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## Hajime Kaji <br> On the Gauss maps of space curves in characteristic p

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# On the Gauss maps of space curves in characteristic p 

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## §0. Introduction

In this article, we discuss the extensions of function fields defined by Gauss maps of space curves in positive characteristic $p$. As is well-known, if $p=0$, then the Gauss maps of space curves are always birational onto its image. However, if $p>0$, then this is not true.

Precisely speaking, our purpose is to investigate the sets $\mathscr{K}^{\prime}, \mathscr{K}_{\text {un }}^{\prime}$ and $\mathscr{K}_{\text {imm }}^{\prime}$ defined as follows: for a smooth connected complete curve $C$ over an algebraically closed field of positive characteristic $p$, we define $\mathscr{K}^{\prime}$ as the set of subfields $K^{\prime}$ of $K(C)$ such that there exists a morphism $t$ from $C$ to some projective space $\mathbb{P}$, birational onto its image, such that $l(C)$ is not a line in $\mathbb{P}$ and the extension of function fields defined by the Gauss map coincides with $K(C) / K^{\prime}$; replacing the word "morphism" above by "unramified morphism" and "closed immersion", we moreover define $\mathscr{K}_{\text {un }}^{\prime}$ and $\mathscr{K}_{\text {imm }}^{\prime}$, respectively.

Our main results are

## Theorem 0.1:

(a) If $C$ is an ordinary elliptic curve, then $\mathscr{K}_{\mathrm{imm}}^{\prime}$ contains any subfield $K^{\prime}$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K^{\prime}$ and the separable closure of $K^{\prime}$ in $K(C)$ is a cyclic extension over $K^{\prime}$ with degree indivisible by $p$.
(b) If $C$ is a supersingular elliptic curve, then

$$
\mathscr{K}_{\mathrm{imm}}^{\prime}= \begin{cases}\left\{K(C)^{p}, K(C)^{p^{2}}\right\} & \text { if } p=2 \\ \left\{K(C), K(C)^{p}\right\} & \text { otherwise } .\end{cases}
$$

(see Section 5)

Theorem 0.2:
(a) If $C$ is a curve of genus $g \geqslant 2$, then an arbitrary element $K^{\prime}$ of $\mathscr{K}_{\text {imm }}^{\prime}$ is of the form $K(C)^{p^{\prime}}$ for some integer $l \geqslant 0$.
(b) Moreover, for an integer $l>0$, let

$$
C \rightarrow C^{(p)} \rightarrow \cdots \rightarrow C^{\left(p^{\prime}\right)}
$$

be a sequence of Frobenius morphisms of C. Then, the following conditions are equivalent:
(1) $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K(C)^{p^{l}}$;
(2) there exists a rank 2 vector bundle $\mathscr{E}$ on $C^{\left(p^{l}\right)}$ such that $\mathscr{E}_{C^{(p)}}$ is stable and $\mathscr{E}_{C}$ is isomorphic to the bundle $\mathscr{P}_{C}^{1}(\mathscr{L})$ of principal parts of $\mathscr{L}$ of first order for some line bundle $\mathscr{L}$ on $C$, where $\mathscr{E}_{C^{(p)}}$ and $\mathscr{E}_{C}$ are the pull-backs of $\mathscr{E}$ to $C^{(p)}$ and $C$, respectively.
(see Corollaries 4.4 and 6.2)

We shall show also the following results: $\mathscr{K}^{\prime}$ (respectively, $\mathscr{K}_{\text {imm }}^{\prime}$ ) contains $K(C)$ if and only if $p \neq 2$ (see Corollary 2.3 ); for a proper subfield $K^{\prime}$ of $K(C), \mathscr{K}^{\prime}$ contains $K^{\prime}$ if and only if $K(C)$ is finite, inseparable over $K^{\prime}$ (see Corollaries 2.2 and 3.4 ); if $C$ is rational, then $\mathscr{K}^{\prime}=\mathscr{K}_{\text {imm }}^{\prime}$ (see Corollary 3.6); and, $\mathscr{K}_{\text {un }}^{\prime}=\mathscr{K}_{\text {imm }}^{\prime}$ for any $C$ (see Corollary 4.3).

As a corollary to our results, one can show that a smooth plane curve has separable degree 1 over the dual curve via the Gauss map (see Corollary 4.5 and [16, p. 342]; compare with [24, Proposition 4.2]), and that if $C$ is a Tango-Raynaud curve (see, for example, [2] or [27]), then $\mathscr{K}_{\mathrm{imm}}^{\prime \prime}$ contains $K(C)^{p}$ (see Corollary 6.5).

The fundamental results in our study are Theorems 2.1, 3.1 and 4.1, from which we shall deduce all the results above.

## §1. Gauss maps

Throughout this article, we shall work over an algebraically closed field $k$ of positive characteristic, denoted by $p$.

Let $C$ be a smooth, connected, complete curve defined over $k$. For a morphism $i$ from $C$ to a projective space $\mathbb{P}$, birational onto its image, let $V$ be a vector space $H^{0}\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)\right)$, and $\mathscr{P}_{C}^{1}\left(\imath^{*} \mathcal{O}_{\mathbb{P}}(1)\right)$ the bundle of principal parts of $\imath^{*} \mathcal{O}_{\mathbb{P}}(1)$ of first order on $C$ (see, for example, [14, IV, A], [19, §1] or [25, $\S \S 2$ and 6]). We have a natural map

$$
a^{1}: V \otimes_{k} \mathcal{O}_{C} \rightarrow \mathscr{P}_{C}^{1}\left(\imath * \mathcal{O}_{\mathrm{P}}(1)\right)
$$

which is surjective if and only if $l$ is unramified. The image of $a^{1}$ is a quotient bundle of $V \otimes_{k} \mathcal{O}_{C}$ of rank 2.

Let $\mathbb{G}$ be a Grassmann manifold consisting of 2-dimensional quotient spaces of $V$ with the universal quotient bundle

$$
V \otimes_{k} \mathcal{O}_{G} \rightarrow \mathscr{2}
$$

DEFINITION: By virtue of the universality of $\mathbb{G}$, one obtains from $a^{1}$ a morphism

$$
C \rightarrow \mathbb{G}
$$

We call this morphism the Gauss map of $C$ in $\mathbb{P}$ via $l$ (see, for example, $[3,2, \S 4],[4,1,(e)]$ or $[14$, IV, B]).

In particular, the image of $a^{1}$ is isomorphic to the pull-back of $\mathscr{2}$ to $C$, denoted by $\mathscr{2}_{c}$. We identify these bundles.

If one considers $\mathbb{G}$ to be consisting of lines in $\mathbb{P}$, then, for a general point $x$ of $C$, the image of $x$ in $\mathbb{G}$ under the Gauss map is corresponding to the tangent line to $l(C)$ at $l(x)$.

We always assume that $l(C)$ is not a line in $\mathbb{P}$, and denote by $C^{*}$ the image of $C$ in $\mathbb{G}$ under the Gauss map, which is called the dual curve of $C$ if $\mathbb{P}=\mathbb{P}^{2}$ (see, for example, [5, I, Exercise 7.3] or [14, I, C]).

DEFInition: For a curve $C$, we define $\mathscr{K}^{\prime}$ as the set of subfields $K^{\prime}$ of $K(C)$ as follows: there exists a morphism $i$ from $C$ to some projective space $\mathbb{P}$, birational onto its image, such that $t(C)$ is not a line in $\mathbb{P}$ and the extension $K(C) / K\left(C^{*}\right)$ defined by the Gauss map coincides with $K(C) / K^{\prime}$. Replacing the word "morphism" above by "unramified morphism" and "closed immersion", we moreover define $\mathscr{K}_{\text {un }}^{\prime}$ and $\mathscr{K}_{\text {imm }}^{\prime}$, respectively.

