \xi}{\partial t^2} \\ &= \frac{2}{k_0}\left\{\frac{1}{k_0^2}(k_0' k_3 - k_0 k_3' + k_0 k_1 k_3)Y - \frac{k_3{}^2}{k_0} Z\right\} + \frac{2}{k_0^2}(k_3^2 t + k_0 k_1)\left(-\frac{k_3}{k_0} Y\right) \\ &= -\frac{2}{k_0}\left(\frac{k_3}{k_0}\right)' Y + 2\left(\frac{k_3}{k_0}\right)^2 \xi. \end{aligned} \tag{3.1}$$

This implies that if the mean curvature $H = (k_3/k_0)$ is constant, then the Gauss map ξ of M is of 1-type:

$$\Delta\xi = 2H^2\xi,$$

namely, the Gauss map $\xi : M \to S_1^2(1)$ is harmonic (cf. [10]). Thus the Gauss map satisfied the following formula in Theorem

$$\Delta\xi = A\xi, \quad A \in gl(3, \boldsymbol{R}). \tag{3.2}$$

Now let us consider the converse. Assume that the Gauss map ξ of the null scroll M satisfies (3.2). Then, for the matrix A we have by (2.4), (3.1) and (3.2)

$$\frac{k_3}{k_0}tAY + AZ = 2\left\{\frac{1}{k_0}\left(\frac{k_3}{k_0}\right)' + \left(\frac{k_3}{k_0}\right)^3 t\right\}Y + 2\left(\frac{k_3}{k_0}\right)^2 Z$$

for the parameter t. Then we have

$$\frac{k_3}{k_0}AY = 2\left(\frac{k_3}{k_0}\right)^3 Y, \tag{3.3}$$

$$AZ = \frac{2}{k_0}\left(\frac{k_3}{k_0}\right)' Y + 2\left(\frac{k_3}{k_0}\right)^2 Z. \tag{3.4}$$

We put $k = (k_3/k_0)$. Differentiating (3.3) with respect to the parameter s, we get

$$k'AY + k(AY)' = 2(3k^2k'Y + k^3Y'). \tag{3.5}$$

On the other hand, the Frenet equation (2.1) gives

$$(AY)' = AY' = -k_1AY + k_3AZ.$$

From this together with (3.3), (3.4) and (3.5) we have $kk'Y = 0$, which implies that k^2 is constant. It completes the proof of Theorem.

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