Remark: It is well-known that if $p=0$, then $\mathscr{K}^{\prime}=\{K(C)\}$.

## §2. Generic projections

Our first result is
ThEOREM 2.1: Let $\mathfrak{l}: C \rightarrow \mathbb{P}$ be a morphism birational onto its image as before, let $\Pi: \mathbb{P} \rightarrow \mathbb{P}_{1}$ be a projection of $\mathbb{P}$ from a general point in $\mathbb{P}$, and let $l_{1}: C \rightarrow \mathbb{P}_{1}$ be a composition of $\Pi$ with $l$. Let $C^{*}$ and $C_{1}^{*}$ be the images of $C$
under the Gauss maps via 1 and $t_{1}$, respectively, and let $K\left(C^{*}\right)_{s}$ and $K\left(C_{1}^{*}\right)_{s}$ be the separable closures of $K\left(C^{*}\right)$ and $K\left(C_{1}^{*}\right)$ in $K(C)$, respectively. If $\operatorname{dim} \mathbb{P} \geqslant 3$, then

$$
K(C) / K\left(C^{*}\right)_{s}=K(C) / K\left(C_{1}^{*}\right)_{s} .
$$

Proof: Denote by $P$ the centre of $\Pi$, let $\mathbb{G}$ and $\mathbb{G}_{1}$ be Grassmann manifolds of lines in $\mathbb{P}$ and $\mathbb{P}_{1}$, respectively, and let $\pi: \mathbb{G} \rightarrow \mathbb{G}_{1}$ be a rational map naturally induced by $\Pi$. Then, we have a commutative diagram


We note that, via the Plücker embeddings of $\mathbb{G}$ and $\mathbb{G}_{1}$ into some projective spaces, $\pi$ is compatible with a linear projection of the projective space, denoted by $\wedge \pi$. Writing $\sigma(P)$ for the subset of $\mathbb{G}$ consisting of lines in $\mathbb{P}$ passing through $P$, we see that the base locus of $\pi$ is equal to $\sigma(P)$, which coincides with the centre of $\wedge \pi$. Choose a smooth point $x$ of $C^{*}$, and consider the embedded tangent space $T_{x} C^{*}$ to $C^{*}$ at $x$. Counting the dimensions, one proves that $T_{x} C^{*}$ and $\sigma(P)$ do not meet for a general $P$, which implies that $C^{*}$ is separable over $C_{1}^{*}$ (see, for example, [28, §3]). This completes our proof.

Remark: Similarly, one can prove that if $\operatorname{dim} \mathbb{P} \geqslant 4$, then $K(C) / K\left(C^{*}\right)=$ $K(C) / K\left(C_{1}^{*}\right)$. Moreover, it can be shown that when $\operatorname{dim} \mathbb{P}=3$, this equality does not hold if and only if $l(C)$ is strange and not contained in any plane in $\mathbb{P}$, where $t(C)$ is called strange if there exists a point in $\mathbb{P}$ which lies on all the tangent lines at smooth points of $i(C)$ (see, for example, [5, IV, §3] or $[29$, II] $)$.

Corollary 2.2: For an element $K^{\prime}$ of $\mathscr{K}^{\prime}$, if $K(C)$ is separable over $K^{\prime}$, then $K(C)=K^{\prime}$.

Proof: For a morphism $i: C \rightarrow \mathbb{P}$ associated to $K^{\prime}$, using Theorem 2.1, one can reduce the problem to the case $\operatorname{dim} \mathbb{P}=2$. In this case, the result is known (see, for example, [9, §9.4] or [14, p. 310]).

Corollary 2.3: For a curve $C$, the following conditions are equivalent:
(1) $\mathscr{K}^{\prime}$ contains $K(C)$;
(2) $\mathscr{K}_{\mathrm{un}}^{\prime}$ contains $K(C)$;
(3) $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K(C)$;
(4) $p \neq 2$.

Proof: The implications (3) $\Rightarrow(2) \Rightarrow(1)$ are obvious.
For a morphism $t: C \rightarrow \mathbb{P}$, birational onto its image, we denote by $m$ the intersection multiplicity of $l(C)$ and a general tangent line to $l(C)$ at a general point. We note that $m+1$ is equal to the third gap of the linear system defining $i$. It follows from Theorem 2.1 and [6, Proposition 4.4] that $K(C) / K\left(C^{*}\right)$ is separable if and only if $p \neq 2$ and $m=2$. This proves the implication (1) $\Rightarrow$ (4).

Moreover, it follows from the Riemann-Roch theorem that if $t$ is a closed immersion defined by a complete linear system with sufficiently large degree, then $m=2$. Thus, the implication (4) $\Rightarrow(3)$ follows from Corollary 2.2.

Remark: Combining Theorem 2.1 with [6, Proposition 4.4], one can prove that if $K(C) / K\left(C^{*}\right)$ is not separable, then its inseparable degree is equal to $m$.

Remark: Using Theorem 2.1, one can prove that if the Gauss map of a curve $C$ in $\mathbb{P}$ via $t$ gives a separable extension $K(C) / K\left(C^{*}\right)$, then $t(C)$ is not strange in $\mathbb{P}$.

## §3. Gauss maps and ruled surfaces

The following plays a key role throughout the rest of this article.
Theorem 3.1: For a subfield $K^{\prime}$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K^{\prime}$, the following conditions are equivalent:
(1) $\mathscr{K}^{\prime}$ contains $K^{\prime}$;
(2) there exist a rank 2 vector bundle $\mathscr{E}$ on a curve $C^{\prime}$ with $K\left(C^{\prime}\right)=K^{\prime}$ and a morphism $h: C \rightarrow \mathbb{P}(\mathscr{E})$, birational onto its image, such that the extension $K(h(C)) / K\left(C^{\prime}\right)$ defined by the projection of $\mathbb{P}(\mathscr{E})$ over $C^{\prime}$ coincides with $K(C) / K^{\prime}$.
This is true for $\mathscr{K}_{\text {un }}^{\prime}$ (respectively, $\mathscr{K}_{\text {imm }}^{\prime}$ ) if one assumes moreover that $h$ is unramified (respectively, a closed immersion) in (2).

We first verify the implication (1) $\Rightarrow$ (2).
Lemma 3.2: Consider a trivial extension ( $\varepsilon_{0}$ ) and a unique non-trivial extension $\left(\varepsilon_{1}\right)$ of $\mathcal{O}_{C}$ by $\Omega_{C}^{1}$. Then, for a line bundle $\mathscr{L}$ on $C$, we have a natural extension

$$
0 \rightarrow \Omega_{C}^{1} \otimes \mathscr{L} \rightarrow \mathscr{P}_{C}^{1}(\mathscr{L}) \rightarrow \mathscr{L} \rightarrow 0
$$

which coincides with $\left(\varepsilon_{0}\right) \otimes \mathscr{L}$ if the degree of $\mathscr{L}$ is divisible by the characteristic $p$. Otherwise, the extension coincides with $\left(\varepsilon_{1}\right) \otimes \mathscr{L}$.

Proof: See $[11, \S 1]$.

For a morphism $t: C \rightarrow \mathbb{P}$, birational onto its image, combining the Euler sequence on $\mathbb{P}$ with the extension above, we have a commutative diagram

with exact rows. Let $C^{\prime}$ be the normalization of the image $C^{*}$ of $C$ under the Gauss map, and let $C_{0}$ be a section of the ruled surface $\mathbb{P}\left(\mathscr{Q}_{C}\right)$ over $C$ associated to the quotient $t^{*} \mathcal{O}_{\mathrm{p}}(1)$ of $\mathscr{Q}_{C}$, where $\mathscr{Q}_{C}$ is the image of $a^{1}$. We have a commutative diagram (see $[11, \S 1]$ )

where $X$ is an image of $\mathbb{P}\left(\mathscr{2}_{C}\right)$ in $\mathbb{P}$, and $f$ is a natural morphism induced by the Gauss map. Intuitively, $C_{0}$ is consisting of points of contact on $\mathbb{P}\left(\mathscr{Q}_{C}\right)$. We see that the induced morphism $C_{0} \rightarrow \mathbb{P}$ above coincides with $t$ via $C_{0} \simeq C$.

Now, let $\mathscr{E}$ be the pull-back $\mathscr{2}_{C^{\prime}}$ and $h$ the composition of $\left.f\right|_{C_{0}}$ with the isomorphism $C \rightarrow C_{0}$. Since $t$ is birational onto its image, so is $h$, and we have $K(h(C)) / K\left(C^{\prime}\right)=K(C) / K\left(C^{*}\right)$. Moreover, if $\iota$ is unramified (respectively, a closed immersion), then so is $h$. This completes the proof of $(1) \Rightarrow(2)$.

Remark: The curve $l(C)$ is strange if and only if the morphism $\mathbb{P}\left(\mathscr{D}_{C^{\prime}}\right) \rightarrow X$ above is not finite. It follows from the proof of [18, Proposition 3] that if $l$ is unramified, then $t(C)$ is not strange except for the case when $p=2$ and $l(C)$ is a conic.

The converse (2) $\Rightarrow$ (1) follows from

Lemma 3.3: Let $\mathscr{E}$ be a rank 2 vector bundle on a curve $C^{\prime}$, and let $h: C \rightarrow \mathbb{P}(\mathscr{E})$ be a morphism birational onto its image. If $h(C)$ is finite, inseparable over $C^{\prime}$ via the projection of $\mathbb{P}(\mathscr{E})$, then there exists a morphism $t: C \rightarrow \mathbb{P}$, birational onto its image, such that the extension $K(C) / K\left(C^{*}\right)$ defined by the Gauss map coincides with $K(C) / K\left(C^{\prime}\right)$. Moreover, if $h$ is unramified (respectively, a closed immersion), then so is $t$.

Proof: Take a sufficiently very ample line bundle $\mathscr{M}$ on $C^{\prime}$, and let $\varrho$ be an embedding of $\mathbb{P}(\mathscr{E})$ into a projective space $\mathbb{P}$ as a scroll defined by a complete linear system associated to $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^{*} \mathscr{M}$, where $\pi$ is the projection of $\mathbb{P}(\mathscr{E})$. We define $t$ to be a composition of $\varrho$ with $h$. Then, the image of each fibre of $\mathbb{P}(\mathscr{E})$ under $\varrho$ is a line in $\mathbb{P}$, tangent to $t(C)$ since $h(C)$ is inseparable over $C^{\prime}$. Denoting by $V$ the vector space $H^{0}\left(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^{*} \mathscr{M}\right)$, we have a commutative diagram

$$
\begin{aligned}
V_{\mathrm{P}(\mathscr{\delta})} & \rightarrow \mathcal{O}_{\mathrm{P}(\mathscr{E})}(1) \otimes \pi^{*} \cdot \mathscr{M} \\
\downarrow & \downarrow \\
V_{h(C)} & \rightarrow \mathcal{O}_{\mathrm{P}(\tilde{\delta})}(1) \otimes \pi^{*} \cdot \mathscr{M}_{h(C)}
\end{aligned}
$$

where all the arrows are surjective, $\varrho$ is determined by the upper arrow, and $l$ is determined by the lower arrow. Let $\mathscr{L}$ be a quotient line bundle of $(\pi \circ h)^{*} \mathscr{E}$ associated to $h$ (see, for example, [5, II, Proposition 7.12]). Taking the inverse image by $h$ and a morphism $\mathbb{P}\left((\pi \circ h)^{*} \mathscr{E}\right) \rightarrow \mathbb{P}(\mathscr{E})$, taking the direct image by the projection of $\mathbb{P}\left((\pi \circ h)^{*} \mathscr{E}\right)$ over $C$, we get

where all the arrows are surjective. We find that this diagram coincides with

$$
\begin{array}{ccc}
V_{C} & \rightarrow & \mathscr{Q}_{C} \\
\| & \downarrow \\
V_{C} & \rightarrow & \imath^{*} \mathcal{O}_{\mathbb{P}}(1)
\end{array}
$$

which is obtained from the commutative diagram under Lemma 3.2 for our $t$, and the extension of function fields defined by the Gauss map of $C$ via $t$ is equal to $K(C) / K\left(C^{\prime}\right)$.

This completes the proof of Theorem 3.1.

Corollary 3.4: The set $\mathscr{K}^{\prime}$ contains any subfield $K^{\prime}$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K^{\prime}$.

Proof: Let $K^{\prime}$ be a subfield of $K(C)$ as above, and let $C^{\prime}$ be a curve with $K\left(C^{\prime}\right)=K^{\prime}$. According to Lemma 3.5 below, there is a primitive element $\psi$
of $K(C)$ over $K^{\prime}$. From $\psi$ and a morphism $C \rightarrow C^{\prime}$ defined by $K(C) / K^{\prime}$, we obtain a morphism

$$
h: C \rightarrow \mathbb{P}^{1} \times C^{\prime} .
$$

Then, we find that $h$ is birational onto its image, and the result follows from Theorem 3.1.

Lemma 3.5: An arbitrary finite extension of function fields of dimension 1 over $k$ is simple.

Proof: Let $K / K^{\prime}$ be an extension as above, and let $K_{s}^{\prime}$ be the separable closure of $K^{\prime}$ in $K$. A finite, separable extension $K_{s}^{\prime} / K^{\prime}$ is simple. On the other hand, an arbitrary intermediate field of the purely inseparable extension $K / K_{s}^{\prime}$ is of the form $K^{p^{l}}$ for some integer $l>0$ (see, for example, [5, IV, Proposition 2.5]), whose number is finite. So, it follows from [10, Theorem 15, p. 55] that $K / K_{s}^{\prime}$ is simple. Thus, according to [10, Theorem 14, p. 54], the extension $K / K^{\prime}$ is simple.

From now on, we shall focus our attention on $\mathscr{K}_{\text {un }}^{\prime}$ and $\mathscr{K}_{\text {imm }}^{\prime}$.
Corollary 3.6: If $C$ is a rational curve, then $\mathscr{K}^{\prime}=\mathscr{K}_{\text {imm }}^{\prime}$.

Proof: For a proper subfield $K^{\prime}$ in $\mathscr{K}^{\prime}$, let $C^{\prime}$ be a curve with $K\left(C^{\prime}\right)=K^{\prime}$, and let $h$ be a graph morphism of $C \rightarrow C^{\prime}$ defined by $K(C) / K^{\prime}$ :

$$
h: C \rightarrow C \times C^{\prime}
$$

which is a closed immersion. The result follows from Theorem 3.1 and Corollary 2.3.

Remark: A similar argument to the above can be found in [26, Example 2.13].

Corollary 3.7: For an element $K^{\prime}$ of $\mathscr{K}_{\text {un }}^{\prime}$ (respectively, $\mathscr{K}_{\text {imm }}^{\prime}$ ), let $K_{1}^{\prime}$ be an intermediate field of $K(C) / K^{\prime}$. If $K(C) / K_{1}^{\prime}$ is not separable, then $\mathscr{K}_{\mathrm{un}}^{\prime}$ (respectively, $\mathscr{K}_{\text {imm }}^{\prime}$ ) contains $K_{1}^{\prime}$.

Proof: Let $C^{\prime}$ be a curve with $K\left(C^{\prime}\right)=K^{\prime}$. According to Theorem 3.1, we have a vector bundle $\mathscr{E}$ of rank 2 on $C^{\prime}$ and an unramified morphism $C \rightarrow \mathbb{P}(\mathscr{E})$, birational onto its image, such that $K(h(C)) / K\left(C^{\prime}\right)=K(C) / K^{\prime}$.

Let $C_{1}^{\prime}$ be a curve with $K\left(C_{1}^{\prime}\right)=K_{1}^{\prime}$, and let $\mathscr{E}_{1}$ be a pull-back of $\mathscr{E}$ by a morphism $C_{1}^{\prime} \rightarrow C^{\prime}$ defined by $K_{1}^{\prime} / K^{\prime}$. Using $h$ and a morphism $C \rightarrow C_{1}^{\prime}$ defined by $K(C) / K_{1}^{\prime}$, we get an unramified morphism $C \rightarrow \mathbb{P}\left(\mathscr{E}_{1}\right)$, birational onto its image, such that $K\left(h_{1}(C)\right) / K\left(C_{1}^{\prime}\right)=K(C) / K_{1}^{\prime}$. Then, the result for $\mathscr{K}_{\text {un }}^{\prime}$ follows from Theorem 3.1. If $h$ is a closed immersion, then so is $h_{1}$. This completes the proof.

Corollary 3.8: For an element $K^{\prime}$ of $\mathscr{K}_{\mathrm{un}}^{\prime}$ (respectively, $\mathscr{K}_{\mathrm{imm}}^{\prime}$ ), let $K_{1}^{\prime}$ be a subfield of $K^{\prime}$ such that $K^{\prime} / K_{1}^{\prime}$ is finite, purely inseparable, let $C^{\prime}$ and $C_{1}^{\prime}$ be curves with function fields $K^{\prime}$ and $K_{1}^{\prime}$, respectively, and let $f: C^{\prime} \rightarrow C_{1}^{\prime}$ be a morphism defined by $K^{\prime} / K_{1}^{\prime}$. Then, the following conditions are equivalent:
(1) $\mathscr{K}_{\text {un }}^{\prime}$ (respectively, $\mathscr{K}_{\text {imm }}^{\prime}$ ) contains $K_{1}^{\prime}$;
(2) there exists a rank 2 vector bundle $\mathscr{E}_{1}$ on $C_{1}^{\prime}$ such that

$$
f \mathscr{E}_{1} \simeq \mathscr{E}
$$

for some vector bundle $\mathscr{E}$ on $C^{\prime}$ associated to $K^{\prime}$ as in Theorem 3.1.

Proof: (1) $\Rightarrow$ (2). This follows from Theorem 3.1.
(2) $\Rightarrow$ (1). Let $h$ be the morphism $C \rightarrow \mathbb{P}(\mathscr{E})$ associated to $K^{\prime}$ as in Theorem 3.1, and let $h_{1}$ be the morphism $h$ followed by a morphism $\mathbb{P}(\mathscr{E}) \rightarrow \mathbb{P}\left(\mathscr{E}_{1}\right)$ induced from $f$. By virtue of Theorem 3.1, it suffices to show that $h_{1}$ is unramified, birational onto its image (respectively, a closed immersion). Since $f$ is purely inseparable, the morphism $\mathbb{P}(\mathscr{E}) \rightarrow \mathbb{P}\left(\mathscr{E}_{1}\right)$ is injective. So, it is sufficient to verify that a tangent vector to $h(C)$ in $\mathbb{P}(\mathscr{E})$ at each point of $h(C)$ is not vanished under $\mathbb{P}(\mathscr{E}) \rightarrow \mathbb{P}\left(\mathscr{E}_{1}\right)$. But, this is true since a fibre of $\mathbb{P}(\mathscr{E})$ over $C^{\prime}$ is tangent to $h(C)$, and mapped isomorphically into $\mathbb{P}\left(\mathscr{E}_{1}\right)$ by $\mathbb{P}(\mathscr{E}) \rightarrow \mathbb{P}\left(\mathscr{E}_{1}\right)$.

## §4. Calculation on ruled surfaces

The third of our fundamental results is

Theorem 4.1: Let $l: C \rightarrow \mathbb{P}$ be a morphism birational onto its image, let $g$ and $g^{\prime}$ be the genera of $C$ and $C^{\prime}$, respectively, and denote by $p_{a}\left(f\left(C_{0}\right)\right)$ the arithmetic genus of $f\left(C_{0}\right)$, with the same notations as before. If $l$ is unramified, then we have

$$
p_{a}\left(f\left(C_{0}\right)\right)=g=g^{\prime}
$$

We denote by $d$ the degree of $\imath^{*} \mathcal{O}_{\mathfrak{P}}(1)$, that is, the degree of $t(C)$ in $\mathbb{P}$, and by $d^{*}$ the degree of $C^{*}$ in $\mathbb{G}$ via the Plücker embedding, where $C^{*}$ is the image of $C$ in $\mathbb{G}$ under the Gauss map. We write $n$ for the degree of the extension $K(C) / K\left(C^{*}\right)$, that is, the degree of the Gauss map of $C$. We denote by $v$ the morphism $\mathbb{P}\left(\mathscr{2}_{C^{\prime}}\right) \rightarrow X$, and by $\gamma$ the degree of the cokernel of the map $a^{1}$.

One can easily prove the following (see, for example, [11, §1], [19, §3] or [25, §3]).

## Lemma 4.2: With the same notations as before, we have

$$
\begin{aligned}
2 g-2+2 d-\gamma & =\operatorname{deg} \mathscr{2}_{C} \\
& =n d^{*} \\
& =(\operatorname{deg} f)(\operatorname{deg} v)(\operatorname{deg} X) .
\end{aligned}
$$

To prove Theorem 4.1, using a generic projection of $\mathbb{P}$, we may assume that $\mathbb{P}=\mathbb{P}^{2}$.
Writing $D$ for the image $f\left(C_{0}\right)$, we first study the numerical class of $D$ in the ruled surface $\mathbb{P}\left(\mathscr{2}_{C}\right)$. For a general line $L$ in $\mathbb{P}^{2}$, set

$$
H:=v^{*} L
$$

in $\mathbb{P}\left(\mathscr{Q}_{C}\right)$. Then, $H$ is a section of $\mathbb{P}\left(\mathscr{Q}_{C}\right)$ over $C^{\prime}$. Since $X$ coincides with $\mathbb{P}^{2}$, in particular, it has degree 1 , it follows from Lemma 4.2 that $v$ has degree $d^{*}$. Thus, we have $\left(H^{2}\right)=d^{*}$. Since $\left.v\right|_{f\left(C_{0}\right)}$ is birational onto its image, we have $(D . H)=d$. On the other hand, denoting by $F$ the numerical class of a fibre of the projection of $\mathbb{P}\left(\mathscr{V}_{C}\right)$ over $C^{\prime}$, we obtain $(D . F)=n$ from the proof of $(1) \Rightarrow(2)$ in Theorem 3.1. Therefore, using Lemma 4.2, we find that

$$
D \equiv n H-(2 g-2+d-\gamma) F .
$$

Denoting by $K$ the numerical class of a canonical divisor of $\mathbb{P}\left(\mathscr{2}_{C}\right)$, we see

$$
K \equiv-2 H+\left(2 g^{\prime}-2+d^{*}\right) F .
$$

Now, it follows from the adjunction formula that

$$
2 p_{a}(D)-2=n\left\{\left(2 g^{\prime}-2\right)-(2 g-2)\right\}+\left(n d^{*}-2 d+n \gamma\right) .
$$

So, we obtain by Lemma 4.2 that

$$
\begin{aligned}
& \left(2 p_{a}(D)-2\right)-(2 g-2) \\
& \quad=n\left\{\left(2 g^{\prime}-2\right)-(2 g-2)\right\}+(n-1) \gamma
\end{aligned}
$$

We here have $p_{a}(D) \geqslant g \geqslant g^{\prime}, n \geqslant 1$, and $\gamma=0$ since $l$ is unramified. Therefore, the result follows from the equality above. This completes the proof of Theorem 4.1.

Corollary 4.3: $\mathscr{K}_{\text {un }}^{\prime}=\mathscr{K}_{\text {imm }}^{\prime}$.
Proof: It follows from Theorem 4.1 that $f\left(C_{0}\right)$ is smooth, so that the unramified morphism $h$ in Theorem 3.1 is a closed immersion.

Corollary 4.4: If $C$ has genus $g \geqslant 2$, then an arbitrary element $K^{\prime}$ of $\mathscr{K}_{\mathrm{un}}^{\prime}$ is of the form $K(C)^{p^{t}}$ for some integer $l \geqslant 0$.

Proof: Use Hurwitz's formula (see, for example, [5, IV, §2]).
Corollary 4.5: A smooth plane curve has separable degree 1 over the dual curve via the Gauss map.

Proof: If $g \geqslant 2$, then the result follows directly from Corollary 4.4. If $g=0$, then the result clearly follows. We hence assume that $g=1$, so that $d=3$. We obtain from Theorem 4.1 that $d^{*} \geqslant 3$, and the result follows from Corollary 2.2 and Lemma 4.2.

Remark: At the last part of the proof above, we do not need Theorem 4.1 because we always have $n \leqslant d$.

Remark: This result answers a question in [16, p. 342], and improves [24, Proposition 4.2].

## §5. Elliptic curves

This section is devoted to elliptic curves.
Theorem 5.1: Let $C$ be an ordinary elliptic curve in characteristic $p$, and let $K^{\prime}$ be a subfield of $K(C)$ such that $K(C)$ is finite, inseparable over $K^{\prime}$. If the separable closure of $K^{\prime}$ in $K(C)$ is cyclic over $K^{\prime}$ and the separable degree of $K(C) / K^{\prime}$ is not divisible by $p$, then $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K^{\prime}$.

Proof: Let $C^{\prime}$ be a curve with $K\left(C^{\prime}\right)=K^{\prime}$. By virtue of Theorem 3.1, it is sufficient to show that there exist a rank 2 vector bundle $\mathscr{E}$ on $C^{\prime}$ and a closed immersion $h: C \rightarrow \mathbb{P}(\mathscr{E})$ such that $K(h(C)) / K\left(C^{\prime}\right)=K(C) / K^{\prime}$.

Writing $s$ and $q$ respectively for the separable and inseparable degree of $K(C) / K^{\prime}$, we have an exact sequence of group schemes

$$
0 \rightarrow \mathbb{Z} / s \mathbb{Z} \times \mu_{q} \rightarrow C \rightarrow C^{\prime} \rightarrow 0
$$

Taking the dual, we get

$$
0 \rightarrow \mathbb{Z} / s \mathbb{Z} \times \mathbb{Z} / q \mathbb{Z} \rightarrow \hat{C}^{\prime} \rightarrow \hat{C} \rightarrow 0
$$

Since $s$ and $q$ are coprime, one may choose a single element $\mathscr{L}$ of $\hat{C}^{\prime}$ which generates the kernel above.

Now, we put

$$
\mathscr{E}:=\mathcal{O}_{C^{\prime}} \oplus \mathscr{L}
$$

We note that there exist only two sections of $\mathbb{P}(\mathscr{E})$ over $C^{\prime}$ such that the self-intersection number is equal to zero. It follows

$$
\mathscr{E}_{C} \simeq \mathcal{O}_{C}^{\oplus 2}
$$

and we have a commutative diagram

$$
\begin{array}{rr}
\mathbb{P}^{1} \times C \simeq \mathbb{P}\left(\mathscr{E}_{C}\right) \xrightarrow{f} \mathbb{P}(\mathscr{E}) \\
\downarrow & \downarrow \\
C & \longrightarrow C^{\prime}
\end{array}
$$

where $f$ is a morphism induced by $K(C) / K^{\prime}$.
We consider a constant section $C_{0}$ of $\mathbb{P}\left(\mathscr{E}_{C}\right)$ over $C$ which comes from neither of the two sections of $\mathbb{P}(\mathscr{E})$ mentioned above, and let us prove that $\left.f\right|_{c_{0}}$ is a closed immersion.

Let $C^{\prime \prime}$ be the normalization of the image $f\left(C_{0}\right)$. Using a base change by the naturally induced morphisms $C \rightarrow C^{\prime \prime} \rightarrow C^{\prime}$, we get a commutative diagram

where we denote by $C_{0}^{\prime \prime}$ a section of $\mathbb{P}\left(\mathscr{E}_{C^{\prime}}\right)$ over $C^{\prime \prime}$ naturally defined by the base change. Then, we find that $C_{0}$ is a pull-back of $C_{0}^{\prime \prime}$ and the selfintersection number of $C_{0}^{\prime \prime}$ is also equal to zero.

Now, suppose that $\left.f\right|_{C_{0}}$ is not birational. It follows that $C \rightarrow C^{\prime \prime}$ should not be birational. Therefore, $\mathscr{E}_{C^{\prime}}$ is not trivial and $C_{0}^{\prime \prime}$ comes from either of the two sections of $\mathbb{P}(\mathscr{E})$ specified above, so does $C_{0}$. This is a contradiction, and it follows that $\left.f\right|_{C_{0}}$ is birational.

It remains to show that $f\left(C_{0}\right)$ is smooth in $\mathbb{P}(\mathscr{E})$. Computing its arithmetic genus, one can easily deduce this result.

Theorem 5.2: If $C$ is a supersingular elliptic curve in characteristic $p$, then

$$
\mathscr{K}_{\mathrm{imm}}^{\prime}= \begin{cases}\left\{K(C)^{p}, K(C)^{p^{2}}\right\} & \text { if } p=2 \\ \left\{K(C), K(C)^{p}\right\} & \text { otherwise } .\end{cases}
$$

Proof: We show the inclusion $\subseteq$. By Theorem 3.1, we see that the converse follows from the existence of suitable vector bundles, which can be verified simultaneously below (use, for example, [1]).

Take an element $K^{\prime}$ of $\mathscr{K}_{\text {imm }}^{\prime}$, and let $t: C \rightarrow \mathbb{P}$ be a closed immersion associated to $K^{\prime}$. We employ the same notations as before.

Case: $p \neq 2$
If $K(C)=K^{\prime}$, then there is nothing to prove. We may assume that $K(C)$ is not separable over $K^{\prime}$.

Let $C_{1}, C^{\prime}$ and $C^{\prime \prime}$ be curves with function fields $K(C)^{p}, K^{\prime}$ and $K^{1 / 1 / p}$, respectively. We get a commutative diagram

$$
\begin{array}{ll}
C \rightarrow & C_{1} \\
\downarrow & \downarrow \\
C^{\prime \prime} & \rightarrow C^{\prime}
\end{array}
$$

Since $t$ is unramified, using Lemmas 3.2 and 4.2 , we have

$$
\mathscr{2}_{C} \simeq \imath^{*} \mathcal{O}_{p}(1)^{\oplus 2}
$$

and $C_{0}$ is a constant section of $\mathbb{P}\left(\mathscr{Q}_{C}\right)$ over $C$ via an isomorphism $\mathbb{P}\left(\mathscr{Q}_{C}\right) \simeq$ $\mathbb{P}^{1} \times C$.

I claim that $\mathscr{2}_{C_{1}}$ is indecomposable. Suppose that $\mathscr{2}_{C_{1}}$ is decomposed. Since $C_{1}$ is supersingular, a line bundle $\mathscr{L}$ on $C_{1}$ with $\mathscr{L}_{C} \simeq \imath^{*} \mathcal{O}_{\mathrm{P}}(1)$ is uniquely
determined. Thus, $\mathscr{2}_{C_{1}}$ should be of the form $\mathscr{L}^{\oplus 2}$ for some line bundle $\mathscr{L}$. This implies that the map

$$
C_{0} G \mathbb{P}\left(\mathscr{2}_{C}\right) \rightarrow \mathbb{P}\left(\mathscr{2}_{C_{1}}\right)
$$

is not birational onto its image. Then, we find a contradiction because $\left.f\right|_{c_{0}}$ is birational onto its image.

It follows that $\mathscr{2}_{C}$ is indecomposable. I claim moreover that $\mathscr{2}_{C}$, has even degree. Suppose that $\mathscr{Q}_{C^{\prime}}$ has odd degree. It clearly follows that $\mathscr{\mathscr { C }}_{C^{\prime}}$ should have also odd degree. Since $\mathscr{Q}_{C}$ is a direct sum of two line bundles with the same degree, $\mathscr{Q}_{C^{\prime}}$ must be indecomposable. Therefore, $\mathscr{V}_{C^{\prime}}$ and $\mathscr{2}_{c^{\prime}}$ are both indecomposable if they have odd degree. Since $C \rightarrow C^{\prime \prime}$ is isomorphic to $C_{1} \rightarrow C^{\prime}$ as a finite cover of abstract curves, $\mathscr{2}_{C}$ is indecomposable if and only if so is $\mathscr{2}_{C_{1}}$ (see, for example, [1, II]). This is a contradiction.

Since $C^{\prime \prime}$ is supersingular, we have

$$
\mathscr{2}_{C^{n}} \simeq \mathscr{M}^{\oplus 2}
$$

for some line bundle $\mathscr{M}$ on $C^{\prime \prime}$.
If $K(C) / K^{\prime}$ is not purely inseparable of degree $p$, again this contradicts the birationality of $\left.f\right|_{C_{0}}$ because the map

$$
\begin{gathered}
C_{0} \hookrightarrow \mathbb{P}\left(\mathscr{Q}_{C}\right) \\
\downarrow \\
\\
\\
\mathbb{P}\left(\mathscr{2}_{C}\right)
\end{gathered}
$$

should not be birational onto its image. This completes the proof.
Case: $p=2$
According to Corollaries 2.2 and 2.3, $K(C)$ is always inseparable over $K^{\prime}$. Since $l$ is unramified, we have an exact sequence
( $\varepsilon$ ) $0 \rightarrow \iota^{*} \mathcal{O}_{\mathrm{p}}(1) \rightarrow \mathscr{Q}_{C} \rightarrow \iota^{*} \mathcal{O}_{\mathrm{p}}(1) \rightarrow 0$.
It follows from Lemma 3.2 that ( $\varepsilon$ ) splits if and only if $d$ is even.
By the similar way to the former case, let us consider the curves $C_{1}, C^{\prime}$, $C^{\prime \prime}$ and the commutative diagram above.

Subcase: ( $\varepsilon$ ) splits
I first claim that $\mathscr{2}_{C_{1}}$ is indecomposable and has even degree. The indecomposability of $\mathscr{2}_{C_{1}}$ follows by the similar way to the former case. If $\mathscr{2}_{C_{1}}$ has odd
degree, then, according to [23, Theorem 2.16], $\mathscr{Q}_{C}$ should be indecomposable. This is a contradiction.

Next, I claim that $K\left(C_{1}\right) / K\left(C^{\prime}\right)$ is either trivial or not separable. Suppose the contrary. Since $C_{1}$ is supersingular, the degree of $C_{1} \rightarrow C^{\prime}$ should not be divisible by $p$, that is, odd. Since $Q_{C_{1}}$ is indecomposable with even degree, so is $\mathscr{2}_{C^{\prime}}$. Thus, $\mathscr{2}_{C^{\prime}}$ should be of the form $\mathscr{M}^{\oplus 2}$ for some line bundle $\mathscr{M}$ on $C^{\prime \prime}$ because $C^{\prime \prime}$ is supersingular, and this contradicts the birationality of $\left.f\right|_{C_{0}}$ as above.

Now, if $K\left(C_{1}\right)=K\left(C^{\prime}\right)$, then $K(C)$ is purely inseparable of degree $p$ over $K^{\prime}$. We hence consider the case when $C_{1} \rightarrow C^{\prime}$ is not separable. Then, one can take a curve $C_{2}$ between $C_{1}$ and $C^{\prime}$ with $K\left(C_{2}\right)=K(C)^{p^{2}}$, and $C^{\prime \prime \prime}$ between $C$ and $C^{\prime \prime}$ with $K\left(C^{\prime \prime \prime}\right)=K^{\prime 1 / p^{2}}$. We obtain a commutative diagram


I claim that $\mathscr{Q}_{C_{2}}$ has odd degree. Suppose that $\mathscr{Q}_{C_{2}}$ has even degree. Then, $\mathscr{2}_{C_{1}}$ should be decomposed since $C_{1}$ is supersingular. This is a contradiction.

Therefore, $\mathscr{2}_{C^{\prime}}$ is indecomposable with odd degree and $\mathscr{2}_{C^{\prime \prime \prime}}$ is of the form $\mathrm{N}^{\oplus 2}$ for some line bundle $\mathcal{N}$ on $C^{\prime \prime \prime}$.

If $K(C) / K^{\prime}$ is not purely inseparable of degree $p^{2}$, then we find a contradiction by the similar way to the last part of the former case. This completes our proof.

Subcase: ( $\varepsilon$ ) does not split
We see that $C_{0}$ is a unique section of $\mathbb{P}\left(\mathscr{Q}_{C}\right)$ over $C$ such that the selfintersection number is equal to zero, and $\mathscr{2}_{C}$ is indecomposable with even degree.

I claim that $\mathscr{2}_{C_{1}}$ has odd degree. Suppose that $\mathscr{Q}_{C_{1}}$ has even degree. Then, $\mathscr{2}_{C_{1}}$ should have a non-trivial extension

$$
\left(\varepsilon_{1}\right) 0 \rightarrow \mathscr{L} \rightarrow \mathscr{2}_{C_{1}} \rightarrow \mathscr{L} \rightarrow 0
$$

with some line bundle $\mathscr{L}$ on $C_{1}$ because $\mathscr{2}_{C_{1}}$ is indecomposable. We find that the pull-back of $\left(\varepsilon_{1}\right)$ to $C$ must coincide with ( $\varepsilon$ ). Therefore, similarly as above, this contradicts the birationality of $\left.f\right|_{c_{0}}$.

Hence, $\mathscr{2}_{C^{\prime}}$ and $\mathscr{Q}_{C^{\prime}}$ are indecomposable with odd and even degree, respectively, and we have $K\left(C_{1}\right)=K\left(C^{\prime}\right)$ since $\left.f\right|_{C_{0}}$ is birational onto its image. Thus, $K(C) / K^{\prime}$ is purely inseparable of degree $p$.

Remark: In the latter subcase above, we have not used the assumption that $C$ is supersingular.

## §6. Curves of higher genus

The main purpose of this section is to prove

Theorem 6.1: Let $C$ be a curve in characteristic $p$ with genus $g \geqslant 2$, let

$$
f: C \rightarrow C^{\prime}
$$

be a Frobenius morphism of $C$, and let $\mathscr{L}$ be a line bundle on $C$ such that the degree of $\mathscr{P}_{C}^{1}(\mathscr{L})$ is divisible by $p$. Then, the following conditions are equivalent:
(1) $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K(C)^{p}$;
(2) there exists a stable vector bundle $\mathscr{E}$ of rank 2 on $C^{\prime}$ such that

$$
\mathscr{P}_{c}^{\prime}(\mathscr{L}) \simeq f^{*} \mathscr{E}
$$

Combining Theorem 6.1 with Corollary 3.8 , we get
Corollary 6.2: Let $C$ be as above. For an integer $l>0$, let

$$
C \rightarrow C^{(p)} \rightarrow \cdots \rightarrow C^{\left(p^{\prime}\right)}
$$

be a sequence of Frobenius morphisms of C. Then, the following conditions are equivalent:
(1) $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K(C)^{p^{l}}$;
(2) there exists a vector bundle $\mathscr{E}$ of rank 2 on $C^{\left(p^{\prime}\right)}$ such that $\mathscr{E}_{C(p)}$ is stable and $\mathscr{E}_{C} \simeq \mathscr{P}_{C}^{1}(\mathscr{L})$ for some line bundle $\mathscr{L}$ on $C$.

To prove Theorem 6.1, let us use Theorem 3.1. For a rank 2 vector bundle $\mathscr{E}$ on $C^{\prime}$, to give a morphism $h: C \rightarrow \mathbb{P}(\mathscr{E})$, it is equivalent to give a quotient line bundle $\mathscr{L}$ of $f^{*} \mathscr{E}$, and there is a commutative diagram

where $\mathbb{P}(\mathscr{L})$ is a section of $\mathbb{P}\left(f^{*} \mathscr{E}\right)$ over $C$. Then, we have

Lemma 6.3: The following conditions are equivalent:
(1) $h$ is birational onto its image;
(2) the quotient line bundle $\mathscr{L}$ of $f^{*} \mathscr{E}$ does not come from any quotient line bundle of $\mathscr{E}$.

Proof: Since $\mathbb{P}(\mathscr{L})$ is purely inseparable of degree $p$ over $C^{\prime}$, one of the two cases below occurs: $\mathbb{P}(\mathscr{L})$ is purely inseparable of degree $p$ over $h(C)$, and $K(h(C))=K\left(C^{\prime}\right)$, so $h(C)$ is a section of $\mathbb{P}(\mathscr{E})$ over $C^{\prime} ; h$ is birational onto its image, and $K(h(C)) / K\left(C^{\prime}\right)$ is purely inseparable of degree $p$. This completes the proof.

Proposition 6.4: Let $C$ be a curve in characteristic p, let $f: C \rightarrow C^{\prime}$ be a Frobenius morphism of $C$, and let $\mathscr{L}$ be a line bundle on $C$ such that the degree of $\mathscr{P}_{C}^{1}(\mathscr{L})$ is divisible by p. Then, the following conditions are equivalent:
(1) $\mathscr{K}_{\text {imm }}^{\prime}$ contains $K(C)^{p}$;
(2) there exists a vector bundle $\mathscr{E}$ of rank 2 on $C^{\prime}$ such that

$$
\mathscr{P}_{C}^{1}(\mathscr{L}) \simeq f^{*} \mathscr{E}
$$

and the natural quotient line bundle $\mathscr{L}$ of $\mathscr{P}_{C}^{1}(\mathscr{L})$ does not come from any quotient line bundle of $\mathscr{E}$.

Proof: We first show the implication (1) $\Rightarrow(2)$ in case of $p \neq 2$. For a closed immersion $t: C \rightarrow \mathbb{P}$ associated to $K(C)^{p}$ in $\mathscr{K}_{\text {imm }}^{\prime}$, we have

$$
\mathscr{P}_{C}^{1}\left(t^{*} \mathcal{O}_{\mathbb{P}}(1)\right) \simeq f^{*} \mathscr{Q}_{C^{\prime}}
$$

In particular, $\mathscr{L}$ and $\imath^{*} \mathcal{O}_{\mathcal{P}}(1)$ have the same degree modulo $p$ since $p \neq 2$. It follows from Lemma 3.2 that

$$
\mathscr{P}_{C}^{\prime}(\mathscr{L}) \simeq \mathscr{P}_{C}^{\prime}\left(\imath^{*} \mathcal{O}_{\mathbb{P}}(1)\right) \otimes\left(\mathscr{L} \otimes \imath^{*} \mathcal{O}_{\mathbb{P}}(1)^{\vee}\right)
$$

where $\imath^{*} \mathcal{O}_{\mathrm{p}}(1)^{\vee}$ is the dual of $\imath^{*} \mathcal{O}_{\mathrm{p}}(1)$. Thus, there exists a vector bundle $\mathscr{E}$ on $C^{\prime}$ such that $\mathscr{P}_{C}^{1}(\mathscr{L}) \simeq f^{*} \mathscr{E}$, and it follows from Lemma 6.3 (1) $\Rightarrow$ (2) that $\mathscr{E}$ has the required property.

Next, we consider the case $p=2$. For a closed immersion $t: C \rightarrow \mathbb{P}$ such that the degree of $l(C)$ is even (respectively, odd), it follows from Corollaries 2.2 and 2.3 that the extension $K(C) / K\left(C^{*}\right)$ defined by the Gauss map via $t$ is not separable. Using Corollary 3.7 , we obtain a closed immersion $t_{1}: C \rightarrow \mathbb{P}_{1}$ such that the degree of $t(C)$ is even (respectively, odd) and the
extension defined by the Gauss map via $t_{1}$ is purely inseparable of degree $p$. In particular, this implies the condition (1) in case of $p=2$. Now, apply the argument in case of $p \neq 2$ above to this $t_{1}$. We obtain the required $\mathscr{E}$ in (2) when the degree of $\mathscr{L}$ is even (respectively, odd). Thus, we have proved that both (1) and (2) are always satisfied when $p=2$.

Finally, we show the implication (2) $\Rightarrow$ (1). For the natural quotient line bundle $\mathscr{L}$ of $f^{*} \mathscr{E}$ via $f^{*} \mathscr{E} \simeq \mathscr{P}_{C}^{1}(\mathscr{L})$, by virtue of Lemma 6.3 (2) $\Rightarrow$ (1), we have a morphism $h: C \rightarrow \mathbb{P}(\mathscr{E})$ such that $K(h(C)) / K\left(C^{\prime}\right)$ is purely inseparable of degree $p$. Computing the arithmetic genus of $h(C)$, we see that $h(C)$ is smooth, so that $h$ is a closed immersion. Then, the result follows from Theorem 3.1.

We here consider a special case of this situation.

Definition: For a curve $C$ with a Frobenius morphism $f: C \rightarrow C^{\prime}$, if $C^{\prime}$ has a line bundle $\mathcal{N}$ such that:
(1) $f^{*} \mathscr{N} \simeq \Omega_{C}^{1}$, and
(2) $f^{*}: H^{1}\left(C^{\prime}, \mathscr{N}^{\vee}\right) \rightarrow H^{1}\left(C, f^{*} \mathscr{N}^{\vee}\right)$ is not injective, then $C$ is called a Tango-Raynaud curve.

COROLLARY 6.5: If $C$ is a Tango-Raynaud curve in characteristic $p$, then $\mathscr{K}_{\text {imm }}^{\prime \prime}$ contains $K(C)^{p}$.

Proof: It follows from the assumption that there is a non-zero element $\xi$ of $H^{1}\left(C^{\prime}, \mathscr{N}^{\vee}\right)$ for some line bundle $\mathscr{N}$ on $C^{\prime}$ such that $f^{*}(\xi)=0$ in $H^{1}\left(C, f^{*} \mathscr{N}^{\vee}\right)$. Take $\mathscr{E}$ to be the extension of $\mathscr{N}$ by $\mathcal{O}_{C^{\prime}}$ determined by $\xi$, and $\mathscr{L}$ to be $f^{*} \mathcal{O}_{C^{\prime}}$ in the situation above. The result follows from Proposition 6.4.

Remark: If a line bundle $\mathscr{N}$ on $C^{\prime}$ with $\operatorname{deg} f^{*} \mathscr{N}=2 g-2$ satisfies the condition (2), then we have $f^{*} \mathscr{N} \simeq \Omega_{C}^{1}$.

Remark: Consider a curve $C$ of genus $g \geqslant 2$, and denote by $n(C)$ Tango's invariant of $C$ in [32, Definition 11]. Then, $C$ is a Tango-Raynaud curve in our sense if and only if

$$
n(C)=(2 g-2) / p
$$

and our definition above is equivalent to the ordinary one (see, for example, [2] or [27]).

Remark: Neither rational nor ordinary elliptic curve $C$ is a Tango-Raynaud curve. On the other hand, a supersingular elliptic curve $C$ is a TangoRaynaud curve, because $\mathcal{O}_{C^{\prime}}$ enjoys the required properties.

Remark: Any curve $C$ in characteristic 2 is a Tango-Raynaud curve if $C$ is neither rational nor ordinary elliptic, because the cokernel of a natural map $\mathcal{O}_{C^{\prime}} \rightarrow f_{*} \mathscr{O}_{C}$ is a line bundle having the required properties (see, for example, [2, Exemple i), p. 81] or [20, 2.2]).

Now, we go back to the proof of Theorem 6.1. For a rank 2 vector bundle $\mathscr{F}$ on a curve $C$ and a quotient line bundle $\mathscr{L}$ of $\mathscr{F}$, we put

$$
s(\mathscr{F}, \mathscr{L}):=2 \operatorname{deg} \mathscr{L}-\operatorname{deg} \mathscr{F} .
$$

Moreover, we put

$$
s(\mathscr{F}):=\inf _{\mathscr{L}} s(\mathscr{F}, \mathscr{L}),
$$

where $\mathscr{L}$ 's are quotient line bundles of $\mathscr{F}$. The value $s(\mathscr{F})$ is called the stability degree of $\mathscr{\mathscr { F }}$ because of the fact that $\mathscr{F}$ is stable if and only if $s(\mathscr{F})>0$.

Lemma 6.6: For a rank 2 vector bundle $\mathscr{F}$ on a curve, if $s(\mathscr{F})<0$, then a quotient line bundle $\mathscr{L}$ of $\mathscr{F}$ with $s(\mathscr{F}, \mathscr{L}) \leqslant 0$ is uniquely determined.

Proof: Taking a quotient line bundle $\mathscr{L}$ of $\mathscr{F}$ with $s(\mathscr{F}, \mathscr{L})<0$, we denote by $\mathscr{M}$ the kernel of $\mathscr{F} \rightarrow \mathscr{L}$. Let $\mathscr{L}^{\prime}$ be an arbitrary quotient line bundle of $\mathscr{F}$ with $s\left(\mathscr{F}, \mathscr{L}^{\prime}\right) \leqslant 0$, and denote by $\mathscr{M}^{\prime}$ the kernel of $\mathscr{F} \rightarrow \mathscr{L}^{\prime}$. Then, we have that $\operatorname{deg} \mathscr{M}>\operatorname{deg} \mathscr{L}$, and $\operatorname{deg} \mathscr{M}^{\prime} \geqslant \operatorname{deg} \mathscr{L}^{\prime}$. It follows that $\operatorname{deg} \mathscr{M}>\operatorname{deg} \mathscr{L}^{\prime}$, and $\operatorname{deg} \mathscr{M}^{\prime}>\operatorname{deg} \mathscr{L}$. Thus, we see that $\mathscr{L}=\mathscr{L}^{\prime}$ as a quotient line bundle of $\mathscr{F}$.

For a line bundle $\mathscr{L}$ on a curve $C$ with genus $g$, we have $s\left(\mathscr{P}_{C}^{1}(\mathscr{L}), \mathscr{L}\right)=$ $-(2 g-2)$. Moreover, if $g \geqslant 2$, then it follows from Lemma 6.6 that

$$
s\left(\mathscr{P}_{C}^{1}(\mathscr{L})\right)=-(2 g-2) .
$$

Remark: One can say that $\mathscr{P}_{C}^{1}(\mathscr{L})$ is the farthest from a stable bundle on $C$ even if $\mathscr{P}_{C}^{1}(\mathscr{L})$ is indecomposable (see, for example, [5, V, Theorem 2.12]).

Proposition 6.7: Let $f: C \rightarrow C^{\prime}$ be a finite morphism of curves, and let $\mathscr{F}$ be a rank 2 vector bundle on $C^{\prime}$. If the pull-back $f^{*} \mathscr{F}$ has a quotient line bundle $\mathscr{L}$ with $s\left(f^{*} \mathscr{F}, \mathscr{L}\right)<0$, then the following conditions are equivalent:
(1) the quotient line bundle $\mathscr{L}$ of $f^{* \mathscr{F}}$ does not come from any quotient line bundle of $\mathscr{F}$;
(2) $\mathscr{F}$ is stable.

Proof: It suffices to show the implication (1) $\Rightarrow$ (2) since the converse is obvious. Assume that there exists a quotient line bundle $\mathscr{L}^{\prime}$ of $\mathscr{F}$ such that $s\left(\mathscr{F}, \mathscr{L}^{\prime}\right) \leqslant 0$. It clearly follows that $s\left(f^{*} \mathscr{F}, f^{*} \mathscr{L}^{\prime}\right) \leqslant 0$. According to Lemma 6.6 , the quotient line bundle $\mathscr{L}$ must coincide with the pull-back $f^{*} \mathscr{L}^{\prime}$. This completes the proof.

Finally, Theorem 6.1 follows from Propositions 6.4 and 6.7.
Remark: In Theorem 6.1, if the degree of $\mathscr{L}$ is not divisible by $p$ and there exists a vector bundle $\mathscr{E}$ on $C^{\prime}$ such that $f^{*} \mathscr{E} \simeq \mathscr{P}_{C}^{\prime}(\mathscr{L})$, then $\mathscr{E}$ is stable.

Remark: In characteristic 2 , the line bundle $\imath^{*} \mathcal{O}_{\mathbb{p}}(1)$ is not necessarily related to $\mathscr{L}$ in Theorem 6.1 and the condition (1) is always satisfied, as we have seen in the proof of Proposition 6.4.

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